A Brief History of Kähler Geometry

by Shing-Tung Yau*

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* Department of Mathematics, Harvard University, Cambridge, Massachusetts, U.S.A.
E-mail: yau@math.harvard.edu

1. The Work of Riemann

Riemann was one of the founders of complex analysis, along with Cauchy. Riemann pioneered several directions in the subject of holomorphic functions:

1. The idea of using differential equations and variational principle. The major work here is the Cauchy-Riemann equation, and the creation of Dirichlet principle to solve the boundary value problem for harmonic functions. (It took several great mathematicians, such as David Hilbert, to complete this work of Riemann.)
2. He gave the proof of the Riemann mapping theorem for simply connected domains. This theory of uniformization theorems has been extremely influential. There are methods based on various approaches, including methods of partial differential equations, hypergeometric functions and algebraic geometry. A natural generalization is to understand the moduli space of Riemann surfaces where Riemann made an important contribution by showing that it is a complex variety with dimension $3g-3$.
3. The idea of using geometry to understand multivalued holomorphic functions, where he looked at the largest domain that a multivalued holomorphic function can define. He created the concept of Riemann surfaces, where he studied their topology and their moduli space. In fact, he introduced the concept of connectivity of space by cutting Riemann surface into pieces. The concept of Betti number was introduced by him for spaces in arbitrary dimension.
   The idea of understanding analytic problems through topology or geometry has far-reaching consequences. It influenced the later works of Poincaré, Picard, Lefstchetz, Hodge and others.

Important examples of Riemann's research is to use monodromy groups to study analytic functions. Such study has deep influence on the development of discrete groups in the 20th century. The Riemann-Hilbert problem was inspired by this and up to now, is still an important subject in geometry and analysis. The study of ramified covering and the Riemann-Hurwitz formula gave an efficient technique in algebraic geometry and number theory.

4. The discovery of Riemann-Roch formula over algebraic curve. The generalizations by Kodaira, Hirzebruch, Grothendieck, Atiyah-Singer have led to tremendous progress in mathematics in the twentieth century.
5. His study of period integrals related to Abel-Jacobi map and the hypergeometric equations:

$$z(1-z)y'' + \left[c-(a+b+c)z\right]y' - aby = 0.$$

6. The study of Riemann bilinear relations, the Riemann forms and the theta functions. During his study of the periods of Riemann surfaces, he found that the period matrix must satisfy period relations with a suitable invertible skew symmetric integral matrix which is called Riemann matrix later. Riemann realized that the period relations give necessary and sufficient condition for the existence of non-degenerate Abelian functions.
(According to Siegel [68], his formulation was incomplete and he did not supply a proof. Later, Weierstrass also failed to establish a complete proof despite his many efforts in this direction. Complete proofs were finally attained by Appell for the case $g = 2$ and by Poincaré for arbitrary g.)

It should be noted that Riemann spent most of his last four years in Italy because he contracted Tuberculosis and needed to avoid the severe winter in Germany. But as a result, he inspired a large group of differential geometers and projective algebraic geometers in Italy. Their works influenced the development of geometry and physics in the 20th century.

First of all, we should say that Riemann was the mathematician that brought us a new concept of space that was not perceived by any mathematician before him. I believe that was the reason that Gauss was so touched by his famous address on the foundations of geometry in 1854. I could not read German and was only able to read this address recently after it was translated into English. I was rather surprised that Riemann had rather liberal view about what geometry is supposed to be.

His guiding principle was nature itself (B. Riemann, *On the Hypotheses Which Lie at the Foundation of Geometry*, 1854.):

> The theorems of geometry cannot be deduced from the general notion of magnitude alone, but only from those properties which distinguished space from other conceivable entities, and these properties can only be found experimentally.... This takes us into the realm of another science – physics.

He thinks a deep understanding of geometry should be based on concepts of physics. And this is indeed the case as we experienced in the past century, especially in the past 50 years development of geometry. Although he was the one who introduced the concept of Riemann surface, which is the largest domain that a multivalued holomorphic function lives in, the precise modern concept was developed much later through the efforts of Klein, Poincaré and others.

While Felix Klein [38] already used atlas to describe Riemann surface, it has to wait until Hermann Weyl [80] who first gave the modern rigorous definition of Riemann surface, in terms of coordinate charts.

It was rather strange that a formal introduction of the concept of complex manifold was quite a bit later. Historically, generalization of one complex variable to several complex variables began by the study of functions on domains in $\mathbb{C}^n$. There were fundamental works of Levi, Oka, and Bergman.

The natural generalization of the concept of two dimensional surfaces to higher dimensional manifolds was done by O. Veblen and J. H. C. Whitehead in 1931–32. H. Whitney (1936) clarified the concept by proving that differentiable manifolds can be embedded into Euclidean space.

However, it was only in 1932 at the International Congress of Mathematicians in Zurich, did Caratheodory study "four dimensional Riemann surface" for its own sake. In 1944, Teichmüller mentioned "komplexe analytische Mannigfaltigkeit" in his work on "*Veränderliche Riemannsce Flächen*".

Chern was perhaps the first to use the English name "complex manifold" in his work [19].

The general abstract concept of almost complex structure was introduced by Ehresmann and Hopf in the 1940s. In 1948, Hopf [34] proved that the spheres S^4 and S^8 cannot admit almost complex structures.

The concept of Kähler geometry was introduced by Kähler [35] in 1933 where he demanded the Kähler form (which was first constructed by E. Cartan) to have a Kähler potential. Kähler had already observed special properties of such metric. He knew that the Ricci tensor associated to the metric tensor $g_{i\bar{j}}$ can be written rather simply as

$$R_{k\bar{l}} = -\frac{\partial^2}{\partial z_k \partial \bar{z}_l}(\log \det g_{i\bar{j}}),$$

which gave a globally defined closed form on the manifold.

He knew that it defines a topological invariant for the geometry. It defines a cohomology class independent of the metric. It was found later that, after normalization, it represents the first chern class of the manifold. The simplicity of the Ricci form allows Kähler to define the concept of Kähler-Einstein metric and he wrote down the equation locally in terms of the Kähler potential. He gave examples of the Kähler metric of the ball.

Slightly afterwards, Hodge developed Hodge theory, without knowing the work of Kähler, based on the induced metric from projective space to the algebraic manifolds. He studied the theory of harmonic forms with special attention to algebraic manifolds. The (p,q) decomposition of the differential forms have tremendous influence on the global understanding of Kähler manifolds. A very important observation is that the Hodge Laplacian commutes with the projection operator to the (p,q)-forms and hence the (p,q) decomposition descends to the de Rham cohomology. The theory was soon generalized to cohomology with twisted coefficients.

A very important cohomology with twisted coefficient is cohomology with coefficient in the tangent bundle or cotangent bundle, and their exterior powers. For the first cohomology with coefficient in tangent bundle, Kodaira and Spencer developed the fundamental theory of deformation of geometric structures, which gave far reaching generalization of the works of Riemann, Klein, Teichmüller and others on parametrization of complex structures over Riemann surfaces. They realize that the first cohomology with coefficient on tangent bundle, denoted by $H^1(T)$, parametrize the complex structure infinitesimally and that the second cohomology with coefficient on tangent bundle, denoted by $H^2(T)$, gives rise to obstruction to the deformation. The last statement was made very precisely by Kurinishi using Harmonic theory of Hodge-Kodaria. It describes the singular structure of the moduli space locally. Kodaira-Spencer studied how elements in $H^1(T)$ acts on other cohomology, which leads to study of variation of Hodge structures. The Hodge groups can be group in an appropriate way to form a natural filtration of the natural de Rham group. The Kodaira-Spencer map plays a very important role in understanding the deformation of such filtrations. Cohomology with coefficient of cotangent bundle or wedge product of cotangent bundle gives to hodge (p,q) forms. The duality of tangent bundle and cotangent bundle gives rise to something called mirror symmetry studied extensively in the last thirty years in relation to the theory of Calabi-Yau manifolds.

A very important tool in complex geometry was the introduction of Chern classes to complex bundles over a manifold and the representation of such classes by curvature of the bundle.

When Chern introduced the concept of Chern classes, he was influenced by the works of Pontryagin classes. In the course of defining Chern classes by de Rham forms given by symmetric polynomial of the curvature form, Chern defined the Chern connection for holomorphic bundles. He also proved that Chern classes of holomorphic bundles are represented by algebraic cycles on algebraic manifolds. This has been the major evidence of the Hodge conjecture: That every (p,p) class can be represented by algebraic cycles.

Chern proved that three different ways to define Chern classes are equivalent. In particular, he proved they are integral classes. Weil explained how they are related to Lie algebra invariant polynomials. Weil remarked that the integrality of Chern classes should play a role in quantum theory. Chern-Weil theory forms a bridge between topology, geometry, and mathematical physics.

The desire to generalize Riemann-Roch formula to higher dimensional algebraic manifolds has been relative slow, until the very powerful methods of sheaf theory was introduced by Leray, and important inputs were given by Weil, Borel and Serre. These basic techniques enabled Hirzebruch to arrive at the important Hirzebruch-Riemann-Roch formula in his 1954 paper [32], which can be stated in the following way:

$$\chi(V,E) = \int_V \mathrm{ch}(E)\mathrm{td}(V),$$

where E is a holomorphic vector bundle over a projective variety V.

The formulation of this formula by itself is remarkable. Hirzebruch developed the splitting principle and the theory of multiplicative sequences to find important power series of Chern classes. The Todd class is such a power series which is found by Hirzebruch to represent the arithmetic genus of the algebraic manifold, generalizing some old works of Todd in lower dimension. The Chern character $\mathrm{ch}(E)$ was invented by him to be a homomorphism from space of holomorphic vector bundles to even dimensional cohomology. The left-hand side of the formula is the Euler characteristic of cohomology with coefficient in E. This beautiful formula was observed by Serre when the algebraic manifold is two dimensional.

In the other direction, Kodaira was the first major mathematician who developed Hodge theory of harmonic forms right after its announcement by Hodge, and he generalized the theory of harmonic forms to manifolds with boundaries, where various boundary conditions have to be imposed.

Perhaps his most important work was his deep understanding that the Bochner argument in Riemannian geometry can be used to prove a vanishing theorem for cohomology classes under curvature condition of the manifold. He realized that the natural place for such vanishing theorem is to deal with cohomology with coefficient on bundle or sheaf. The vanishing theorem of Kodaira says that for positive line bundle L on a compact complex manifold M:

$$H^q(M, K_M \otimes L) = 0$$

for $q > 0$.

Coupled with the following theorem of Serre duality:

$$H^q(M, E) \cong H^{n-q}(M, K \otimes E^*),$$

Kodaira vanishing theorem implies that the Euler characteristic of cohomology with coefficients in a holomorphic vector bundle E with $E \otimes K^*$ positive, is simply the dimension of the group of holomorphic sections of E.

The above mentioned Hirzebruch-Riemann-Roch theorem then gives a formula to compute the dimension of the sections of the holomorphic bundle in terms of Chern numbers defined by Chern classes of the manifold and the bundle. This creates the most basic tool to understand algebraic manifolds.

Kodaira also showed that by blowing up points on the manifold, one can find enough holomorphic sections to separate points of the original manifolds and in fact gives an embedding of the manifold into complex projective space by using holomorphic sections of the bundle.

In particular, he proved that any Kähler manifold, whose Kähler class is defined by the Chern class of a holomorphic line bundle, can be holomorphically embedded into the complex projective space. The theorem of Chow then implies the manifold is in fact defined by an ideal of homogeneous polynomials, and hence an algebraic manifold.

What Kodaira has proved is one of the most spectacular theorems in geometry, and a glorious generalization of the work of Riemann on the condition of a complex torus to be abelian. More importantly the method of proving the Kodaira vanishing theorem has far reaching consequences in complex geometry. It was generalized to noncompact complex manifold, by various mathematicians including C. Morrey, Hörmander, Kohn, Vessintini, and others.

The Kodaira embedding theorem requires a high enough power of the ample line bundle to accomplish the embedding into projective space. An upper bound of this power of the line bundle is not clear from his argument.

Later on, Matsusaka [58, 59] (improved by Kollár-Matsusaka [39]) proved the very-ampleness of mL for an ample line bundle L on an n-dimensional projective variety X, when m is no less than a bound, depending only on the intersection numbers L^n and $K_X \cdot L^{n-1}$ on X.

In 1980s, Kawamata proved his famous basepoint freeness theorem about the pluricanonical systems of minimal models in [36, 37]. This is very important in the study of abundance conjecture. He proved that under the assumption that the numerical Kodaira dimension of a minimal variety X is equal to its Kodaira dimension, the pluricanonical system $|mK_X|$ is basepoint free for large m. This implies the basepoint freeness for minimal models of general type varieties. Later on, in a series paper of Miyaoka and Kawamata, they settled the proof of abundance conjecture for threefolds.

An important unsolved conjecture was proposed by Fujita in 1985, $mL + K_X$ is base-point free for $m \geq n+1$ and is very ample for $m \geq n+2$. Many mathematicians did important work on Fujita's conjecture, including Reider, Ein-Lazarsfeld, Kawamata, and many others. Demailly proved an effective formula for the bound on very ampleness [21]. Angehrn and Siu proved a quadratic bound for basepoint freeness [4].

There are many other contributioins to algebraic geometry made by Japanese algebraic geometers. Mori first introduced the ingenious idea of "bend and break" argument in his proof of Hartshorne conjecture [64]. This leads to his proof of cone theorem in birational geometry and had deep influences in minimal model program. Mukai introduced the Fourier-Mukai transform in 1981 [65]. This became an important tool in the study of derived categories.

2. Calabi Conjecture and Kähler-Einstein Metrics

The theorems by Kodaira, Matsusaka, Kawamata provide abundance of holomorphic sections for the holomorphic line bundle to embed the manifold into complex projective with higher dimension. An interesting important problem is the zero codimension case where we want to embed X to complex projective space with the same dimension. Hirzebruch and Kodaira [33] conjectured that every algebraic manifold that is homeomorphic to $\mathbb{CP}^n$ is actually biholomorphic to it. They used Hirzebruch-Riemann-Roch formula, but they could only treat the case of odd dimensional manifolds due to the indeterminacy of the sign of the first Chern class. The even dimensional case was finally settled by me [84] in 1976. While the arguments of Kodaria are based on Hilbert space theory, which depends on linear analysis, the argument that I used was nonlinear in nature. It has became an important new tool in complex geometry in the past forty years.

My argument depends on the existence of Kähler-Einstein metrics assuming the first Chern class is either positive, zero or negative. Although the Kähler-Einstein metric was already discussed by Kähler in his

1933 paper [35], where he wrote the equation explicitly, it wasn't until 1954 when Calabi [11] made a formal proposal to prove the existence of Kähler metric with prescribed volume form.

This could be used to proved the existence of Ricci-flat Kähler metric for any polarization if the first Chern class of the manifold is zero. Then Calabi asked the question when the first Chern class of the manifold is either negative or positive. The questions of Calabi were believed to be too good to be true in the old days, as nobody was able to construct an explicit Kähler-Einstein metric on any compact Kähler manifolds with no symmetries.

On 1976, I settled the cases when the first Chern class is either trivial or negative. (Aubin did the work independently for negative first Chern class.) I also considered the case when the manifold can have singularities, as was announced in my talk [87] at 1978 ICM in Helsinki.

2.1 Kähler-Einstein Metrics on Fano Manifolds

When the first Chern class is positive, it is called a Fano manifold. There are many interesting properties about Fano manifolds. Kollár, Mori and Miyaoka in [40] showed that smooth Fano varieties are rationally connected, in the sense that any two points are connected by a rational curve with (effectively) bounded degree. This implies an effective bound for the degree of the Fano n-fold, with respect to its anti-canonical bundle. Based on the work of Kollár and Matsusaka, it also implies that Fano n-folds form a bounded family.

In this case, there is an obstruction for the existence of Kähler-Einstein metric due to Matsushima [60]: the Lie algebra of the automorphism group of the manifold must be reductive. On [28], Futaki introduced his beautiful invariant defined on this Lie algebra. The Futaki invariant soon became a fundamental tool to study Kähler-Einstein metric on Fano manifolds. On the other hand, It took a long while to find a necessary and sufficient condition for the existence of Kähler-Einstein metric on Fano manifolds. Many people, including Calabi, was misled to believe that the non-existence of nonzero holomorphic vector fields is enough for the existence of Kähler-Einstein metric on Fano manifolds.

Right after I proved the Calabi conjecture on the existence of Kähler metric with prescribed volume form, I tried to work on the problem of the existence of Kähler-Einstein metric on Fano manifolds.

It is clear that based on the (nontrivial) higher order estimates that I had (independently due to Aubin for second order estimate) in the proof of the Calabi conjecture [86], the only missing point is some integral estimate of the Kähler potential. I found it is useful to set up the continuity argument

$$\det\left(g_{i\bar{j}}+\frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}\right)=\exp(h-tu)\det(g_{i\bar{j}}),$$

where $t=0$ correspond to a Kähler metric with positive Ricci curvature, as was given by the Calabi conjecture.

A simple calculation shows that the Ricci curvature of all members in the family have positive lower bound. This simplifies the analysis quite a bit as we have experiences with compact manifolds with Ricci curvature bounded from below by positive constant. In 1978, I returned to Stanford from my visit of Berkeley. At that time, I succeeded to convince Stanford mathematics department to hire Y.-T. Siu to come to Stanford from Yale.

We started to think about a proof of the existence of Kähler-Einstein metric by finding some integral estimate of the Kähler potential. Many estimates were found, but they are short of proving the existence of the metric. Some of those estimates can be sharpened if there are symmetries on the manifold, a procedure similar to the way that Moser sharpened the Trudinger inequality on the sphere when there is antipodal symmetry.

In the meanwhile, in 1977, I realized that Bogomolov [8] used the concept of stability of bundles to prove Chern number inequalities for algebraic surfaces which were sharpened by Miyaoka [61] and myself [84] independently. I started to believe there has to be links between the concept of stability with the existence of Hermitian Yang-Mills connections on bundles. When I proved the Calabi conjecture in 1976, I was at UCLA, and had had a fruitful discussion with David Giesecker, who is a great expert on the stability of bundle theory. He re-proved the Chern number inequality of Bogomolov over characteristic p.

The fact that a holomorphic bundle admits a Hermitian Yang-Mills connection if the bundle is polystable was proved by Uhlenbeck-Yau on arbitrary compact Kähler manifolds, and by Simon Donaldson for algebraic surfaces. Simpson observed that the proof of Uhlenbeck-Yau can be used to settle the case when there is a Higg's field. (Up to now, the argument of Uhlenbeck-Yau has been the only argument to prove existence of the Yang-Mills-Higgs equation.)

The Bogomolov inequality is optimal for general stable bundles. But it is not as sharp as the Miyaoka-Yau inequality when applied to the tangent bundle of the manifold. Hence I suspected that existence of Kähler-Einstein metric should be considered as a nonlinear version of the existence of Hermitian Yang-Mills connection, and the stability of bundle should be replaced by manifold stability. Therefore, only in 1980, I realized that the right condition for existence of Kähler-Einstein metric is the stability of the algebraic manifold.

I made the conjecture that the existence of Kähler-Einstein metric is equivalent to stability of manifold. I told all my graduate students about this conjecture, especially to Gang Tian who showed interest in the problem of Kähler-Einstein metric. But it took a long time to convince him of the validity of my conjecture.

There are many ways to define stability of manifolds including the concepts of Chow stability or Hilbert stability. I was not sure which one is correct. But I started to explore it with my students in my seminars. First of all, one had to make sure that algebraic stability, which is defined by embeddings of algebraic manifolds into complex projective space, can be linked to existence of Kähler-Einstein metric.

In fact, in order to link stability condition to algebraic geometry, I [88, p. 139] proposed to prove any Hodge metric on an algebraic manifold can be approximated by normalized Fubini-Study metric induced on the manifold through embedding of the manifold into complex projective space by high powers of an ample line bundle.

I asked Tian to follow this line of argument to finish the first step of my conjecture on the equivalence of stability of Fano manifolds with the existence of Kähler-Einstein metrics.

I suggested Tian to use my method with Siu [70] on the uniformization of Kähler manifolds to produce peak functions to achieve such a goal. (The purpose of that paper with Siu was also embedding of Kähler manifolds.)

The proof was reasonably transparent using technology from my paper with Siu. This became Tian's thesis at Harvard.

The method can be said to be an understanding of the works of Kodaira in the analytic setting. The work was carried out as I expected and it was strengthened by Catlin [13], Zelditch [92] and by Lu [53].

So, we know that we can approximate any Hodge metric by the induced metric of the projective embedding of the manifold into some complex projective spaces. However, there is an ambiguity due to the action of complex projective group. This is of course what geometric invariant theory studies.

It turns out that when I studied first eigenvalue of the Laplacian with Bourguignon and Peter Li [10], we need to find a good position for the embedding upon action by the projective group, which we called the balanced condition. It can be written in the following form:

$$\int_{\sigma(M)} \frac{z_i \bar{z}_j}{|z_0|^2 + \cdots + |z_N|^2} \omega^n = \frac{\mathrm{vol}(M)}{N+1} \delta_{ij}$$

for some $\sigma \in SL(N+1, \mathbb{C})$.

With such a condition, we can use the embedding to give a good estimate of the first eigenvalue in terms of the total volume and the degree defined by the Chern form wedge with the Kähler classes.

I suggested this condition as a starting point to my former student Luo to understand the concept of stability required to prove my conjecture on the existence of Kähler-Einstein metric based on stability.

Luo [54] found it effective to change the measure in the above formula defined by the induced measure of the ambient projective space. And it turns out that for a polarized manifold (M,L) if there exists a metric on L such that the Bergman function of L^k is constant for some k, then it is Chow stable.

A theorem of Shouwu Zhang [93] says that the existence of a unique balanced embedding is equivalent to the manifold being Chow-Mumford stable.

My conjecture that the existence of Kähler-Einstein metric is equivalent to stability was announced several times in several conferences and was explicitly written in my article [89] for the proceedings of UCLA conference on differential geometry in 1990.

I also communicated to Tian in detail on how to understand the Futaki invariant in this setting. The final conjecture of mine was solved recently by Chen-Donaldson-Sun [16, 17, 18] based on earlier works of Donaldson including the right algebro-geometric definition of K-stability.

According to Donaldson [23], a Fano manifold is called K-stable if all its non-trivial test configurations (which describe certain degeneration of Kähler manifolds by flat families) have positive Futaki invariants. For a test configuration $\mathcal{X} \to \mathbb{C}$ with $\mathbb{C}^*$ action, the Futaki invariant F_1 can be found from the total weight w_k of $\mathbb{C}^*$ acting on $H^0(X_0, L^k)$, using

$$\frac{w_k}{kd_k} = F_0 + F_1 k^{-1} + O\left(k^{-2}\right)$$

where d_k is the dimension of $H^0(X_0, L^k)$.

But the condition of K-stability is not easy to check, even in the case of surfaces. It would therefore be interesting to prove the existence of balanced condition for high power embeddings of a Fano manifold implies existence of Kähler-Einstein metrics. It is highly desirable to clarify the condition of K-stability so that it can be checked effectively.

2.2 Balanced Metric and Strominger System

Kähler-Einstein metrics are very useful in biregular geometry. We shall discuss it later. However, it cannot answer the important question whether an algebraic manifold is rational or not. The existence of Kähler metric is not a concept that is invariant under birational transformations, while the existence of balanced metric is. The concept of balanced metric was introduced by Michelson [56]. A Hermitian metric is called balanced if its Kähler form satisfied the following equation:

$$d\left(\omega^{n-1}\right) = 0$$

and it was proved by Alessandrini and Bassanelli [1] that its existence is invariant under birational transformations. However, there is much more freedom to deform a balanced metric than a Kähler metric. Just demanding that Ricci curvature equal to zero is not enough to determine a unique Balanced metric within the $(n-1,n-1)$ class.

On the other hand, balanced metric comes up naturally in the theory of Heterotic string theory in complex 3-dimension. And (this) balanced condition is related to the concept of supersymmetry. When there is a nonwhere vanishing top dimensional holomorphic 3-form, we look for an Hermitian metric which is balanced, and a stable holomorphic bundle (stable with respect to the balanced metric) whose second Chern Class is equal to the second Chern Class given by the Hermitian metric. Altogether, the following equations of the Strominger system need to be satisfied:

(1) $d(\|\Omega\|_\omega \Omega^2) = 0$
(2) $F_h^{2,0} = F_h^{0,2} = 0,\ F_h \wedge \omega^2 = 0$
(3) $\sqrt{-1}\partial\bar{\partial}\omega = \frac{\alpha'}{4}(\mathrm{tr}(R_\omega \wedge R_\omega) - \mathrm{tr}(F_h \wedge F_h))$

It provides a natural generalization of the Calabi-Yau geometry, which couples Hermitian metrics with Hermitian Yang-Mills theory. my belief is that the above system of equations can be solved when the obvious conditions hold. Jun Li and I [45] solved this system on any Calab-Yau manifold by making a deformation from the original Calabi-Yau metric.

For some intrinsically non-Kähler manifold, Fu and I [27] solved the Strominger system based on some ansätz for a 3-dimensional complex manifolds obtained from the Calabi-Eckmann construction. (The construction of the non-Kähler manifolds based on Calabi-Eckmann construction was also observed by Goldstein and Prokushkin [31].) It is a nonsingular complex torus bundle over the K3 surface. The proof of existence of nonsingular solution to the Strominger system given by Fu-Yau [27] is based on nontrivial estimates related to complex Monge-Ampère equations. In order to understand the significance of Strominger system, Tseng and I [76, 77], and later with Tsai [75], developed a new theory of symplectic cohomology which we expect to be dual to this kind of geometry.

Note that the existence of Ricci-flat Kähler metric provides a reduction of holonomic group to a subgroup of $SU(n)$, and according to the work of Candelas-Horowitz-Strominger-Witten [12], provides a supersymmetric model for vacuum solutions for Type II string theory. They called such manifolds to be Calabi-Yau manifolds. The Strominger system was introduced by Strominger to study Heterotic string where the vacuum is a warped product instead of a direct product.

2.3 Questions of Kähler-Einstein Metrics in Algebraic Geometry

There are several interesting consequences of the existence of Kähler-Einstein metric.

2.3.1 Understanding of Kähler-Einstein Metrics near Singularities

A corollary of the above mentioned theorem of Chen-Donaldson-Sun is that the K-stability of such manifold implies that the tangent bundle is stable with respect to the polarization given by the anti-canoncial line bundle. This is an interesting statement that is purely algebraic geometric, for which it would be nice to have a proof based only on algebraic geometry.

Also it implies that a K-stable Fano manifold is biregular to $\mathbb{CP}^n$ if the ratio of its two Chern numbers $c_2c_1^{n-2}$ and c_1^n is the same as $\mathbb{CP}^n$.

Another interesting question is the following: If a smooth algebraic manifold has Kodaira dimension either equal to the dimension of the manifold or $-\infty$, and if it is minimal in the sense in birational geometry and the ratio of two chern numbers $c_2c_1^{n-2}$ and c_1^n is the same as $\mathbb{CP}^n$, then the manifold is either $\mathbb{CP}^n$ or complex ball quotient.

For the case of general type, this is likely to be true. But it will be good to allow singular minimal models and in the case of singular algebraic manifolds, we need to define the Chern numbers suitably. This is related to the question of what is the best Kähler-Einstein metric on an algebraic manifold with singularity.

Let us look at the simplest case when the singularity is isolated. If the Kähler metric is complete at the singularity, it is not hard to prove that the Kähler-Einstein metric is unique. However, when it is not complete, it is not necessarily unique. It depends on the behavior of the volume form near the singularity.

What kind of volume forms are allowed? We need to know that the Ricci form of this volume form is positive definite and that the n-fold product of the Ricci form is asymptotic to this volume form near the singularity. (We may require that the metric defined by the Ricci form should have lower bound on its bisectional curvature.) It would be interesting to classify the asymptotic models of such volume forms. In principe, each of them will give rise to a canonical Kähler-Einstein metric with the given asymptotic behavior of the volume form.

It would be interesting to calculate the contribution of the singularity towards the Chern numbers. An important case is the canonical singularity appearing in the minimal model theory, which we recall below.

Suppose that Y is a normal variety and $f : X \to Y$ be a resolution of the singularities. Then

$$K_X = f^*(K_Y) + \sum_i a_i E_i$$

where the sum is over the irreducible exceptional divisors and the rational numbers a_i are called the discrepancies.

Then the singularities of Y are called canonical if $a_i \geq 0$ for all i and called terminal if $a_i > 0$ for all i.

A 3-dimensional singularity is terminal of index 1 if and only if it is an isolated composite DuVal (cDV) point in $\mathbb{C}^4$. A 3-dimensional terminal singularity of index $r \geq 2$ is a quotient of an isolated cDV point in $\mathbb{C}^4$.

The important question is to find a good Kähler-Einstein metric in a neighborhood of the cDV singularity which is invariant under the group action. For orbifold singularities, one can use those metrics obtained by pushing down from the nonsingular model before quotient by the group. On the other hand, there may be some other volume form that satisfies the above properties that is distinct from the orbifold construction. The complicated situation is the case that the Ricci form of the volume form may define a metric that is partially going to complete and partially degenerate at the singular point.

It will be important to construct nice model volume form in a neighborhood of the canonical singularities of the manifold whose Ricci form can give rise to a nice metric which is asymptotically Kähler-Einstein.

2.3.2 Kähler-Einstein Metrics on Quasiprojective Varieties and Sasakian-Einstein Metrics

In my first paper on the Calabi conjecture, we know that given any Kähler class, we can find a Kähler metric which may degenerate along a divisor whose volume is given by the unique volume defined by the divisor of the pluricanonical sections. How to calculate the second Chern class related to this divisor would be important. The Chern numbers calculated by the degenerate Ricci flat metrics should have residue from the divisor. It would be important to calculate this contribution.

The noncompact version of complete Ricci-flat metric is more complicated, partially because we lack of a good model space to build a good ansatz. At 1978 ICM in Helsinki, I [87] announced the way to build complete noncompact Ricci-flat manifolds.

I conjectured that the manifold can be written as the complement of a divisor D of a compact Kähler manifold M. (It was pointed out by Michael Anderson et al. [3] that we should assume the finiteness of the topology of the manifold, otherwise Taub-NUT manifolds can provide counterexamples.)

My program was to take D to be an anticanonical divisor of M which cannot be contracted to a codimension two subvarity. There will be a holomorphic volume form on M which has poles along D. I expect that this is close enough to provide a necessary and sufficient condition for $M \backslash D$ to admit complete Kähler metric with zero Ricci curvature. When D is nonsingular, I have worked out the program. The details were written up with Tian in two papers [72, 73].

However, when D has normal crossing singularities, the problem is unsolved, largely because we do not have a good model of complete Ricci-flat metric in a neighborhood of D when D has singularity. An important and interesting case is to allow the complete Kähler metric to have certain type of singularities. Besides quotient singularity, we can allow cone singularity.

In the last case, the interesting examples are metric cones over a Sasakian-Einstein manifold. Important progress was made by Gauntlett, Martelli, Sparks and myself starting with [29] on the existence of Sasakian-Einstein metrics. In [29, 55] we gave several obstructions to their existence by studying the Einstein-Hilbert functional restricted to the space of Sasakian-Einstein metrics where it becomes essentially the volume functional. It can further be shown to be a functional of the Reeb vector field associated to the Sasakian structure alone.

We obtain a useful obstruction from the Lichnerowicz bound on the Laplacian [49] which we could identify preicsely as the physics criterion of a unitarity bound in the conformal field theory associated to the hypersurface singularity. We also show that the first variation of the volume functional is related to the Futaki invariant on the Kähler orbifold, hence volume minimization (and a-maximization in the physics language) implies vanishing of Futaki invariant. This includes the cases of regular and quasi-regular Sasakian structures as classified by Reeb vector orbits. In the irregular case, Collins and Székelyhidi [15] extended the notion of K-semistability to Sasakian structures, showing constant scalar curvature Sasakian metric implies K-semistability and also recovered our results based on the volume functional. The complete classification is still not known, even for complex hypersurfaces with isolated singularity which admits $\mathbb{C}^*$-action.

2.3.3 Compatification of Shimura Varieties

Another very important class of Kähler-Einstein metrics on quasi-projective varieties appears on the compactification of Shimura varieties of noncompact type. There is the work of Mumford on giving a toroidal compactification which is nonsingular. In terms of the divisor at infinity, Yi Zhang and I [90] wrote down the behavior of the volume form of the Hermitian symmetric metric in a neighborhood of the divisor.

Here is a summary of my work with Yi Zhang: The positive cone $C(\mathfrak{F}_0)$ of the standard minimal cusp $\mathfrak{F}_0$ of the Siegel space $\mathfrak{H}_g$ can be regarded as the set of all symmetric postive $g \times g$ real matrices. Let $\Sigma_{\mathfrak{F}_0}$ be any decomposition of $C(\mathfrak{F}_0)$ such that the corresponding Mumford toroidal compactification $\overline{\mathcal{A}}_{g,\Gamma}$ of $\mathcal{A}_{g,\Gamma}$ has normal crossing boundary divisor $D_\infty = \overline{\mathcal{A}}_{g,\Gamma} \setminus \mathcal{A}_{g,\Gamma}$. Let σ be an arbitrary top-dimensional polyhedral cone in $\Sigma_{\mathfrak{F}_0}$ and Let $D_1, \cdots, D_N (N = \dim_{\mathbb{C}} \mathcal{A}_{g,\Gamma})$ be some different irreducible components of D_∞ corresponding to edges of σ.

Then the volume $\Phi_{g,\Gamma}$ on $\mathcal{A}_{g,\Gamma}$ can be represented by

$$\Phi_{g,\Gamma} = \frac{d\mathcal{V}_g}{(\prod_{j=1}^{N} ||s_i||_i^2) F_\sigma^{g+1}(\log ||s_1||_1, \cdots, \log ||s_N||_N)},$$

where

- where $d\mathcal{V}_g$ is a continuous volume form on a partial compactification $\mathcal{U}_{\sigma_{\max}}$ of $\mathcal{A}_{g,\Gamma}$ with $\mathcal{A}_{g,\Gamma} \subset \mathcal{U}_{\sigma_{\max}} \subset \overline{\mathcal{A}}_{g,\Gamma}$,
- the $|| \cdot ||_i$ is a suitable Hermitian metric of the line bundle $[D_i]$ on $\overline{\mathcal{A}}_{g,n}$ for every integer $i \in [1, N]$,
- the s_i is global section of $\mathcal{O}_{\overline{\mathcal{A}}_{g,\Gamma}}(D_i)$ such that $D_i = \{s_i = 0\}$,
- the $F_\sigma \in \mathbb{Z}[x_1, \cdots, x_N]$ is a homogenous polynomial of degree g, and the coefficients of F_σ are integers dependent only on Γ and σ together with marking order of edges.

In fact, Yi Zhang and I computed the volume form of the Hermitian symmetric metric as is represented on the coordinate given by the Toroidal compactification. It shows that $K + D$ is nonnegative and positive on $M \backslash D$.

Whether $D_\infty := \overline{\mathcal{A}}_{g,\Gamma} \setminus \mathcal{A}_{g,\Gamma}$ is normal crossing or not, Yi Zhang and I showed that there is always a local model of partial compactification associated to each maximal regular cone σ in the cusp $\mathfrak{F}_0$.

The quotient manifold $\mathfrak{H}_g / (\Gamma \cap U^{\mathfrak{F}_0}(\mathbb{Q}))$ gives an étale map of the Siegel variety. For each maximal cone σ in the cusp $\mathfrak{F}_0$, we have associate exponential maps of the inclusion $\mathfrak{H}_g \subset U^{\mathfrak{F}_0}(\mathbb{C}) \cong \mathbb{C}^n$, so that these maps endow a local model of partial compactification

$$\mathfrak{H}_g / \big(\Gamma \cap U^{\mathfrak{F}_0}(\mathbb{Q})\big) \subset (\mathbb{C}^*)^n.$$

The exponential map $\mathbb{C}^n \to (\mathbb{C}^*)^n$ is given by

$$z \mapsto w = (w_1, \cdots, w_n) \text{ where } w_i = \exp\big(2\pi\sqrt{-1} l_i(z)\big) \, \forall i,$$

where $\{l_i\}_{i=1}^n$ is the dual base of edges of the cone σ.

The $(w_1, \cdots, w_n)$ gives a local coordinate system of the partial compactification, but it can not be a local coordinate system of $\overline{\mathcal{A}}_{g,\Gamma}$ if the D_∞ is not normal crossing.

The quotient manifold $\mathfrak{H}_g / (\Gamma \cap U^{\mathfrak{F}}(\mathbb{Q}))$ also has an induced Kähler-Einstein metric with volume form

$$\Phi_\sigma = \frac{(\frac{\sqrt{-1}}{2})^n 2^{\frac{g(g-1)}{2}} \mathrm{vol}_\Gamma(\sigma)^2 \bigwedge_{1 \le i \le n} dw_i \wedge d\overline{w_i}}{(\prod_{1 \le i \le n} |w_i|^2)(F_\sigma(\log |w_1|, \cdots, \log |w_n|))^{g+1}}.$$

The coefficients of the polynomial F_σ are integers dependent only on Γ and σ, and the function $H := -\log F_\sigma$ must satisfy the following elliptic Monge-Ampére equation

$$\det\left(\frac{\partial^2 H}{\partial x_i \partial x_j}\right)_{i,j} = 2^{\frac{g(g-1)}{2}} \mathrm{vol}_\Gamma(\sigma)^2 \exp\big((g+1)H\big)$$

on the domain $\{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid x_i \le -C < 0 \forall i\}$.

Note that it was known to me [85] that for a Divisor D in an algebraic manifold M, if $K + D$ is strictly positive, then there is a canonical complete Kähler-Einstein metric on $M \backslash D$ whose volume form behaves like:

$$\Phi \sim \frac{dV}{\prod_{j=1}^{k} ||s_j||^2 (-\log ||s_j||)^2}$$

for some integer $k > 0$, where s_i is a section of the line bundle $[D_i]$ if $D = \sum D_i$, dV is a global smooth volume form on M, and the norm is defined such that its zero set is D and the minus Ricci tensor of M plus the Ricci curvature of the metric on D is positive.

In the above work with Yi Zhang, the volume form is more complicated, because we only know $K + D$ is nonnegative and positive outside D. How to study such quasi-projective manifold? It is important to find the right algebro-geometric and combinatorial conditions on the Chern forms of M and the Chern forms of the various devisors D so that the Ricci curvature of the volume form gives rise to a positive definite Ricci form whose n-fold power is asymptotic to the volume form itself.

An ansatz we propose to contruct complete Kähler-Einstein metric on $M \setminus D$ is to construct a volume form described similar to the above Φ_σ, where H satisfies the above Monge-Ampére equation with $x_i = \log |s_i|$. We need to find the condition on the divisors D_i so that Φ_σ gives rise to a positive Ricci form whose n-flod power is asymptotic to Φ_σ. Hopefully we can find a good existence theorem for complete Kähler-Einstein metric with finite volume on $M \setminus D$, when $K + D$ is nonnegative.

The existence of complete Kähler-Einstein metric on the Shimura varieties comes from the Hermitian symmetric domain that covers it. The tangent bundle and various homogeneous bundles over the symmetric domain are invariant under the discrete group acting on the domain. Hence they can be descended to the Shimura varieties. As was explained by Mumford, these bundles and their connections can be extended naturally to the Mumford Toroidal compactification of the Shimura variety. (Mumford proved that the extension satisfies the property of being "good".)

And the Chern forms defined by the connection were identified by Mumford to represent the Chern class of the extended bundles in the sense of distribution. In the case of the cotangent bundle, the cotangent bundle is extended to be $\Omega^1(\log(D))$, where D is the divisor at infinity.

Since the bundles are homogenous, the Chern numbers of these extended bundles are determined by some numerical combination of its curvature tensor at one point times the volume of the Shimura variety. The existence of Kähler-Einstein metric on the Shimura variety shows that the manifold is stable in various senses and the homogeneous bundles are stable with respect to the polarization $K+D$ of the variety. Besides algebraic geometric characterization of Shimura variety, it would be good to characterize algebraic geometrically those holomorphic bundles that are homogenous.

Incidentally, from my observation in 1978 that the positivity of $K+D$ implies the existence of a unique canonical Kähler-Einstein metric on the complement of D. We can find a map from the space of divisors D such that $K+D$ is ample to the space of stable bundles defined by cotangent $\Omega^1(\log(D))$. It will be nice to find conditions on D so that we can weaken the conclusion $K+D>0$ to $K+D\geq 0$.

For a compact algebraic manifolds M, it can be shown to be a Shimura variety if the canonical line bundle is ample, and if the bundle, formed by symmetric powers of the cotangent (or tangent) bundle tensored by some line bundle so that the determinant bundle is trivial, is irreducible and has nontrivial sections. This is a simple observation (due independently to Kobayashi-Ochiai and myself) [94] because the existence of Kähler-Einstein metric will make this nontrivial section to be a parallel section and hence the holonomic group will be reduced. Algebraic characterization of Shimura varieties allows one to give a simple proof of the theorem of David Kazhdan, that Shimura varieties are invariant under Galois conjugation. Unfortunately our knowledge for noncompact manifold is not good enough to give such a proof in such case.

2.3.4 Explicit Construction of Kähler-Einstein Metrics and Uniformization

For Kähler-Einstein manifolds with negative or zero first Chern class, I proposed [88, p. 139] that the metric can be computed in the following manner: When the canonical line bundle K is ample, we can embed the manifold into the complex projective space by nth power of K.

The embedding can be changed by projective transformation in general. But there was a concept of balanced position (inspired by my work with Bourguignon and Peter Li [10] on first eigenvalue of the Laplacian) that I suggested to my former student Luo [54].

The embedding is unique up to unitary transformation after putting into such balanced position. The induced metric from complex projective space defines a sequence of Kähler metrics on the manifold, which after division by n, will converge to a Kähler-Einstein metric of the manifold.

In the above construction, when the manifold has zero first Chern class (Calabi-Yau manifold), the canonical line bundle should be replaced by any positive line bundle.

The balanced position is achieved by some projective transformation. We expect that the projective transformation depends algebraically on the original embedding of the manifold. The whole procedure should give a reasonable "explicit" form of the Kähler-Einstein metric. Once we obtain an explicit form of the Kähler-Einstein metric, we can compute the uniformization of the manifold.

A simple case is the elliptic curve where we know how to calculate its unique holomorphic 1-form by residue. The absolute value of it gives the Ricci-flat metric on the elliptic curve. We can calculate the uniformization of the elliptic curve, using the period calculation.

Computation of the periods of the holomorphic 1-form is obtained by computing the Picard-Fuchs equation. Once one finds the period, one can obtain a map from the complex line, mod the lattice spanned by the periods, to the elliptic curve. The components of this map is the Weierstrass $\wp$ function and its derivatives. This procedure is classical and went back to Abel, Jacobi and Riemann.

The uniformization of a general algebraic curve: finding a covering holomorphic map from the upper half plane to the curve, is more difficult and is done only for special curves.

Suppose we can calculate the Poincaré metric on the curve, as was explained above, we can calculate this map by studying the periods through the Picard-Fuchs equation. It is of course much more challenging to calculate the uniformization map explicitly in higher dimensions.

Given an algebraic manifold, we know it can be uniformized as a quotient of some classical domain. It is a classical question on how to find such a uniformization.

As mentioned above, we know how to find an algebro-geometric criterion (by using Chern numbers) for an algebraic manifold to be a ball quotient. But we can generalize this criterion to more general manifolds covered by Hermitian symmetric domains.

Once we identify such an algebraic manifold, we need to find suitable multivalued holomorphic map from the manifold into a Hermitian symmetric domain.

Chenglong Yu, Peng Gao and I proposed the following program for the ball quotient:

1. Find a reasonably explicit way to construct the Kähler-Einstein metric on the algebraic manifold. This involves having a good understanding of the right projective embedding for the algebraic manifold.

2. Based on Kähler-Einstein metric, we compute its connection A_h and construct a system of holomorphic linear differential equations

$$ds + \begin{pmatrix} -A_h^T + \frac{1}{n+1}\mathrm{tr}(A_h) - ad\hat{z} & \partial a + A_h^T a - a(d\hat{z}\cdot a) \\ d\hat{z} & d\hat{z}a - \frac{1}{n+1}\mathrm{tr}(A_h) \end{pmatrix} s = 0$$

where

$$s = \left(f^1, f^2 \cdots f^{n+1}\right)^T, \quad d\hat{z} = \left(dz^1, dz^2 \cdots dz^n\right),$$
$$a = (a_1 \cdots a_n)^T, \quad a_i = -\Gamma_i{}^j_j + \frac{1}{n+1}\sum_k \Gamma_k{}^k_i \delta^i_j$$

Here j doesn't depend on i. The value of a is such that it gives a gauge transformation making the connection matrix holomorphic.

3. We find a base for the solution of the this system, given by the span of $\{s_1, s_2, \cdots, s_{n+1}\}$. This allows us to define locally a map to the projective space of dimension n. Up to projective transformations we find a map to the complex ball of dimension n as well.

These maps are multivalued functions. The inverse of this map should be given by automorphic forms and one should be able to find information of the discrete group that acts on the ball based on the information of the algebraic variety and the monodromy of the map. An example in the case of elliptic curve is given by the Weierstrass-$\wp$ function.

For higher genus curve, if the Kähler-Einstein metric is $e^{2u}dz\wedge d\bar{z}$, then the system above becomes

$$d\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} + \begin{pmatrix} 0 & ((u_z)^2 - u_{zz})dz \\ dz & 0 \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = 0,$$

or

$$(f_2)'' + \left((u_z)^2 - u_{zz}\right) f_2 = 0.$$

The coefficient in the differential equation gives the Schwarzian derivative for the uniformization map $S(u) = u_{zz} - u_z^2$.

A complete understanding of this program should create many interesting special functions and should be related to the GKZ system and the tautological system introduced by Bong Lian and myself [48] on computing the period of integrals of holomorphic forms.

2.3.5 Relation with Birational Geometry

The moduli space of algebraic manifolds of general type was studied by Gieseker and Viehweg who proved that they are quasi-projective. The detailed structure of the moduli space is not well understood. However, the canonical Kähler-Einstein metric on the manifold can be useful for such study. It induces a canoncial metric on the moduli space which is called Weil-Petersson metric: A Kähler metric on the manifold gives rise to a metric on $H^1(T)$ which is the tangent space of the Kuranishi space of the manifold.

The Weil-Petersson metric can be computed in some cases. But the properties were best understood only for moduli space of curves. There are several metrics defined on the moduli space of curves: Weil-Petersson metric defined by using the Poincaré metric (or the Bergman metric) on the curve, the Teichmüller metric (which was proved by Royden [66] to be equal to the Kobayashi metric), the Carathéodory metric, the Bergman metric and the Kähler-Einstein metric.

The last three metrics can be defined by general method, not just for moduli space of curves. Hence the computation of them is interesting. Although K. Liu, X. Sun and I [50, 51] showed that the last four metrics on the Teichmüller space are all uniformly equivalent to each other, it is likely that the Carathéodory metric is different from the Kobayashi metric. But the precise statement is not known.

However, Liu-Sun-Yau did calculate the asymptotic behavior of the Kähler-Einstein metric on the Teichmüller space. In fact, the minus Ricci tensor of the Weil-Peterson metric defines a complete metric on the Teichmüller space. The Kähler-Einstein metric could be obtained by perturbing from this metric. The Teichmüller metric and the Weil-Petersson metric are computable based on local information of the Riemann surface.

The other metrics are defined by global means. Hence the remarkable theorem of Royden, proving that a metric defined by global means to be equal to locally defined metric, has provided powerful information. It will be very useful to compute all these metrics in the simplest Teichmüller space of genus two curves.

The second class of canonical metrics are those manifolds with zero Kodaira dimension. By taking the absolute value of the pluriholomorphic n-form, and taking roots, we obtain a canonical volume form which may degenerate along a divisor. We can solve the Monge-Ampère equation to obtain some Kähler metric which may be degenerate along the divisor. It will be interesting to know the birational class of these manifolds.

Let us now consider the possibility of using metrics to understand birational geometry. First of all, for two classes of manifolds, there are natural measures that are birational invariant:

For manifolds with Kodaira dimensional equal to zero, we have a canonical volume form by using the absolute value of the pluricanonical form. For manifolds of general type, I introduced 40 years ago [83] an intrinsic measure that is invariant under birational transformations. It was a generalization of the construction given by Kobayashi and Eisenman in 1970s.

In both cases, we can pick any Kähler class and solve the Calabi conjecture with this volume form as prescribed. In the first case, such metric was studied in the second part of my paper on Calabi conjecture. The metric can be proved to be unique and smooth outside the divisor defined by the pluricanonical form. In the second case, the metric also exists uniquely. But smoothness depends on the measure that I constructed.

In any case, given two birational manifolds of general type M and M', we can find M'' and smooth rational maps from M'' to M and M' respectively. If the pullbacks of polarizations on M and M' are the same on M'', then they are isometric to each other. It is easy to derive from this fact that the group of automorphisms of a manifold of general type is finite.

It is not hard to prove that any algebraic manifold of general type admits a Kähler-Einstein metric with singularity (as was demonstrated by Tsuji and myself thirty years ago). However, in order for such metrics to be useful, one needs to know the singular behavior of the metric. Kähler-Einstein metrics do not respect rational maps. However, the Bergman metric has better behavior under birational transformation.

Let us look at the line bundle K^m where K is the canonical line bundle. For any holomorphic section s of K^m, we can take $2/m$ power of its absolute value, which defines a pseudo-norm on the canonical line bundle. If we normalize its integral to be one and maximize the pseudo-norms among all such s, we obtain a canonical pseudo-norm on the canonical line bundle. It defines a birational invariant volume form. The curvature form of this volume form should define a pseudo-Kähler metric on the manifold. See [20] for a detailed discussion.

We can deform this pseudo-Kähler metric within its class to obtain a pseudo-Kähler metric which is Einstein when it is smooth. When m is large, the Kähler-Einstein metric should be less singular and if we know the singular behavior of the original pseudo-Kähler metric, we should have a way to control the singularity of this pseudo-Kähler-Einstein metric. It should be useful to study birational geometry.

For example, when the Kodaira dimension of the manifold is zero, and the nonzero form s is a section of K^m. The volume form is the absolute value of s to the power of $2/m$. At the nonsingular point of the divisor of s, the local model of the Ricci-flat metric should be the push-forward of the Kähler metric on an m-fold branch cover of the manifold branched along the divisor $s=0$.

Based on this, one can compute the second Chern form of the Ricci-flat metric degenerate along the divisor $s=0$. The second Chern form of this degenerate metric wedge with Kähler class to the top dimension is positive unless it is flat. This should give interesting information for manifolds with Kodaira dimension zero.

Many years ago, I conjectured that there are only a finite number of deformation types for compact Kähler manifolds with $c_1=0$ at each dimension. The question is still unknown and is getting more and more important in string theory. The minimal model of algebraic manifold with Kodaira dimension zero should play an important role if we want to ask similar questions for such manifolds.

3. Hermitian-Yang-Mills Connections

Hermitian metric on a complex manifold has a natural generalization to Holomorphic bundles over complex manifolds. Given a Hermitian metric on the bundle, there is a natural connection which preserves the metric and also the $(0,1)$ part of the covariant derivative would be the same as the naturally defined $\bar\partial$ operator that depends only on the complex structure of the bundle and the complex manifold. The curvature is a $(1,1)$-form with values in the endomorphism of the bundle.

3.1 Donaldson-Uhlenbeck-Yau Correspondence

There is a natural generalization of the Kähler-Einstein condition to this setting by wedging the curvature 2-form with the Kähler form to the top dimension and require it to be a scalar multiple of identity tensor with the volume form. This equation is the natural generalization of anti-self-dual equations for bundles over a Kähler surface.

In fact, around 1977, C. N. Yang [95] was trying to solve the anti-self-dual Yang-Mills equation on $\mathbb{R}^4$, and he showed that it can be reduced to Cauchy-Riemann equations. And therefore he demonstrated that the above equation is part of Yang-Mills equations. It is therefore natural to call such connection to be Hermitian Yang-Mills connection.

The equation became rather well known in the math community after 1977, when people recognized the importance of applications of Kähler-Einstein metric to complex geometry. The proof of the existence of such connections would be clearly different as the Calabi-Yau theorem was based on the complex Monge-Amperè equation which depends only on a scalar. The Hermitian Yang-Mills connection is a vector-valued equation.

In December of 1977, when I was preparing the talk for the ICM in Helsinki, I thought about the possible conditions for existence of Hermitian Yang-Mills connections. I concluded that it had to be related to the slope stability of the holomorphic bundle, as was motivated by the work of Bogomolov and Miyaoka on Chern number inequalities. I was informed much later that this possibility was also believed to be true by Hitchin and Kobayashi.

However, the proof would have to be quite tough as there is no good way to handle such a nonlinear system of elliptic equations. It turns out that Donaldson and Uhlenbeck-Yau were working on this problem independently. I learnt from Hitchin during a trip to England that Donaldson was able to prove the existence for Hermitian connections of any holomorphic vector bundle that can be deformed to the tangent bundle of a K3 surface. (Note that the Ricci-flat metric on a K3 surface provides a natural solution of the Hermitian Yang-Mills connection on the tangent bundle.) This is of course encouraging as it indicates the possibility of the conjecture.

It turns out that Donaldson [22] was concentrated on algebraic surfaces and Uhlenbeck-Yau [79] on arbitrary dimensional Kähler manifolds. While Donaldson used the Bott-Chern form and the Hermitian Yang-Mills flow, Uhlenbeck-Yau constructed a destabilizing sheaf assuming the nonexistence of Hermitian Yang-Mills connection.

The proof of regularity of the destabilizing subsheaf took nontrivial effort and as a result, our paper appeared later than the work of Donaldson's proof for algebraic surfaces. After we published our work, Donaldson found that some of our formula can be used to re-prove the Uhlenbeck-Yau theorem for algebraic manifolds by restriction of the bundle to hyperplane sections of the algebraic manifold. (It was proved by Maruyama and Mehta-Ramanathan that a stable bundle is stable on a generic hyperplane section.)

This later argument of Donaldson depends intrinsically on the manifold being projective for higher dimensional manifolds. As was acknowledged by Donaldson, the argument of Uhlenbeck-Yau is most natural and in fact, all the later development for Hermitian Yang-Mills connections for higher dimensional manifolds are based on the procedure of Uhlenbeck-Yau.

Some later paper such as the one by Bando-Siu [5] used the Hermitian Yang-Mills flow to generalize our result, but the essential feature of Uhlenbeck-Yau procedure is still needed in an essential manner. It should also be pointed out that the continuity argument used by Uhlenbeck-Yau is just as convenient as the Hermitian Yang-Mills flow.

A few years later, Carlos Simpson [69] generalized the Uhlenbeck-Yau argument to establish similar theorem when the Higgs field was introduced. Hermitian Yang-Mills connections were proposed by me to Edward Witten in 1984 to study heterotic string, which had since became an important subject in mathematical physics. But from the very beginning, we knew the importance of Hermitian Yang-Mills connections, as it provides important Chern number inequalities, and also the conditions for the bundle to be projectively flat.

3.2 Chern Number Inequalities and Characterization of Flat Bundles

The very first applications was the sharpening of the Chern number inequality of Bogomolov and a very important generalization of the theorem of Seshadri-Narasimhan (1965) that every stable bundle over an algebraic curve is flat if the degree of the bundle is zero. A very remarkable corollary of the existence of Hermitian Yang-Mills connection for stable holomorphic bundle is that such bundle must be projective flat, if the Bogomolov inequality $2rc_2(E) \geq (r-1)c_1(E)^2$ becomes equality.

This can be considered as a generalization of my theorem that the equality of certain Chern numbers can be used to characterize ball quotients. In fact, Carlos Simpson observed that by generalizing this theorem to the Hermitian Yang-Mills-Higgs connection, one can reproduce my previous theorem that an algebraic surface of general type is covered by the ball if the ratio of the two Chern numbers is the same as the projective plane. In fact, by choosing the Higgs field carefully, one can generalize the theorem to characterize quotient of general Hermitian symmetric space assuming suitable stability.

Characterization of flat bundles based on Hermitian Yang-Mills-Higgs connection also allows Simpson to construct variation of Hodge structures. This is remarkable and led me to believe that there is a good connection with the characterization of quotients of more general Hermitian symmetric domains based on the existence of Kähler-Einstein metrics.

It is really remarkable that the construction of stable bundles satisfying certain Chern number equality gives rise to nontrivial projective representation of the fundamental group of the manifold, which we know little about.

In particular, if some natural bundle constructed from the tangent bundle of the manifold is stable with respect to certain polarization and if the numbers defined by wedging the second Chern class of the natural bundle with the polarization $n-2$ times, and the square of the first Chern class of the natural bundle wedged with the polarization $n-2$ times are equal to zero, then the natural bundle admits a flat Hermitian connection, which means that the fundamental group of the manifold has a nontrivial unitary representation, unless the natural bundle is trivial. Note that we do not need to assume existence of Kähler-Einstein metric on the manifold in this setting. Natural bundles are bundles constructed from natural decomposition of tensor product of tangent and cotangent bundles.

It raises an interesting question in this regard: given an algebraic manifold M with a fixed Kähler class, we consider all holomorphic bundles with trivial first Chern class over M which are polystable with respect to this Kähler class. We consider two such bundles equivalent if they become isomorphic to each other after adding trivial bundles. They form a ring consisting of countable number of algebraic subvarities which are moduli space of the bundles with a fixed Hilbert polynomial.

There is a subring formed by those stable bundles whose second Chern class wedged with the Kähler class to $n-2$ times vanishes. Does the structure of this subring determine the algebraic fundamental group of the manifold? (It is quite likely that we need to consider bundles with Hilbert space fiber in order to obtain information for the full fundamental group.) What is the structure of this subring for Shimura varieties? Can they determine the Shimura variety?

3.3 Generalization to Non-Kähler and Non-Compact Manifolds

Since the theory of Uhlenbeck-Yau was generalized by Jun Li and myself to general complex manifolds, we are able to apply it to handle some interesting non-Kähler manifolds. The most notable one was the class VII surfaces of Kodaira. They were studied by Kodaira, Inoue and Bombieri. Kobayashi and Ochiai realized the importance of holomorphic connections for such manifolds. Bogomolov claimed that for such manifolds without curves, they are given by the examples constructed by Inoue. The proof by Bogomolov [6, 9] is not clear.

Jun Li, Fangyang Zheng and I [46] gave a clear proof based on the existence of Hermitian Yang-Mills connections. It should be possible to generalize our argument to handle those class VII surfaces with finite number of curves also. Many years ago, I proposed to study those connections mod the curves long ago. If this proposal is successful, it should complete the Kodaira classification of complex non-Kähler surfaces.

The study of Hermitian Yang-Mills connections over quasi-projective curve was discussed by Simpson. The generalization to the case when the base pair is (M,D) with D nonsingular, is not hard. The case when D is normal crossing divisor is more difficult, and was studied by Takuro Mochizuki [63].

3.4 Analytic Criterions for Various Stability Conditions

There is no simple criterion to check whether a bundle is stable or not. In many cases, the existence of Hermitian Yang-Mills connection helps to understand properties of stability of bundles. Slope stability is only one kind of stability that appeared in algebraic geometry. A natural class of stability was introduced by David Gieseker in early 1970s. He compared Hilbert polynomials of the subsheaves.

The analytic analog of Gieseker stability is not well understood, although Conan Leung studied this problem in his PhD thesis [44], under my guidance about 20 years ago. There are a sequence of differential equations which can be considered as a natural generalization of the Hermitian Yang-Mills equations. (Todd classes are part of the equations as Hilbert polynomial need to be expressed.) Assuming the curvature is uniformly bounded, Leung proved that the existence of the equations is equivalent to Gieseker stability of the bundle. This bound of the curvature has not been proved and whether this set of equations is the most natural set of equations is not clear.

As was proposed by me [89], the existence of a Kähler Einstein metric or metrics with constant scalar curvature on an algebraic manifold is related to stability of the algebraic manifold. My former student Luo, followed my suggestion of using the concept of balanced condition to study stability of manifolds. It would be good to relate manifold stability to bundle stability. Now Chen-Donaldson-Sun proved that K-stability of the manifold implies the existence of Kähler-Einstein metric. It implies, in particular, the stability of the tangent bundle of the manifold.

In order to relate two concepts of stability, I propose to define a bundle to be balanced if the sections of the bundle, after twisted by a very ample line bundle, can embed the manifold into a balanced submanifold of the Grassmannian.

Hermitian Yang-Mills bundles are mirror to special Lagrangian submanifolds in the theory of mirror symmetry under the program of Strominger-Yau-Zaslow. Gieseker stability is slightly weaker than slope stability. It may be interesting to know which class of Lagrangian cycles will be their mirror images. By studying stability question for Lagrangian cycles carefully and applying mirror symmetry, Mike Douglas found new concepts of stability of bundles. Based on his work and the works of F. Denef, Douglas-Reinbacher-Yau [24] proposed a conjecture on the existence of stable bundles based on Chern classes of the bundle which can be stated as follows:

Consider an ample class D on a simply connected Calabi-Yau threefold X and an integer $r > 1$ and three classes

$$c_i \in H^{2i}(D, \mathbb{Z}),\ i = 1, 2, 3$$

such that

$$\left(2rc_2 - (r-1)c_1^2 - \frac{r^2}{12}c_2(D)\right) = 2r^2 D^2$$

and

$$\left(c_1^3 + 3r(rch_3 - ch_2 c_1)\right) < 8\sqrt{2} \cdot r^3 D^3$$

Then there exists a rank r reflexive sheaf V on X stable with respect to some ample class such that

$$c_i(V) = c_i,\ i = 1, 2, 3$$

Prior to the SYZ program on mirror symmetry, Kontsevich introduced the concept of homological mirror symmetry, where he introduced the derived category over algebraic manifolds. It was realized later to correspond to branes in string theory. This has been developed into a rich theory. Bridgeland studied the concept of stability of derived category and it is now called Bridgeland stability. It would be interesting to find a suitable analytic counterpart of Bridgeland stability.

4. Mirror Symmetry

Supersymmetry provides powerful tools to understand Calabi-Yau manifolds. The intuitions from physics have been powerful. The important concepts introduced by string theorists have deep influence on the geometry of such manifolds. The most important one was the idea of mirror symmetry. It called for the existence of another Calabi-Yau manifold (which we call the mirror manifold) whose Hodge diamond for cohomology is the transpose of the Hodge diamond of the original Calabi-Yau manifold.

4.1 Counting of Curves

More importantly the conformal field theory based on one Calabi-Yau manifold is dual to that of its mirror manifold. The Type IIA conformal field theory of Calabi-Yau manifold is isomorphic to the Type IIB theory of the mirror manifold. This is a remarkable theory predicted by Vafa, Dixon and others. But it was Greene-Plesser and Candelas et al. who developed the details of such theory. The most remarkable consequence is that it solved an old problem in enumerative geometry.

The reason is that the Type IIB theory can be computed by deformation theory of Kodaria-Spencer while the type IIA has quantum corrections. The quantum corrections are provided by the rational curves on the Calabi-Yau manifold. Since Type IIA theory of one manifold is isomorphic to the type IIB of the mirror manifold, we can compute the number of rational curves on the Calabi-Yau manifolds by the variation of Hodge structure for its mirror family.

The initial theory was mostly based on physical intuition. But two groups of mathematicians proved such statements rigorously in 1996, by Lian-Liu-Yau [47] and Givental [30] independently. Despite that the proof is rigorous, the intuition from string theory played the most important role. The idea of supersymmetry has become one of the most fundamental philosophies underlying the current modern development of algebraic geometry.

The simplest and most elegant examples of geometric structures showing up in string theory studies are the Calabi-Yau manifolds, where rich structures related to deep string theory and Quantum Field Theory dualities are discovered. This includes the so called Gromov-Witten invariants related to the counting of rational curves mentioned above.

The counting of algebraic curves of higher genus is far more complicated. One approach was initiated by Bershadsky, Cecotti, Ooguri and Vafa (BCOV), who developed a theory called Kodaria-Spencer theory of gravity. The computation gives beautiful predictions based on some master equations. But it suffers from an ambiguity which is called holomorphic ambiguity. Up to now, we still have difficulty to overcome this ambiguity, although some important progress was made by Zinger, Jun Li and their coauthors for genus one or low genus curves.

Modularity of partition functions of the so-called topological strings contains non-trivial arithmetic information of the Gromov-Witten invariants and it is still a mystery to understand the appearance of modular forms completely. Yamaguchi and I were able to demonstrate some polynomial structure on such partition functions [82], which suggested that there is rich algebraic structure behind them. Due to especially its impressive power in enumerative geometry, there was a great desire to understand mirror symmetry mathematically.

4.2 Mathematical Approaches to Mirror Symmetry

As mentioned above, two different approaches were proposed. One is the famous Kontsevich's homological mirror symmetry conjecture [41] which says that the derived category of coherent sheaves of a Calabi-Yau manifold is equivalent to the Fukaya category of its mirror manifold. Fukaya pioneered the research to study the extensive complicated structure of the Florer theory of Lagrangian cycles through A_∞-algebra [26, 25], which is an important ingredient in the conjecture of Konsevich. Another approach was proposed by Strominger-Yau-Zaslow [71] that the Calabi-Yau manifold is fibered by special Lagrangian torus and the mirror manifold is obtained by replacing the torus by its dual torus. Much progress was made by Auroux, Seidel et al. in this direction. It is important that singularities are allowed in the fibration for both topological and more subtle reasons. The SYZ conjecture has much evidence to be true. Gross and Siebert made a lot of progress in the last few years using tropical methods.

Mirror symmetry has inspired many important developments in Kähler geometry. The program of SYZ calls for close relationship between special Lagrangians with bundles. In the paper of Leung-Yau-Zaslow, we explained how, under the SYZ map, equation for special Lagrangian cycle which intersects the SYZ torus at one point can be transformed to an equation for a holomorphic line bundle. The equation turns out to be studied by M. Mariño, R. Minasian, G. Moore and Strominger [57], recently Tristan Collins, Adam Jacob, and myself [14] studied this equation and we can prove the existence for many important cases assuming some form of stability for the $(1,1)$-class. The equation has the form:

$$\mathrm{Im}\big(J+\sqrt{-1}\omega\big)^n = \tan\big(\hat{\theta}\big)\mathrm{Re}\big(J+\sqrt{-1}\omega\big)^n$$

where $\hat{\theta}$ is a topological constant determined by J and $[\omega]$.

The equation admits supersymmetry and the pair consists of the Kähler class J and the closed $(1,1)$-class ω can be looked as a natural complexfication of the Kähler class. It defines an open set in the complexified Kähler cone. This may give a good candidate for the mirror of the moduli space of polarized complex structures of its mirror manifold. Note that we like to see the "Kähler moduli" to be isomorphic to the moduli space of the complex structure on its mirror manifold. It is also important to find a suitable discrete group acting on this open set in the Kähler cone. The mirror symmetric version of the special Lagrangian that intersects the SYZ torus for more than a point is supposed to be a higher rank bundle. The equation defined on it is being explored.

One should note that the homological mirror conjecture of Kontsevich has inspired a great deal of study of derived category in geometry. While we may not be used to abstract reasoning of category theory in geometry, we hope that its relationship to SYZ construction may eventually broaden the scope of geometry.

5. Future Directions in Mathematical Physics and Arithmetic Geometry

In conclusion, we should say that the beautiful subject initiated by Riemann in the nineteenth century on Riemann surfaces had deep influence on the development of complex geometry in the 20th century. While Hodge provided the fundamental structure relating complex analysis with topology via Hodge groups, Kodaira provided fundamental methods to construct holomorphic sections of bundles. With the works of Chern classes and Hirzebruch-Riemann-Roch formula, the works of Hodge and Kodaira have been developed to be most powerful tools in understanding Kähler geometry. The modern development has been emphasizing the use of nonlinear elliptic equations, relating the concept of Kähler-Einstein metrics and Hermitian Yang-Mills equations to various fundamental concepts of stability introduced to study moduli spaces.

The most recent development on Calabi-Yau space due to cooperations between mathematicians and string theorists has been spectacular. Ideas of many fields in mathematics were used. We hope to see some more ideas of number theory in this beautiful subject. For many Calabi-Yau manifolds, the partition functions related to conformal field theory are related to modular forms. For example, it was observed in 1996 by Zaslow and I [91] that the partition function counting rational curves of various degrees in K3 surfaces can be written in terms of η-functions. This was the first time that such modular function appears in counting curves in algebraic geometry. This motivated Göttsche to generalize the Yau-Zaslow formula to general surfaces and for curves of arbitrary genus. This was recently first proved by Y. Tzeng [78] and later by Kool-Shende-Thomas [42]. For Calabi-Yau manifolds with higher dimension, these formulas are much more complicated and their organization is still being explored. In recent work with Zhou and others [2], we were able to show the ring generated by quasi-modular forms associated to $PSL(2,\mathbb{Z})$ or a congruent subgroup therein is isomorphic to the ring of higher genus Gromov-Witten invariants for certain non-compact Calabi-Yau geometries based on the projective plane and also del Pezzo surfaces. This was later extended to the orbifold case [67] and also to include open Gromov-Witten theory in [43].

For more classical arithmetic geometry, we may point out that Serge Lang has noticed long time ago that the importance of Kobayashi hyperbolicity is relevant to the question of Diophanitine problem, such as the Mordell conjecture proved by Faltings for algebraic curves of higher genus. Kobayashi conjectured that for an algebaic manifold of general type, the Kobayashi metric should be non-degenerate in a Zariski open set. In particular, there is a subvariety of the manifold such that all rational curves and elliptic curves are subset of this algebraic subvariety. This is sometimes called the Lang conjecture.

Lang also conjectures that if the manifold is defined over integers, the rational points of the manifold should all be in this subvariety. There was little progress on the Kobayashi-Lang conjecture except in the case of surfaces where Bogomolov [7] and Miyaoka [62] made important contributions. Steven Lu and I studied the differential geometric aspect of it [52].

For algebraic surfaces with positive index, one can find a Finsler metric with strongly negative holomorphic sectional curvature, but the metric may degenerate in some subvarieties. This statement implies the Kobayashi-Lang conjecture. Therefore one would like to make the following conjecture: an algebraic manifold is of general type if and only if it admits a complex Finsler metric which may be degenerate along a subvariety which has strongly negative holomorphic sectional curvature. It is quite possible that Finsler metric may be replaced by Kähler metric. The converse was asked by me, there were some progress due to several people, but only recently Damin Wu and I [81] were able to prove that if an algebraic manifold admits a Kähler metric with strongly negative holomorphic sectional curvature, its canonical line bundle must be ample. (Our original argument assumes manifold to be algebraic, but it was pointed out by Valentino Tosatti and Xiaokui Yang [74] that our argument, which is based on solving Monge-Amperè equation, can work for Kähler manifold also.) It is not hard to generalize the theorem to complete non-compact Kähler manifolds whose holomorphic sectional curvature is bounded by two negative constants. A natural question is that a compact manifold admits a pseudo-Kähler metric (Kähler metric that may degenerate along some subvarities) with strongly negative holomorphic sectional curvature iff the manifold is of general type. Manin has conjectured that Kähler-Einstein metric will play important role in arithmetic geometry. I believe that is the case. There is still much to learn about the relation between complex geometry, algebra and number theory.

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(Note. I intend to give Takagi lectures based on parts of this paper.—Shing-Tung Yau)