Open Problems

compiled by Xuhua He*

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Open Problem 2015001 (Lie theory). *Proposed by Xuhua He, University of Maryland, USA and HKUST, Hong Kong.*

A Coxeter group W is an abstract group defined in terms of reflections, that is, a group with presentation

$$W = \langle s_1, \cdots, s_n \mid (s_i s_j)^{m_{ij}} = 1 \rangle,$$

where $m_{ii} = 1$ and $m_{ij} = m_{ji} \geq 2$ for $i \neq j$. Here $m_{ii} = 1$ implies that $s_i^2 = 1$. The generators s_i are called simple reflections.

Let J be a subset of $\{1, 2, \ldots, n\}$. We denote by W_J the subgroup of W generated by s_i for $i \in J$ and call it a parabolic subgroup of W. The group W_J is again a Coxeter group.

A conjugacy class $\mathcal{O}$ of W is called elliptic if it does not contain an element in any proper parabolic subgroup. An element in an elliptic conjugacy class of W is called an elliptic element of W.

The set of elliptic conjugacy classes forms a "building block" of the set of all conjugacy classes of W. Namely, for any conjugacy class $\mathcal{O}$ of W, there exists a parabolic subgroup W_J such that $\mathcal{O}$ contains an elliptic element of W_J.

In [2, Theorem 3.2.11], Geck and Pfeiffer showed that for finite Coxeter groups, the elliptic conjugacy classes never fuse.

Theorem 0.1. *Let W be a finite Coxeter group and W_J a parabolic subgroup of W. If $\mathcal{O}$ is a conjugacy class of W that contains an elliptic element of W_J, then $\mathcal{O} \cap W_J$ is a single conjugacy class of W_J.*

In other words, if w and w' are elliptic elements in W_J and are conjugate by an element in W, then they are conjugate by an element in W_J.

This result is useful to "build" conjugacy classes from elliptic ones. Namely, it implies that for finite Coxeter groups W, there is a natural bijection between the conjugacy classes of W and the pairs (J, C), where J is a subset of $\{1, 2, \ldots, n\}$ (up to conjugation) and C is an elliptic conjugacy class of W_J.

This result is also useful to study the induction/restriction functors, cocenters and representations of finite and affine Hecke algebras. We refer to [1, Theorem 5.2.2 & Proposition 6.1.1] for more details.

The proof of the theorem, however, uses the characterization of elliptic conjugacy classes via characteristic polynomials and involves a case-by-case analysis (see [2, Theorem 3.2.7 (P3)]. Now I propose two problems:

Problem 1. *Find a case-free proof of Theorem 0.1.*

Problem 2. *Does the similar result hold for arbitrary Coxeter groups?*

Elliptic conjugacy classes of finite Coxeter groups play an important role in the representation of Lie groups, as well as the geometry of unipotent classes. And new proof (even for known results) without case-by-case analysis improves our understanding of the theory and is well worth pursuing.

The study of similar problems for arbitrary Coxeter groups is very interesting too. It is expected to play a role in the study of Kac-Moody groups.

* University of Maryland, College Park, Maryland, U.S.A.; and Hong Kong University of Science and Technology, Hong Kong

References

[1] D. Ciubotaru and X. He, *The cocenter of graded affine Hecke algebra and the density theorem*, arXiv:1208.0914.

[2] M. Geck and G. Pfeiffer, *Characters of finite Coxeter groups and Iwahori-Hecke algebras*, London Mathematical Society Monographs. New Series, vol. 21, The Clarendon Press, Oxford University Press, New York, 2000.