3D Technology and Its Impact on the Future of Technology

A Lecture Given at Kumning, China in June, 2014

by Shing-Tung Yau*

Introduction

Thank you, Mr. President. And thanks to all the organizers and students. It is a pleasure for me to be here, at the old site of Southwest Associated University in Yunnan Province today. When I visited the Museum of SAU this morning, I was deeply impressed by the scholars who have persisted in their scientific research and educating young students despite the hardships they have experienced in their own lives. I am especially touched by the words of the SAU School Anthem sung by several of your students just now. I believe that all overseas Chinese scholars will feel the same, because we all expect the speedy rise of our motherland, particularly among academics.

One of the main reasons that Chinese has endured many hardships in the past 100 years is that its science and technological development has not reached the level that it ought to have reached. SAU nurtured many young students during the period of the Anti-Japanese War, and they had a great impact on the science and technological development of China in the previous decades. This is really worthy of our reminiscence and admiration. However, based on the tradition we inherited from SAU, we need to move forward and embark on further development. We shall ask ourselves what path we can take and which directions we can choose for science and technological development for China. Can the Chinese people, especially the young ones, pioneer new directions and make China one of the world's leaders in science and technology? Today, I shall explore this topic.

Applied Mathematics and Myself

I began as a pure mathematician 45 years ago. I studied basic science: the topics that interested me ranged from differential geometry to differential equations, and from general relativity to string theory. I always studied with passion, hoping to see that the advancement of fundamental science bring contributions to mankind and further our understanding of nature. Now I also realize that fundamental science is actually very helpful in studying social science and industrial technology. Therefore, I have taken part in much research in engineering sciences over the past two decades. Besides research on the technology and geometry of 3D, I have also participated in other applied science such as control theory. My speech today is mainly focused on 3D.

In recent years, I'm really happy to see that some of my previous work in pure mathematics has had some real life applications. However, I also notice that quite a few engineering and applied mathematics majors don't pay enough attention to theoretical science. As a result, their achievements are less important compared with their peers in the past. Theoretical science is the mother of applied science, and if we don't acquire sufficient understanding of theoretical science, we cannot fully develop the potential applications in industry.

For over two thousand years, Chinese scientists have made many important contributions to science

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and technology, but more towards the applied side. This is very much different from the development of science and technology in the western world where they paid a great deal of attention to the development of basic science. But their technological development moved rapidly during and after the Renaissance period. Unfortunately, the Chinese have not had enough understanding of theoretical science in the past 400 years, and consequently our technological development have fallen behind that of our western counterparts. So, I hope the young Chinese who major in engineering and applied science can pay more attention to theoretical science.

Today, I am going to lecture on a branch of applied science that may demonstrate how theoretical science can play a decisive role in applied science. For the past fifteen years, my team and I have witnessed the rapid development of imaging technology that is moving from two dimensions to three dimensions. We believe that this trend will change the future of modern technology.

Let us now elaborate on the psychological significance in our lives of the expansion from a two- to three-dimensional orientation brought about by certain advances in applied science. Recall that humans were actually very much confined to a two-dimensional space until airplanes were invented. Aviation, at the dawn of the twentieth century, freed us from the horizontal plane of movement, and gave us an unprecedented and previously unimagined sense of physical freedom—the freedom of movement in three dimensions. When the Wright brothers taught us how to fly, most scientists did not believe that we could fly like birds, using a man-made machine. This was due to some miscalculation in fluid dynamics on the part of applied physicists, which had persisted till the end of the 19th century. This change from two-dimensional to three-dimensional orientation has meant a gain a degree of freedom in our day-to-day life. Nowadays, we can see our planet Earth from outer space. And the view is much better than the view from the surface of the Earth. The panoramic views of the Great Wall and city of Beijing taken from satellites are spectacular. The scientific and economic benefits of 3D technology are potentially enormous. Within ten years of the Wright brothers' first flight, aircraft were already being used to carry passengers, and in warfare. Commercial aircraft are essential to our day-to-day life.

Despite of all these great advances, automobiles, ships, and aircraft cannot be designed and built by the average person. But the new three-dimensional design and imaging technology is available to anyone with an ordinary personal computer.

The debut of the personal computer brought with it so many changes—in the technologies of communications, computation and data storage—which could never have been expected half a century ago. Few people at the time could conceive of the ways in which the PC would change the conduct of the world's industry and economy. This trend of change cannot be stopped, and a country that cannot adapt to such modern technology will face financial problems soon.

In the 1980s, IBM was the greatest computer manufacturer of the world. An IBM-made mainframe computer, which all big companies and big government operations needed, cost over ten million dollars at the time, so that IBM made billions in profit from selling it. But IBM's leading status in the computer industry was soon challenged by the newly emerged PC industry, and IBM was brought almost at the verge of bankruptcy. Fortunately, IBM changed their focus quickly and recovered.

Let us take another example. Twenty years ago, the photographic film industry was monopolized by the Eastman Kodak Company. Its film was used all over the world, and it was the first company to bring out a digital camera. However, it might be said that Kodak did not sufficiently promote its digital cameras, perhaps because of satisfactory profits from selling film. It's said that for quite a long time 80% of Kodak's revenue was from selling film. It so happened that other companies then cornered the digital camera market, and Kodak finally went under. Both IBM and Kodak were large corporations that had dominated the world for several decades, but they brought grave difficulties down upon themselves, by focusing on short-term profits without paying attention to the changing directions in technology.

It is important to digitize cameras. Nowadays we can even use cell phones to take good-quality photos. Because of the continuing technological advancements in computers, a digital camera's data storage capacity and other capabilities are much greater than before, which has brought fundamental change in the camera industry. I remember that, only about ten years ago, we had great difficulty in watching movies on personal computers, but now we can even do so on a cell phone.

Nowadays, we can view 3-dimensional images on a cell phone, but the pictures have relatively low resolution. Why have we run into difficulties in developing 3-dimensional camera and imaging technologies? One of the main reasons is that we still have many technical problems using computers to process 3-dimensional data. From 2D to 3D, lots of software technologies—in addition to hardware technologies—needed to be developed. There are many technological issues yet to be resolved in the industry, and I will try to address some of them in today's talk.

Laptops can process some 3-dimensional information now, but have not reached the point of doing so with ease. When the 3D technology reaches maturity, it will be an important era of transition in the industrial sector. 2-dimensional technology can be processed and executed almost perfectly using modern computers. I still remember that it was difficult to process 2-dimensional images on computers twenty years ago; but it is not so difficult now because of the breakthroughs that have come in computer hardware and software technologies. In the near future, the industrial sector will march into the realm of 3D technology. Just as the invention of aircraft changed the world, the 2D industries will be replaced by 3D ones; that will be a significant change for the industrial sector, but this process has not been completed. We may not have seen the tremendous revolution of 3D technology that will appear in industry yet because we are still in transition from 2D to 3D. The coming 3D revolution might be compared to a tsunami, which, just formed, moves across the deep ocean as a swell barely a meter high, but ultimately strikes the shore as an enormous, powerful wave. We do not see the popularization of 3D technology while 2D still predominates, for we are still on the coastline and have not yet seen the huge wave that will strike. But once 3D technology has eventually penetrated most high-technology industries globally, then Chinese industries—if they have fallen behind—will find it hard to catch up. Recently, President Obama of the United States has claimed that 3D technology will lead to a third industrial revolution. He must have been informed by experts to have made such a bold claim.

Therefore, I think that 3D technology is worthy of our attention and study. Meanwhile, young Chinese engineers and scientists have to pay attention to this technology now, before 3D technology has fully developed.

Take movies, for example. Disney's 3D animation films have had outstanding performance at the box office over the past seven or eight years; indeed the whole animation film industry has been greatly affected by 3D technology. When Avatar was at the production stage, its technical adviser invited our team to help them with 3-dimensional technology. I had no idea what Avatar was about at that time, so I didn't agree to participate in their technological development, and I didn't expect it to become such a blockbuster film with record-breaking audiences. It's clear that 3-dimensional technology can play a role in television, film and animation because it presents more realistic images that appeal to many people.

Recently, people are very interested in 3D printing—a rapid prototyping technology developed in the 1990s. A real-sized object can be produced by stacking layers of materials according to an objects' 3D information. Here are many 3D figurines produced by a 3D printer. Above is the figurine of a girl (Figure 1), produced by our team with a 3D printer. How can we get a 3D printing like this? The first step is to take 3D photos and store the data in a computer. Then we let the computer instruct the printer regarding how to lay down layers of materials. The raw materials can be plastic or other substances. Layers of materials are slowly accumulated to build the 3D object.

Here is another beautiful figurine (Figure 2): it is a reproduction of a stone statue of a bull that fronts

Figure 1.

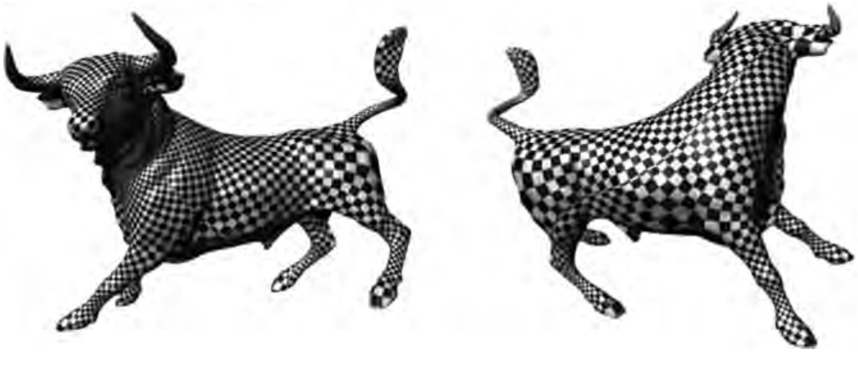


Figure 2.

the tomb of the great General of Han, Huo Qubing. We took the image of the original stone statue using a 3D camera, and then reconstructed the image as a figurine, using a 3D printer. We can likewise replicate any number of real physical objects from 3D image data—and using a variety of materials. Besides good-looking figurines, we can also produce some machine components and molds. 3D printing is a multi-directional industrial technology. It continues to accommodate itself to new fabrication materials, which currently include metal, ceramic, plastic, sand, plaster, wax, and more. 3D printing produces an object by stacking the raw material—layer by layer—under the control of a computer. 3D printers can reproduce objects whose pictures were taken with a 3D camera, and they can also turn engineering blueprints into real objects. A 3D printer has many other applications besides. For example, if we want to build a toy for our kids based on the knowledge that we have, we can make a model and take a 3D pictures of it, then reproduce the object with a 3D printer. 3D printing technology could even be applied to the manufacture of automated robots; to the preservation of cultural relics; in architecture, dentistry, medicine, and education; and in the creative industries for—as an example—the design of jewelry accessories. And so on. We can send our electronic blueprints across the world, and ask our friends to duplicate objects for us. 3D technology will also be helpful in military technology, and in medicine. Within ten years of today, we can expect 3D technology to be applied to the production of replacement body parts. We hope to build a new *kind* of industry—the flexible reproduction of many diverse physical objects—and that 3D technology will be the foundation of this industry.

In the forthcoming transition to the use of 3D technology, the desired close integration of informatization and industrialization poses, for industry, a great challenge, but it also presents a huge opportunity. But we need to learn how to better process 3-dimensional information, and to develop the computer technology that processes geometric computations. So, we have to develop hardware technology as well as software technology. I remember that, twelve years ago, our team had great difficulty finding real 3-dimensional data with which to test our software. Moreover, we needed a 3D camera as well, but that cost 750 thousand dollars at that time, too expensive for a lab to purchase, not to mention for an ordinary person. However, the price is much lower now.

3D hardware technology has improved a lot today, and 3D printing is frequently in the news, yet 3D printing, all things considered, is only a small part of 3-dimensional technology. And there are still many problems with 3D software that await resolution. Although modern computing is advanced enough to solve many difficult computational problems, yet many problems of 3D software engineering still remain unsolved.

How to Deal with 3D Computation

Let's talk about how to solve some of those problems.

In the beginning, Xianfeng Gu and Song Zhang were the key players on our team. Gu was one of my PhD students from the computer science department of Harvard University, and Zhang was one of my postdoctoral fellows. Xianfeng Gu and I applied some differential geometry theory to study 3D images. One of the important tools is the uniformization theory in conformal geometry introduced by Riemann 150 years ago. The mapping that we use to process images must keep the angles unchanged. Conformal geometry is a classical mathematical theory, and it is still an important theory in pure mathematics. But we also find it to be very powerful in applied mathematics, especially in depicting 3D images. In recent years, we have begun to process images via a classical equation called the Monge-Ampère equation. The mapping obtained from the solution to this equation can locally keep the area of the images unchanged. Shiu Yuan Cheng and I studied this equation 30 years ago only because it had a beautiful theory behind it. We did not expect that it might someday be applied to 3D numerical computation.

After our team developed a set of software to process 3D images, we needed experiments to validate the methodology used. First, we needed a 3-dimensional camera to obtain 3D information. 3D cameras were expensive then, but fortunately we had Song Zhang, who had joined our team in the beginning. He is an expert in structured-light theory. We began to develop 3-dimensional camera technology that can reduce the cost. We reduced the cost of a 3D camera—which captures more than 60 frames of 3D images per second—to just a few thousand dollars. 3D cameras produced by companies other than ours take one frame every two seconds. In the past, by comparison, cameras costing 750 thousand dollars took only one frame per hour.

Why do we need high-speed 3D cameras? It is because we couldn't capture the best images of the constantly changing facial expressions if the speed of a camera were not fast enough, especially when we are trying to capture 3D photos of athletes doing diving and running exercises. Those movements are much faster than those of facial expressions, so that 60 frames per second is in fact not fast enough. That need is indeed met by a far more expensive sort of 3D camera, which can capture hundreds of frames per second.

Better camera software can help us extract more data that is more useful. We have lots of outstanding young doctors working with our research team, including Feng Luo, Yalin Wang, Ronald Lui, Wenwei Lin, Mei–Heng Yueh, and others. And joining us are also many distinguished mathematicians and numerical analysts from the Chinese University of Hong Kong, Tsinghua University of Beijing, Hsinchu National Chiao Tung University, and National Taiwan University. We study 3D technology based on the achievements of modern mathematics, such as Teichmüller Theory and other profound theories. We also have made some breakthroughs in processing medical images, with good results in 3D imaging of the human brain, liver, lung, and kidney.

Introduction to 3-Dimensional Technology

At this time, we are developing simultaneously along two different lines of in the computation theory of 3D technology: one is the theory of conformal geometry and the Riemann surface discovered by Riemann 150 years ago; the other is the optimal transport method, a computation method that uses the Monge-Ampère equation. Recently our team successfully presented the method of solving this equation, on computers developed by Shiu-Yuan Cheng and myself 30 years ago.

Our work has included studies of: Riemannian geometry on the Internet, transport theory of resource allocation of cloud computing, geometric approximation of 3D printing, numerical control tool CAD, 3D human face detection, diagnosis and prevention of tumor, and more—all of which use the technology mentioned above.

How does a 3D camera capture a 3-dimensional image? It captures a large set of points. This set of points forms a discrete space, which typically contains almost a million points. the aim is to connect some of these points in order to find the object's geometric information. Therefore, we have to study the mathematics of a discrete surface to obtain the object's geometry. Starting from a discrete set of points, we need to do a lot of computation on many abstract geometric concepts, such as the curvature, the metric of the object, and many other different structures.

Internet and Discrete Riemannian Geometry

The topology and geometry of the large-scale Internet is the fundamental issue in the field of networks. The classical Riemannian geometry assumes that our faces are smooth, and the concepts of Riemannian metric, connection, and curvatures are all derived from a smooth differential structure. We want to generalize the classical harmonic analysis theory to the network theory. Why? When we try to reproduce a real picture, a camera can only catch a set of points. Yet we want to get useful information from these sets. The information is represented by different concepts of geometry. For example, we want to define the curvature of discrete manifolds that may approximate the original smooth manifolds. The hope is that these concepts can help us describe the original smooth spaces. For much more complicated object such as networks, we also want to find a suitable definition of curvature or other important features to give a deeper understanding of their structure. These concepts can help us solve problems related to congestion of the core network and the stability of network flow.

In fact, our team studied the net produced by the Internet. We found that their structure can be studied by geometric methods. Those points with negative curvature form the network backbone and those points with positive curvature form the partial network cluster that connects them. Network congestions on the Internet are places with negative curvatures. The geodesics on network regions with negative curvature are relatively stable, and the local perturbation of network structure has little influence. So the study of these mathematical concepts of the Internet will have a significant role in the future development of computer networks.

Resource Allocation of Cloud Calculation

In performing cloud calculations, we need to do study the allocation of tasks and the problem of allocating a large amount of producers and consumers, and there are two measure spaces: one is the space of the consumer, and the other is the space of the producer. How to match them with each other? We have to guarantee the balance between production and selling, and at the same time match these spaces by minimizing the cost of transportation. This is called the problem of optimal transport, and we can solve this problem with the help of a non-linear partial differential equation called the Monge-Ampère equation. This equation was studied by the Russian geometers, and by Cheng and I in the early seventies in relation to convex surfaces in Euclidean space. It is gratifying to see that this work can be used in cloud computation. We did not expect that the beautiful geometry would have such a practical use.

3-Dimensional Printing

Now let's take a look at some pictures. Using our 3D camera, we obtained a head-shot of the fa-

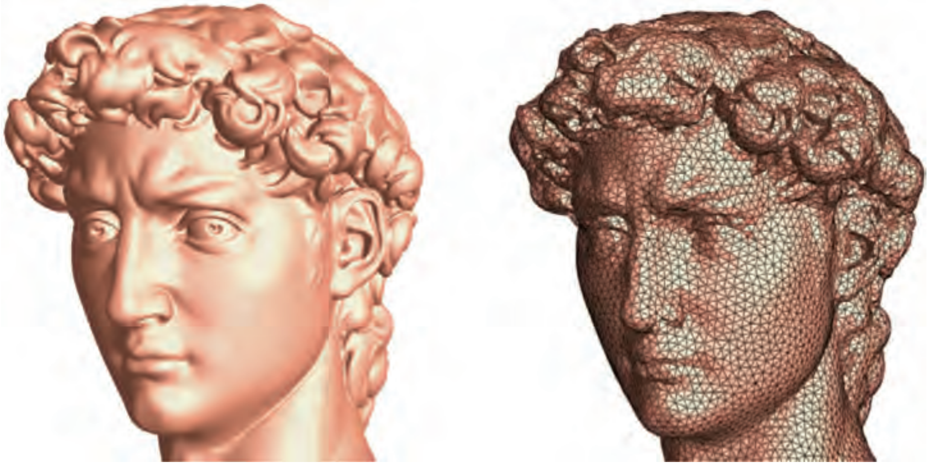


Figure 3.

mous sculpture "David" (Figure 3, at left). Three-dimensional photography printing technology relies on the geometric approximation theory: Because a 3D camera captures, in fact, only a set of points, and we have to "de-noise" it when the camera is not focused and then do post-production on the left hand side head shot by our graphic software. We construct a triangulation of the 3D picture by connecting the point cloud with line segments (Figure 3, at right). The triangles are formed in a rather homogeneous manner.

Triangulation is an important issue. Because the lengths and the angles of the triangles keep changing, we connect appropriate points with straight lines to form triangles which are fairly symmetric. This is very important for computations. We can do this is because the way we flatten 3D images is locally conformal. We get a very robust numerical convergence result by using our triangulation.

Geometric Model CAD

The core technology of geometric modeling is the computation of the affine structure of a surface, and we need to effectively control its singular points. With, for example, the 3D image of a human head and face, there are many important points that we might control. See Figure 4. Whether we are creating animations, or producing dolls or toys having a human head and face, manipulation at some feature points of the head image can make facial expressions richer. By applying the theory of Ricci flow and topological obstruction, we invented a theory and method of manifold spline, which can be used to achieve global modeling and to reduce the number of singular points, as well as to more effectively control the positions of these singular points so as to manipulate facial expressions. (Look at the parts near the eyes, mouth and nose: these are all important control points that

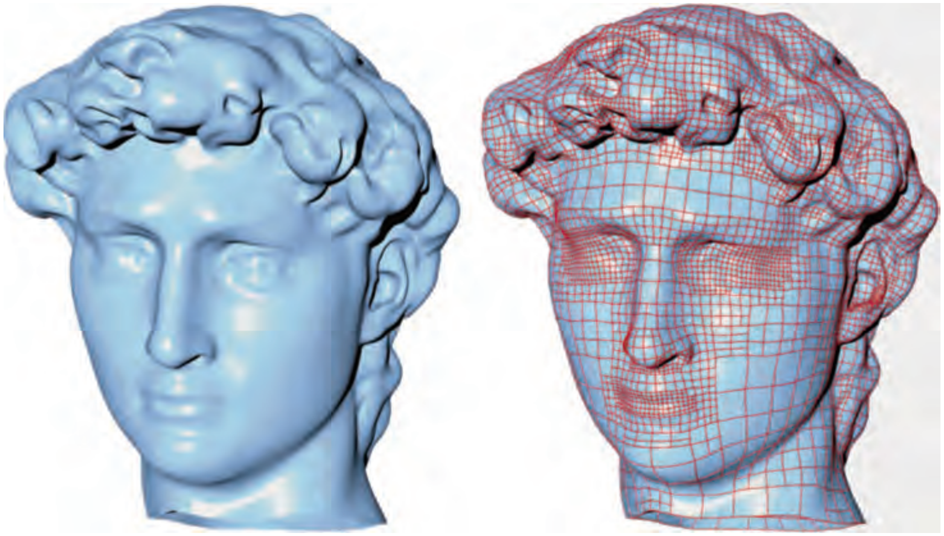


Figure 4.

we call singular points). Our approach is to control how they change, so that, subsequently, we can efficiently process images in, say, video games, or elsewhere.

3-Dimensional Facial Recognition

Our approach to facial recognition, which differs from the general approaches, is that we make use of conformal maps. Look at the four images in Figure 5. The top two, taken by a 3D camera, are the faces of two different people. The bottom two pictures are the images on a disc of the top two by conformal maps. The top two faces are the images in a three-dimensional space. Using conformal maps flattens the faces as the images on a disc—as two-dimensional objects, in other words. The image of a conformal map is essentially unique. Therefore, when we try to match two faces, it is much more efficient to compare faces on discs.

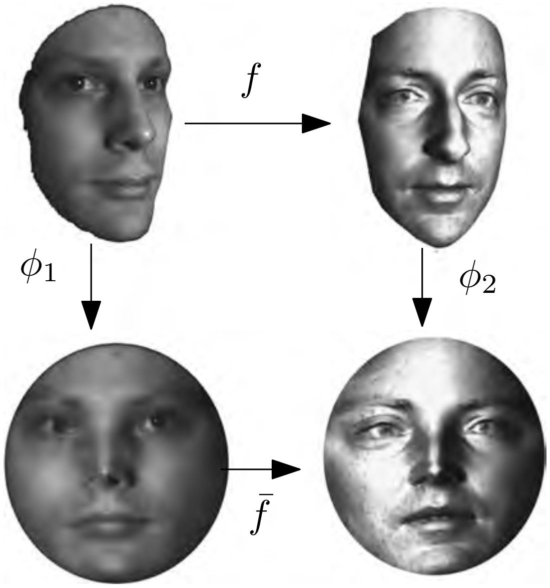


Figure 5.

The same human face can have different expressions on different occasions, sometimes with a smile and sometimes without. We have to determine whether the face we obtain is your face or not, with a smile or without one. Of course, we want to make sure that it is not your brother's face. To do so, we need a very good technology. The fundamental issue is to identify the commonality even when there are subtle differences. Our method is to transform 3D images into 2D ones, just as the two bottom images on discs. Our 2D images retain all the original 3-dimensional information, so that the differentiation of similarities and differences of 3D images can be resolved by 2D technology.

Cancer Prevention

Here we will talk about our medical technology. As we all know, colonoscopy involves the insertion of an endoscope, a flexible tube with a mirror (or fiber optic camera), into a patient's large intestine to find out whether there are tumors or not. But there is always a risk of puncturing the large intestine when the endoscope moves through the intestines, especially for elderly patients. Recently, a friend of mine had a colonoscopy and died from a punctured large intestinal inflammation on the same day. Nowadays, we can use, instead of colonoscopy, X-ray and MRI technology to get slices of pictures of the patient's large intestine, and then compose these pictures into a 3D image using our 3D imaging techniques. See Figure 6. We can then examine the 3D image for signs of a tumor. By this method, there will be no risk of puncturing the patient's large intestine, and it is much more effective than an ordinary colonoscopy.

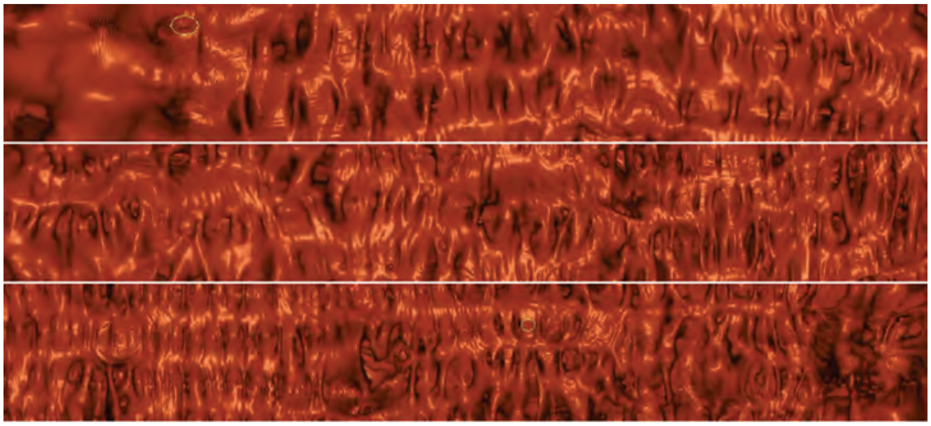


Figure 6.

Theory and Method of Conformal Geometry Computation

The Advantages of Conformal Geometry

Conformal geometry theory has been developed by mathematicians for more than 100 years now. There are several major advantages when we use it in 3D-image processing technology:

First of all, let us discuss the concept of uniformization. Any complicated surface can be conformally deformed into a unified surface, in order to be processed. The unified surface can be a sphere or a surface locally like the Euclidean plane or a hyperbolic plane.

The second advantage is dimension reduction. As I have just described, we can flatten a 3-dimensional human face to an image on a two-dimensional disc; hence we can encode all information of a 3D picture into a 2D image.

Thirdly, data retention. When we deform 3D surfaces into 2D surfaces, all the original geometric forms are kept. We can reconstruct original 3D images by using 2D images without losing any information. Note that, as there are 1.4 billion people in China, it would be much more convenient to store 3D images as 2D data.

Also, our method offers universality because we can present and process all images with conformal geometry.

We can build new technology to analyze a huge number of images. For 1.4 billion people, this is not a trivial task. We have to organize and put all faces together to form a shape space. We can also study the mapping spaces among them, which provides a way to compare different faces.

Conformal Geometry

In our experience, we have found that many profound and diverse mathematical theories can be very useful in industry. For example, the method of conformal mapping can be used in computational hydrodynamics, in computational hydromechanics, in computational aerodynamics, or in computational electromagnetics. Most of these theories had been done efficiently on the plane. Now, based on our theory, we can carry the calculation of curved surfaces in 3D. For example, we compute the movement of tears on human faces.

Conformal geometry can also be used in computational geometry, computer graphics, and computer vision, and in image processing and information technology.

Computational Conformal Geometry

When we compute graphs, there are many quantities which we call conformal invariants. The computation of these quantities can help us process 3D images much more effectively. For example, when processing human faces, we may want to decide which two images are identical to each other. In such cases, it is convenient for us to compute conformal invariants to help identify features of the faces. This theory was applied by our team to study problems in medical imaging and engineering applications.

Conformal Map

Let's take a look at what a conformal map is. It is a one-to-one correspondence that preserves angles between two objects. In this mapping, length and area might be different, but angles must remain unchanged. In our work, we exploit the very important fact that every surface has a unique conformal structure.

Let's take a look at this image of a human face (Figure 7). We can flatten it onto a disc or a square. There are sets of grids like a chessboard made of

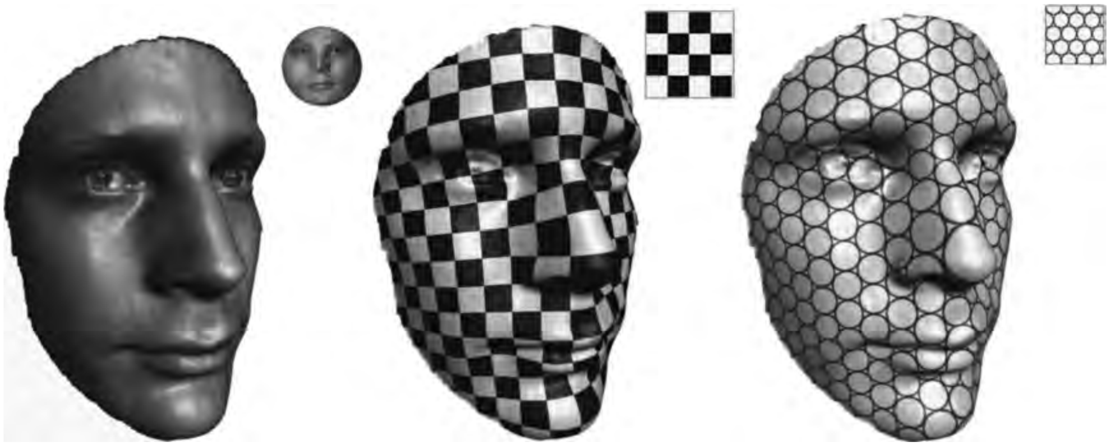


Figure 7.

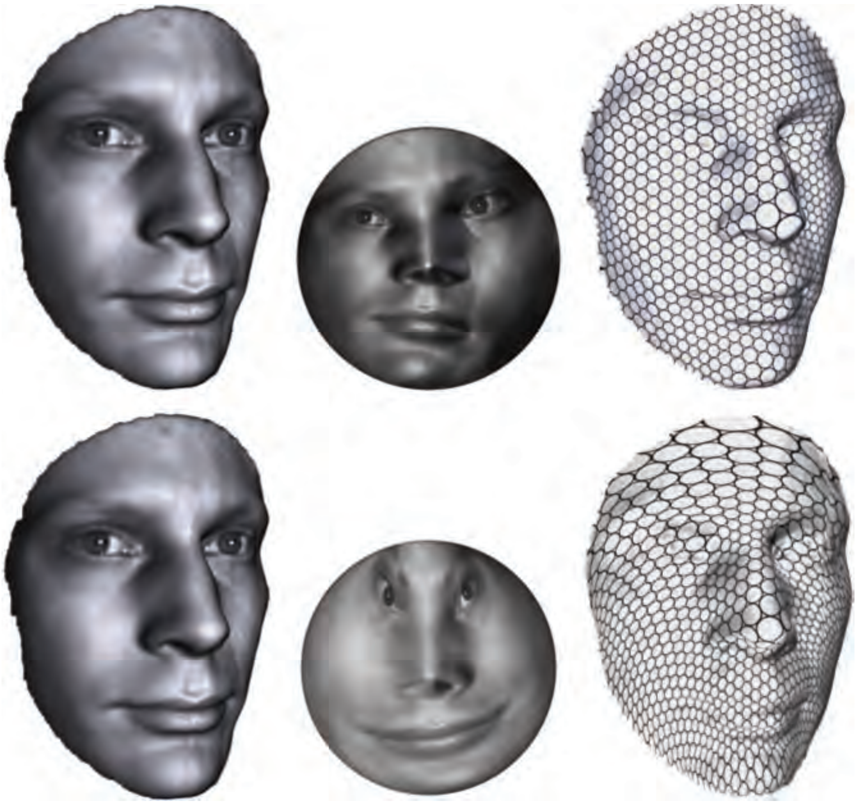


Figure 8.

squares, and we can pull those grids one by one back to the human face; therefore we see the allocation of grids on the human face. If you look closely, you will notice that these mutually perpendicular lines on the square remain perpendicular when pulling back the human face, which is because we used a conformal map when mapping the human face onto the square.

Now take a look at the picture on the right-hand side. There are some small circles on the plane and they remain the same on the human face because the mapping is conformal. Why do we draw the grids? Because we can paint something on each grid of the square and then use the conformal map to pull them back to the face. Since the map is conformal, the pictures on the face are quite similar to the ones that are drawn on the square. This is a very useful technology for animation.

Diffeomorphism

Now let us compare the two sets of facial images in Figure 8. The facial images in the top row can be covered by small circles which remain the same after mapping. That's because we do the mapping by a conformal map. However, the facial images in the bottom row are not mapped by a conformal map but an arbitrary one, so the images of circle tiles on the square are no longer circles, and are of different sizes. This shows that it is not the best way to process images. Thus we can readily see the advantages of conformal mapping.

Graphics Model Representation

In our computations, we make use of the famous theorem due to Poincare *et al*, called the uniformization theorem. Poincaré was a great mathematician in the late 19th century who proved an important theorem: any compact 2-dimensional surface can be conformally mapped to a surface of constant curvature. In Figure 9, we have three images taken by a 3D camera.

The image on the left hand side can be conformally mapped onto the sphere. This is a picture of a child, and we deform it by conformal map onto the sphere. This technique is possible because the famous theorem of Poincaré states that any 2-dimensional surface can be deformed into an image on the sphere by a conformal map so long as it is homeomorphic to a sphere.

The second picture of a cat is different from the one on the left-hand side. It has a handle and cannot be shrunk into a ball. According to Poincaré's theorem, it can be conformally deformed into a flat 2D image.

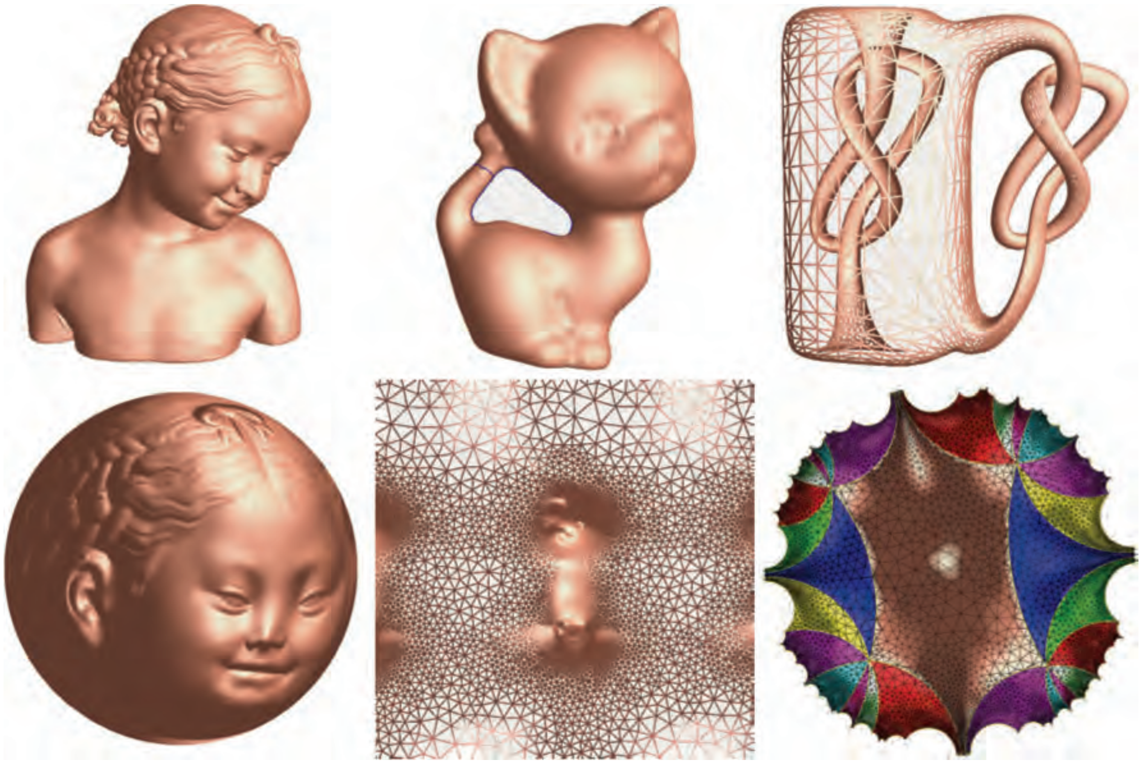


Figure 9.

The third picture, on the right side, is much more complicated than the two on the left, and has a richer topology. We call it a hyperbolic surface.

In this way, the image on the left side, taken by a 3D camera, is mapped to the right-hand side through a harmonic map.

We also develop other graphical representation methods, one of which is to apply the harmonic forms method from Hodge theory. Hodge discovered this theory in the beginning of 20th century, and he was inspired by ideas from fluid mechanics. These new geometric concepts can be applied to image processing.

You may take a look at this relatively complicated image on which we apply Hodge's harmonic 1-form method. Using different harmonic 1-forms, we have found that many different graphs can be drawn on the surfaces. All these beautiful graphs on the surfaces can be used in animation. These methods are supported by elegant mathematical theories. So far, our team alone is able to process and compute images with complicated topologies.

Optimal Transport Theory and Algorithm

Conformal maps keep the angles of an image unchanged. The advantage of conformality is that a tiny rectangle and tiny circle can remain the same, so that either can be well displayed on a human face. Yet we have also found many other important methods besides the conformal map method. One such method is to construct a mapping that can keep the local area unchanged.

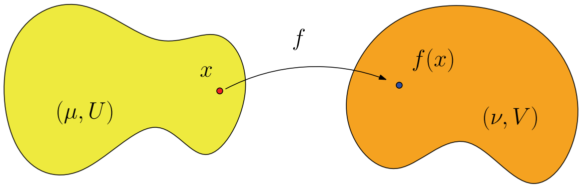


Figure 10.

Here is a one-to-one correspondence mapping from the left hand side region to the right hand side region in which the area remains unchanged (Figure 10). To construct this mapping, we need to solve equations of the following type:

$$det\left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)\nu\left(\nabla u(\mathbf{x})\right)=\mu(\mathbf{x}).$$

This equation has appeared many times in differential geometry; geometric methods can be used in constructing the mapping that keeps the local area unchanged. Also, we can derive this equation by the optimal transport theory. Cheng and I studied this equation forty years ago. And now our team can make use of the method mentioned in our paper to produce solutions of that equation.

Application

Medical Imaging

We reconstruct the shapes of organs from medical images, including the brain and heart, and from them we can compute their geometric characteristics. We can compare different images of a brain in different time periods. Generally speaking, the fundamental issue is to study how the organs change. When a doctor diagnoses a patient, he or she may have to see the changes in the patient's organs for the duration of 10 days or for two-month period. We can compare the earlier with the later images, and if the comparison method is good, doctors can find the cause of disease much more easily.

Conformal Brain Map

The surface of a brain is very complicated, so we have difficulty in dealing with the geometric configuration of a brain. See Figure 11. We have mapped the brain picture on the left-hand side to the sphere of the right-hand side, and the advantage is that the sphere has longitude and latitude on it just as a globe does. Hence, any point on the brain can be defined by longitude and latitude, just as a captain reports that the location of his or her ship to the harbor controls is, say, 20 degrees east longitude and 18 degrees north latitude while sailing in the sea. And the ship can be located.

Now, even though the configuration of a brain is complicated, we can assign a longitude and latitude for each point on its conformal mapping image on the sphere. For example, if there is a tumor in the brain, we only need to know its longitude and latitude and observe its changes after a month, which can be very helpful for medical treatment.

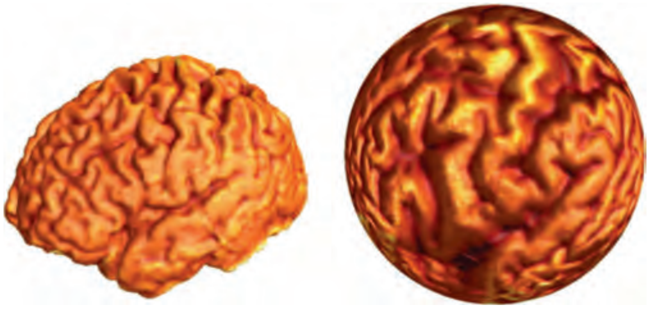


Figure 11.

Automatic Selection of Brain Sulci and Gyri Curve

Usually, when we study an image of the brain, we not only examine its surface but also the curves in its folds. Doctors pay particular attention to the location of the curves among the folds. Our method is to deform the picture of the brain with a conformal map to a region surrounded by a few circles. See Figure 12. This circle represents the line above, so we can see how these curves are distributed in the folds of the brain clearly.

Figure 12.

By using conformal structure, we can compute partial differential equations on a Riemann surface, and its solution can be relatively easily found. For example, we can study the flow of tears on a face by this method.

Detection of HIV Virus

We can detect the presence of the HIV virus by looking for changes in the cerebral cortex. See Figure 13. This change can be spotted by our method, which involves the Beltrami coefficient through a conformal map, so that we can locate the parts of the brain most impacted by the HIV virus.

Detection of Alzheimer's Disease

There are all sorts of diseases that can be detected by our method. After scanning a patient's brain at different times to locate the relatively thick cortical area of the brain, we can diagnose the cause of disease. See Figure 14.

There are different functional regions in the brain, and we can distinguish between them and find out, through our methods, how a brain has changed. See Figure 15.

By processing pictures of the patient's cerebral cortex, taken at different times, through area-preserving mappings, we can measure the area

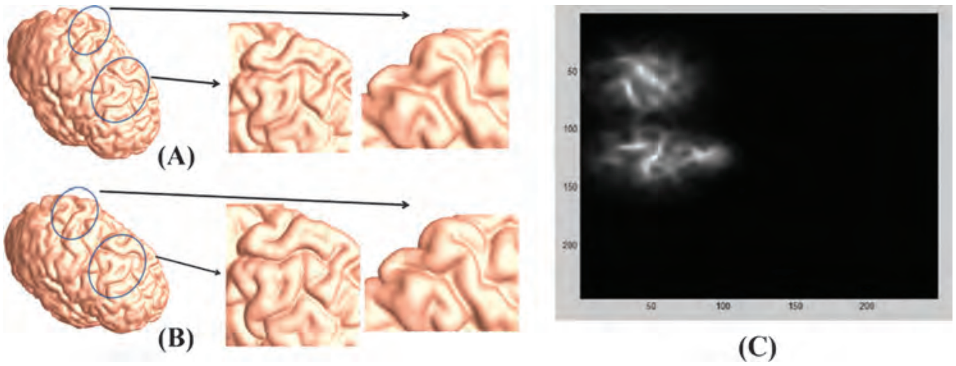


Figure 13.

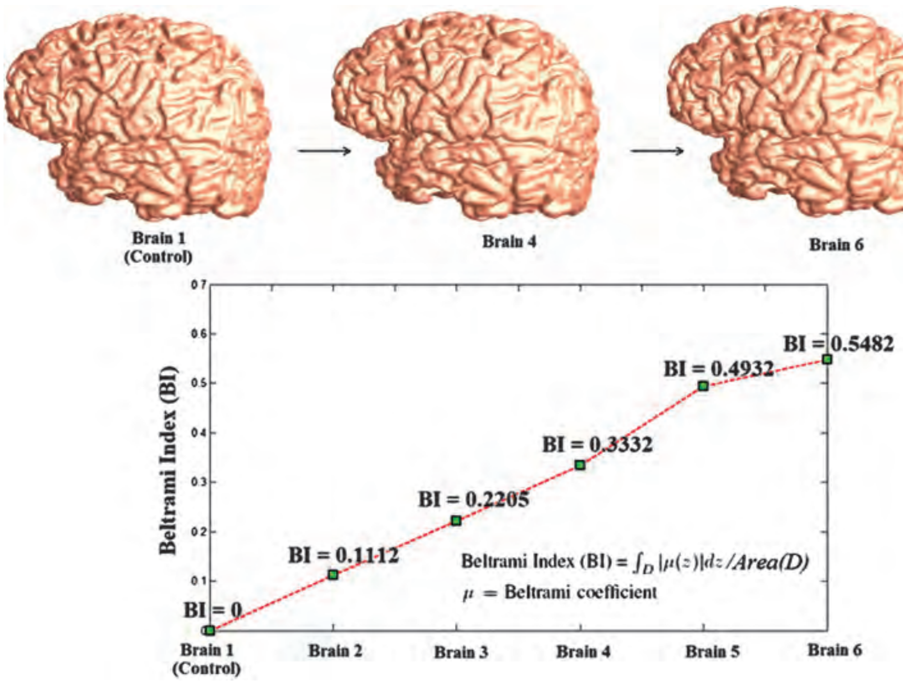
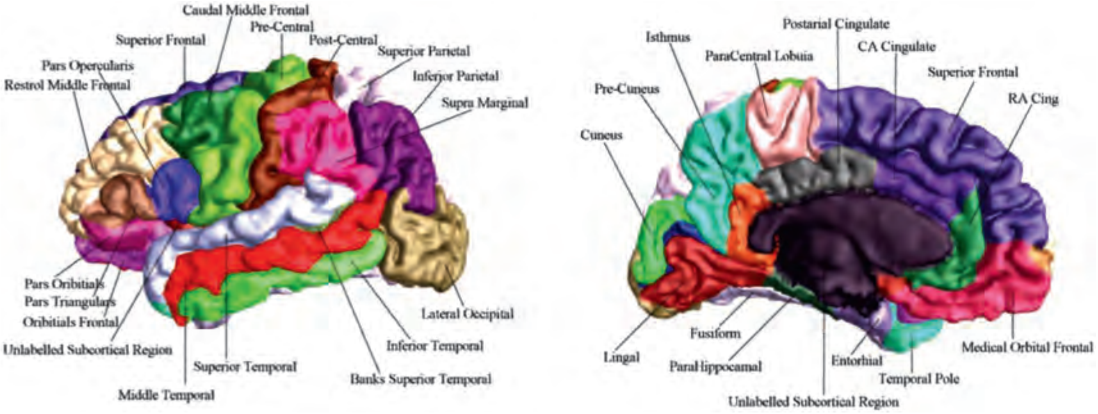


Figure 14.



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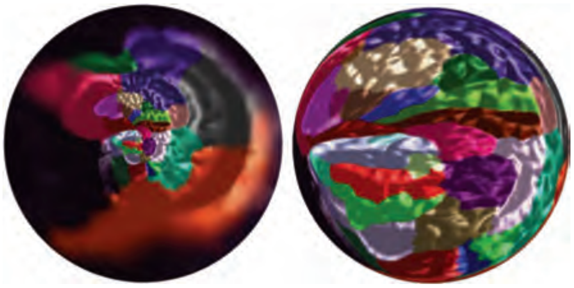


Figure 16.

changes of every functional region, so as to determine the degree of atrophy suffered. See Figure 16. With this information, a diagnosis can be reached as to whether Alzheimer's disease has set in.

Visual Colonoscopy

We draw pictures of the intestine with many grids in order to find the locations of tumors and neoplasias. There is a great advantage in this noninvasive approach in that it eliminates the need for anesthesia and requires no direct contact with patients.

Ordinarily, during a colonoscopy, anesthesia is necessary because there is a risk of complications in the process of inserting the endoscope into intestines. Our method is much better. See Figure 17.

Within our larger team, the group led by Xianfang Gu has developed conformal map software that can be used to locate colon polyps, by flattening the image of the intestinal wall, taken by MRI or X-ray, into a plane. See Figure 18.

Siemens Inc. has applied our method with their large machines, and obtained good results. We observe the changes of polyps by typical maps. We do X-ray mapping of intestines from both their front and back, so that the images can crosscheck each other, and we obtain good 3D images in this way. See Figure 19.

The Application of Computer Vision

Our computer vision application is also important. Computing the geometric characteristics, analyzing the shapes, and cross-tabulating at the same time, is important to the tracking of surfaces. We project the surfaces on a plane and establish their cross-tracking method. See Figure 20.

After constructing a series of images—see Figure 21—of various stages of facial change expressions, from the top one with a closed mouth to the bottom one with an open mouth, we arrived at the

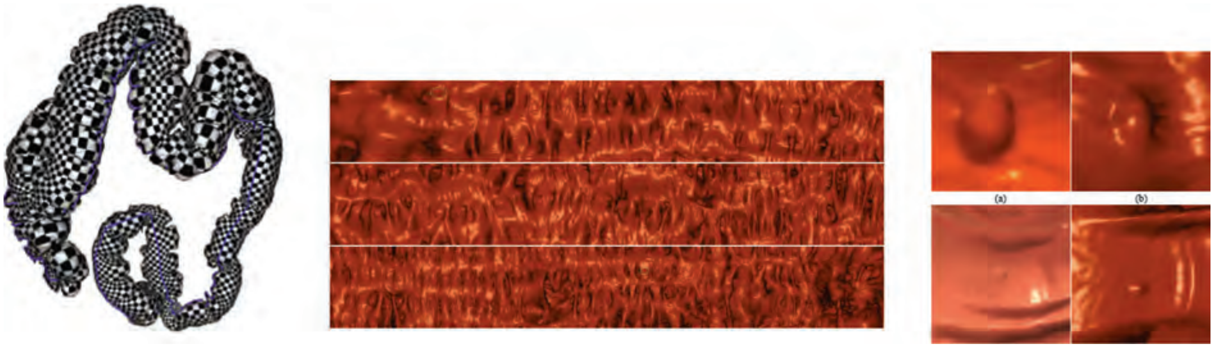


Figure 17.

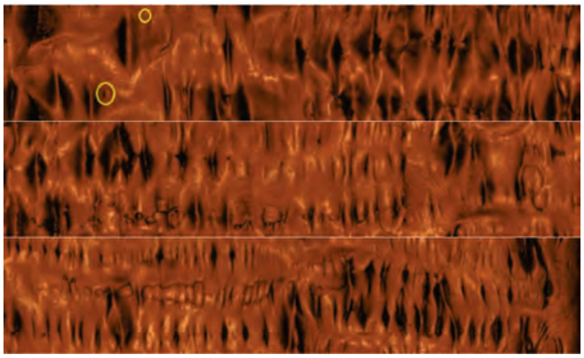


Figure 18.

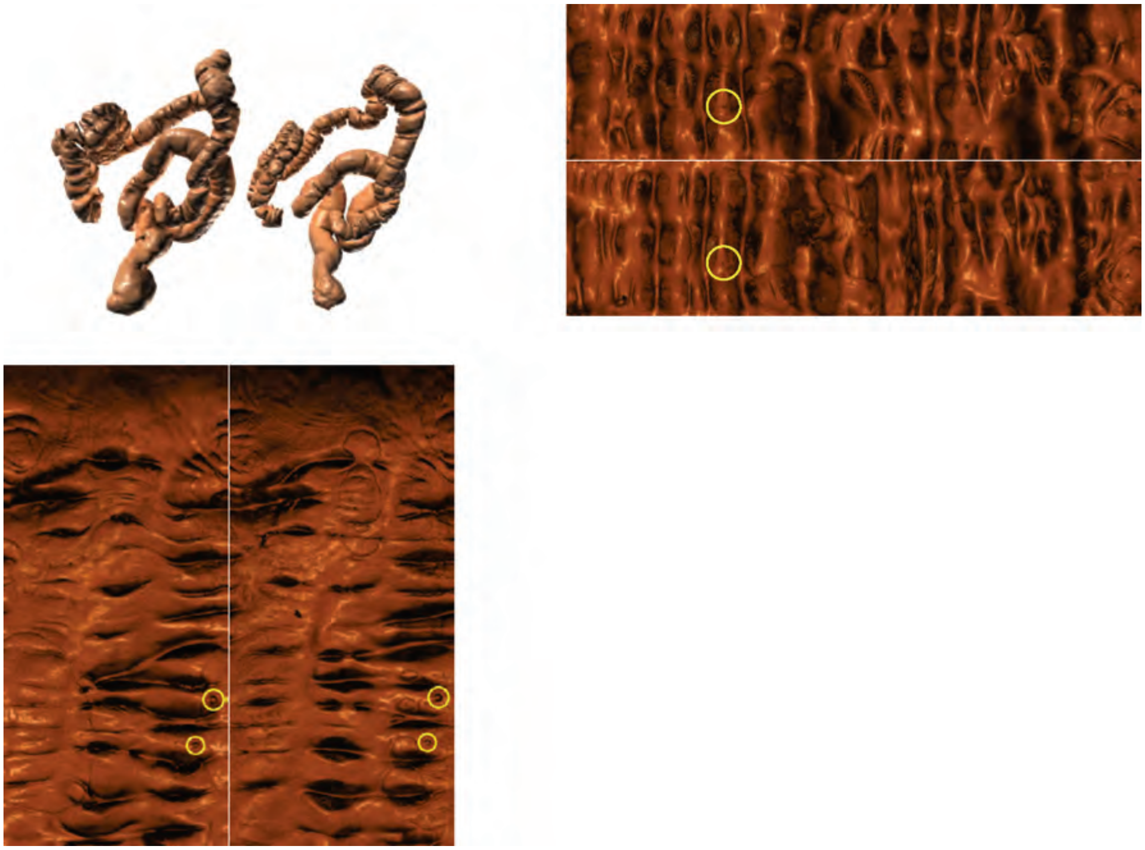


Figure 19.

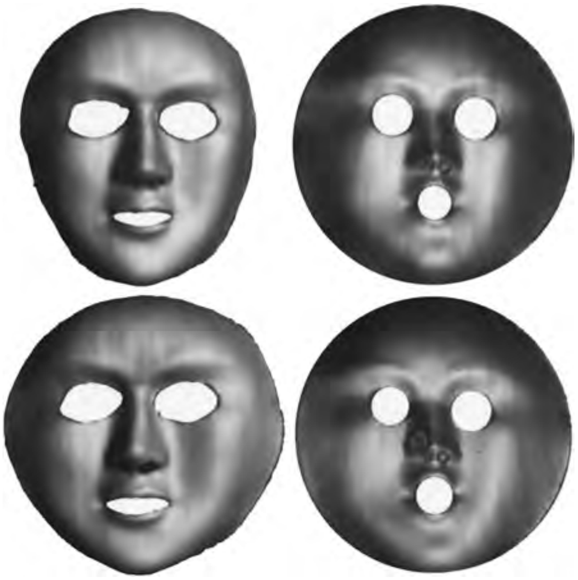


Figure 20.

Figure 21.

right-hand side image. These two images can be compared, by using image processing mode and machine learning method, to achieve facial recognition. This technique helps us in the cataloging of human faces in a computer.

The dynamic performance of facial expressions is very important in animation technology. Let's look at this girl who is smiling (Figure 22). By capturing and naturally segmenting her facial expressions, we can enhance her facial expressions using the production technology in animation.

In the past, the animation of human facial expression was done by hand and took many days to complete, but our method is essentially automatic and

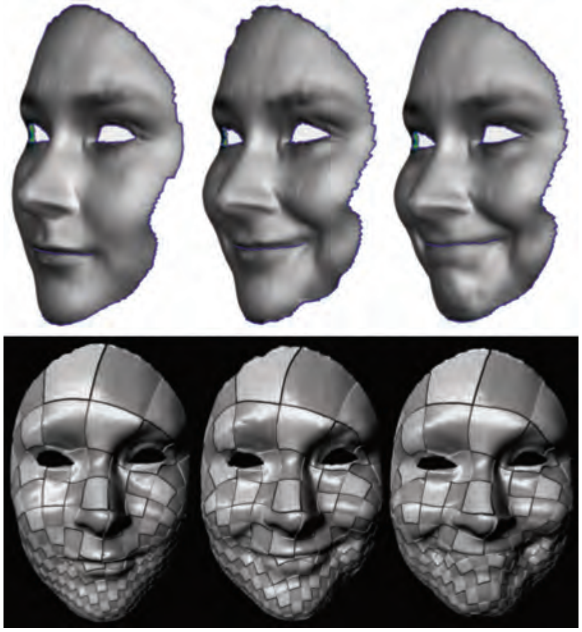
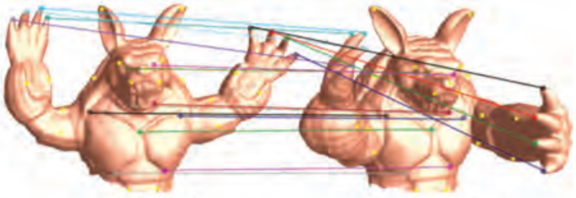


Figure 22.

Figure 23.

can be done in a short period of time. Let us look at the facial expressions that we obtain—see Figure 23—from left to right. Comparing the expressions of the faces from the top row to the bottom row, we see that we can get image tracking. This is our method of capturing dynamic human facial expressions. We can see how the facial expressions change. We can also transfer these expressions to a virtual character in animated films (Figure 24). We can use our own facial expressions to drive other people' facial expressions and we can also use the human facial expressions to drive the facial expressions of animals, which is quite interesting. The results we obtained are very good. With our automated methods, we can use human facial expressions to drive facial expressions of cats or dogs, letting them laugh and cry with a human face. These techniques can be seen in animated films, but film images which were once done by hand but can now be automatically done by our method. We connect the images together and let them transform, and our method is to deform them from 3D to 2D, which is very efficient.



(a) Armadillo #1 (b) Armadillo #2

(c) APP map #1 (d) APP map #2

(e) Conformal map #1 (f) Conformal map #2

Figure 24.

Computer Graphics

There are many problems in graphics that can be solved by our methods. Whether it is surface parameterization, texture mapping, texture synthesis and transfer, vector field design, or other computer technologies, our approach can provide important help. We project a graph to the flat plane to reduce its variation and can then quickly display images with different textures (Figure 25).

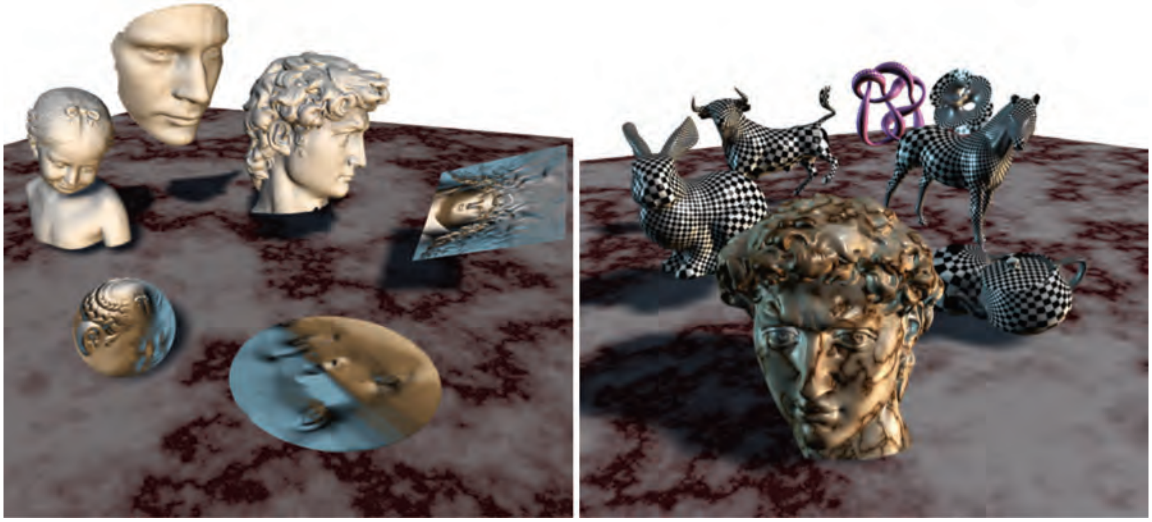
This technique can be used in fashion design. The user can decide the location and index of singular points, so as to simulate the ground color of paintings and clothes of designers (Figure 26).

If you see a sculpture that you like and want to find out how it was made, you can compute it by our methods. We can also turn a surface into a knitted image to produce beautiful artworks (Figure 27).

Texture Transfer

We can transfer one surface texture to another surface to get different textures (Figure 28).

We can use the methods of conformal map and area-preserving map to process this. Manifold spline can help to solve problems on machine tools, and it also has important applications in industry. As for polyhedral mesh conversion, our method of spline surface can produce many different kinds of results (Figure 29).



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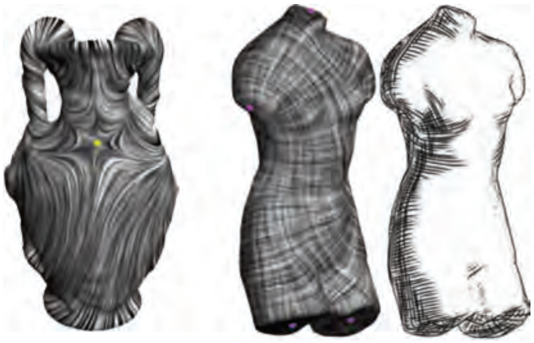

Figure 26.

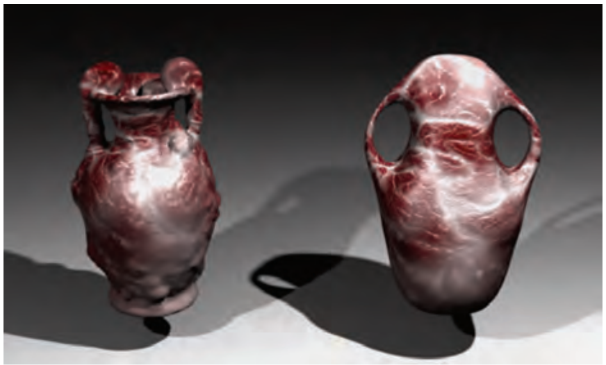

Figure 28.

Figure 27.

These results are important for industries, animation and films. After transforming to spline surface, many things would be possible in industry. One of them is to manufacture machines, which we can remanufacture if we want to make a knockoff. We can make such a knockoff by 3-dimensional mapping through one-piece reproduction. Also, we can use 3-dimensional scanned data to convert them into surface spline. See Figure 30.

We can also make sensors that are important for the internet. In addition, there are important applications in graph theory, but I will not discuss them because of my limited time today.

Surface Triangulation

The graphical presentation of a surface will be clear if it has a good triangulation. We can plot one million points on a human face and connect them with triangles, and the triangles we get are close to equilateral triangles, which is important for computational accuracy.

The usual image processing cannot get complete computation of curvature, so that the general methods can be less than adequate. However, by plotting many points on the image, and at the same time de-

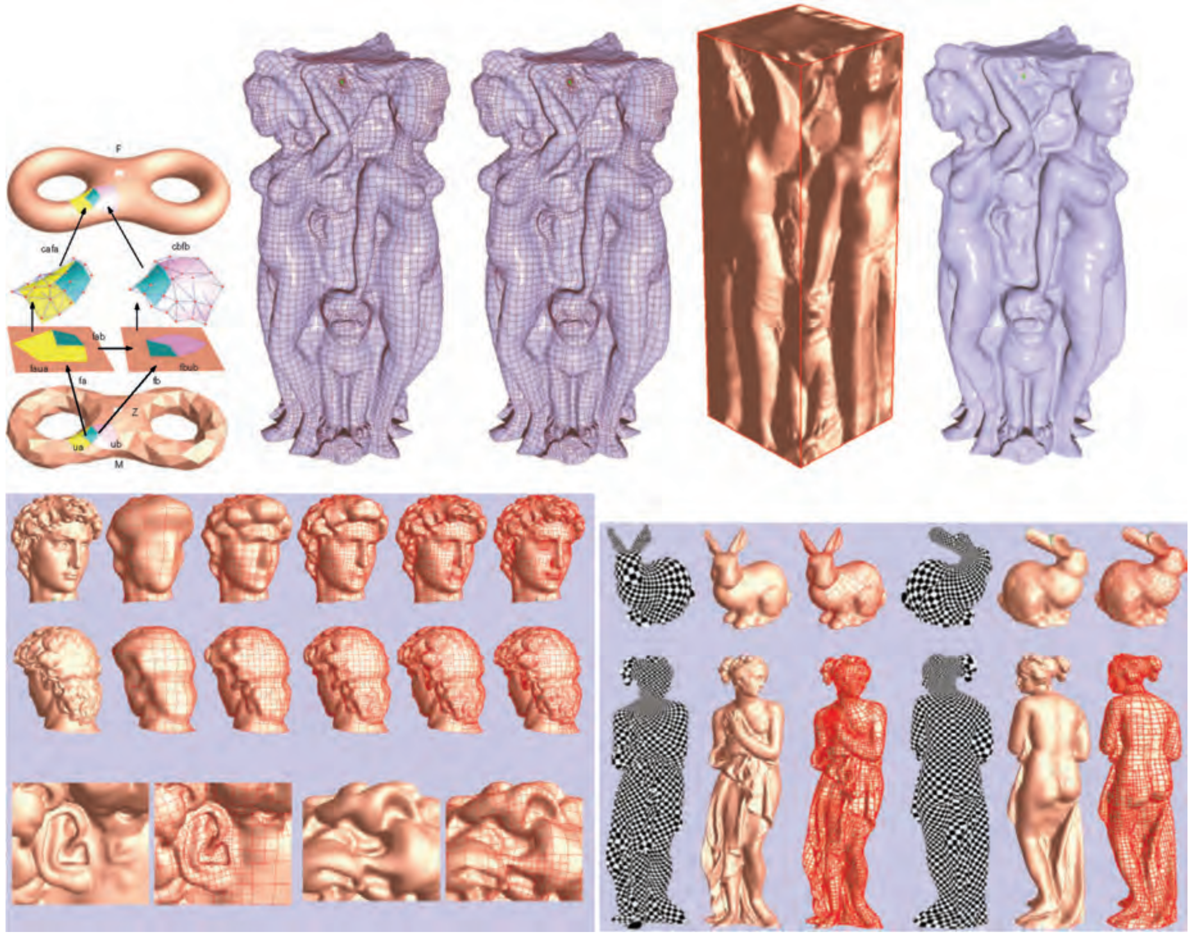

Figure 29.

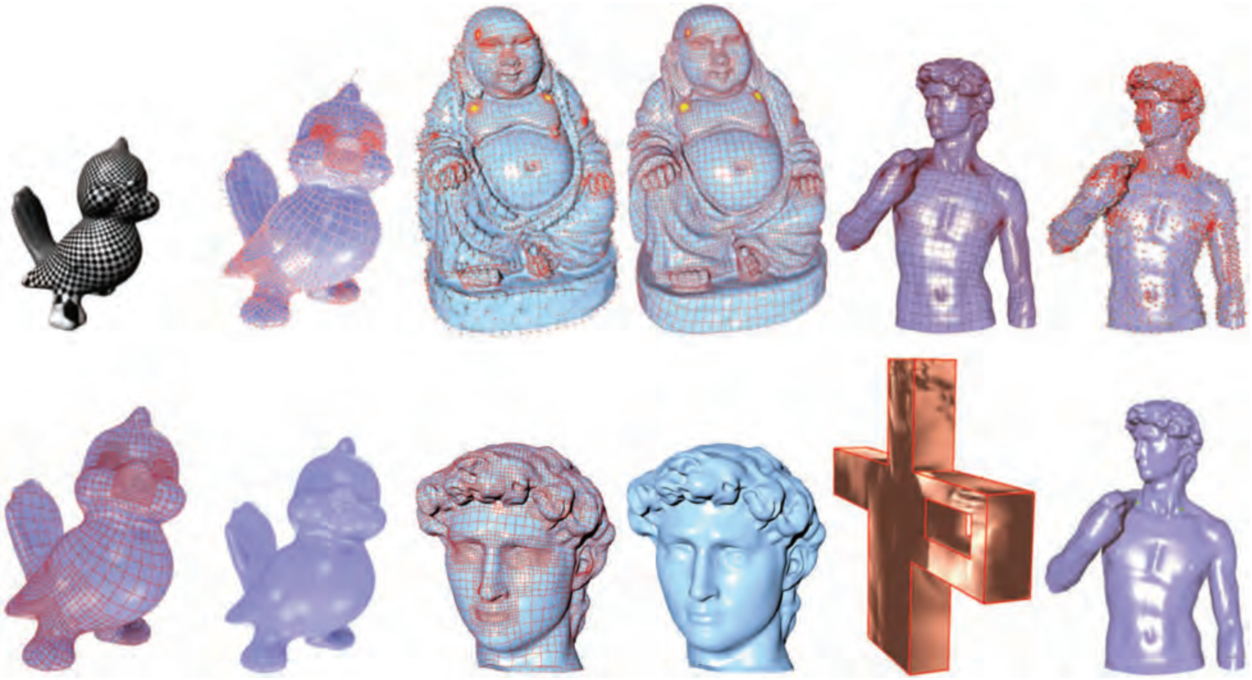

Figure 30.

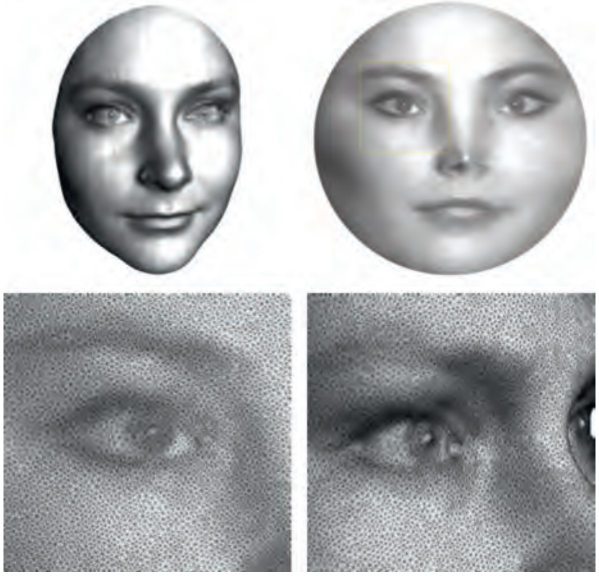

Figure 31.

signing the triangulation well, we can get good results. See Figure 31.

Summary

The development of 3D image processing technology provides challenges for mathematics, which when successfully met can not only promote the development of mathematics but also those of various fields such as engineering and medicine. These methods have become important tools in engineering and medicine, and we're looking forward to their successful development in China.

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