Open Problems

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Note. The readers are welcome to propose the solutions. The authors should send their solutions to liu@math.ucla.edu and post their solutions in arXiv. The correct solutions will be announced and some souvenirs will be presented to the solvers.

Problem 2013013 (Differential geometry) *Proposed by Shing-Tung Yau, Harvard University, USA.*

Prove that any Calabi-Yau manifold with finite fundamental group cannot support nontrivial topological action by compact continuous group. The same should hold for algebraic manifolds with negative first Chern class.

Problem 2013014 (Arithmetic geometry) *Proposed by Ching-Li Chai, Academia Sinica, Taipei/ University of Pennsylvania, USA.*

Let $g \geq 4$ be an integer. Show that there exists a g-dimensional abelian variety over $\overline{\mathbb{F}}_p$ which is not isogenous to the Jacobian of any smooth proper connected curve over $\overline{\mathbb{F}}_p$.

Remark This is a folklore problem and remains a challenge.

Problem 2013015 (Algebraic geometry) *Proposed by Baohua Fu, Chinese Academy of Sciences, China.*

Classify germs of isolated symplectic singularities, i.e., germs of normal varieties with an isolated singularity such that there exists a holomorphic symplectic form on the smooth locus.

Are they all given by a finite quotient of either $\bar{O}_{min}$ or $\mathbb{C}^{2n}$? Note that a complete answer to this problem would imply the long-standing conjecture (due to LeBrun-Salomon) that a Fano contact manifold is always homogeneous. Furthermore, this could give a proof of the folklore conjecture that an isolated symplectic singularity of dimension ≥ 4 admitting a symplectic resolution is locally analytically isomorphic to $\bar{O}_{min}$ in $\mathfrak{sl}_n$ for some n, which has been solved in dimension 4 by Wierzba-Wisniewski.

Problem 2013016 (Number theory, Automorphic form) *Proposed by Wei Zhang, Columbia University, USA.*

Consider a Maass-Hecke eigenform f for a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Suppose that all Hecke eigenvalues are real numbers. Denote its L-function by $L(s,f)$, normalized so that it satisfies a functional equation with center at $s = 1/2$. Prove that the first derivative at the center $s = 1/2$ is non-negative: $L'(1/2, f) \geq 0$.

Remark This is a consequence of the generalized Riemann hypothesis for $L(s,f)$. Similar positivity for a weight two modular form follows from Gross-Zagier formula and positivity of Neron-Tate height pairing.

Problem 2013017 (Topology) *Proposed by Feng Luo, Rutgers University, USA.*

Suppose M^3 is a non-simply connected compact 3-manifold with or without boundary. Show that for any $\gamma \in \pi_1(M) - \{1\}$, there exist a finite commutative ring K with unit and a group homomorphism $\phi : \pi_1(M) \to PGL(2,K)$ so that $\phi(\gamma) \neq id$.

Remark Residually finiteness of the fundamental group of 3-manifolds is known due to the geometrization conjecture. This problem specifies the types of finite groups which can detect the non-trivial group elements.

Problem 2013018 (Nonlinear Elliptic PDEs) *Proposed by Jun-Cheng Wei, The Chinese University of Hong Kong, China.*

Consider the following Allen-Cahn equation

$$\Delta u + u - u^3 = 0 \quad \text{in} \quad \mathbb{R}^N. \tag{1}$$

When $N = 1$, the solution (up to translation) is given by $w(t) = \tanh(\frac{t}{\sqrt{2}})$. For any unit vector $a \in \mathbb{R}^N$, $b \in \mathbb{R}$, the following function

$$u_{a,b} = w(a \cdot x - b) \tag{2}$$

is also a solution. In fact, it is stable.
Question: Are all stable solutions to (1) given by the form (2)?

Remark It is known that for $N = 2$ it is true and for $N \geq 8$ it is no longer true. So we may assume that $3 \leq N \leq 7$.

Problem 2013019 (PDEs) *Proposed by Tong Yang, City University of Hong Kong, China.*

How to analyze the interaction/superposition of boundary layer and wave patterns for the Boltzmann equation?

Problem 2013020 (Fractal) *Proposed by Jiaxin Hu, Tsinghua University, China.*

How to construct a local, conservative, regular, self-similar Dirichlet form on Sierpiski carpet in an analytic way, and to calculate its walk dimension as well? (A two-decade open problem)

Problem 2013021 (The spectrum, Rayleigh's conjecture) *Proposed by Qing-Ming Cheng, Fukuoka University, Japan.*

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain in an n-dimensional Euclidean space $\mathbb{R}^n$. Assume that Γ_1 is the first eigenvalue of the clamped plate problem:

$$\begin{cases} \Delta^2 u = \Gamma u & \text{in } \Omega \\ u = \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

where Δ is the Laplacian in $\mathbb{R}^n$ and ν denotes the outward unit normal. Prove

$$\Gamma_1(\Omega) \geq \Gamma_1(\Omega^*)$$

for $n \geq 4$, where Ω^* denotes the ball in $\mathbb{R}^n$ with the same volume as Ω.

Remark When $n = 2, 3$, Nadirashvili (Arch. Rational Meth. Anal. 129(1995), 1-10) and Ashbaugh and Benguiria (Duke Math. J. 78(1995), 1-17) proved the above inequality. For $n \geq 4$, a partial result was obtained by Ashbaugh and Benguiria.

Problem 2013022 (Complex geometry) *Proposed by Shing-Tung Yau, Harvard University.*

Given an algebraic manifold M of general type, there is the finite dimensional space of holomorphic sections of the pluricanonical bundle mK. There is a birational invariant called pseudonorm. It was proved in the thesis of Chi and later extended by Chi-Yau that this pseudonorm space determines the birational geometry of M completely when m is large enough so that the pluricanonical map is birational to its image. Hence it is important to calculate the pseudonorm for explicit algebraic manifolds of general type, especially for curves of higher genus. The pseudonorm can be calculated in the following way. Given a section of mK, we can look at the inverse image under the m-th power map from the canonical line bundle to mK. One gets a multivalued holomorphic n forms. By taking a base change, or a branch cover of M, one can find a single valued holomorphic n form. The periods of this form can be calculated by the Picard-Fuchs type equations. This will give the pseudonorm of the original problem. Carry out this procedure explicitly for curves of genus two. Find the singular points of the pseudonorms, which amounts to pick out some special automorphic forms. What is the structure of these automorphic forms? It may be interesting to carry out the calculation for meromorphic form whose pseudonorm is finite.