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# $K$-THEORY OF REAL GRASSMANN MANIFOLDS

SUDEEP PODDER AND PARAMESWARAN SANKARAN

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*Abstract*

Let $G_{n,k}$ denote the real Grassmann manifold of $k$-dimensional vector subspaces of $\mathbb{R}^n$. We compute the complex $K$-ring of $G_{n,k}$, up to a small indeterminacy, for all values of $n, k$ where $2 \leqslant k \leqslant n-2$. When $n \equiv 0 \pmod 4, k \equiv 1 \pmod 2$, we use the Hodgkin spectral sequence to determine the $K$-ring completely.

## 1. Introduction

Let $G_{n,k}$ denote the real Grassmann manifold consisting of all $k$-dimensional vector subspaces in the real vector space $\mathbb{R}^n$. We put the standard inner product on $\mathbb{R}^n$. We have the identification of $G_{n,k}$ with the homogeneous space

$$\mathrm{SO}(n)/S(\mathrm{O}(k) \times \mathrm{O}(n-k)),$$

where $\mathrm{O}(k) \times \mathrm{O}(n-k)$ is the subgroup of the orthogonal group $\mathrm{O}(n)$ that stabilises the subspace $\mathbb{R}^k$ spanned by the first $k$ standard basis vectors, and

$$S(\mathrm{O}(k) \times \mathrm{O}(n-k)) = \mathrm{SO}(n) \cap (\mathrm{O}(k) \times \mathrm{O}(n-k)).$$

In this note our aim is to compute the complex $K$-ring of $G_{n,k}$.

Recall that the oriented Grassmann manifold $\widetilde{G}_{n,k} \cong \mathrm{SO}(n)/(\mathrm{SO}(k) \times \mathrm{SO}(n-k))$ is the double cover of $G_{n,k}$ and is simply-connected, except in the case of $\widetilde{G}_{2,1} \cong \mathbb{S}^1$. The description of the $K$-ring of $\widetilde{G}_{n,k}$ goes back to work of Atiyah and Hirzebruch [**AH**] when $n$ is odd or $k$ is even. Note that in each of these cases, the subgroup $\mathrm{SO}(k) \times \mathrm{SO}(n-k)$ is connected and has rank equal to that of the whole group $\mathrm{SO}(n)$. When $n$ is even and $k$ odd the $K$-ring was computed by Sankaran and Zvengrowski [**SZ1**].

The fact that $S(\mathrm{O}(k) \times \mathrm{O}(n-k))$ is not connected makes the determination of the ring $K(G_{n,k})$ difficult and, to the best of our knowledge, has not been carried out for $2 \leqslant k \leqslant n-2$. Note that since $G_{n,k} \cong G_{n,n-k}$, it suffices to consider the case when $k \leqslant n/2$. When $k = 1$, $G_{n,1}$ is the same as the real projective space $\mathbb{R}P^{n-1}$, whose $K$-ring had been determined by Adams [**A**].

Our aim is to express $K^*(G_{n,k}) = K^0(G_{n,k}) \oplus K^1(G_{n,k})$ in terms of generators and relations. However, we have thus far only met with partial success. We obtain complete results only under the assumption that $n \equiv 0 \pmod 4$ and $k$ odd. In the remaining

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cases, our description is complete up to a small indeterminacy. See Theorem 1.2 below and Proposition 5.5.

We now state the two main results of this paper. The proofs will be given in §4 and §5.

**Theorem 1.1.** *Let $n = 2m, k = 2s+1, n-k = 2t+1$ and suppose that $m = s+t+1$ is even. Let $S$ denote the polynomial algebra $\mathbb{Z}[\lambda_1, \ldots, \lambda_s; \mu_1, \ldots, \mu_t, \theta]$ in $s+t+1$ variables. Then*

$$K^0(G_{n,k}) = S/\mathcal{I} = \mathbb{Z}[\lambda_1, \ldots, \lambda_s; \mu_1, \ldots, \mu_t, \theta]/\mathcal{I},$$

*where the ideal $\mathcal{I}$ is generated by the following elements:*

*(i)* $\theta^2 - 1, 2^{m-1}(\theta - 1)$,

*(ii)* $\sum_{0\leqslant p\leqslant j} \lambda_p \mu_{j-p} - \binom{n}{j}\theta^j, 1 \leqslant j \leqslant m-1$, *where* $\lambda_{k-p} = \lambda_p, \mu_{n-k-q} = \mu_q$.

*The $K^0(G_{n,k})$-module $K^1(G_{n,k})$ is the ideal generated by $\theta + 1$ in the ring $S/\widetilde{\mathcal{I}}$, where $\widetilde{\mathcal{I}}$ is generated by elements listed in (ii) above together with $\theta^2 - 1$.*

The element $[\theta]$ in the above theorem corresponds to the complexification of the Hopf line bundle $\xi = \xi_{n,k}$ over $G_{n,k}$, which is associated to double cover $\tilde{G}_{n,k} \to G_{n,k}$. Note that since $\theta^2 - 1 \in \widetilde{\mathcal{I}}$ we have $(\theta - 1)\cdot y = 0$ for all $y \in K^1(G_{n,k})$. It follows that the $S/\widetilde{\mathcal{I}}$-module $K^1(G_{n,k})$ is indeed a module over $S/\mathcal{I} = K^0(G_{n,k})$-module.

Let $\gamma_{n,k}$ be the canonical (real) $k$-plane bundle over $G_{n,k}$. Denote by $\mathcal{K}_{n,k}$ the $\lambda$-subring of $K(G_{n,k})$ generated by the class $[\gamma_{n,k} \otimes \mathbb{C}]$. An algebraic description of $\mathcal{K}_{n,k}$ will be given in §5.

**Theorem 1.2.** *Let $2 \leqslant k \leqslant n/2$. With the above notation, the inclusion*

$$\mathcal{K}_{n,k} \hookrightarrow K(G_{n,k})$$

*has finite cokernel.*

The main tool needed in the proof of Theorem 1.1 is the Hodgkin spectral sequence. This will be recalled in §2. We need to compute the complex representation ring $RH_{n,k}$ of a certain subgroup $H_{n,k}$ of the spin group $\mathrm{Spin}(n)$ and determine its structure as a module over $R\mathrm{Spin}(n)$. The relevant subgroup $H_{n,k}$ is such that $G_{n,k} \cong \mathrm{Spin}(n)/H_{n,k}$. This is carried out in §4 when $n \equiv 0 \pmod 4$ and $k$ is odd. This seems rather complicated for arbitrary values of $n, k$. As an application we obtain bounds for the order of the element $[\xi \otimes \mathbb{C}] - 1 \in K(G_{n,k})$ for any $n, k,\ 2 \leqslant k \leqslant n/2$.

Our proof of Theorem 1.2 uses standard arguments involving the Chern character.

The Hodgkin spectral sequence had been used to determine the $K$-theory of many compact homogeneous manifolds. Hodgkin [**Ho**, §12] applied it to determine the $K$-ring of most of the compact simple Lie groups which are not necessarily simply connected. Roux [**R**] used it to compute the $K$-ring of real Stiefel manifolds, independently of Gitler and Lam [**GL**], who had determined the same using a different approach. Antoniano, et al. [**AGUZ**] and Barufatti and Hacon [**BH**] used the Hodgkin spectral sequence for computing the $K$-ring of real projective Stiefel manifolds, and Minami [**Mi**] for simply connected compact symmetric spaces. See also [**SZ1, SZ2**].

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## 2. The Hodgkin spectral sequence

We briefly recall the Hodgkin spectral sequence here. Let $H$ be a proper closed subgroup of a compact Lie group $G$. We denote the complex representation ring of $G$ by $RG$. Let $\rho\colon RG \to RH$ denote the restriction homomorphism and regard $RH$ as an $RG$-module via $\rho$. Hodgkin [**Ho**] established the existence of a spectral sequence, whose $E_2$-diagram is given by $\mathrm{Tor}^*_{RG}(RH, \mathbb{Z})$, which converges to $K^*(G/H)$ when $\pi_1(G)$ is torsion-free. Here $\mathrm{Tor}^p_A(B, M)$ denotes $\mathrm{Tor}^A_{-p}(B, M)$. In particular, $\mathrm{Tor}^*_{RG}(RH, \mathbb{Z})$ is graded by non-positive integers. *We define the degree of an element* $x \in \mathrm{Tor}^p_A(B, M)$ *to be* $p$.

When the rings $RG, RH$, and $\mathbb{Z}$ are given the trivial $\mathbb{Z}_2$ grading, we obtain a $\mathbb{Z}_2$-grading on $E_2^{p,q}$, where $E_2^{p,q} = \mathrm{Tor}^p_{RG}(RH, \mathbb{Z})$ if $q$ is even and is zero if $q$ is odd. In particular, $0 = E_2^{p,q} = E_\infty^{p,q}$ if $q$ is odd. The differential $d_r\colon E_r^{p,q} \to E_r^{p+r,q-r+1}$ vanishes when $r$ is even.

Using the multiplication in $RH$, one then obtains a $\mathbb{Z}_2$-graded ring structure on $\mathrm{Tor}^*_{RG}(RH, \mathbb{Z})$. The differential in the spectral sequence is an anti-derivation, leading to a $\mathbb{Z}$-graded ring structure on $E_\infty^*$ which is compatible with the $\mathbb{Z}_2$-graded ring $K^*(G/H)$.

If $\mathrm{Tor}^*_{RG}(RH, \mathbb{Z})$ is generated by elements of degree at least $-2$, then the spectral sequence collapses at the $E_2$-stage and we have $\mathrm{Tor}^*_{RG}(RH, \mathbb{Z}) \cong K^*(G/H)$. See [**R**].

Pittie [**P**] has shown that $RH$ is stably free over $RG$ if $H$ is connected, $\pi_1(G)$ is torsion-free, and the rank of $H$ equals the rank of $G$, i.e., if $H$ has a maximal torus $T \subset H$ which is maximal in $G$. Moreover, if $|W(G,T)|/|W(H,T)| > 1 + \dim T$, then $RH$ is a free $RG$-module. (Here $W(G,T)$ denotes the Weyl group of $G$ with respect to $T$.) Consequently the Hodgkin spectral sequence collapses and we have $K(G/H) = \mathrm{Tor}^0_{RG}(RH; \mathbb{Z}) = RH \otimes_{RG} \mathbb{Z}$. In case $G$ is prime to the exceptional Lie groups of type $E_6, E_7, E_8$, this was proved by Atiyah and Hirzebruch [**AH**], who conjectured its validity for any $G$ with $\pi_1(G)$ torsion-free.

### 2.1. Change of rings spectral sequence

Suppose that $G$ is simply connected so that $RG$ is a polynomial ring $\mathbb{Z}[x_1, \ldots, x_m]$. When $RH$ is not a free $RG$-module (via the restriction homomorphism), but is free over a subring $\Lambda = \mathbb{Z}[x_1, \ldots, x_r]$, then it is possible to use the change of rings spectral sequence due to Cartan and Eilenberg [**CE**] to compute $\mathrm{Tor}^*_{RG}(RH, \mathbb{Z})$. See [**R, AGUZ**, §6] and also [**BH**, §6] for a more detailed discussion on the use of the change of rings spectral sequence in the computation of $K(G/H)$. We now recall the Cartan–Eilenberg change of rings theorem.

Let $K$ be any ring. A $K$-algebra $\Lambda$ together with a $K$-homomorphism $\varepsilon\colon \Lambda \to K$ is called a *supplemented $K$-algebra* with augmentation $\varepsilon$. Let $(\Lambda, \varepsilon), (\Gamma, \eta)$ be supplemented $K$-algebras, and let $\varphi\colon \Lambda \to \Gamma$ be a $K$-algebra homomorphism such that $\eta \circ \varphi = \varepsilon$. Denote $\ker(\varepsilon)$ by $I(\Lambda)$. A $K$-algebra homomorphism $\varphi\colon \Lambda \to \Gamma$ is *normal* if the left ideal, denoted $\Gamma \cdot I(\Lambda)$, of $\Gamma$ generated by $\varphi(I(\Lambda))$, is also a right ideal of $\Gamma$

(always the case when $K$ is commutative). Then $\Omega := \Gamma/(\Gamma \cdot I(\Lambda))$ is a supplemented $K$-algebra.

We shall state the theorem in the special case of *commutative* augmented $K$-algebras. So if $\Gamma,\ \Lambda$ are supplemented, any augmentation preserving $K$-homomorphism $\Gamma \to \Lambda$ is normal. In our applications, $K = \mathbb{Z}$, $\Gamma = RG$, $\Lambda$ will be a subring of $\Gamma$, and $A = RH$, where the $\Gamma$-module structure is given via the restriction homomorphism $\rho\colon RG \to RH$. Also, the $\Omega$-module $C$ in the statement of the theorem below will be $\mathbb{Z}$ (via the augmentation).

**Theorem 2.1.** ([**CE**, Theorem 6.1, Chapter XVI]) *We keep the above notations. Suppose that $K$ is commutative. Suppose that $\varphi\colon \Lambda \to \Gamma$ is normal and that $\Gamma$ is projective as a $\Lambda$-module (via $\varphi$). Then, for any $\Gamma$-module $A$ and $\Omega$-module $C$, there exists a spectral sequence $\mathrm{Tor}^{\Omega}_*(\mathrm{Tor}^{\Lambda}_*(A,K),C)$ that converges to $\mathrm{Tor}^{\Gamma}_*(A,C)$.*

The $\Omega$-module structure on $\mathrm{Tor}^{\Lambda}_q(A,K)$ arises from the functorial isomorphism $\mathrm{Tor}^{\Gamma}_q(A,\Omega) = \mathrm{Tor}^{\Gamma}_q(A,\Gamma \otimes_\Lambda K) \cong \mathrm{Tor}^{\Lambda}_q(A,K)$. (See [**CE**] for details.)

## 3. The representation ring of $H_{n,k}$

We follow the notations of Husemoller's book [**H**] closely in our description of the representation rings of the groups $\mathrm{SO}(n)$ and $\mathrm{Spin}(n)$.

Let $2 \leqslant k \leqslant \lfloor n/2 \rfloor$. Recall that $H_{n,k}$ is the inverse image of $S(\mathrm{O}(k) \times \mathrm{O}(n-k))$ under the double cover $\pi\colon \mathrm{Spin}(n) \to \mathrm{SO}(n)$. The identity component of $H_{n,k}$ is the group $H^0_{n,k} := \mathrm{Spin}(k) \cdot \mathrm{Spin}(n-k) \subset \mathrm{Spin}(n)$ with quotient $H_{n,k}/H^0_{n,k} \cong \mathbb{Z}_2$. Although the representation ring of $H^0_{n,k}$ has been worked out in [**SZ1**], we shall give most of the details here in order to make the exposition self-contained. Note that $H^0_{n,k}$ is the quotient of $\mathrm{Spin}(k) \times \mathrm{Spin}(n-k)$ by the cyclic subgroup of order 2 generated by $(-1,-1)$. The canonical surjection $\mathrm{Spin}(k) \times \mathrm{Spin}(n-k) \to H^0_{n,k}$ induces a ring monomorphism $RH^0_{n,k} \to R(\mathrm{Spin}(k) \times \mathrm{Spin}(n-k))$ which we regard as an inclusion. The image is generated as an abelian group by representations of $\mathrm{Spin}(k) \times \mathrm{Spin}(n-k)$ on which $(-1,-1)$ acts as identity. Likewise, the projection $H^0_{n,k} \to \mathrm{SO}(k) \times \mathrm{SO}(n-k)$ induces a monomorphism

$$R(\mathrm{SO}(k) \times \mathrm{SO}(n-k)) \to RH^0_{n,k},$$

which we regard as an inclusion, whose image is generated by representations of $H^0_{n,k}$ on which the kernel of the projection acts as the identity. This allows us to describe $RH^0_{n,k}$ in a straightforward manner. The ring $R(\mathrm{SO}(k) \times \mathrm{SO}(n-k))$ is a polynomial ring when $n$ is even and $k$ is odd. The ring homomorphism

$$R\mathrm{SO}(2r+1) \to R\mathrm{SO}(2r) \ \text{ induced by the inclusion } \ \mathrm{SO}(2r) \hookrightarrow \mathrm{SO}(2r+1)$$

is a monomorphism. Moreover, $R\mathrm{SO}(2r+1)$ is a polynomial ring in $r$ indeterminates. The ring $R\mathrm{SO}(2r)$ is not isomorphic to a polynomial algebra; it is known that $R\mathrm{SO}(2r)$ is generated over $R\mathrm{SO}(2r+1)$ by an element $\lambda^+_r$ which satisfies a monic quadratic equation. As such $R\mathrm{SO}(2r)$ is a free $R\mathrm{SO}(2r+1)$-module of rank 2. So, for all parities of $k,n$, $R(\mathrm{SO}(k) \times \mathrm{SO}(n-k))$ is a free module of finite rank over a polynomial ring generated by $\lfloor k/2 \rfloor + \lfloor (n-k)/2 \rfloor$ indeterminates. We will show in this section that the same statement holds for $RH_{n,k}$ as well.

Before proceeding further in describing $RH^0_{n,k}, RH_{n,k}$, we need to introduce notations for certain natural representations of the spin and special orthogonal groups.

Set

$$k = 2s + \varepsilon, \quad n - k = 2t + \eta, \quad \varepsilon, \eta \in \{0, 1\} \text{ where } s, t \text{ are integers.}$$

Now $n = 2s + 2t + 1$ if $n$ is odd. When $n$ is even, both $k$ and $n - k$ are of the same parity and $n = 2s + 2t$ or $n = 2s + 2t + 2$ according as $k$ is even or odd. Let $\lambda_1$ denote the standard $k$-dimensional complex representation of $\mathrm{SO}(k)$. We then denote by $\lambda_j \in R\mathrm{SO}(k)$ the $j$th exterior power $\Lambda^j_{\mathbb{C}}(\lambda_1), j \leqslant k$. (It is understood that $\lambda_0 = 1$, the trivial representation).[1] We have the equality

$$\lambda_j = \lambda_{k-j} \text{ in } R\mathrm{SO}(k).$$

When $k$ is even, the Hodge star operator $*$ yields a splitting $\lambda_s = \lambda_s^+ + \lambda_s^-$, where $\lambda_s^+, \lambda_s^- \in R\mathrm{SO}(2s)$ are the classes of $+1, -1$-eigenspaces when $k \equiv 0 \pmod 4$ and are the $i, -i$-eigenspaces when $k \equiv 2 \pmod 4$ respectively. In the case of $\mathrm{Spin}(k)$ we have the spin representation $\Delta_s$. When $k$ is even, it splits as a sum of two half-spin representations $\Delta_s^+, \Delta_s^-$; they are distinguished by the way an element $z_0$ in the centre of $\mathrm{Spin}(k)$ acts. (This will be made precise later.) We have the following theorem proved in [**H**, §10, Chapter 13]. In the case of $R\mathrm{SO}(2s)$, our description is slightly different from the one given in Husemoller's book *op. cit.*, but it is readily seen that the two descriptions are equivalent.

**Theorem 3.1.** *With the above notations, we have*

*(i)* $R\mathrm{Spin}(2s) = \mathbb{Z}[\lambda_1, \dots, \lambda_{s-2}, \Delta_s^+, \Delta_s^-]$,
*(ii)* $R\mathrm{Spin}(2s+1) = \mathbb{Z}[\lambda_1, \dots, \lambda_{s-1}, \Delta_s]$,
*(iii)* $R\mathrm{SO}(2s+1) = \mathbb{Z}[\lambda_1, \dots, \lambda_s]$, *and,*
*(iv)* $R\mathrm{SO}(2s) = \mathbb{Z}[\lambda_1, \lambda_2, \dots, \lambda_s][\lambda_s^+]/\sim$

*where the ideal of relations is generated by* $(\lambda_s^+)^2 - a\lambda_s^+ - b$ *for suitable polynomials* $a, b$ *in* $\lambda_j$, $1 \leqslant j \leqslant s$ *(with $\mathbb{Z}$-coefficients).*

As the notation suggests, the rings $R\mathrm{Spin}(2s), R\mathrm{Spin}(2s+1), R\mathrm{SO}(2s+1)$ are polynomial rings in the indicated variables. Also, the elements $\lambda_j, 1 \leqslant j \leqslant s$, are algebraically independent in $R\mathrm{SO}(2s)$.

*Remark 3.2.* The quadratic relation that $\lambda_s^+$ satisfies over $\mathbb{Z}[\lambda_1, \dots, \lambda_s]$ can be explicitly written down as follows: Set $\lambda_s^- := \lambda_s - \lambda_s^+$. From [**H**, Theorem 10.3, Chapter 13], we have the relation

$$\lambda_s^+ \cdot \lambda_s^- = (\lambda_{s-1} + \lambda_{s-3} + \cdots)^2 - \lambda_s(\lambda_{s-2} + \lambda_{s-4} + \cdots) - (\lambda_{s-2} + \lambda_{s_4} + \cdots)^2$$

in $\mathbb{Z}[\lambda_1, \dots, \lambda_s]$. Denoting the *negative* of the right hand side of the last equality by $b$ and setting $a := \lambda_s$, we have

$$(\lambda_s^+)^2 = \lambda_s^+(\lambda_s - \lambda_s^-) = a\lambda_s^+ + b.$$

The inclusion $\mathrm{Spin}(2s) \hookrightarrow \mathrm{Spin}(2s+1)$ induces an injective ring homomorphism

$$\rho\colon R\mathrm{Spin}(2s+1) \to R\mathrm{Spin}(2s) \text{ where } \rho(\Delta_s) = \Delta_s^+ + \Delta_s^-, \rho(\lambda_i) = \lambda_i + \lambda_{i-1},$$

$1 \leqslant i \leqslant s$. The homomorphism $R\mathrm{Spin}(2s) \to R\mathrm{Spin}(2s-1)$ induced by the inclusion

[1] We shall often use the same notation for a representation and its class in the representation ring.

$\mathrm{Spin}(2s-1) \hookrightarrow \mathrm{Spin}(2s)$ is given by $\lambda_j \mapsto \lambda_j + \lambda_{j-1}, 1 \leqslant j < s, \Delta_s^{\pm} \mapsto \Delta_{s-1}$. These restriction homomorphisms also yield the restrictions $R\mathrm{SO}(k) \to R\mathrm{SO}(k-1)$ for any parity of $k$.

Recall that given any two compact Lie groups $H_1, H_2$, we have $R(H_1 \times H_2) = RH_1 \otimes RH_2$. We have the natural quotient homomorphisms

$$\pi_0\colon \mathrm{Spin}(k) \times \mathrm{Spin}(n-k) \to H^0_{n,k} \quad \text{and} \quad \pi\colon H^0_{n,k} \to \mathrm{SO}(k) \times \mathrm{SO}(n-k),$$

where $\ker(\pi_0) \cong \mathbb{Z}_2$ is generated by $(-1,-1) \in \mathrm{Spin}(k) \times \mathrm{Spin}(n-k)$ and $\ker \pi \cong \mathbb{Z}_2$, by $\pi_0(1,-1) \in H^0_{n,k}$. We shall regard the ring homomorphisms

$$\pi_0^*\colon RH^0_{n,k} \to R(\mathrm{Spin}(k) \times \mathrm{Spin}(n-k)), \quad \pi^*\colon R(\mathrm{SO}(k) \times \mathrm{SO}(n-k)) \to RH^0_{n,k},$$

which are injective, as inclusions. It is easy to see that $RH^0_{n,k}$ is generated as an $R(\mathrm{SO}(k) \times \mathrm{SO}(n-k))$-algebra by elements $xy \in R(\mathrm{Spin}(k) \times \mathrm{Spin}(n-k))$ where $x, y$ vary over the $R(\mathrm{SO}(k) \times \mathrm{SO}(n-k))$-algebra generators of $R(\mathrm{Spin}(k) \times \mathrm{Spin}(n-k))$. The following description, in Proposition 3.3, of $RH^0_{n,k}$ is an immediate consequence of Theorem 3.1.

We shall use the notation $\mu_j \in R\mathrm{SO}(n-k)$ for the element represented by the $j$th exterior power of the standard representation of $\mathrm{SO}(n-k)$. Also $\Delta'_t$, and $\Delta'^{\pm}_t$ will denote the spin and half-spin representations of $\mathrm{Spin}(n-k)$ respectively. Thus $R(\mathrm{SO}(k) \times \mathrm{SO}(n-k))$ contains the polynomial subring $\mathbb{Z}[\lambda_1, \dots, \lambda_s, \mu_1, \dots, \mu_t]$.

**Proposition 3.3.** *We keep the above notations. Let $R := R(\mathrm{SO}(k) \times \mathrm{SO}(n-k))$. Then*

$$RH^0_{n,k} = \begin{cases} R[\Delta_s \Delta'_t], & \text{if } k = 2s+1, n-k = 2t+1, \\ R[\Delta_s (\Delta'_t)^{\pm}], & \text{if } k = 2s+1, n-k = 2t, \\ R[\Delta_s^{\pm} \Delta'_t], & \text{if } k = 2s, n-k = 2t+1, \\ R[\Delta_s^{\pm} (\Delta'_t)^{\pm}, \Delta_s^{\pm} (\Delta'_t)^{\mp}], & \text{if } k = 2s, n-k = 2t. \end{cases}$$

*Moreover, the squares of the indicated generators belong to $R$.*

**Notations 3.4.** We shall denote by $\Delta_{s,t}$ the element

$$\Delta_s \Delta'_t \in R(\mathrm{Spin}(k) \times \mathrm{Spin}(n-k)).$$

Also $\Delta_{s,t}^{\varepsilon,\eta}$ will denote $\Delta_s^{\varepsilon} \cdot (\Delta'_t)^{\eta}, \varepsilon, \eta \in \{+,-\}$. Also, we shall use upper case letters $\Lambda_j, 1 \leqslant j \leqslant m$, etc., to denote the generators of $R\mathrm{Spin}(n)$ and similarly $\lambda_1, \dots, \lambda_s$ (resp. $\mu_1, \dots, \mu_t$) to denote generators of $R\mathrm{Spin}(k)$ (resp. $R\mathrm{Spin}(n-k)$) as in Theorem 3.1.

Next we turn our attention to the representation ring of $H_{n,k}$. Recall that we have $2 \leqslant k \leqslant n/2$ and so $n \geqslant 4$. First we analyse when the exact sequence

$$1 \to H^0_{n,k} \to H_{n,k} \to Z \to 1 \tag{1}$$

splits. Evidently, the sequence splits if and only if there exists an order 2 element $z_0 \in H_{n,k} \subset \mathrm{Spin}(n)$ such that $z_0 \notin H^0_{n,k}$. Taking $z_0 := e_1 e_2 e_3 e_n \in C_n$, we see that $z_0^2 = 1$ and $z_0 \in H_{n,k} \setminus H^0_{n,k}$, so the short exact sequence (1) splits. Here $C_n$ denotes the Clifford algebra of the quadratic space $(\mathbb{R}^n, -|| \cdot ||^2)$ and $e_1, \dots, e_n$ denote the standard basis vectors of $\mathbb{R}^n$. So $H_{n,k} \cong H^0_{n,k} \rtimes \mathbb{Z}_2$.

Suppose that $H_{n,k} = H^0_{n,k} \times Z$ and let $z_0$ be the generator of $Z \cong \mathbb{Z}_2$. Then

$$\pi(H^0_{n,k}) \times \pi(Z) = \pi(H_{n,k}) = S(\mathrm{O}(k) \times \mathrm{O}(n-k))$$

is isomorphic to the product $\mathrm{SO}(k) \times \mathrm{SO}(n-k) \times \{\pm I_n\}$. In particular, $n$ is even and $k$ is odd and $z_0 \in Z$ maps to $-I_n$. So, the order 2 element $z_0$ is in the centre of $\mathrm{Spin}(n)$. It follows that $n \equiv 0 \pmod 4, k \equiv 1 \pmod 2$.

When $n \equiv 0 \pmod 4$, $k \equiv 1 \pmod 2$, we may take $z_0 = e_1 e_2 \cdots e_n \in H_{n,k}$. Then $z_0$ is in the centre of $H_{n,k}$ and $z_0 \notin H^0_{n,k}$ and so $H_{n,k}$ is the direct product $H^0_{n,k} \times Z$.

Thus $H_{n,k} \cong H^0_{n,k} \times \mathbb{Z}_2$ if and only if $n \equiv 0 \pmod 4, k \equiv 1 \pmod 2$.

Using Proposition 3.3, we obtain the following.

**Proposition 3.5.** *We keep the above notations. Let $k = 2s+1, n-k = 2t+1$, and $s+t$ odd. Let $f_{s,t} \in R := R(\mathrm{SO}(k) \times \mathrm{SO}(n-k))$ be the element such that $\Delta^2_{s,t} = f_{s,t}$ and let $\theta$ be the class of the unique non-trivial one-dimensional representation of $H_{s,t}$. Then*

$$RH_{n,k} = RH^0_{n,k} \otimes RZ = R[\Delta_{s,t}, \theta]/\langle \theta^2 - 1, \Delta^2_{s,t} - f_{s,t} \rangle. \tag{2}$$

*In particular, $RH_{n,k}$ is a free $R$-module with basis $\{1, \theta, \Delta_{s,t}, \theta\Delta_{s,t}\}$.*

Writing $\lambda_0 = 1 = \mu_0$, $f_{s,t} \in R$ can be expressed as a polynomial in $\lambda_p, \mu_q \in R$, for $0 \leqslant p \leqslant s, 0 \leqslant q \leqslant t$ as follows (see [**H**, Theorem 10.3, Chapter 14].)

$$f_{s,t} = \Delta^2_{s,t} = \Delta^2_s \cdot (\Delta'_t)^2 = (\sum_{0 \leqslant p \leqslant s} \lambda_p)(\sum_{0 \leqslant q \leqslant t} \mu_q) = \sum_{0 \leqslant r \leqslant s+t} (\sum_{p+q=r} \lambda_p \mu_q). \tag{3}$$

## 4. The restriction homomorphism $R\mathrm{Spin}(n) \to RH_{n,k}$

Throughout this section we assume that $k = 2s+1, n-k = 2t+1$ so that $n = 2m$, where $m := s+t+1$. Also we shall assume that $s+t$ is odd so that $n \equiv 0 \pmod 4$. Hence $H_{n,k} = H^0_{n,k} \times Z$ where $Z \cong \mathbb{Z}_2$ is generated by $z_0 = e_1 \cdots e_n \in \mathrm{Spin}(n)$.

The double covering $\phi\colon \mathrm{Spin}(n) \to \mathrm{SO}(n)$ is defined as $\phi(u)(x) = uxu^*$, $x \in \mathbb{R}^n$, where $*$ is (the restriction to $\mathrm{Spin}(n)$ of) the anti-involution of the Clifford algebra $C_n$, uniquely defined by the requirement: $v^* = v$, $v \in \mathbb{R}^n$. We refer the reader to [**H**] concerning the spin group and its representation ring.

**Maximal tori**

Set $\omega(\theta_1, \ldots, \theta_m) := \prod_{1 \leqslant j \leqslant m} (\cos 2\pi\theta_j + \sin 2\pi\theta_j . e_{2j-1} e_{2j}) \in \mathrm{Spin}(n)$ for $\theta_j \in \mathbb{R}$. Then

$$\widetilde{T} := \{\omega(\theta_1, \ldots, \theta_m) \in \mathrm{Spin}(n) \mid \theta_j \in \mathbb{R}, 1 \leqslant j \leqslant m\} \cong (\mathbb{S}^1)^m$$

is a maximal torus of $\mathrm{Spin}(n)$. Its image in $\mathrm{SO}(n)$ is the standard maximal torus $T := \mathrm{SO}(2) \times \cdots \times \mathrm{SO}(2)$ whose elements restrict to rotations on $\mathbb{R}e_{2j-1} + \mathbb{R}e_{2j}$, whenever $1 \leqslant j \leqslant m$. In fact $\phi(\omega(\theta_1, \ldots, \theta_m)) = D(2\theta_1, \ldots, 2\theta_m) \in T$ where $D(t_1, \ldots, t_m)$ restricts to the positive rotation by angle $2\pi t_j$ on the oriented vector subspace $\mathbb{R}e_{2j-1} + \mathbb{R}e_{2j}, 1 \leqslant j \leqslant m$, the orientation being given by the ordering $e_{2j-1}, e_{2j}$ of the basis elements.

Let $\mathbb{T}$ be the 'standard torus' $(\mathbb{S}^1)^m = (\mathbb{R}/\mathbb{Z})^m$. One has a homomorphism

$$\omega\colon \mathbb{T} \to \widetilde{T} \text{ defined by } (\theta_1, \ldots, \theta_m) \mapsto \omega(\theta_1, \ldots, \theta_m).$$

Note that $\omega(\theta_1+\varepsilon_1/2,\ldots,\theta_m+\varepsilon_m/2)=(-1)^\varepsilon\omega(\theta_1,\ldots,\theta_m)$ where $\varepsilon_j\in\{0,1\}$ for all $j$, and $\varepsilon=\sum_{1\leqslant j\leqslant m}\varepsilon_j$. In particular $\ker(\omega)\cong(\mathbb{Z}_2)^{m-1}$. The kernel of $\phi\circ\omega\colon\mathbb{T}\to T$ is readily seen to be $\mathbb{Z}_2^m\cong\{-1,1\}^m\subset\mathbb{T}$.

Since $n$ is even and $k$ is odd, the rank of $H^0_{n,k}$ equals $m-1=\operatorname{rank}(\operatorname{Spin}(n))-1$. In this case,

$$\widetilde{T}_0:=H^0_{n,k}\cap\widetilde{T}=\{\omega(\theta_1,\ldots,\theta_m)\in\widetilde{T}\mid\theta_{s+1}=0\}$$

is a maximal torus of $H^0_{n,k}$. Also, we observe that the element $z_0=e_1\ldots e_n$, the generator of $Z$, belongs to $\widetilde{T}$. Let $T_0=\pi(\widetilde{T}_0)=T\cap(\mathrm{SO}(k)\times\mathrm{SO}(n-k))$ which is a maximal torus of $\mathrm{SO}(k)\times\mathrm{SO}(n-k)$.

The representation rings of $\widetilde{T},\widetilde{T}_0,T,T_0$ are viewed as subrings of $R\mathbb{T}$ as follows: Let $u_j\colon\mathbb{T}\to\mathbb{S}^1$ be the $j$th projection, regarded as a character. We also denote the corresponding 1-dimensional representation of $\mathbb{T}$ by the same symbol $u_j$. Then

$$R\mathbb{T}=\mathbb{Z}[u_1^{\pm1},\ldots,u_m^{\pm1}],R\widetilde{T}=\mathbb{Z}[u_1^{\pm2},\ldots,u_m^{\pm2},u_1\cdots u_m],\ \text{ and }\ RT=\mathbb{Z}[u_1^{\pm2},\ldots,u_m^{\pm2}],$$

both regarded as subrings of $R\mathbb{T}$. Also $H_{n,k}\cap\widetilde{T}=\widetilde{T}_0\times Z$. We have

$$RT_0=\mathbb{Z}[u_1^{\pm2},\ldots,u_s^{\pm2},v_1^{\pm2},\ldots,v_t^{\pm2}]\subset R\mathbb{T},$$

where $v_j:=u_{s+j+1},1\leqslant j\leqslant t$, and,

$$R\widetilde{T}_0=\mathbb{Z}[u_1^{\pm2},\ldots,u_s^{\pm2},v_1^{\pm2},\ldots,v_t^{\pm2},u_1\cdots u_sv_1\cdots v_t]\subset R\widetilde{T}.$$

In order to determine the restriction homomorphism $\rho\colon R\mathrm{Spin}(n)\to RH_{n,k}$, we first consider the homomorphism $R\mathrm{Spin}(n)\to R\mathrm{Spin}(n)\otimes RZ$ induced by the homomorphism $\mu\colon\mathrm{Spin}(n)\times Z\to\mathrm{Spin}(n)$ defined by multiplication: $(g,z)\mapsto gz$. Note that the restriction of $\mu$ to $H^0_{n,k}\times Z$ is an isomorphism $H^0_{n,k}\times Z\to H_{n,k}$. The homomorphisms

$$H^0_{n,k}\times Z\to H_{n,k},\quad\widetilde{T}\times Z\to\widetilde{T},\quad\widetilde{T}_0\times Z\to\widetilde{T}\quad\text{and}\quad\widetilde{T}_0\times Z\to H_{n,k},$$

each of which is obtained from $\mu$ by appropriately restricting its domain and codomain, will all be denoted by the same symbol $\mu$ by an abuse of notation. These group homomorphisms induce homomorphisms of rings

$$\mu^*\colon R\widetilde{T}\to R\widetilde{T}\otimes RZ,\quad\mu^*\colon R\widetilde{T}\to R\widetilde{T}_0\otimes RZ,$$
$$\mu^*\colon RH_{n,k}\to R\widetilde{T}_0\otimes RZ,\quad\mu^*\colon RH_{n,k}\xrightarrow{\cong}RH^0_{n,k}\otimes RZ,\quad\text{and}$$
$$\mu^*\colon R\mathrm{Spin}(n)\to R\mathrm{Spin}(n)\otimes RZ.$$

Let $\sigma\colon\widetilde{T}\hookrightarrow\mathrm{Spin}(n)$ be the inclusion. We have the following commutative diagram where the homomorphisms in the first row are induced by respective inclusions of groups.

$$\begin{array}{ccccccc}
R\mathrm{Spin}(n)\otimes RZ & \hookrightarrow & R\widetilde{T}\otimes RZ & \to & R\widetilde{T}_0\otimes RZ & \hookleftarrow & RH^0_{n,k}\otimes RZ\\
\uparrow\mu^* & & \mu^*\uparrow & & \uparrow id & & \uparrow\mu^*\\
R\mathrm{Spin}(n) & \overset{\sigma^*}{\hookrightarrow} & R\widetilde{T} & \xrightarrow{\mu^*} & R\widetilde{T}_0\otimes RZ & \overset{\mu^*}{\hookleftarrow} & RH_{n,k},
\end{array}\tag{4}$$

The inclusion $\sigma^*\colon R\mathrm{Spin}(n)\hookrightarrow R\widetilde{T}$ is via the identification of $R\mathrm{Spin}(n)$ with the invariant subgroup of $R\widetilde{T}$ under the action of the Weyl group $W(\mathrm{Spin}(n),\widetilde{T})$. Similarly we have the inclusion $RH^0_{n,k}\hookrightarrow R\widetilde{T}_0$ which in turn induces $RH_{n,k}\hookrightarrow R\widetilde{T}_0\otimes RZ$.

Moreover, $\mu^*(R\mathrm{Spin}(n))$ is contained in $RH_{n,k} \subset R\widetilde{T}_0 \otimes RZ$ since $H_{n,k} \subset \mathrm{Spin}(n)$. This allows one to describe the restriction homomorphism $\rho\colon R\mathrm{Spin}(n) \to RH_{n,k}$ easily, once $\mu^*\colon R\widetilde{T} \to R\widetilde{T}_0 \otimes RZ$ is determined. This we shall carry out below, with $\theta$ as in Proposition 3.5.

Routine computation, using $n = 2m, m$ even, yields that

$$u_1 \cdots u_m(z_0) = \begin{cases} 1 & \text{if } n \equiv 0 \pmod 8, \\ \theta(z_0) & \text{if } n \equiv 4 \pmod 8. \end{cases} \tag{5}$$

When $t \in \widetilde{T}_0$, we have $u_{s+1}^2(t) = 1$ and so $u_1 \cdot \ldots \cdot u_m$ restrict to $u_1 \cdots u_s \cdot v_1 \cdots v_t$ on $\widetilde{T}_0$. Therefore

$$\mu^*(u_j^{\pm 2}) = \begin{cases} \theta u_j^{\pm 2}, & 1 \leqslant j \leqslant s, \\ \theta, & j = s+1, \\ \theta v_{j-s-1}^{\pm 2}, & s+1 < j \leqslant m, \end{cases} \tag{6}$$

and,

$$\mu^*(u_1 \cdots u_m) = \begin{cases} \prod_{1 \leqslant j \leqslant s} u_j \cdot \prod_{1 \leqslant j \leqslant t} v_j, & n \equiv 0 \pmod 8, \\ \theta \prod_{1 \leqslant j \leqslant s} u_j \cdot \prod_{1 \leqslant j \leqslant t} v_j, & n \equiv 4 \pmod 8. \end{cases} \tag{7}$$

Let $e_j(x_1, \ldots, x_r)$ denote the $j$th elementary symmetric polynomial in $x_1, \ldots, x_r$. Recall that $\sigma^*(\Lambda_j) = e_j(u_1^2, u_1^{-2}, \ldots, u_m^2, u_m^{-2})$. So, for $1 \leqslant j \leqslant m$, we have

$$\begin{aligned} \rho(\Lambda_j) &= \mu^*(e_j(u_1^2, u_1^{-2}, \ldots, u_m^2, u_m^{-2})) \\ &= \theta^j e_j(u_1^2, u_1^{-2}, \ldots, u_s^2, u_s^{-2}, 1, 1, v_1^2, v_1^{-2}, \ldots, v_t^2, v_t^{-2}) \\ &= \theta^j \textstyle\sum_{p+q=j} e_p(u_1^2, u_1^{-2}, \ldots, u_s^2, u_s^{-2}, 1) \cdot e_q(v_1^2, v_1^{-2}, \ldots, v_t^2, v_t^{-2}, 1) \\ &= \theta^j \cdot \textstyle\sum_{p+q=j; 0 \leqslant p \leqslant k, 0 \leqslant q \leqslant n-k} \lambda_p \mu_q, \\ &= \theta^j f_j \end{aligned} \tag{8}$$

for a suitable element

$$f_j = f_j(\lambda_1, \ldots, \lambda_s, \mu_1, \ldots, \mu_t) \in R$$

since $\lambda_p = \lambda_{k-p}, \mu_q = \mu_{n-k-q}$.

Using Equations (6) and (7) we obtain that if $\varepsilon_j \in \{1, -1\}$, then

$$\mu^*(u_1^{\varepsilon_1} \cdots u_m^{\varepsilon_m}) = \theta^{\varepsilon} u_1^{\varepsilon_1} \cdots u_s^{\varepsilon_s} \cdot v_1^{\eta_1} \cdots v_t^{\eta_t}, \tag{9}$$

where $\eta_j = \varepsilon_{s+1+j}$, and the value of $\varepsilon \in \{0, 1\}$ is obtained as follows:

$$\begin{aligned} \varepsilon &\equiv \sum_{1 \leqslant j \leqslant m} \varepsilon_j \pmod 2 \quad \text{if} \quad n \equiv 0 \pmod 8 \quad \text{and} \\ \varepsilon &\equiv 1 + \sum_{1 \leqslant j \leqslant m} \varepsilon_j \pmod 2 \quad \text{if} \quad n \equiv 4 \pmod 8. \end{aligned}$$

The following proposition now follows immediately from equations (8), (9), and the definitions of $\Delta_m^{\pm},\ \Delta_{s,t}$.

**Proposition 4.1.** *Let* $n = 2m \equiv 0 \pmod 4$, $k = 2s+1$, $n-k = 2t+1$. *With the above notations, the restriction homomorphism* $\rho\colon R\mathrm{Spin}(n) \to RH_{n,k}$ *is defined by*

$$\rho(\Lambda_j) = \Lambda'_j = \theta^j \sum_{p+q=j} \lambda_p \mu_q = \theta^j f_j, \quad 1 \leqslant j \leqslant m-1,$$
$$\rho(\Delta_m^+) = \theta^\varepsilon \Delta_{s,t}, \quad \rho(\Delta_m^-) = \theta^{1+\varepsilon} \Delta_{s,t},$$

*where* $\varepsilon = 0, 1$ *according as* $n \equiv 0 \pmod 8$ *or* $n \equiv 4 \pmod 8$ *respectively.*

The ring $R' := \mathbb{Z}[\theta^p\lambda_p, \theta^q\mu_q; 1 \leqslant p \leqslant s, 1 \leqslant q \leqslant t] \subset RH_{n,k}$ is mapped to the polynomial ring $\mathbb{Z}[\lambda_p, \mu_q; 1 \leqslant p \leqslant s, 1 \leqslant q \leqslant t] = R = R(\mathrm{SO}(k) \times \mathrm{SO}(n-k))$ by an automorphism of the ring $R[\theta]$ since $\theta$ is invertible. It follows that $R'$ is a polynomial ring in $s + t = m - 1$ indeterminates. Evidently, $R'[\theta] = R[\theta]$.

**Lemma 4.2.** *Let* $n = 2m \equiv 0 \pmod 4, k = 2s + 1, n - k = 2t + 1$. *Let*

$$R'[\theta] = R[\theta] \subset RH_{n,k}.$$

*Then* $R'[\theta]$ *is a free* $\Lambda'$*-module of rank* $2\binom{m-1}{s}$ *where*

$$\Lambda' := \mathbb{Z}[\Lambda'_1, \ldots, \Lambda'_{m-1}] \subset RH_{n,k}.$$

*In particular,* $\Lambda'_1, \ldots, \Lambda'_{m-1}$ *are algebraically independent. Also* $RH_{n,k} = R[\theta, \Delta_{s,t}]$ *is a free module of rank* $4\binom{m-1}{s}$ *over* $\mathbb{Z}[\Lambda_1, \ldots, \Lambda_{m-1}]$ *via* $\rho$.

*Proof.* Since $R = R(\mathrm{SO}(k) \times \mathrm{SO}(n-k))$ is a polynomial algebra in $s + t = m - 1$ indeterminates, the algebraic independence of $\Lambda'_1, \ldots, \Lambda'_{m-1}$ would follow once we show that $R[\theta] \cong R \oplus R$ is a finitely generated free $\Lambda'$-module.

First note that $\Lambda'[\theta]$ is free as a $\Lambda'$-module with basis $\{1, \theta\}$.

Next we will show that $R[\theta] \subset RH_{n,k}$ is free as a $\Lambda'[\theta]$-module of rank $\binom{m-1}{s}$. Let $\Lambda_0 = \mathbb{Z}[f_1, \ldots, f_{m-1}]$. Then $\Lambda'[\theta] = \Lambda_0[\theta] = \Lambda_0 \otimes_{\mathbb{Z}} \mathbb{Z}[\theta]$. Since $R[\theta] = R \otimes_{\mathbb{Z}} \mathbb{Z}[\theta]$, it suffices to show that $R$ is free as a module over $\Lambda_0 \subset R$, of rank $\binom{m-1}{s}$.

Denote by $\rho_0 \colon R\mathrm{Spin}(n) \to RH_{n,k} \to RH^0_{n,k}$ the restriction homomorphism induced by the inclusion $H^0_{n,k} \hookrightarrow H_{n,k} \hookrightarrow \mathrm{Spin}(n)$. Then $\Lambda_0 = \rho_0(\Lambda)$ and $\rho_0(\Lambda) \subset R \subset R[\Delta_{s,t}]$. Then $R$ is free as a $\Lambda_0$-module (see [**SZ1**, Lemma 2.6]). We give a proof for the sake of completeness.

Let

$$z_j = e_j(u_1^2 + u_1^{-2}, \ldots, u_m^2 + u_m^{-2}),$$
$$x_p = e_p(u_1^2 + u_1^{-2}, \ldots, u_s^2 + u_s^{-2}), \quad \text{and}$$
$$y_q = e_q(v_1^2 + v_1^{-2}, \ldots, v_t^2 + v_t^{-2}).$$

Then $\mathbb{Z}[z_1, \ldots, z_m] = \mathbb{Z}[\Lambda_1, \ldots, \Lambda_m]$. Indeed, since $\Lambda_1, \ldots, \Lambda_m$ are expressible as symmetric polynomials in $u_j^2 + u_j^{-2}, 1 \leqslant j \leqslant m$, they are expressible as polynomials in $z_1, \ldots, z_m$. Conversely, since $z_1, \ldots, z_m \in \mathbb{Z}[u_1^2, u_1^{-2}, \ldots, u_m^2, u_m^{-2}]$ are invariant under the permutations of the variables $u_1^2, \ldots, u_n^2$ as well as the involutions $u_j^2 \mapsto u_j^{-2}$ for every $j$, we see that the $z_j$ belong to the subring of $\mathbb{Z}[u_1^2, u_1^{-2}, \ldots, u_n^2, u_n^{-2}]$ fixed by the group $\mathbb{Z}_2^n \rtimes S_n$. This fixed subring equals $\mathbb{Z}[\Lambda_1, \ldots, \Lambda_m]$; see [**H**, §10, Ch. 13]. So each $z_j$ is expressible as a polynomial in the $\Lambda_i$.

The same argument shows that $\mathbb{Z}[\lambda_1, \ldots, \lambda_s] = \mathbb{Z}[x_1, \ldots, x_s]$ and $\mathbb{Z}[\mu_1, \ldots, \mu_t] = \mathbb{Z}[y_1, \ldots, y_t]$. Consequently, $R = \mathbb{Z}[\lambda_p, \mu_q; 1 \leqslant p \leqslant s, 1 \leqslant q \leqslant t] \subset RH^0_{n,k}$.

Now using Equation (6) we obtain

$$\rho_0(z_j) = \sum_{p+q=j} x_p y_q + 2 \sum_{p+q=j-1} x_p y_q, 1 \leqslant j \leqslant m-1, \tag{10}$$

and $\rho_0(z_m) = 2x_s.y_t$ where it is understood that $z_0 = x_0 = y_0 = 1$. Set $z'_1 := z_1 - 2$, and, inductively, $z'_r := z_r - 2z'_{r-1}, 2 \leqslant r < m$, so that

$$\rho_0(z'_r) = \sum_{p+q=r} x_p y_q, \quad 1 \leqslant r \leqslant m-1.$$

Then $\mathbb{Z}[z'_1, \ldots, z'_{m-1}] = \mathbb{Z}[z_1, \ldots, z_{m-1}] = \Lambda_0$. Moreover, we have

$$\rho_0(z'_j) = \sum_{p+q=j} x_p.y_q, 1 \leqslant j \leqslant m-1. \tag{11}$$

The proof that $R$ is a free $\Lambda_0$-module of rank $\binom{m-1}{s}$ is now completed using some well-known facts concerning the cohomology of classifying spaces $B\mathrm{U}(s)$ of the unitary group $\mathrm{U}(s)$, as we shall now explain. We regard $R = \mathbb{Z}[x_1, \ldots, x_s, y_1, \ldots, y_t]$ as a *graded* ring where $|x_p| = 2p, |y_q| = 2q$. Then $\Lambda_0 = \mathbb{Z}[z'_1, \ldots, z'_{m-1}]$ is a graded subring where $|z'_r| = 2r$. We may identify $R$ with $H^*(B(\mathrm{U}(s) \times \mathrm{U}(t)); \mathbb{Z})$ and $\Lambda_0$ with $H^*(B\mathrm{U}(s+t); \mathbb{Z})$ so that the inclusion $\Lambda_0 \hookrightarrow R$ corresponds to the homomorphism induced by the the projection of the fibre bundle $B(\mathrm{U}(s) \times \mathrm{U}(t)) \to B\mathrm{U}(s+t)$ with fibre the complex Grassmann manifold $\mathbb{C}G_{s+t,s} = U(s+t)/U(s) \times U(t)$. The Grassmann manifold bundle is totally non-cohomologous to zero (with $\mathbb{Z}$-coefficients) and so by the Leray–Hirsch theorem $H^*(B(\mathrm{U}(s) \times \mathrm{U}(t)); \mathbb{Z})$ is a free $H^*(B\mathrm{U}(s+t); \mathbb{Z})$-module of rank equal to $\mathrm{rank}(H^*(\mathbb{C}G_{s+t,t}; \mathbb{Z})) = \binom{s+t}{s}$.

Since $RH_{n,k}$ is a free $R[\theta]$-module (with basis $\{1, \Delta_{s,t}\}$) by Proposition 3.5, the last assertion of the lemma follows. □

*Remark 4.3.* (i) We shall denote by $\mathcal{B}_0$ a basis of $R = \mathbb{Z}[\lambda_p, \mu_q; 1 \leqslant p \leqslant s, 1 \leqslant q \leqslant t]$ over $\Lambda_0$ and assume that $1 \in \mathcal{B}_0$. Then a

$$\mathbb{Z}[\Lambda_1, \ldots, \Lambda_{m-1}]\text{-basis for } \ RH_{n,k} \ \text{ is } \ \mathcal{B}_0 \cup \mathcal{B}_0\theta \cup \mathcal{B}_0\Delta_{s,t} \cup \mathcal{B}_0\theta\Delta_{s,t}.$$

(ii) The argument in the last paragraph of the above proof is valid irrespective of the parity of $m = s+t+1$. It follows that $R = \mathbb{Z}[x_1, \ldots, x_s, y_1, \ldots, y_t]$ is a free $\Lambda_0 = \mathbb{Z}[z'_1, \ldots, z'_{s+t}]$-module for any $s, t \geqslant 1$. Moreover, the quotient ring $R/I$, being isomorphic to $H^*(\mathbb{C}G_{s+t,s}; \mathbb{Z})$, is a free abelian group of rank $\binom{s+t}{s}$ where $I$ is the ideal $\langle z'_1, \ldots, z'_{s+t} \rangle \subset R$.

Next we note that irrespective of whether $n \equiv 0$ or $4 \pmod 8$, we have

$$\rho((\Delta^+_m)^2 - (\Delta^-_m)^2) = 0 \ \text{ and } \ \rho(\Delta^+_m \Delta^-_m) = \theta\Delta^2_{s,t} = \theta f_{s,t}.$$

We have the following consequence of Lemma 4.2.

**Lemma 4.4.** *The elements $\Lambda'_1, \ldots, \Lambda'_{m-2}, \rho(\Delta^+_m) \in RH_{n,k}$ are algebraically independent. As a module over $\Lambda := \mathbb{Z}[\Lambda_1, \ldots, \Lambda_{m-2}, \Delta^+_m] \subset R\mathrm{Spin}(n)$, $RH_{n,k}$ is free of rank $2\binom{m-1}{s}$ with basis $\mathcal{B}_0 \cup \mathcal{B}_0\theta$.*

*Proof.* Since $\rho(\Delta^+_m)^2 = \Delta^2_{s,t} = f_{s,t}$, it suffices to show that $\Lambda'_1, \ldots, \Lambda'_{m-2}, f_{s,t}$ are algebraically independent in $RH_{n,k}$. Note that $\Delta^+_m \cdot \Delta^-_m = \Lambda_{m-1} + \Lambda_{m-3} + \cdots + \Lambda_1$ in $R\mathrm{Spin}(n)$; see [**H**, Theorem 10.3, Chapter 14]. So

$$f_{s,t} = \theta\rho(\Delta^+_m \cdot \Delta^-_m) = \Lambda'_{m-1} + \Lambda'_{m-3} + \cdots + 1.$$

Since $\Lambda'_1, \ldots, \Lambda'_{m-1}$ are algebraically independent, it follows that $\Lambda'_1, \ldots, \Lambda'_{m-2}, f_{s,t}$ are also algebraically independent. Moreover, we have $\Lambda'[\rho(\Delta^+_m)] = \rho(\Lambda) \cong \Lambda$.

Let $\mathcal{B}$ be a basis for $R'[\theta] = R[\theta]$ over $\Lambda' = \mathbb{Z}[\Lambda'_1, \ldots, \Lambda'_{m-1}]$. Note that we may take $\mathcal{B}$ to be $\mathcal{B}_0 \cup \mathcal{B}_0\theta$ by Remark 4.3. Then $\mathcal{B}$ is a basis for $R[\theta, \rho(\Delta^+_m)] = RH_{n,k}$ over $\Lambda'[\rho(\Delta^+_m)] \cong \Lambda$. In view of Lemma 4.2, we conclude that $RH_{n,k}$ is a free module over $\Lambda$ of rank $2\binom{m-1}{s}$. □

Let $\delta_m = \Delta^+_m - \Delta^-_m$. Then $R\mathrm{Spin}(n) = \Lambda[\delta_m]$ with $\Lambda$ as in Lemma 4.4. Note that $\rho((\Delta^+_m)^2 - (\Delta^-_m)^2) = 0$ and $\rho(\Delta^+_m \cdot \Delta^-_m) = \theta\Delta^2_{s,t} = \theta f_{s,t}$. So the following equations hold in $RH_{n,k}$:

$$\rho((\Delta^+_m)^2) = \rho(\delta^2_m - 2\Delta^+_m\delta_m) = 0, \text{ and } \rho(\Delta^+_m)\rho(\delta_m) + (\theta - 1)\cdot f_{s,t} = 0. \tag{12}$$

### 4.1. Computation of $\mathrm{Tor}^*_{R\mathrm{Spin}(n)}(RH_{n,k}, \mathbb{Z})$

We shall apply the change of rings spectral sequence (§2.1) in order to compute $\mathrm{Tor}^*_{R\mathrm{Spin}(n)}(RH_{n,k}, \mathbb{Z})$. In the notation of Theorem 2.1, we let $\Gamma = R\mathrm{Spin}(n)$, with $A = RH_{n,k}$, $K = C = \mathbb{Z}$ and $\Lambda = \mathbb{Z}[\Lambda_1, \ldots, \Lambda_{m-2}, \Delta^+_m] \subset \Gamma = R\mathrm{Spin}(n)$. Then $A$ is a free $\Lambda$-module via the restriction homomorphism, in view of Lemma 4.4. Hence setting

$$B := \mathrm{Tor}^\Lambda_*(RH_{n,k}, \mathbb{Z}),$$

we have, with $\varepsilon \in \{0, 1\}$ as in Proposition 4.1,

$$B_q = \mathrm{Tor}^\Lambda_q(RH_{n,k}, \mathbb{Z}) = \begin{cases} RH_{n,k}/\langle \Lambda'_j - \binom{n}{j}, 1 \leqslant j \leqslant m-2; \theta^\varepsilon\Delta_{s,t} - 2^{m-1}\rangle, & q = 0, \\ 0, \text{ if } q \neq 0. \end{cases} \tag{13}$$

Thus

$$B = B_0 = RH_{n,k}/\langle \Lambda'_j - \binom{n}{j}, 1 \leqslant j \leqslant m-2;\ \theta^\varepsilon\Delta_{s,t} - 2^{m-1}\rangle.$$

Recall the basis $\mathcal{B} = \mathcal{B}_0 \cup \mathcal{B}_0\theta$ of $RH_{n,k}$ over $\Lambda$ given in Lemma 4.4. (See Remark 4.3 for the definition $\mathcal{B}_0$.) Under the natural projection $\eta\colon RH_{n,k} \to B$, the subring $\rho(\Lambda)$ maps to $\mathbb{Z}$ and $\mathcal{B}$ to a $\mathbb{Z}$-basis $\overline{\mathcal{B}} = \overline{\mathcal{B}}_0 \cup \overline{\mathcal{B}}_0\theta$ where $\overline{\mathcal{B}}_0 = \eta(\mathcal{B}_0)$. It is readily seen that $|\overline{\mathcal{B}}| = |\mathcal{B}|$. We summarise this observation as a lemma.

**Lemma 4.5.** *The set $\overline{\mathcal{B}}$ is a $\mathbb{Z}$-basis for $B$. Thus $B$ is free abelian of rank $2\binom{m-1}{s}$.*

By Theorem 2.1, the change of rings spectral sequence collapse and we have $\mathrm{Tor}^\Gamma_q(A, \mathbb{Z}) \cong \mathrm{Tor}^\Omega_q(B, \mathbb{Z})$, where $\Omega = R\mathrm{Spin}(n)/\langle \Lambda_j - \binom{n}{j}, \Delta^+_m - 2^{m-1}\rangle = \mathbb{Z}[\delta_m]$ and $\delta_m = \Delta^+_m - \Delta^-_m$.

Since $\Omega$ is a polynomial ring, one can use the Koszul resolution to compute $\mathrm{Tor}^\Omega_q(B, \mathbb{Z})$. The $\Omega$-module structure on $B$ is obtained via the algebra homomorphism $\bar\rho\colon \Omega \to B$ defined by $\rho\colon R\mathrm{Spin}(n) \to RH_{n,k}$. In view of Proposition 4.1, we have $\bar\rho(\delta_m) = \epsilon'(\theta - 1)\Delta_{s,t}$, where the value of $\epsilon' \in \{1, -1\}$ depends on the value of $n$ modulo 8. The Koszul resolution of $\mathbb{Z}$ is

$$0 \to \Omega \cdot \delta \xrightarrow{d} \Omega \xrightarrow{\varepsilon} \mathbb{Z} \to 0.$$

Here $\varepsilon$ is the augmentation defined by $\varepsilon(\delta_m) = 0$ and $d(\delta) = \delta_m$. Tensoring with the $\Omega$-module $B$ we obtain the following chain complex whose homology is $\mathrm{Tor}^\Omega_*(B, \mathbb{Z})$:

$$0 \to B\delta \xrightarrow{\bar{d}} B \to 0,$$

where

$$\bar{d}(\delta) = \bar{d}(1 \cdot \delta) = \bar{\rho}(\delta_m) = \epsilon'(\theta - 1)\Delta_{s,t} \in B.$$

In particular,

$$\mathrm{Tor}_q^{\Omega}(B, \mathbb{Z}) = 0 \;\text{ if }\; q \geqslant 2, \quad \mathrm{Tor}_1^{\Omega}(B, \mathbb{Z}) = \ker(\bar{d}), \quad \mathrm{Tor}_0^{\Omega}(B, \mathbb{Z}) = B/\langle(\theta - 1)\Delta_{s,t}\rangle.$$

We set

$$\bar{B} := \mathrm{Tor}_0^{\Omega}(B, \mathbb{Z}) = B/\langle(\theta - 1)\Delta_{s,t}\rangle. \tag{14}$$

Recall from Equation (8) that

$$\Lambda'_j = \theta^j f_j \;\text{ where }\; f_j = \sum_{0 \leqslant p \leqslant j} \lambda_p \mu_{j-p} \in RH_{n,k}, \quad 1 \leqslant j \leqslant m - 1,$$

while

$$\lambda_p = \lambda_{k-p} \;\text{ and }\; \mu_q = \mu_{n-k-q} \;\text{ when }\; p > s, q > t.$$

Denote by $\eta\colon RH_{n,k} \to B$ the canonical quotient map and by $\bar{\eta}\colon RH_{n,k} \to \bar{B}$ the composition $RH_{n,k} \xrightarrow{\eta} B \to \bar{B}$ where $B \to \bar{B}$ is the canonical quotient map. If we have $x \in RH$, we shall denote $\eta(x) \in B$ by the same symbol $x$ and we shall denote $\bar{\eta}(x) \in \bar{B}$ by $[x]$.

**Lemma 4.6.** *We keep the above notations. The following relations hold in $\bar{B}$:*

*(a)* $2^{m-1}([\theta] - 1) = 0$, $[\Delta_{s,t}] = 2^{m-1}$,

*(b)* $\sum_{0 \leqslant p \leqslant j} [\lambda_p][\mu_{j-p}] = [f_j] = \binom{n}{j}[\theta^j], 1 \leqslant k \leqslant m - 1,$
*(where* $[\lambda_p] = [\lambda_{k-p}], [\mu_q] = [\mu_{n-k-q}]$*),*

*(c)* $[\Delta_{s,t}^2] = (\sum_{0 \leqslant p \leqslant s} [\lambda_p])(\sum_{0 \leqslant q \leqslant t} [\mu_q]) = [f_{s,t}] = 2^{2m-2}$.

*Proof.* (a). We have, by Proposition 4.1, $\rho(\Delta_m^+) = \theta^{\varepsilon}\Delta_{s,t}$, in $RH_{n,k}$ where $\varepsilon \in \{0, 1\}$ depending on the value of $n$ modulo 8. Since $([\theta] - 1)[\Delta_{s,t}] = 0$ in $\bar{B}$, irrespective of the value of $\varepsilon$ we have $\bar{\eta} \circ \rho(\Delta_m^+) = [\Delta_{s,t}]$ in $\bar{B}$. On the other hand, since $\Delta_m^+ = 2^{m-1}$ in $\Omega$, we obtain that $2^{m-1} = \eta\rho(\Delta_m^+) = \theta^{\varepsilon}\Delta_{s,t}$ in $B$. It follows that $[\Delta_{s,t}] = 2^{m-1}$ and so $2^{m-1}([\theta] - 1) = 0$.

(b). It is clear that, when $1 \leqslant j \leqslant m - 2$, the relation $f_j = \bar{\rho}(\Lambda_j)\theta^j = \binom{n}{j}\theta^j$ holds in $B$ and hence in $\bar{B}$ using $\theta^2 = 1$. Since $\Delta_m^+\Delta_m^- = \sum_{1 \leqslant j \leqslant m} \Lambda_{2j-1}$ in $R\mathrm{Spin}(n)$, and since $\bar{\eta} \circ \rho(\Delta_m^{\pm}) = [\Delta_{s,t}] = [\theta][\Delta_{s,t}] = 2^{m-1}$ in $\bar{B}$, applying $\bar{\eta} \circ \rho$ we obtain the following equations in $\bar{B}$:

$$\begin{aligned} 2^{2(m-1)} &= \bar{\eta} \circ \rho(\Delta_m^+\Delta_m^-) \\ &= \bar{\eta} \circ \rho(\textstyle\sum_{1 \leqslant j \leqslant m} \Lambda_{2j-1}) \\ &= [f_{m-1}] - \binom{2m}{m-1} + \textstyle\sum_{1 \leqslant j < m/2} \binom{2m}{2j-1} \\ &= [f_{m-1}] - \binom{2m}{2m-1} + 2^{2m}/4 \end{aligned}$$

since $\sum_{1 \leqslant j < m/2} \binom{2m}{2j-1} = (1/2)\sum_{1 \leqslant j \leqslant m} \binom{2m}{2j-1} = 2^{2m}/4$. Hence $[f_{m-1}] = \binom{2m}{m-1}$.

(c). Since $\Delta_{s,t}^2 = f_{s,t}$ holds in $B$, and since $[\Delta_{s,t}] = 2^{m-1}$ holds in $\bar{B}$, we see that $[f_{s,t}] = 2^{2m-2}$ in $\bar{B}$. □

*Remark 4.7.* It turns out that the relation (c) is a consequence of relations (a), (b). Indeed, recalling that $[\lambda_p] = [\lambda_{k-p}], [\mu_q] = [\mu_{n-k-q}]$ in $\bar{B}$, in addition to knowing that $k = 2s+1, n-k = 2t+1$, we have

$$\begin{aligned} f_{s,t} = [\Delta^2_{s,t}] &= (\textstyle\sum_{0\leqslant p\leqslant s}[\lambda_p])(\sum_{0\leqslant q\leqslant t}[\mu_q]) \\ &= (1/4)(\textstyle\sum_{0\leqslant p\leqslant k}[\lambda_p])(\sum_{0\leqslant q\leqslant n-k}[\mu_q]) \\ &= (1/4)\textstyle\sum_{0\leqslant r\leqslant n}(\sum_{0\leqslant j\leqslant r}[\lambda_j][\mu_{r-j}]) \\ &= (1/4)\textstyle\sum_{0\leqslant r\leqslant n}\binom{n}{r}[\theta]^r, \quad \text{using(b)}, \\ &= (1/4)(1+[\theta])^n. \end{aligned}$$

Since $[\theta]^2 = 1$, we have $(1+[\theta])^2 = 2(1+[\theta])$. So $(1+[\theta])^r = 2^{r-1}(1+[\theta])$ whenever $r \geqslant 1$. Therefore, since $n = 2m \geqslant 4$, we have

$$\begin{aligned} (1/4)(1+[\theta])^n &= (1/4)(1+[\theta])^3 \cdot (1+[\theta])^{n-3} \\ &= (1+[\theta]) \cdot (1+[\theta])^{2m-3} \\ &= (1+[\theta])^{2m-2} \\ &= 2^{2m-3}(1+[\theta]) \\ &= 2^{2m-2}, \end{aligned}$$

using $2^{2m-3}[\theta] = 2^{2m-3}$. Therefore $f_{s,t} = 2^{2m-2}$.

**Lemma 4.8.** *With the above notations, the rank of the abelian group $\bar{B}$ equals $\binom{m-1}{s}$. Moreover the torsion subgroup of $\bar{B}$ is generated as a $B$-module by $(\theta - 1)$. In particular, any torsion element has order $2^r$ for some $r \leqslant m-1$.*

*Proof.* In view of Lemma 4.5, the set $\overline{\mathcal{B}}_0 \cup \overline{\mathcal{B}}_0(\theta - 1)$ is a basis for $B$. Under the quotient map $B \to \bar{B}$, the abelian group $\bar{B}_0$ generated by $\overline{\mathcal{B}}_0$ projects isomorphically onto a summand of $\bar{B}_0$. Since $2^{m-1}([\theta]-1) = 0$, the subgroup $C$ of $\bar{B}$ is generated by $([\theta]-1)\overline{\mathcal{B}}_0$ consists only of elements whose (additive) order divides $2^{m-1}$. This completes the proof. □

We now turn to $\mathrm{Tor}_1^{\Omega}(B, \mathbb{Z}) = \ker(\bar{d}\colon B\delta \to B)$. Since $\bar{d}(\delta) = \pm(\theta-1)\Delta_{s,t}$, $\ker(\bar{d})$ is the $B$-submodule $J \cdot \delta$ where $J \subset B$ is the annihilator ideal of $(\theta-1)\Delta_{s,t} \in B$. It is clear that $(\theta+1) \in J$ since $\theta^2 - 1 = 0$. We claim that $J$ *equals* the ideal generated by $\theta + 1$. In order to see this, let $x \in J$ and let $\bar{\mathcal{B}}_0 = \{b_j\}$. Write

$$x = \sum y_j b_j + \theta \sum z_j b_j \quad \text{where} \quad y_j, z_j \in \mathbb{Z}.$$

Since $x \in J$, multiplying by $(\theta-1)\Delta_{s,t}$, and using the relations $\Delta_{s,t} = 2^{m-1}\theta^{\varepsilon}$ (where the value of $\varepsilon \in \{0,1\}$ depends on the parity of $m$) and $\theta(\theta-1) = 1-\theta$ in $B$, we obtain that

$$2^{m-1}(\theta-1)\theta^{\varepsilon}\sum y_j b_j - 2^{m-1}\theta^{\varepsilon}(\theta-1)\sum z_j b_j = 0.$$

Since $B$ is a free abelian group, and since $\theta^{\varepsilon}$ is invertible in $B$, the above equation can be rewritten as $-(\sum(y_j - z_j)b_j) + \theta\sum(y_j - z_j)b_j = 0$. This implies that $y_j = z_j$ for all $j$. Therefore $x = (\theta+1)(\sum y_j b_j) \in J$.

We are now ready to prove Theorem 1.1.

*Proof of Theorem 1.1.* The Hodgkin spectral sequence $\mathrm{Tor}^*_{R\mathrm{Spin}(n)}(RH_{n,k}, \mathbb{Z})$ converges to $K^*(G_{n,k})$. Since $\mathrm{Tor}^*_{R\mathrm{Spin}(n)}(RH_{n,k}, \mathbb{Z}) \cong \mathrm{Tor}^*_{\Omega}(B, \mathbb{Z})$, and since $\mathrm{Tor}^*_{\Omega}(B, \mathbb{Z})$

is generated by degree $-1$ elements, by the discussion in §2 we obtain that

$$K^0(G_{n,k}) = \mathrm{Tor}_\Omega^0(B,\mathbb{Z}) = \bar{B} \text{ and } K^{-1}(G_{n,k}) = \mathrm{Tor}_1^\Omega(B,\mathbb{Z}) = \mathrm{Ann}(\theta-1) \subset B.$$

The theorem now follows from Equation (14), Lemma 4.6, and the above discussion that describes $\mathrm{Ann}((\theta-1)\Delta_{s,t})$. □

Let $\xi = \xi_{n,k}$ be the Hopf line bundle over $G_{n,k}$. It is associated to the double cover $\widetilde{G}_{n,k} \to G_{n,k}$. If $\eta$ is a real vector bundle, we denote by $\eta^{\mathbb{C}}$ the complexification of $\eta$. Note that $\eta^{\mathbb{C}}$, regarded as a real vector bundle via restriction of scalars, is isomorphic to $\eta \oplus \eta$. See [**MS**, p. 176].

**Proposition 4.9.** *Let $n = 2m, k = 2s+1$. If $n \equiv 0 \pmod 4, k \equiv 1 \pmod 2$ as well as $k(n-k) < 2^m$, then*

$$2^m\xi \cong 2^m\epsilon_{\mathbb{R}} \quad \textit{where} \quad n = 2m, \quad \textit{but} \quad [2^{m-2}\xi] \neq 2^{m-2} \quad \textit{in} \quad KO(G_{n,k}).$$

*If $n \equiv 0 \pmod 8$ and $k(n-k) < 2^{m-1}$, then $2^{m-1}\xi \cong 2^{m-1}\epsilon_{\mathbb{R}}$.*

*Proof.* Since

$$2^{m-1}[\xi^{\mathbb{C}}] = 2^{m-1}\theta = 2^{m-1} \in K(G_{n,k}),$$

it follows that $2^m[\xi] = 2^m \in KO(G_{n,k})$. If $\dim G_{n,k} = k(n-k) < 2^m = \mathrm{rank}(2^m\xi)$, then equality of the classes of the vector bundles $[2^m\xi]$ and $[2^m\epsilon_{\mathbb{R}}] = 2^m$ in $KO(G_{n,k})$ implies the *isomorphism* of the vector bundles: $2^m\xi \cong 2^m\epsilon_{\mathbb{R}}$. See [**H**, Theorem 1.5, Chapter 8].

When $n \equiv 0 \pmod 8$, the representations $\Delta_m^+, \Delta_m^- \in R\mathrm{Spin}(n)$ are real, that is, they arise as complexification of *real* representations $\Delta_{m,\mathbb{R}}^+, \Delta_{m,\mathbb{R}}^-$ of $\mathrm{Spin}(n)$. See [**H**, §12, Chapter 13]. Evidently $\theta$ is real. Indeed $\theta = \chi \otimes_{\mathbb{R}} \mathbb{C}$ of $H_{n,k}$ where

$$\chi\colon H_{n,k} \to \mathrm{O}(1) \text{ is defined by the projection } H_{n,k} \to H_{n,k}/H_{n,k}^0 \cong \mathrm{O}(1).$$

The line bundle associated to $\chi$ is isomorphic to $\xi$ whereas the bundle associated to $\Delta_{m,\mathbb{R}}^-$ equals the trivial real vector bundle of rank $2^{m-1}$. This can be shown to imply that $2^{m-1}[\xi] = 2^{m-1} \in KO(G_{n,k})$. As before, this leads to the isomorphism $2^{m-1}\xi \cong 2^{m-1}\epsilon_{\mathbb{R}}$ when $k(n-k) < 2^{m-1}$. □

As for the torsion part of $K^0(G_{n,k})$, it has no $p$-torsion for any odd prime $p$. For any $n,k$, the element $[\Lambda^k(\gamma_{n,k}^{\mathbb{C}})] - 1 = [\xi^{\mathbb{C}}] - 1 \in K(G_{n,k})$ generates a finite cyclic subgroup of order $2^r$ for some $r$. There are the obvious inclusions

$$i\colon G_{n,k} \hookrightarrow G_{n+1,k+1}, \quad j\colon G_{n,k} \hookrightarrow G_{n+1,k},$$

which have the property that $i^*(\gamma_{n+1,k+1}) \cong \gamma_{n,k} \oplus \epsilon_{\mathbb{R}}$ and $j^*(\gamma_{n+1,k}) = \gamma_{n,k}$.

**Theorem 4.10.** *Suppose that $n = 4l+j, k = 2s+\varepsilon, 1 \leqslant j \leqslant 3, \varepsilon \in \{0,1\}$. Let $2^r$ be the order of $[\xi^{\mathbb{C}}] \in K(G_{n,k})$. Then $2l-1 \leqslant r \leqslant 2l+1$.*

*Proof.* Suppose $\varepsilon = 1$. Then we have inclusions $G_{4l,k} \overset{j_0}{\hookrightarrow} G_{4l+j,k} \overset{j_1}{\hookrightarrow} G_{4l+4,k}$ where $j_1^*(\xi_{4l+4,k}) = \xi_{n,k},\ j_0^*(\xi_{n,k}) = \xi_{4l,k}$. The bounds for $r$ now follow from Theorem 1.1.

When $\varepsilon = 0$, we use the inclusions $G_{4l,2s-1} \overset{i_0}{\hookrightarrow} G_{n,k} \overset{i_1}{\hookrightarrow} G_{4l+4,2s+1}$. When $s = 1$, $G_{4l,2s-1} = \mathbb{R}P^{4l-1}$ and the order of the bundle $[\xi^{\mathbb{C}}] - 1$ is known to be $2^{2l-1}$ from the work of Adams [**A**, Theorem 7.3]. Now we proceed exactly as in the case $\varepsilon = 1$. □

## 5. $K$-theory of $G_{n,k}$ for arbitrary values of $n, k$

In this section we shall prove Theorem 1.2. The proof will make use of the Chern character $\mathrm{ch}\colon K^*(G_{n,k})\otimes\mathbb{Q}\to H^*(G_{n,k};\mathbb{Q})$. We begin by recalling, in Theorem 5.1 and the following paragraph, the rational cohomology algebra of the Grassmann manifolds. We refer the reader to [**MS**, §15] for the definition and properties of Pontrjagin classes. We shall write $k=2s+\varepsilon, n-k=2t+\eta$ where $\varepsilon,\eta\in\{0,1\}$ so that $n=2s+2t+\varepsilon+\eta$.

We denote by $\beta_{n,k}$ the canonical $(n-k)$-plane bundle over $G_{n,k}$ whose fibre over $L\in G_{n,k}$ is the vector space $L^\perp\subset\mathbb{R}^n$. We have $\gamma_{n,k}\oplus\beta_{n,k}\cong n\epsilon_{\mathbb{R}}$, and, (denoting the complexification $\gamma_{n,k}\otimes\mathbb{C}$ by $\gamma_{n,k}^{\mathbb{C}}$ etc.,) we obtain

$$\gamma_{n,k}^{\mathbb{C}}\oplus\beta_{n,k}^{\mathbb{C}}=n\epsilon_{\mathbb{C}}. \tag{15}$$

Let $p_j=p_j(\gamma_{n,k})\in H^{4j}(G_{n,k};\mathbb{Z}[1/2])$, $1\leqslant j\leqslant s$, be the $j$th (rational) Pontrjagin class of $\gamma_{n,k}$, and let $q_j=p_j(\beta_{n,k}), 1\leqslant j\leqslant t$. Since $\gamma_{n,k}\oplus\beta_{n,k}\cong n\epsilon_{\mathbb{R}}$, we have, for $1\leqslant r\leqslant s+t$,

$$\sum_{0\leqslant j\leqslant s} p_jq_{r-j}=0, \tag{16}$$

where it is understood that $p_0=q_0=1, p_i=0, q_j=0$ if $i>s, j>t$. In fact, the cohomology algebra $H^*(G_{n,k};\mathbb{Z}[1/2])$ has the following description. It can be derived from the known description of $H^*(\widetilde{G}_{n,k};\mathbb{Z}[1/2])$ as the fixed subring under the action of the deck transformation group of the double covering $\widetilde{G}_{n,k}\to G_{n,k}$. We refer the reader to [**MS**, Theorem 15.9] for the description of $H^*(\widetilde{G}_{n,k};\mathbb{Z}[1/2])$.

**Theorem 5.1.** *With the above notations, we have*

$$H^*(G_{n,k};\mathbb{Z}[1/2])=\mathbb{Z}[1/2][p_1,\ldots,p_s;q_1,\ldots,q_t,v_{n-1}]/J, \tag{17}$$

*where degree of $v_{n-1}=n-1$, and the ideal $J$ is generated by the following elements:*

*(i)* $\sum_{0\leqslant j\leqslant r}p_jq_{r-j}, 1\leqslant r\leqslant s+t,$

*(ii)* $v_{n-1}$ *if $n$ is odd or $k$ is even;* $v_{n-1}^2$ *if $n$ is even and $k$ odd.*

As a consequence we note that $H^*(G_{n,k};\mathbb{Z})$ has no $p$-torsion except when $p=2$.

Denote by $P_{n,k}\subset H^*(G_{n,k};\mathbb{Q})$ the even-graded subalgebra, namely,

$$H^{\mathrm{ev}}(G_{n,k};\mathbb{Q})=\oplus_{r\geqslant 0}H^{2r}(G_{n,k};\mathbb{Q})=\mathbb{Q}[p_1,\ldots,p_s;q_1,\ldots,q_t]/\!\sim.$$

Then $P_{n,k}$ depends only on $s$, $t$ and not on the values of $\varepsilon,\eta\in\{0,1\}$, along with $\dim_{\mathbb{Q}}P_{n,k}=\binom{s+t}{s}$. Moreover, $P_{n,k}=H^*(G_{n,k};\mathbb{Q})$, except when $n=2s+2t+2$ is even and $k=2s+1$ is odd. When $n=2s+2t+2$, $k=2s+1$, we have

$$H^{\mathrm{odd}}(G_{n,k};\mathbb{Q})=v_{n-1}P_{n,k}\cong P_{n,k}\quad\text{as a}\quad P_{n,k}\text{-module}.$$

We have a natural $\mathbb{Z}_2$-gradation on $H^*(G_{n,k};\mathbb{Q})$ defined by the parity of the degree.

Recall the Chern character map $\mathrm{ch}\colon K^*(G_{n,k})\otimes\mathbb{Q}\to H^*(G_{n,k};\mathbb{Q})$, which is an isomorphism of $\mathbb{Z}_2$-graded rings. So $K^0(G_{n,k})$ has rank equal to $\dim_{\mathbb{Q}}P_{n,k}=\binom{s+t}{s}$. In case $n$ is odd or $k$ is even, we have $H^{\mathrm{odd}}(G_{n,k};\mathbb{Q})=0$ and so $K^1(G_{n,k})$ is a finite abelian group. When $n$ is even and $k$ is odd, $K^1(G_{n,k})$ has rank equal to that of $K^0(G_{n,k})$.

We now turn to the proof of Theorem 1.2. We shall denote by $\phi$ the inclusion map $\mathcal{K}_{n,k} \hookrightarrow K(G_{n,k})$.

*Proof of Theorem 1.2.* The inclusion $\phi\colon \mathcal{K}_{n,k} \hookrightarrow K(G_{n,k})$ induces an inclusion

$$\phi \otimes 1\colon \mathcal{K}_{n,k} \otimes \mathbb{Q} \to K(G_{n,k}) \otimes \mathbb{Q}.$$

We need to show that the composition $\mathrm{ch} \circ (\phi \otimes 1)\colon \mathcal{K}_{n,k} \otimes \mathbb{Q} \to P_{n,k}$ is surjective. Note that, in view of Equation (16), $P_{n,k}$ is generated by $p_j, 1 \leqslant j \leqslant s$. So we need only show that the $p_j \in P_{n,k}$ are in the image of $\mathrm{ch} \circ (\phi \otimes 1)$.

We have a formal expression of $p_j = p_j(\gamma_{n,k})$ in terms of the Chern 'roots'

$$x_j, -x_j, \quad 1 \leqslant j \leqslant s, \quad \text{of} \;\; \gamma^{\mathbb{C}}_{n,k}$$

given as $p_j = (-1)^j e_j(x_1^2, \ldots, x_k^2), 1 \leqslant j \leqslant s$, where $e_j$ denotes the $j$th elementary symmetric polynomial in the indicated arguments. (See [**MS**, §15].) From the definition of Chern character we have

$$\mathrm{ch}(\gamma^{\mathbb{C}}_{n,k}) = k + 2 \sum_{m \geqslant 1} \sum_{1 \leqslant j \leqslant s} x_j^{2m}/(2m)! = k + 2 \sum_{m \geqslant 1} u_m/(2m)!,$$

where $u_m := \sum_{1 \leqslant m \leqslant s} x_j^{2m}$ for $m \geqslant 1$. The symmetric polynomials can be expressed as polynomials in the power sums over $\mathbb{Q}$ and so we have

$$(-1)^j p_j = u_j/j + F_j(u_1, \ldots, u_{j-1}), \quad 1 \leqslant j \leqslant s, \tag{18}$$

where $u_0 = k$ and

$$F_j(u_1, \ldots, u_{j-1}) \in H^{4j}(G_{n,k}; \mathbb{Q}) \;\; \text{is a suitable polynomial in} \;\; u_1, \ldots, u_{j-1}.$$

So it suffices to show that the $u_j$ are in the image of $\mathrm{ch} \circ (\phi \otimes 1)$. To see this, it is convenient to use the Adams operations $\psi^r$. Note that $\mathcal{K}_{n,k}$ contains $\Lambda_j(\gamma^{\mathbb{C}}_{n,k})$ and so it also contains $\psi^r(\gamma^{\mathbb{C}}_{n,k})$ for all integers $r \geqslant 1$ since the $\psi^r$ can be expressed (with $\mathbb{Z}$-coefficients) in terms of the exterior power operations. Although $\psi^r(\gamma^{\mathbb{C}}_{n,k})$ is only a virtual bundle, its Chern characters are easy to compute since $rx_j, -rx_j$ are its Chern roots. Thus, writing $d = \lfloor k(n-k)/2 \rfloor$, we have, for $r \in \mathbb{Z}$,

$$\begin{aligned} v_r \;&:=\; \mathrm{ch}([\psi^r(\gamma^{\mathbb{C}}_{n,k})] - k) \\ &=\; 2 \textstyle\sum_{m \geqslant 1} (\sum_{1 \leqslant j \leqslant s} r^{2m} x_j^{2m}/(2m)!) \\ &=\; 2 \textstyle\sum_{1 \leqslant m \leqslant d} r^{2m} u_m/(2m)!. \end{aligned} \tag{19}$$

We obtain the equation $2uM = v$ where $M = (m_{ij})$ is the $d \times d$ matrix defined as $m_{ij} = j^{2i}$, and $u = (u_1/2!, u_2/4!, \ldots, u_d/(2d)!), v = (v_1, \ldots, v_d)$ are regarded as (row) vectors in the $d$-fold direct sum $(H^{\mathrm{ev}}(G_{n,k}; \mathbb{Q}))^d$. Since $M$ is invertible and since the $v_j$ are in the image of $\mathrm{ch} \circ (\phi \otimes 1)$, it follows that the $u_j/(2j)!$ are also in the image of $\mathrm{ch} \circ (\phi \otimes 1)$ for $1 \leqslant j \leqslant d$. So $u_1, \ldots, u_s$ are in the image of $\mathrm{ch} \circ (\phi \otimes 1)$. This completes the proof. □

We conclude by giving, in Proposition 5.5, a description of $\mathcal{K}_{n,k}$ as a quotient of a ring $K_{n,k}$, explicitly described in terms of generators and relations, with finite kernel. It seems plausible that $K_{n,k}$ is isomorphic to $\mathcal{K}_{n,k}$ but we have not been able to prove this.

The operator $\Lambda_t = \sum_{r \geqslant 0} \Lambda^r t^r$, which is a formal power series in the indeterminate

$t$ whose coefficients are exterior power operators, has the property $\Lambda_t(\omega_0 \oplus \omega_1) = \Lambda_t(\omega_0) \cdot \Lambda_t(\omega_1)$ for any two complex vector bundles $\omega_0, \omega_1$. So we have

$$\Lambda_t(\gamma_{n,k}^{\mathbb{C}}) \cdot \Lambda_t(\beta_{n,k}^{\mathbb{C}}) = (1+t)^n \quad \text{since} \quad \Lambda_t(\epsilon_{\mathbb{C}}) = (1+t).$$

Equivalently, for any $r \geqslant 1$, we have

$$\sum_{p+q=r} \Lambda^p(\gamma_{n,k}^{\mathbb{C}}) \otimes \Lambda^q(\beta_{n,k}^{\mathbb{C}}) = \binom{n}{r}. \tag{20}$$

We know that $2^r \xi^{\mathbb{C}}$ is stably trivial for some $r$ where $\xi = \xi_{n,k}$ denotes the Hopf line bundle over $G_{n,k} = \mathrm{SO}(n)/S(\mathrm{O}(k) \times \mathrm{O}(n-k))$. By Theorem 4.10, one may take $r = m+1$. We let $\nu$ be the least positive integer for which this happens. Then $(1-[\xi^{\mathbb{C}}])^{\nu+1} = 2^\nu(1-[\xi^{\mathbb{C}}]) = 0$ in $K(G_{n,k})$. Note that $\xi = \Lambda^k(\gamma_{n,k}) = \Lambda^{n-k}(\beta_{n,k})$ is associated to the character

$$\chi \colon S(\mathrm{O}(k) \times \mathrm{O}(n-k)) \to \mathrm{O}(1) \;\; \text{defined as} \;\; \left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) \mapsto \det(A).$$

We let $\theta$ be the complexification of $\chi$ so that $\xi^{\mathbb{C}}$ is associated to $\theta$. We shall denote $[\xi^{\mathbb{C}}] \in K(G_{n,k})$ by $[\theta]$.

For any real vector space $V$ of dimension $k$, one has a functorial non-degenerate bilinear pairing $\Lambda^p(V) \times \Lambda^{k-p}(V) \to \Lambda^k(V)$ defined as $(u,v) \mapsto u \wedge v$. If $V$ is an inner product space, then we have the induced inner product

$$\Lambda^q(V) \times \Lambda^q(V) \to \mathbb{R} \;\; \text{defined as} \;\; (u_1 \wedge \cdots \wedge u_q, v_1 \wedge \cdots \wedge v_q) \mapsto \det((u_i, v_j)).$$

Thus, we obtain a natural isomorphism $\Lambda^p(V) \cong \Lambda^{k-p}(V) \otimes \Lambda^k(V)$. This yields an isomorphism $\Lambda^p(\gamma_{n,k}) \cong \Lambda^{k-p}(\gamma_{n,k}) \otimes \xi$ of real vector bundles. See [**MS**, §2]. A similar isomorphism holds for $\beta_{n,k}$ as well. Complexifying we obtain the following isomorphisms for $1 \leqslant p \leqslant k, 1 \leqslant q \leqslant n-k$:

$$\Lambda^p(\gamma_{n,k}^{\mathbb{C}}) \cong \xi^{\mathbb{C}} \otimes \Lambda^{k-p}(\gamma_{n,k}^{\mathbb{C}}), \;\; \Lambda^q(\beta_{n,k}^{\mathbb{C}}) \cong \xi^{\mathbb{C}} \otimes \Lambda^{n-k-q}(\beta_{n,k}^{\mathbb{C}}). \tag{21}$$

We are now ready to define the ring $K_{n,k}$.

**Definition 5.2.** Let $A = \mathbb{Z}[\theta]/\langle \theta^2 - 1, 2^\nu(1-\theta)\rangle$. Then $A \cong \mathbb{Z} \oplus \mathbb{Z}_{2^\nu}(1-\theta)$. Write $k = 2s + \varepsilon, n-k = 2t + \eta$ where $\varepsilon, \eta \in \{0,1\}$ so that $n = 2s + 2t + \varepsilon + \eta$. We define $K_{n,k} := A[\lambda_1, \ldots, \lambda_k, \mu_1, \ldots, \mu_{n-k}]/I$, the quotient of the polynomial algebra over $A$ where the ideal $I$ is generated by the following elements:

(i) $\lambda_{k-p} - \theta\lambda_p, \; \mu_{k-q} - \theta\mu_q$ for $1 \leqslant p \leqslant k, \; 1 \leqslant q \leqslant n-k$,

(ii) $Q_r(\lambda,\mu) - \binom{n}{r}$ for $1 \leqslant r \leqslant n$ where $Q_r(\lambda,\mu) := \sum_{p+q=r, 0 \leqslant p \leqslant k, 0 \leqslant q \leqslant n-k} \lambda_p\mu_q$, for $1 \leqslant r \leqslant n$.

*Remark 5.3.*

(a) The $A$-algebra $K_{n,k}$ is generated by $\lambda_1, \ldots, \lambda_s, \mu_1, \ldots, \mu_t$. This is immediate from the relations 5.2(i).

(b) In fact, using the relations 5.2 (ii), (and (a)), we see that $\mu_1 = n - \lambda_1$, and, if $2 \leqslant r \leqslant t$, then $\mu_r$ can be expressed in terms of the $\lambda_1, \ldots, \lambda_s, \mu_1, \ldots, \mu_{r-1}$ (with coefficients in $A$). So, by induction, the $\mu_r$ can be expressed in terms of $\lambda_1, \ldots, \lambda_s$. Hence $K_{n,k}$ is generated by $\lambda_p, 1 \leqslant p \leqslant s$.

(c) One has a ring homomorphism $A \to \mathbb{Z}$ which maps $\theta$ to 1 with kernel the ideal $A(1-\theta)$.

Set

$$\bar{K}_{n,k} := K_{n,k} \otimes_A \mathbb{Z} = K_{n,k}/(1-\theta)K_{n,k} = \mathbb{Z}[\lambda_1, \ldots, \lambda_s, \mu_1, \ldots, \mu_t]/I_0,$$

where $I_0$ is the ideal generated by the elements listed in Definition 5.2 (ii), and where $\lambda_p = \lambda_{k-p}, \mu_q = \mu_{n-k-q}$ for $p > s, q > t$.

**Lemma 5.4.** *One has the following isomorphisms of rings:*

$$\bar{K}_{2s+2t+2,2s+1} \stackrel{\alpha_0}{\to} \bar{K}_{2s+2t+1,2s+1} \stackrel{\alpha_1}{\to} \bar{K}_{2s+2t,2s}, \tag{22}$$

*where,* $\alpha_0(\lambda_p) = \lambda_p,\ \alpha_0(\mu_q) = \mu_q + \mu_{q-1}$, *and,* $\alpha_1(\lambda_p) = \lambda_p + \lambda_{p-1},\ \alpha_1(\mu_q) = \mu_q$, *for all* $p \leqslant k, q \leqslant n-k$. *(It is understood that* $\lambda_0 = 1 = \mu_0$*.) As an abelian group* $\bar{K}_{n,k}$ *is free of rank* $\binom{s+t}{s}$ *where*

$$(n,k) = (2s+2t+2, 2s+1),\ (2s+2t+1, 2s+1),\ (2s+2t, 2s).$$

*Proof.* It is readily verified that $\alpha_0, \alpha_1$ are surjective homomorphisms. We need to show that they are injective as well.

Consider $\beta_0\colon \bar{K}_{2s+2t+1,2s+1} \to \bar{K}_{2s+2t+2,2s+1}$, and, $\beta_1\colon \bar{K}_{2s+2t,2s} \to \bar{K}_{2s+2t+1,2s+1}$ defined as follows: for $p \leqslant s, q \leqslant t$,

$$\beta_0(\lambda_p) = \lambda_p, \quad \beta_0(\mu_q) = \sum_{0 \leqslant j \leqslant q} (-1)^{q-j}\mu_j, \quad \text{and}$$

$$\beta_1(\lambda_p) = \sum_{0 \leqslant j \leqslant p} (-1)^{p-j}\lambda_j, \quad \beta_1(\mu_q) = \mu_q.$$

Straightforward verification, using the identity $\sum_{0 \leqslant j \leqslant r} (-1)^j \binom{n}{r-j} = \binom{n-1}{r}$, shows that $\beta_0$ and $\beta_1$ are well-defined homomorphisms of rings. Again, these are surjective, since the generators $\lambda_p$ (resp. $\mu_q$) are in the image of $\beta_0$ (resp. $\beta_1$).

We claim that $\alpha_0, \beta_0$ (resp. $\alpha_1, \beta_1$) are inverses of each other. Indeed,

$$\beta_0 \circ \alpha_0(\lambda_p) = \lambda_p \ \text{ for all } \ p \leqslant s \ \text{ and } \ \alpha_0 \circ \beta_0(\lambda_p) = \lambda_p \ \text{ for all } \ p.$$

By Remark 5.3(b) above, our claim follows. Similarly $\alpha_1, \beta_1$ are inverses of each other.

For the last assertion, we need only consider the case $(n,k)=(2s+2t+2, 2s+1)$. The ring $\bar{K}_{2s+2t+2,2s+1}$ is isomorphic to the quotient ring $R/I \cong H^*(\mathbb{C}G_{s+t,s}; \mathbb{Z})$ considered in Remark 4.3(ii). Hence $\bar{K}_{2s+2t+2,2s+1}$ is a free abelian group of rank $\binom{s+t}{s}$. □

**Proposition 5.5.** *One has a surjective homomorphism of rings* $\kappa\colon K_{n,k} \to \mathcal{K}_{n,k}$ *with finite kernel, defined as* $\kappa(\lambda_j) = [\Lambda^j(\gamma^{\mathbb{C}}_{n,k})], 1 \leqslant j \leqslant k$.

*Proof.* In view of Equations (20) and (21), $\kappa$ is a well-defined ring homomorphism. Clearly $\kappa(\lambda_j) = [\Lambda^j(\gamma^{\mathbb{C}}_{n,k})]$ for all $j$ and so, by the definition of $\mathcal{K}_{n,k}$, $\kappa$ is surjective. Since both $K_{n,k}, \mathcal{K}_{n,k}$ have the same (finite) rank, it follows that $\ker(\kappa)$ is finite. □

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Sudeep Podder sudeep@smail.iitm.ac.in

Department of Mathematics, Indian Institute of Technology Madras, Chennai 600036, India

Parameswaran Sankaran sankaran@cmi.ac.in

Chennai Mathematical Institute, SIPCOT IT Park, Siruseri, Kelambakkam, 603103, India