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LIFESPAN FUNCTORS AND NATURAL DUALITIES IN PERSISTENT HOMOLOGY

ULRICH BAUER AND MAXIMILIAN SCHMAHL

(*communicated by Peter Bubenik*)

Abstract

We introduce lifespan functors, which are endofunctors on the category of persistence modules that filter out intervals from barcodes according to their boundedness properties. They can be used to classify injective and projective objects in the category of barcodes and the category of pointwise finite-dimensional persistence modules. They also naturally appear in duality results for absolute and relative versions of persistent (co)homology, generalizing previous results in terms of barcodes. Due to their functoriality, we can apply these results to morphisms in persistent homology that are induced by morphisms between filtrations. This lays the groundwork for the efficient computation of barcodes for images, kernels, and cokernels of such morphisms.

1. Introduction

Persistent homology, the homology of a filtration of simplicial complexes, is a cornerstone in the foundations of topological data analysis. It has found numerous applications in a variety of disciplines, including for example computer vision, neuroscience, materials science, and evolutionary biology [**7**, **13**, **18**, **27**, **30**]. The most common setting studied in the mathematical literature is as follows: Given a filtration of simplicial complexes

$$K_\bullet \colon \emptyset = K_{-\infty} = \cdots = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_N = \cdots = K_\infty = K,$$

by applying homology with coefficients in a field to each space and to each inclusion map one obtains a diagram of vector spaces

$$H_*(K_\bullet)\colon \; \cdots \to H_*(K_0) \to H_*(K_1) \to \cdots \to H_*(K_N) \to \cdots .$$

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Such a diagram is called a *persistence module*, and it decomposes into a direct sum of indecomposable diagrams, each supported on an interval [1, 17, 37]. The collection of these intervals, called the *persistence barcode*, has proven to be a powerful invariant of the filtration [21, 31].

A natural question to ask is how the barcode changes when the filtration changes. This leads to the seminal *stability theorem* of Cohen-Steiner et al. [14], which asserts that passing from filtrations to barcodes is a 1-Lipschitz map. One way to approach the stability theorem is via *induced matchings*, which were introduced in [3]. Given a morphism of filtrations $f_\bullet \colon L_\bullet \to K_\bullet$, the homology functor induces a morphism $H_*(f_\bullet) \colon H_*(L_\bullet) \to H_*(K_\bullet)$ of persistence modules. From this morphism, the induced matching construction yields a partial bijection between the barcodes of $H_*(L_\bullet)$ and $H_*(K_\bullet)$, which can be used to bound the distance between these two barcodes from above. The induced matching is defined in terms of the barcode of $\operatorname{im} H_*(f_\bullet)$, motivating the problem of computing this barcode. A first algorithm for this problem has been given by Cohen-Steiner et al. [15] for the special case where $f_\bullet$ is of the form $L_\bullet = K_\bullet \cap L \hookrightarrow K_\bullet$. In addition to their algorithm for image persistence, Cohen-Steiner et al. [15] present algorithms for computing barcodes of the kernel and cokernel of the morphism $H_*(f_\bullet)$. All of their algorithms rely on the standard reduction of boundary matrices.

The barcode of the image of a morphism of persistence modules has various applications besides the construction of the induced matching. Cohen-Steiner et al. [15] propose applications of the image barcode to recovering the persistent homology of a noisy function on a noisy domain; see also the related work by Chazal et al. [11]. Very recently, Reani and Bobrowski [33] proposed a method that includes the computation of induced matchings in order to pair up common topological features in different data sets, with applications to statistical bootstrapping. Furthermore, the computation of image barcodes is used in a distributed algorithm for persistent homology based on the Mayer–Vietoris spectral sequence by Torras Casas [35].

Despite the usefulness of image persistence, there are a few aspects that have prevented these techniques from being widely used in applications so far. Specifically, to the best of our knowledge, there is no publicly available implementation at this moment. Furthermore, computation using the known algorithms is slow in comparison to modern algorithms for a single filtration. Indeed, computing usual persistent homology for larger data sets arising in real-world applications only became feasible in recent years due to optimizations that exploit various structural properties and algebraic identities of the problem [2, 12, 20]. One of our motivations is to develop a theory allowing for the adaption of these speed-ups to the computation of images and induced matchings.

One of the most important improvements for barcode computations relies on the use of cohomology based algorithms. These were first studied by de Silva et al. in [20] and justified by certain duality results. In summary, the authors provide correspondences between the barcodes for persistent homology and for *persistent cohomology*, as well as the barcodes for *persistent relative homology*

$$H_*(K, K_\bullet) \colon H_*(K, K_0) \to H_*(K, K_1) \to \cdots \to H_*(K, K)$$

and similarly for *persistent relative cohomology*. The homology persistence modules simply have the same barcode as their cohomology counterparts [20, Proposition 2.3].

For the absolute-relative correspondence [20, Proposition 2.4], it turns out that the bounded intervals in the barcodes of $H_{d-1}(K_\bullet)$ and $H_d(K, K_\bullet)$ are also exactly the same, and there is a one-to-one correspondence between intervals of the form $[a, \infty)$ in the barcode of $H_d(K_\bullet)$ and intervals of the form $(-\infty, a)$ in the barcode of $H_d(K, K_\bullet)$.

The original proof for the absolute-relative correspondence uses a decomposition of filtered chain complexes. This strategy relies on a non-canonical choice, which does not extend to the functorial setting. We thus adopt a different point of view based on the long exact sequence of a pair in homology. Applying this functorial construction to a filtration $K_\bullet$, we obtain a long exact sequence of persistence modules

$$\cdots \longrightarrow \Delta H_d(K) \xrightarrow{\epsilon_d} H_d(K, K_\bullet) \xrightarrow{\partial} H_{d-1}(K_\bullet) \xrightarrow{\eta_{d-1}} \Delta H_{d-1}(K) \longrightarrow \cdots,$$

where Δ denotes constant persistence modules. As it turns out, the first two of the short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{im}\partial & \longrightarrow & H_{d-1}(K_\bullet) & \longrightarrow & \operatorname{im}\eta_{d-1} & \longrightarrow & 0 \\ 0 & \longrightarrow & \operatorname{im}\epsilon_d & \longrightarrow & H_d(K, K_\bullet) & \longrightarrow & \operatorname{im}\partial & \longrightarrow & 0 \\ 0 & \longrightarrow & \operatorname{im}\eta_d & \longrightarrow & \Delta H_d(K) & \longrightarrow & \operatorname{im}\epsilon_d & \longrightarrow & 0 \end{array}$$

split (as a special case of Corollary 4.8), showing that $\operatorname{im}\partial$ is a summand of both $H_d(K, K_\bullet)$ and $H_{d-1}(K_\bullet)$. Its barcode consists of the bounded intervals of either persistence module. Moreover, the third short exact sequence has a constant persistence module $\Delta H_d(K)$ in the middle, implying that the persistence modules $\operatorname{im}\eta_d$ and $\operatorname{im}\epsilon_d$ determine each other. Together this shows that the barcodes of $H_*(K_\bullet)$ and $H_*(K, K_\bullet)$ completely determine each other. For details see Section 6.1.

By observing that $\Delta H_d(K) \cong \Delta \operatorname{colim} H_d(K) \cong \Delta \lim H_d(K, K_\bullet)$ and that ϵ and η are the counit and the unit of the adjunctions $\Delta \dashv \lim$ and $\operatorname{colim} \dashv \Delta$, respectively, we can generalize this discussion to arbitrary persistence modules, indexed by totally ordered sets. Taking images, kernels and cokernels of the morphisms ϵ and η yields endofunctors on the category of persistence modules, which we call *lifespan functors* (see Figure 1); in particular, the *mortal part* $(-)^\dagger = \ker \eta_{(-)}$ and the *immortal part* $(-)^\infty = \operatorname{im}\eta_{(-)}$ determine death in the persistence module, while the *nascent part* $(-)^* = \operatorname{coker}\epsilon_{(-)}$ and the *ancient part* $(-)^{-\infty} = \operatorname{im}\epsilon_{(-)}$ determine birth.

This general definition of lifespan functors also works in the category of *matching diagrams*, since these diagrams admit limits and colimits (Proposition 4.1). The category of matching diagrams is equivalent to the category of barcodes [4]. The effect of the lifespan functors on persistence modules and matching diagrams is best described in terms of barcodes, as for the above example $\operatorname{im}\partial \cong \ker \eta_{d-1} \cong \operatorname{coker}\epsilon_d$, whose barcode corresponds to the bounded intervals; see also Figure 1 for an illustration and Corollary 4.7 for a precise statement.

As an application of lifespan functors, we give a simple characterization of the projective and injective objects both in the category of barcodes (or matching diagrams) as well as the category of *pointwise finite-dimensional* persistence modules (Theorems 5.1 and 5.5). In both cases, the projective objects are those with vanishing mortal part, while the injective objects are those with vanishing nascent part.

The lifespan functors allow us to succinctly express the above absolute-relative correspondence in terms of natural isomorphisms and correspondences (Theorem 6.2):

$$H_{d-1}(K_\bullet)^\dagger \cong \operatorname{im}\partial \cong H_d(K,K_\bullet)^*, \quad H_d(K_\bullet)^\infty \cong \operatorname{im}\eta_d \leftrightarrow \operatorname{im}\epsilon_d \cong H_d(K,K_\bullet)^{-\infty}.$$

Here, naturality is inherited from the construction of the long exact homology sequence. In particular, from the morphism of filtrations $f_\bullet\colon L_\bullet \to K_\bullet$ inducing a map $f\colon L \to K$, we get an isomorphism $H_{d-1}(f_\bullet)^\dagger \cong H_d(f, f_\bullet)^*$. We also get a morphism

$$\begin{tikzcd}
\cdots \arrow[r] & \Delta H_d(K) \arrow[r, "\epsilon_d"] & H_d(K,K_\bullet) \arrow[r, "\partial"] & H_{d-1}(K_\bullet) \arrow[r, "\eta_{d-1}"] & \Delta H_{d-1}(K) \arrow[r] & \cdots \\
\cdots \arrow[r] & \Delta H_d(L) \arrow[r, "\epsilon_d"] \arrow[u] & H_d(L,L_\bullet) \arrow[r, "\partial"] \arrow[u] & H_{d-1}(L_\bullet) \arrow[r, "\eta_{d-1}"] \arrow[u] & \Delta H_{d-1}(L) \arrow[r] \arrow[u] & \cdots
\end{tikzcd}$$

of long exact sequences. Note, however, that the induced sequences of kernels, images, and cokernels are no longer exact in general, so the rest of the proof of the absolute-relative-correspondence for a single filtration does not carry over completely to this setting. In order to still obtain a useful absolute-relative correspondences for $H_*(f_\bullet)$, we develop conditions for when the lifespan functors commute with taking images, kernels, and cokernels of morphisms (Theorem 3.10), so that, for example if $H_*(f)$ is an isomorphism, we get

$$(\operatorname{im} H_{d-1}(f_\bullet))^\dagger \cong \operatorname{im}\left(H_{d-1}(f_\bullet)^\dagger\right) \cong \operatorname{im}\left(H_d(f,f_\bullet)^*\right) \cong (\operatorname{im} H_d(f,f_\bullet))^*,$$

meaning that, as in the single filtration case, the bounded intervals in the barcodes of the images of the absolute and relative morphism agree. Furthermore, we will also state a functorial version of the correspondence between persistent homology and cohomology in terms of vector space duality (Proposition 6.4) and analyze how the lifespan functors behave with respect to dualization (Section 4.2).

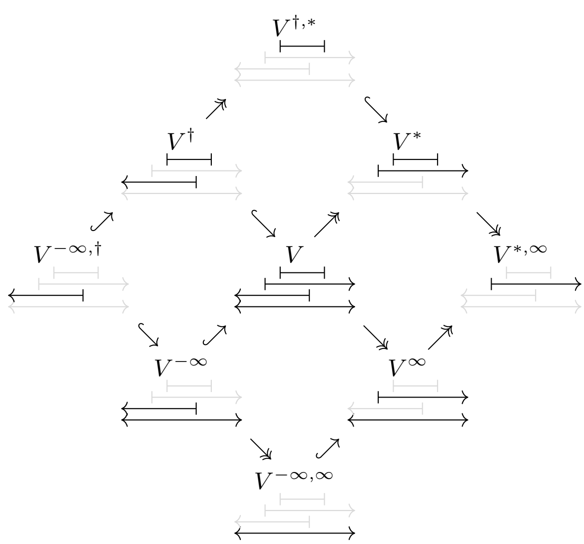

Figure 1: Lifespan functors applied to a finite type $\mathbb{R}$-indexed persistence module V, visualized via their barcode according to Corollary 4.7.

As explained in [2], computing cohomology instead of homology is particularly relevant in conjunction with the *clearing* optimization, introduced by Chen and Kerber in [12], and also used implicitly in the cohomology algorithm by de Silva et al. [20]. There is also an adaptation of this optimization for the computation of barcodes of images, which we aim to formalize in future work. We already provide an implementation of the resulting method for computing barcodes of images of maps between the persistent homologies of Vietoris–Rips filtrations [5].

Outline

We start off by reviewing some preliminaries on persistence including the theory of matching diagrams in Section 2. In Section 3, we present the lifespan functors and some of their relevant properties in a general setting. The lifespan functors are then specialized to persistence modules in Section 4. As an application, we then use the lifespan functors in Section 5 to classify injective and projective objects in the categories of barcodes and p.f.d. persistence modules. We finish by proving functorial dualities in persistent homology involving our lifespan functors in Section 6.

Notation and conventions

Throughout the paper, we fix a totally ordered set $(T, \leqslant)$ and write $\mathbf{T}$ for the corresponding category. Note that $\mathbf{T}^{\mathrm{op}}$ is then the category corresponding to $(T, \geqslant)$. We also fix a field $\mathbb{F}$ and write $\mathbf{Vec}$ for the category of vector spaces over $\mathbb{F}$. The full subcategory of finite dimensional vector spaces is denoted by $\mathbf{vec}$. We write $\mathbf{Top}$ for the category of topological spaces. If $\mathbf{A}$ is an abelian category, we write $\mathbf{Ch}(\mathbf{A})$ for the category of chain complexes over $\mathbf{A}$. If $\mathbf{C}$ and $\mathbf{D}$ are categories, we write $\mathbf{C}^{\mathbf{D}}$ for the category of functors $\mathbf{C} \to \mathbf{D}$.

2. Preliminaries

We recall the basic definitions of absolute and relative persistent (co)homology in Section 2.1. They are examples of persistence modules, which we go over together with matching diagrams in Section 2.2. Subsequently, we present a formal framework for barcodes in Section 2.3. The equivalence between barcodes and matching diagrams is recalled in Section 2.4. We then collect some basic categorical properties of barcodes and matching diagrams in Section 2.5. Finally, we recall some facts about dualization of persistence modules in Section 2.6.

2.1. Persistent homology

Definition 2.1. The category of *persistence modules indexed by* $\mathbf{T}$ is defined as the functor category $\mathbf{Vec}^{\mathbf{T}}$. The category of *pointwise finite dimensional (p.f.d.) persistence modules indexed by* $\mathbf{T}$ is defined as $\mathbf{vec}^{\mathbf{T}}$.

The full subcategory $\mathbf{vec}$ is closed under taking kernels, cokernels and finite direct sums in the abelian category $\mathbf{Vec}$, so it is also abelian. Moreover, since $\mathbf{T}$ is a small category, the functor categories $\mathbf{vec}^{\mathbf{T}}$ and $\mathbf{Vec}^{\mathbf{T}}$ are again abelian, with kernels, cokernels, direct sums, etc. given pointwise.

The most commonly studied example of a persistence module is the persistent homology of a diagram of spaces. To define it, we start by observing that there is a purely formal identification of the categories $\mathbf{Ch}(\mathbf{Vec}^{\mathbf{T}})$ and $\mathbf{Ch}(\mathbf{Vec})^{\mathbf{T}}$. Thus, if we have a chain complex of persistence modules, we can interpret it as a diagram of chain complexes, and vice versa. Objects in these identified categories will be called *persistent chain complexes indexed by* $\mathbf{T}$.

Let $C_*\colon \mathbf{Top} \to \mathbf{Ch}(\mathbf{Vec})$ be the functor assigning to a topological space its singular chain complex with coefficients in $\mathbb{F}$. Similarly, let $C^*\colon \mathbf{Top} \to \mathbf{Ch}(\mathbf{Vec})^{\mathrm{op}}$ denote the singular cochain complex functor with coefficients in $\mathbb{F}$.

If $X\colon \mathbf{T} \to \mathbf{Top}$ is a filtration, or any $\mathbf{T}$-indexed diagram of topological spaces, composing with C_* and C^* yields a persistent chain complex $C_*(X)$ indexed by $\mathbf{T}$ and a persistent cochain complex $C^*(X)$ indexed by $\mathbf{T}^{\mathrm{op}}$. Moreover, recall that every diagram X in $\mathbf{Top}$ has a colimit, and the natural map $X \to \Delta \operatorname{colim} X$ induces morphisms $C_*(X) \to C_*(\Delta \operatorname{colim} X)$ and $C^*(\Delta \operatorname{colim} X) \to C^*(X)$. We write

$$C_*(\operatorname{colim} X, X) = \operatorname{coker}(C_*(X) \to C_*(\Delta \operatorname{colim} X)),$$
$$C^*(\operatorname{colim} X, X) = \ker(C^*(\Delta \operatorname{colim} X) \to C^*(X)).$$

We then define the *d-th persistent homology* of X as $H_d(X) = H_d(C_*(X))$ and the *d-th persistent relative homology* of X as $H_d(\operatorname{colim} X, X) = H_d(C_*(\operatorname{colim} X, X))$, and similarly for cohomology.

Note that the relative versions are intrinsic to the diagram X, and that persistent homology is indexed by $\mathbf{T}$, while persistent cohomology is indexed by $\mathbf{T}^{\mathrm{op}}$.

2.2. Structure of persistence modules

A natural way to construct persistence modules is to specify a basis for each index $t \in T$ and then to specify the linear maps by matching the basis elements of different indices in a compatible way. This can be formalized as a functor from a category of sets and matchings to the category of vector spaces. In fact, this construction already generates all possible p.f.d. persistence modules up to isomorphism, providing a structure theorem for p.f.d. persistence modules. We now introduce the requisite definitions and recall the fundamental results from the literature.

Definition 2.2. If A and B are sets, a subset $\sigma \subseteq A \times B$ is called a *matching* if for each $a \in A$ there is at most one $b \in B$ with $(a, b) \in \sigma$ and for each $b \in B$ there is at most one $a \in A$ with $(a, b) \in \sigma$. If $\tau \subseteq B \times C$ is another matching, we define the composition $\tau \circ \sigma \subseteq A \times C$ as

$$\tau \circ \sigma = \{(a, c) \mid \text{ there exists } b \in B \text{ with } (a, b) \in \sigma \text{ and } (b, c) \in \tau\}.$$

The resulting category, with sets as objects, matchings as morphisms, and the above composition, will be denoted by $\mathbf{Mch}$. If $\sigma \subseteq A \times B$ is a matching, we define its *opposite* matching

$$\sigma^\circ = \{(b, a) \mid (a, b) \in \sigma\} \subseteq B \times A.$$

This construction makes the category $\mathbf{Mch}$ self-dual, i.e., it yields an isomorphism between $\mathbf{Mch}$ and its opposite category. We define the category of *matching diagrams indexed by* $\mathbf{T}$ as the functor category $\mathbf{Mch}^{\mathbf{T}}$.

We can now functorially assign a persistence module to a matching diagram.

Definition 2.3. We define the functor $\mathcal{F}\colon \mathbf{Mch} \to \mathbf{Vec}$ by sending a set A to the free vector space generated by A and sending a matching $\sigma \subseteq A \times B$ to the linear extension of the map

$$a \mapsto \begin{cases} b & \text{if } (a,b) \in \sigma \\ 0 & \text{otherwise.} \end{cases}$$

By a slight abuse of notation, we also define the *matching module* functor of the form $\mathcal{F}\colon \mathbf{Mch}^{\mathbf{T}} \to \mathbf{Vec}^{\mathbf{T}}$ by applying $\mathcal{F}$ pointwise, i.e., $\mathcal{F}(D) = \mathcal{F} \circ D$.

In certain cases we can also go from persistence modules to matching diagrams.

Definition 2.4. We say that a persistence module M is *interval-decomposable* if there exists a matching diagram D with $\mathcal{F}(D) \cong M$. A choice of isomorphism $\mathcal{F}(D) \cong M$ is called an *interval decomposition.*

Interval decompositions are typically described in terms of barcodes, which are collections of intervals in $\mathbf{T}$. We will formally introduce barcodes in the next section, and develop their equivalence to matching diagrams $\mathbf{Mch}^{\mathbf{T}}$ in a categorical sense.

The most commonly used existence result asserts that every p.f.d. persistence module admits an interval decomposition [**17**, Theorem 1.1]; see also [**8**, Theorem 1.2]. While there may be many different interval decompositions for a single persistence module, by a version of the Krull–Schmidt–Azumaya Theorem the structure of the decomposition is still unique, which can be conveniently phrased in the language of matching diagrams as follows.

Theorem 2.5 ([**32**], Theorem 2.7; see also [**32**], Section 4.8)**.** *The matching module functor $\mathcal{F}\colon \mathbf{Mch}^{\mathbf{T}} \to \mathbf{Vect}^{\mathbf{T}}$ reflects the property of being isomorphic: If D and D' are matching diagrams with $\mathcal{F}(D) \cong \mathcal{F}(D')$, then already $D \cong D'$.*

2.3. Barcodes

An equivalent description for a matching diagram can be given in terms of a collection of intervals, called barcode. The intervals encode the index range of matched elements in the matching diagram. A barcode should be thought of as a multiset of intervals, that is, the same interval may appear multiple times.

Definition 2.6. We denote the set of all intervals in T as $\mathfrak{I}(T)$, or simply as $\mathfrak{I}$ when the index set is clear from the context. If A is an arbitrary set, we call any subset $B \subseteq \mathfrak{I} \times A$ a *barcode in* T.

The purpose of the set A in this definition is to distinguish multiple instances of the same interval, as in the standard construction of a disjoint union. If clear from the context, we sometimes suppress this index from the notation. If $B \subseteq \mathfrak{I} \times A$ is a barcode and I an interval in T, the cardinality of the set $\{a \in A \mid (I,a) \in B\}$ measures how many copies of I are in B.

Barcodes form a category, which is equivalent to the category of matching diagrams [**4**]. We introduce some terminology used to give an explicit description of the morphisms in the category of barcodes.

Definition 2.7. Let I and J be intervals in T. We say that I *bounds* J *above* if for all $s \in J$ there exists $t \in I$ such that $s \leqslant t$. We say that I *bounds* J *below* if for all

$u \in J$ there exists $t \in I$ such that $t \leqslant u$. We say that I *overlaps* J *above*, or that J *overlaps* I *below*, if their intersection is non-empty, I bounds J above, and J bounds I below.

Definition 2.8. For barcodes B and B', we call a matching $\sigma \subseteq B \times B'$ an *overlap matching* if for each $((I, a), (I', a')) \in \sigma$ the interval I overlaps the interval I' above. If $\sigma \subseteq B \times B'$ and $\tau \subseteq B' \times B''$ are overlap matchings, we define their *overlap composition* as

$$\tau \bullet \sigma = \{((I, a), (I'', a'')) \in \tau \circ \sigma \mid I \text{ overlaps } I'' \text{ above}\}.$$

The resulting category with barcodes as objects, overlap matchings as morphisms and overlap composition will be denoted by $\mathbf{Barc}(\mathbf{T})$.

Note that two barcodes $B \subseteq \mathfrak{I} \times A$ and $B' \subseteq \mathfrak{I} \times A'$ are isomorphic if and only if there is a bijection $f \colon B \to B'$ such that for all $(I, a) \in B$ there is a' in A' with $f(I, a) = (I, a')$. In other words, B and B' are isomorphic if and only if the sets $\{a \in A \mid (I, a) \in B\}$ and $\{a' \in A' \mid (I, a') \in B'\}$ have the same cardinality for every interval $I \in \mathfrak{I}$.

2.4. Equivalence of barcodes and matching diagrams

As we have mentioned before, the two categories $\mathbf{Mch}^{\mathbf{T}}$ and $\mathbf{Barc}(\mathbf{T})$ are equivalent. We will now review the construction of an explicit equivalence following [4].

Definition 2.9. Let D be a matching diagram. We define its *components* as the set of equivalence classes

$$\mathcal{C}(D) = \left(\bigcup_{t \in T} \{t\} \times D_t\right) /{\sim},$$

where the equivalence relation $\sim$ is defined as follows: For $t \leqslant u \in T$, $d \in D_t$, and $d' \in D_u$, we set $(t, d) \sim (u, d')$ if and only if $(d, d') \in D_{t,u}$. Note that each component $Q \in \mathcal{C}(D)$ can also be regarded as a matching diagram such that $Q_t \subseteq D_t$ has at most one element for each $t \in T$. For a component $Q \in \mathcal{C}(D)$, we define its *support* as the range of indices in T spanned by the component,

$$\operatorname{supp}(Q) = \{t \in T \mid (t, d) \in Q \text{ for some } d \in D_t\}.$$

Note that the component set construction may be regarded as a functor of the form $\mathcal{C} \colon \mathbf{Mch}^{\mathbf{T}} \to \mathbf{Mch}$. Even more than that, it can not only be used to pass from matching diagrams to the matching category, but it can also be used to pass from matching diagrams to barcodes.

Definition 2.10. We define a functor $\mathcal{B} \colon \mathbf{Mch}^{\mathbf{T}} \to \mathbf{Barc}(\mathbf{T})$ by setting

$$\mathcal{B}(D) = \{(I, Q) \in \mathfrak{I} \times \mathcal{C}(D) \mid I = \operatorname{supp}(Q)\}$$

for any matching diagram D and

$$\mathcal{B}(\psi) = \{((I, Q), (I', R)) \in \mathcal{B}(D) \times \mathcal{B}(E) \mid Q_t \times R_t \subseteq \psi_t \text{ for all } t \in I \cap I'\}$$

for any morphism of matching diagrams $\psi \colon D \to E$.

As shown in [4], the support of a component is indeed an interval, a morphism of matching diagrams is mapped to an overlap matching by the above construction, and we indeed get a functor. Conversely, we can also pass from barcodes to matching diagrams.

Definition 2.11. We define a functor $\mathcal{D}\colon \mathbf{Barc}(\mathbf{T}) \to \mathbf{Mch}^{\mathbf{T}}$ by setting $\mathcal{D}(B)$ for any barcode B to be the matching diagram given by

$$\begin{aligned}\mathcal{D}(B)_t &= \{(I,a) \in B \mid t \in I\},\\ \mathcal{D}(B)_{t,u} &= \{((I,a),(I',a')) \in \mathcal{D}(B)_t \times \mathcal{D}(B)_u \mid (I,a) = (I',a')\}.\end{aligned}$$

For an overlap matching σ, we let $\mathcal{D}(\sigma)$ be the morphism of matching diagrams with

$$\mathcal{D}(\sigma)_t = \{((I,a),(I',a')) \in \sigma \mid t \in I \cap I'\}.$$

Again, we refer to [4] for the fact that $\mathcal{D}$ is a well-defined functor.

Theorem 2.12 ([4])**.** *The functors*

$$\mathcal{B}\colon \mathbf{Mch}^{\mathbf{T}} \to \mathbf{Barc}(\mathbf{T}) \quad \textit{and} \quad \mathcal{D}\colon \mathbf{Barc}(\mathbf{T}) \to \mathbf{Mch}^{\mathbf{T}}$$

defined above are quasi-inverses. In particular, the categories $\mathbf{Mch}^{\mathbf{T}}$ *and* $\mathbf{Barc}(\mathbf{T})$ *are equivalent.*

Note that in [4], the equivalences were denoted by E and F. Using this equivalence, we can give an explicit description of the composite functor $\mathcal{F} \circ \mathcal{D}\colon \mathbf{Barc}(\mathbf{T}) \to \mathbf{Vec}^{\mathbf{T}}$ constructing a persistence module with a given barcode as follows.

Definition 2.13. Let $I \subseteq T$ be an interval. The *interval module* $C(I)$ is the persistence module obtained from the barcode consisting of a single instance of I:

$$C(I)_t = \begin{cases} \mathbb{F} & \text{if } t \in I,\\ 0 & \text{otherwise,}\end{cases} \qquad \text{with structure maps} \qquad C(I)_{t,u} = \begin{cases} \mathrm{id}_{\mathbb{F}} & \text{if } t,u \in I,\\ 0 & \text{otherwise.}\end{cases}$$

If I and J are intervals such that I overlaps J above, there exists a canonical morphism $\varphi(I,J)\colon C(I) \to C(J)$ defined by

$$\varphi(I,J)_t = \begin{cases} \mathrm{id}_{\mathbb{F}} & \text{if } t \in I \cap J,\\ 0 & \text{otherwise.}\end{cases}$$

We define the *barcode module functor* $\mathcal{M}\colon \mathbf{Barc}(\mathbf{T}) \to \mathbf{Vec}^{\mathbf{T}}$ by sending a barcode B to the direct sum of interval modules $\bigoplus_{(I,a)\in B} C(I)$ and sending an overlap matching $\sigma \subseteq B \times B'$ to the direct sum of the morphisms $\varphi(I,I')\colon C(I) \to C(I')$ for all pairs $((I,a),(I',a')) \in \sigma$. If a persistence module M satisfies $\mathcal{M}(B) \cong M$ for some barcode $B \in \mathbf{Barc}(\mathbf{T})$, we say that B is a *barcode of* M.

The following proposition is straightforward to verify from the definitions.

Proposition 2.14. *There are natural isomorphisms* $\mathcal{F} \cong \mathcal{M} \circ \mathcal{B}$ *and* $\mathcal{M} \cong \mathcal{F} \circ \mathcal{D}$.

2.5. Categorical properties of matching diagrams

One can use large parts of the theory of homological algebra in the categories $\mathbf{Mch}$ and $\mathbf{Mch}^{\mathbf{T}}$ since they have the following property.

Definition 2.15. A category is called *Puppe-exact* or *p-exact* if it has a zero object, it has all kernels and cokernels, every mono is a kernel and every epi is a cokernel, and every morphism has an epi-mono-factorization.

Put informally, a Puppe-exact category is an abelian category that need not have (co)products. Recall that in any category with kernels and cokernels, monos have vanishing kernels and epis have vanishing cokernels. While the converse is not true in general, it is true in p-exact categories.

Lemma 2.16 ([9, Korollar 2.4.4])**.** *A morphism in a p-exact category is mono if and only if its kernel vanishes and it is epi if and only if its cokernel vanishes.*

We will use this lemma throughout without explicit reference. In particular, we will use it for barcodes and matching diagrams, which form p-exact categories.

Proposition 2.17 ([22, Section 1.6.4])**.** **Mch** *is Puppe-exact. For a choice of matching $\sigma \subseteq A \times B$ we have*

$$\begin{aligned} \ker \sigma &= \{a \in A \mid (a,b) \notin \sigma \textit{ for all } b \in B\}, \\ \operatorname{im} \sigma &= \{b \in B \mid (a,b) \in \sigma \textit{ for some } a \in A\}, \\ \operatorname{coker} \sigma &= \{b \in B \mid (a,b) \notin \sigma \textit{ for all } a \in A\}, \\ \operatorname{coim} \sigma &= \{a \in A \mid (a,b) \in \sigma \textit{ for some } b \in B\}. \end{aligned}$$

$\mathbf{Mch}^{\mathbf{T}}$ *is also Puppe-exact, with kernels, cokernels etc. given pointwise.*

Importantly, these constructions are compatible with the passage to vector spaces and persistence modules, as expressed in the following statement.

Proposition 2.18. *The functor $\mathcal{F}$ preserves and reflects exactness, i.e., a sequence of matchings $V \to V' \to V''$ is exact if and only if the corresponding sequence of vector spaces $\mathcal{F}(V) \to \mathcal{F}(V') \to \mathcal{F}(V'')$ is exact. The same holds for $\mathcal{F}$ as a functor $\mathbf{Mch}^{\mathbf{T}} \to \mathbf{Vec}^{\mathbf{T}}$.*

Using the equivalence between $\mathbf{Mch}^{\mathbf{T}}$ and $\mathbf{Barc}(\mathbf{T})$, we can translate the constructions in Proposition 2.17 to describe the kernels, cokernels, and images of overlap matchings explicitly as barcodes.

Definition 2.19. For an overlap matching $\sigma \subseteq B \times B'$ and $(I,a) \in B$, $(I',a') \in B'$, we set

$$\ker(\sigma,(I,a)) = \begin{cases} (I \setminus I', a) & \text{if } ((I,a),(I',a')) \in \sigma, \\ (I,a) & \text{otherwise;} \end{cases}$$

$$\operatorname{coker}(\sigma,(I',a')) = \begin{cases} (I' \setminus I, a') & \text{if } ((I,a),(I',a')) \in \sigma, \\ (I',a') & \text{otherwise.} \end{cases}$$

Proposition 2.20 ([4])**.** *Let $B \subseteq \mathfrak{I} \times A$ and $B' \subseteq \mathfrak{I} \times A'$ be barcodes. Any overlap matching $\sigma \subseteq B \times B'$ has a kernel, coimage, image and cokernel in* $\mathbf{Barc}(\mathbf{T})$*, with*

$$\begin{aligned} \ker \sigma &= \{(J,a) \in \mathfrak{I} \times A \mid J = \ker(\sigma,(I,a)) \textit{ for } (I,a) \in B\} \\ \operatorname{coim} \sigma &= \{(J,a) \in \mathfrak{I} \times A \mid J = I \cap I' \textit{ for } ((I,a),(I',a')) \in \sigma\} \\ \operatorname{im} \sigma &= \{(J,a') \in \mathfrak{I} \times A' \mid J = I \cap I' \textit{ for } ((I,a),(I',a')) \in \sigma\} \\ \operatorname{coker} \sigma &= \{(J,a') \in \mathfrak{I} \times A' \mid J = \operatorname{coker}(\sigma,(I',a')) \textit{ for } (I',a') \in B'\}. \end{aligned}$$

Using the p-exact structure on barcodes, we will later consider exact sequences of barcodes and translate them to exact sequences of persistence modules. We will also further study the categorical structure on barcodes later on and classify injective and projective objects. In both of these settings, the following characterization of split mono and epi overlap matchings will be important.

Lemma 2.21. *Let σ be an overlap matching and assume that σ is mono or epi. Then σ is split if and only if $((I,a),(I',a')) \in \sigma$ implies $I = I'$.*

Proof. If $((I,a),(I',a')) \in \sigma$ implies $I = I'$ for some overlap matching σ, then its opposite matching σ° is again an overlap matching. If σ is epi, this yields a right inverse and if σ is mono, this yields a left inverse.

If on the other hand σ is split mono or split epi, there needs to be an overlap matching τ with $((I',a'),(I,a)) \in \tau$ whenever $((I,a),(I',a')) \in \sigma$. Since both σ and τ are overlap matchings, I and I' overlap each other above, so we have $I = I'$ whenever $((I,a),(I',a')) \in \sigma$. □

2.6. Dualization for persistence modules

We have seen persistent homology and cohomology as examples of persistence modules indexed by either $(T,\leqslant)$ or $(T,\geqslant)$. In general, the following yields a way of translating between the two.

Definition 2.22. We define the contravariant *dualization* functor

$$(-)^{\vee}\colon \mathbf{Vec}^{\mathbf{T}} \to \mathbf{Vec}^{\mathbf{T}^{\mathrm{op}}}$$

by applying vector space dualization pointwise, i.e., for a $\mathbf{T}$-indexed persistence module M, its *dual* $M^{\vee}$ is the $\mathbf{T}^{\mathrm{op}}$-indexed persistence module given by the formula $M_t^{\vee} = \mathrm{Hom}(M_t, \mathbb{F})$ for all $t \in T$.

Note that a subset $I \subseteq T$ is an interval with respect to $\leqslant$ if and only if it is an interval with respect to $\geqslant$. This yields an obvious contravariant isomorphism between $\mathbf{Barc}(\mathbf{T})$ and $\mathbf{Barc}(\mathbf{T}^{\mathrm{op}})$ which maps each barcode to itself. Thus, we can compare barcodes of persistence modules indexed by $(T,\leqslant)$ with barcodes of persistence modules indexed by $(T,\geqslant)$. As such, we have the following well-known fact.

Lemma 2.23. *Let M be a p.f.d. persistence module. Then B is a barcode for M if and only if it is a barcode for $M^{\vee}$.*

Recall that $\mathbb{F}$ is injective as a module over itself, which means that the contravariant functor $\mathrm{Hom}(-,\mathbb{F})\colon \mathbf{Vec} \to \mathbf{Vec}$ is exact. As pointwise application of an exact functor yields an exact functor of diagram categories, we get the following.

Lemma 2.24. *The dualization functor $(-)^{\vee}$ is exact. In particular, a morphism $\varphi\colon M \to N$ of persistence modules yields isomorphisms*

$$(\ker\varphi)^{\vee} \cong \operatorname{coker}\varphi^{\vee}, \qquad (\operatorname{im}\varphi)^{\vee} \cong \operatorname{im}\varphi^{\vee}, \qquad (\operatorname{coker}\varphi)^{\vee} \cong \ker\varphi^{\vee}.$$

3. Lifespan functors

In Section 3.1, we will construct what we call the lifespan functors based on (co)units of the adjunction of the diagonal functor with (co)limits. We then establish conditions for when the lifespan functors commute with the image, kernel, and cokernel functors in Section 3.2.

3.1. Defining lifespan functors

Let $\mathbf{A}$ be any category with $\mathbf{T}$-shaped limits and colimits, so that we get functors $\lim\colon \mathbf{A}^{\mathbf{T}} \to \mathbf{A}$ and $\operatorname{colim}\colon \mathbf{A}^{\mathbf{T}} \to \mathbf{A}$. As for any functor category, we also have a diagonal functor $\Delta\colon \mathbf{A} \to \mathbf{A}^{\mathbf{T}}$, mapping each object to the corresponding constant diagram. Of course, this setting includes the case where $\mathbf{A} = \mathbf{Vec}$.

For each object V in $\mathbf{A}^{\mathbf{T}}$, the canonical maps $V_t \to \operatorname{colim} V$ for $t \in T$ form a natural transformation $\eta_V\colon V \to \Delta \operatorname{colim} V$. Recall that colim is left adjoint to the diagonal functor Δ, and the morphism η_V is the component at V for the unit

$$\eta\colon \operatorname{id}_{\mathbf{A}^{\mathbf{T}}} \to \Delta \circ \operatorname{colim}$$

of the adjunction $\operatorname{colim} \dashv \Delta$. Similarly, the canonical maps $\lim V \to V_t$ give a natural transformation $\epsilon_V\colon \Delta \lim V \to V$, which is the counit $\epsilon\colon \Delta \circ \lim \to \operatorname{id}_{\mathbf{A}^{\mathbf{T}}}$ of the adjunction $\Delta \dashv \lim$. We thus get the diagram

$$\Delta \lim V \xrightarrow{\epsilon_V} V \xrightarrow{\eta_V} \Delta \operatorname{colim} V.$$

From now on, we assume that $\mathbf{A}$ is Puppe-exact, so that we can form kernels, cokernels, and images.

Definition 3.1. We define the following functors $\mathbf{A}^{\mathbf{T}} \to \mathbf{A}^{\mathbf{T}}$.

1. The *mortal part* functor is defined as $(-)^{\dagger} = \ker \eta_{(-)}$.
2. The *immortal part* functor is defined as $(-)^{\infty} = \operatorname{im} \eta_{(-)}$.
3. The *nascent part* functor is defined as $(-)^{*} = \operatorname{coker} \epsilon_{(-)}$.
4. The *ancient part* functor is defined as $(-)^{-\infty} = \operatorname{im} \epsilon_{(-)}$.

By definition, for each object V in $\mathbf{A}^{\mathbf{T}}$ we get a natural diagram

$$\begin{tikzcd} V^{\dagger} \arrow[dr, hook] & & V^{*} \\ & V \arrow[ur, two heads] \arrow[dr, two heads] & \\ V^{-\infty} \arrow[ur, hook] & & V^{\infty} \end{tikzcd}$$

with diagonal short exact sequences. We also get composite natural transformations

$$(-)^{\dagger} \to \operatorname{id}_{\mathbf{A}^{\mathbf{T}}} \to (-)^{*} \qquad \text{and} \qquad (-)^{-\infty} \to \operatorname{id}_{\mathbf{A}^{\mathbf{T}}} \to (-)^{\infty}$$

and can again form kernels, cokernels and images to get new functors.

Definition 3.2. We define the following functors $\mathbf{A}^{\mathbf{T}} \to \mathbf{A}^{\mathbf{T}}$.

1. The *finite part* functor is defined as $(-)^{\dagger,*} = \operatorname{im}((-)^{\dagger} \to (-)^{*})$.
2. The *constant part* functor is defined as $(-)^{-\infty,\infty} = \operatorname{im}((-)^{-\infty} \to (-)^{\infty})$.

Remark 3.3. The universal property of epi-mono-factorizations implies that we have a canonical isomorphism $V^{-\infty,\infty} \cong \operatorname{im}(\Delta \lim V \to \Delta \operatorname{colim} V)$ for all objects V in $\mathbf{A}^{\mathbf{T}}$.

We will also form kernels and cokernels of the above composite morphisms. In the cases we are interested in, these turn out to coincide: by [**22**, Lemma 2.2.4], pullbacks of monos and pushouts of epis exist in p-exact categories, and we have canonical isomorphisms

$$\ker(V^\dagger \to V^*) \cong V^\dagger \times_V V^{-\infty} \cong \ker(V^{-\infty} \to V^\infty)$$
$$\operatorname{coker}(V^\dagger \to V^*) \cong V^* +_V V^\infty \cong \operatorname{coker}(V^{-\infty} \to V^\infty)$$

for any V. Using this fact, we can make the following well-posed definition.

Definition 3.4. We define the following functors $\mathbf{A}^{\mathbf{T}} \to \mathbf{A}^{\mathbf{T}}$.

1. The *ancient mortal part* functor is defined as
$$(-)^{-\infty,\dagger} = \ker((-)^\dagger \to (-)^*) = \ker((-)^{-\infty} \to (-)^\infty).$$
2. The *immortal nascent part* functor is defined as
$$(-)^{*,\infty} = \operatorname{coker}((-)^\dagger \to (-)^*) = \operatorname{coker}((-)^{-\infty} \to (-)^\infty).$$

We give a common name to all the functors defined above.

Definition 3.5. For an object V in $\mathbf{A}^{\mathbf{T}}$, we will call the diagram

$$\begin{tikzcd}
 & & V^{\dagger,*} \arrow[dr, hook] & & \\
 & V^\dagger \arrow[ur, two heads] \arrow[dr, hook] & & V^* \arrow[dr, two heads] & \\
V^{-\infty,\dagger} \arrow[ur, hook] \arrow[dr, hook] & & V \arrow[ur, two heads] \arrow[dr, two heads] & & V^{*,\infty} \\
 & V^{-\infty} \arrow[ur, hook] \arrow[dr, two heads] & & V^\infty \arrow[ur, two heads] & \\
 & & V^{-\infty,\infty} \arrow[ur, hook] & &
\end{tikzcd}$$

the *lifespan diagram of* V. We call the functors at the nodes of the diagram *lifespan functors* and the natural maps between them *lifespan transformations*. We use the notation $(-)^\diamond$ as a wildcard symbol for an arbitrary lifespan functors.

See Figure 1 for an example of a lifespan diagram of persistence modules. Note that the lifespan diagram simplifies to a smaller diagram in many applications. For example, the short exact sequence $V^{-\infty,\dagger} \hookrightarrow V^{-\infty} \twoheadrightarrow V^{-\infty,\infty}$ on the bottom left vanishes if V is bounded below. Similarly, the bottom right sequence vanishes if V is bounded above. For the top left and the top right short exact sequences in the lifespan diagram, we have the following conditions.

Proposition 3.6. *Consider* $\lim, \operatorname{colim} \colon \mathbf{A}^{\mathbf{T}} \to \mathbf{A}$ *and an object* V *in* $\mathbf{A}^{\mathbf{T}}$.

1. *If* colim *is exact, then* $V^\dagger = 0$ *if and only if all structure maps of* V *are mono.*
2. *If* $\lim$ *is exact, then* $V^* = 0$ *if and only if all structure maps of* V *are epi.*

Proof. We only show the first statement since the second one is dual to it. So, assume that taking colimits is exact.

If $V^\dagger = 0$, then $V \to \Delta \operatorname{colim} V$ is mono, i.e., $V_t \to \operatorname{colim} V$ is mono for any $t \in T$. Now, for any structure map $V_t \to V_u$, we obtain that the composition

$$V_t \to V_u \to \operatorname{colim} V$$

is mono since it is equal to the natural map $V_t \to \operatorname{colim} V$. This implies that $V_t \to V_u$ is mono.

Next, assume that all structure maps of V are mono and let $t \in T$. Define an object $\tilde{V}$ in $\mathbf{A}^\mathbf{T}$ by setting $\tilde{V}_s = V_s$ for any $s < t$ and $\tilde{V}_u = V_t$ for any $u \geqslant t$. There is an obvious map $\tilde{V} \to V$ consisting of structure maps of V and because we assume these structure maps to be mono the map $\tilde{V} \to V$ is mono, too. We assume that taking colimits is exact, so the induced map $\operatorname{colim} \tilde{V} \to \operatorname{colim} V$ is still mono. But $\operatorname{colim} \tilde{V}$ is V_t and the induced map is given by the natural map $V_t \to \operatorname{colim} V$. Hence, $V \to \Delta \operatorname{colim} V$ is mono, which implies $V^\dagger = 0$. $\square$

The construction of the lifespan functors involves kernels, cokernels, and images of the natural transformations ϵ and η. Note, however, that we have not used $\ker \epsilon_{(-)}$ and $\operatorname{coker} \eta_{(-)}$ so far. These play a somewhat different role than the lifespan functors, as they do not yield subobjects or quotients of the object we start with. Still, their properties will be of similar importance, especially in Theorem 6.2.

Definition 3.7. We define the following functors $\mathbf{A}^\mathbf{T} \to \mathbf{A}^\mathbf{T}$.

1. The *ghost complement* functor is defined as $(-)^\triangleright = \ker \epsilon_{(-)}$.
2. The *unborn complement* functor is defined as $(-)^\triangleleft = \operatorname{coker} \eta_{(-)}$.

See Figure 2 for an illustration of the complement functors.

3.2. Lifespan functors and images, kernels, and cokernels

One of our overall goals is to study images, kernels, and cokernels of morphisms in persistent homology. For that purpose, we want to study how the lifespan functors appearing in the statement of Theorem 6.2 behave with respect to these operations. The relevant theorems hold in the general setting, so, as before, let $\mathbf{A}$ be p-exact with $\mathbf{T}$-indexed limits and colimits.

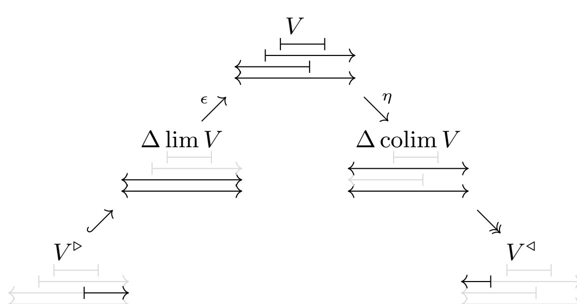

Figure 2: Complement functors applied to a finite type $\mathbb{R}$-indexed persistence module V, visualized via their barcode according to Propositions 4.9 and 4.10.

Example 3.8. The following examples show that the nascent and mortal part do not preserve images. For both examples, let the index set be $\mathbb{Z}$.

1. Consider a morphism $\varphi\colon C([0,+\infty)) \to C([0,1])$ which has maximal rank everywhere, e.g., by taking φ_0 and φ_1 to be identities and all other maps 0. Clearly, φ is epi and in particular $(\operatorname{im}\varphi)^\dagger = C([0,1])^\dagger = C([0,1])$. However, we have $C([0,+\infty))^\dagger = 0$, so $\operatorname{im}\varphi^\dagger = 0$ and thus $\operatorname{im}\varphi^\dagger \neq (\operatorname{im}\varphi)^\dagger$.
2. Now let $\varphi\colon C([-1,0]) \to C((-\infty,0])$ be of maximal rank everywhere. By a similar argument, we get $\operatorname{im}\varphi^* = 0$ but $(\operatorname{im}\varphi)^* = C([-1,0])$.

While the preservation of images fails in general, there are classes of morphisms for which we get the desired result. We start with a lemma.

Lemma 3.9. *Let V and W be objects in $\mathbf{A}^{\mathbf{T}}$ and $\varphi\colon V \to W$ a morphism.*

1. *If φ is epi, then φ^* is epi; if φ is mono, then $\varphi^\dagger$ is mono.*
2. *If $\lim\varphi$ is epi, then $\varphi^{-\infty}$ is epi; if $\operatorname{colim}\varphi$ is mono, then φ^∞ is mono.*

Proof. Assume φ is epi. Note that the canonical map $P \twoheadrightarrow P^*$ is also epi for any P. We get a commutative diagram

$$\begin{tikzcd} V \arrow[r, two heads, "\varphi"] \arrow[d, two heads] & W \arrow[d, two heads] \\ V^* \arrow[r, "\varphi^*"] & W^*, \end{tikzcd}$$

where the composition $V \to W^*$ is epi. Thus, φ^* must be epi, too. The other assertions can be shown analogously. □

Theorem 3.10. *Let V and W be objects in $\mathbf{A}^{\mathbf{T}}$ and $\varphi\colon V \to W$ a morphism.*

1. *If $\ker\operatorname{colim}\varphi = 0$, we have canonical isomorphisms*

$$\ker\varphi^\dagger \cong \ker\varphi, \qquad \ker\varphi^\infty = 0,$$
$$\operatorname{im}\varphi^\dagger \cong (\operatorname{im}\varphi)^\dagger, \qquad \operatorname{im}\varphi^\infty \cong (\operatorname{im}\varphi)^\infty \cong V^\infty.$$

2. *If $\operatorname{coker}\lim\varphi = 0$, we have canonical isomorphisms*

$$\operatorname{coker}\varphi^* \cong \operatorname{coker}\varphi, \qquad \operatorname{coker}\varphi^{-\infty} = 0,$$
$$\operatorname{im}\varphi^* \cong (\operatorname{im}\varphi)^*, \qquad \operatorname{im}\varphi^{-\infty} \cong (\operatorname{im}\varphi)^{-\infty} \cong W^{-\infty}.$$

Proof. We only show the first part of the theorem, the second one being completely dual. First, assume that $\ker\operatorname{colim}\varphi = 0$, i.e., $\operatorname{colim}\varphi$ is mono. Taking kernels is left exact, so we have an exact sequence

$$0 \longrightarrow \ker\varphi^\dagger \longrightarrow \ker\varphi \longrightarrow \ker\varphi^\infty$$

induced by the corresponding sequences from the lifespan diagrams of V and W. By the second part of Lemma 3.9, our assumption that $\operatorname{colim}\varphi$ is mono implies that φ^∞ is mono. This implies $\ker\varphi^\infty = 0$, and by exactness of the above sequence also $\ker\varphi^\dagger \cong \ker\varphi$. In addition, we obtain $\operatorname{im}\varphi^\infty \cong V^\infty$ because φ^∞ is mono.

For the assertion on images, consider the epi-mono-factorizations

$$V^\dagger \twoheadrightarrow \operatorname{im}\varphi^\dagger \hookrightarrow W^\dagger \qquad \text{and} \qquad V \overset{p}{\twoheadrightarrow} \operatorname{im}\varphi \overset{i}{\hookrightarrow} W$$

of $\varphi^\dagger$ and φ, respectively. Applying the mortal part functor to the second factorization and leaving the first one as is yields a commutative diagram

$$\begin{tikzcd}
V^\dagger \arrow[r, two heads] \arrow[dr, "p^\dagger"'] & \operatorname{im}\varphi^\dagger \arrow[r, hook] & W^\dagger \\
& (\operatorname{im}\varphi)^\dagger \arrow[ur, "i^\dagger"'] &
\end{tikzcd}$$

By the universal property of epi-mono-factorizations, we get $\operatorname{im}\varphi^\dagger \cong (\operatorname{im}\varphi)^\dagger$ if $i^\dagger$ is mono and $p^\dagger$ is epi. Since i is mono, by Lemma 3.9 $i^\dagger$ is mono, too. By assumption, $\operatorname{colim}\varphi = \operatorname{colim} i \circ \operatorname{colim} p$ is also mono, so $\operatorname{colim} p$ is mono. Using the second part of Lemma 3.9, we get that p^∞ is mono as well. Thus, applying the snake lemma (which holds in p-exact categories, see [**22**, Lemma 6.2.8]) to the diagram

$$\begin{tikzcd}
0 \arrow[r] & V^\dagger \arrow[r] \arrow[d, "p^\dagger"] & V \arrow[r] \arrow[d, two heads, "p"] & V^\infty \arrow[r] \arrow[d, hook, "p^\infty"] & 0 \\
0 \arrow[r] & (\operatorname{im}\varphi)^\dagger \arrow[r] & \operatorname{im}\varphi \arrow[r] & (\operatorname{im}\varphi)^\infty \arrow[r] & 0
\end{tikzcd}$$

yields that $p^\dagger$ is epi. Hence, we obtain $\operatorname{im}\varphi^\dagger \cong (\operatorname{im}\varphi)^\dagger$ as claimed.

Moreover, recall that colim preserves epis, so $\operatorname{colim} p$ is not only mono but in fact an isomorphism. Thus, we get a commutative diagram

$$\begin{tikzcd}
V \arrow[r, two heads] \arrow[d, two heads, "p"] & \operatorname{im}\eta_V \arrow[r, hook] & \Delta\operatorname{colim}(V) \\
\operatorname{im}\varphi \arrow[r, two heads] & \operatorname{im}\eta_{\operatorname{im}\varphi} \arrow[r, hook] & \Delta\operatorname{colim}(\operatorname{im}\varphi) \arrow[u, hook, "(\Delta\operatorname{colim} p)^{-1}"']
\end{tikzcd}$$

with the epi-mono-factorizations of η_V and $\eta_{\operatorname{im}\varphi}$ in the rows. Uniqueness of the epi-mono-factorization implies that the middle terms have to agree, so we obtain $V^\infty = \operatorname{im}\eta_V \cong \operatorname{im}\eta_{\operatorname{im}\varphi} = (\operatorname{im}\varphi)^\infty$. We have already observed that $\operatorname{im}\varphi^\infty \cong V^\infty$, so we obtain $\operatorname{im}\varphi^\infty \cong (\operatorname{im}\varphi)^\infty$. $\square$

4. Lifespan of persistence modules

We will look at the special case of the lifespan functors for persistence modules and describe their effect at the level of barcodes in Section 4.1. We then discuss how lifespan functors behave under dualization of persistence modules in Section 4.2.

4.1. Lifespan functors and barcodes

In order to give an explicit description of how our lifespan functors change the barcode of an interval-decomposable persistence module, we will take a detour via matching diagrams. We can apply the theory of lifespan functors to them because they have limits and colimits, as we will show using the component set from Definition 2.9.

Proposition 4.1. *Every matching diagram D indexed by $\mathbf{T}$ has a limit and a colimit.*

Proof. The limit is given by

$$\lim D = \{Q \in \mathcal{C}(D) \mid \operatorname{supp}(Q) \text{ is not strictly bounded below}\},$$

with natural maps $\lim D \to D_t$ matching a class Q to its representative in D_t if there is one. We can also explicitly construct the colimit of D as

$$\operatorname{colim} D = \{Q \in \mathcal{C}(D) \mid \operatorname{supp}(Q) \text{ is not strictly bounded above}\}.$$

Here, the natural maps $D_t \to \operatorname{colim} D$ match an element to its equivalence class if this class is contained in the set above. We omit the straightforward verification that these construction satisfy the universal properties of limits and colimits. □

Remark 4.2. The construction above can be adapted to show that **Mch** not only has totally ordered limits and colimits, but all cofiltered limits and filtered colimits.

We will now look at how the lifespan functors behave when being transported to barcodes via the equivalence in Theorem 2.12. We introduce some notation.

Definition 4.3. We define the following subsets of the intervals $\mathfrak{I}$ in T.

$$\begin{aligned} \mathfrak{I}^* &= \{I \in \mathfrak{I} \mid I \text{ is strictly bounded below}\}, & \mathfrak{I}^{-\infty} &= \mathfrak{I} \setminus \mathfrak{I}^*, \\ \mathfrak{I}^\dagger &= \{I \in \mathfrak{I} \mid I \text{ is strictly bounded above}\}, & \mathfrak{I}^{\infty} &= \mathfrak{I} \setminus \mathfrak{I}^\dagger, \\ \mathfrak{I}^{\dagger,*} &= \mathfrak{I}^* \cap \mathfrak{I}^\dagger, & \mathfrak{I}^{-\infty,\infty} &= \mathfrak{I}^{-\infty} \cap \mathfrak{I}^{\infty}, \\ \mathfrak{I}^{-\infty,\dagger} &= \mathfrak{I}^{-\infty} \cap \mathfrak{I}^\dagger, & \mathfrak{I}^{*,\infty} &= \mathfrak{I}^* \cap \mathfrak{I}^{\infty}. \end{aligned}$$

If B is a barcode, we also define

$$B^\diamond = \{(I, a) \in B \mid I \in \mathfrak{I}^\diamond\}$$

for any lifespan functor $(-)^\diamond$.

Theorem 4.4. *Let B be a barcode. We have*

$$\mathcal{B}(\mathcal{D}(B)^\diamond) \cong B^\diamond$$

for all lifespan functors $(-)^\diamond$. Moreover, under these isomorphisms, all lifespan transformations correspond to the respective inclusions and coinclusions.

Proof. From the definitions of $\mathcal{B}$ and $\mathcal{D}$ as well as the explicit constructions of limits and colimits for matching diagrams in the proof of Proposition 4.1, we obtain

$$\begin{aligned} \mathcal{B}(\Delta \lim \mathcal{D}(B)) &\cong \{(T, (I, a)) \in \mathfrak{I} \times B \mid I \in \mathfrak{I}^{-\infty}\}, \\ \mathcal{B}(\Delta \operatorname{colim} \mathcal{D}(B)) &\cong \{(T, (I, a)) \in \mathfrak{I} \times B \mid I \in \mathfrak{I}^{\infty}\}. \end{aligned}$$

The overlap matching

$$\mathcal{B}(\epsilon_{\mathcal{D}(B)}) \colon \mathcal{B}(\Delta \lim \mathcal{D}(B)) \to B$$

matches every interval $(T, (I, a))$ with $I \in \mathfrak{I}^{-\infty}$ to (I, a). Similarly, $\mathcal{B}(\eta_{\mathcal{D}(B)})$ matches every element (I, a) with $I \in \mathfrak{I}^{\infty}$ to $(T, (I, a))$. All lifespan functors are given on the level of barcodes by first forming kernels, cokernels, and images of $\mathcal{B}(\epsilon_{\mathcal{D}(B)})$ and $\mathcal{B}(\eta_{\mathcal{D}(B)})$, and then kernels, cokernels, and images of the resulting composite lifespan transformations. Hence, the claim follows by applying the formulas for kernels, cokernels, and images of overlap matchings from Proposition 2.20 several times. □

Next, we want to show that all the lifespan functors are compatible with the matching module functor $\mathcal{F}$. Since $\mathcal{F}$ is exact, a straightforward proof strategy would be to show that $\mathcal{F}$ also commutes with lim and colim and then use the fact that all lifespan functors are obtained from lim and colim by forming kernels, cokernels, and images. For colimits, this works out.

Lemma 4.5. *The functor* $\mathcal{F}\colon \mathbf{Mch} \to \mathbf{Vec}$ *commutes with* $\mathbf{T}$*-indexed colimits.*

Proof. Recall that in the proof of Proposition 4.1 we constructed the colimit of a matching diagram D as the set of components $Q \in \mathcal{C}(D)$ whose support is in $\mathfrak{I}^\infty$. Further, recall from the definition of the component set that each component can be regarded as a matching diagram. As such, D is canonically isomorphic to the disjoint union (which is not the coproduct, but rather a *butterfly product* in **Mch**, cf. [**22**, Section 2.1.7]) of all its components. Clearly, $\mathcal{F}$ takes disjoint unions to direct sums. Moreover, for each component $Q \in \mathcal{C}(D)$ the colimit of $\mathcal{F}(Q)$ is one-dimensional if $\operatorname{supp} Q \in \mathfrak{I}^\infty$ and trivial else. Altogether, we obtain a natural isomorphism

$$\operatorname{colim} \mathcal{F}(D) \cong \operatorname{colim} \bigoplus_{Q\in\mathcal{C}(D)} \mathcal{F}(Q) \cong \bigoplus_{Q\in\mathcal{C}(D)} \operatorname{colim} \mathcal{F}(Q) \cong \bigoplus_{\substack{Q\in\mathcal{C}(D)\\ \operatorname{supp}(Q)\in\mathfrak{I}^\infty(T)}} \mathbb{F} \cong \mathcal{F}(\operatorname{colim} D),$$

proving the claim. □

In contrast to colimits, $\mathcal{F}$ does not, in general, commute with $\mathbf{T}$-indexed limits: Consider the matching diagram D indexed by the negative integers and given by $D_{-n} = \{1, \dots, n\}$ with structure maps matching each number to itself. Then $\mathcal{F}(\lim D) = \bigoplus_{n\in\mathbb{N}} \mathbb{F}$, but $\lim \mathcal{F}(D) = \prod_{n\in\mathbb{N}} \mathbb{F}$.

Instead, we will use a more explicit argument to show that $\mathcal{F}$ commutes with the ancient part, which, together with the colimit, can also be used as a starting point to construct the other lifespan functors by forming kernels, cokernels, and images.

Theorem 4.6. *Let D be a matching diagram. We have canonical isomorphisms*

$$\mathcal{F}(D)^\diamond \cong \mathcal{F}(D^\diamond)$$

for all lifespan functors $(-)^\diamond$, which commute with the lifespan transformations.

Proof. We start by showing that $\mathcal{F}$ commutes with the ancient part. For this, consider the epi-mono-factorizations

$$\Delta \lim D \twoheadrightarrow D^{-\infty} \hookrightarrow D \quad \text{and} \quad \Delta \lim \mathcal{F}(D) \twoheadrightarrow \mathcal{F}(D)^{-\infty} \hookrightarrow \mathcal{F}(D).$$

Recall that $\mathcal{F}$ preserves exactness and hence also monos and epis. Thus, by applying $\mathcal{F}$ to the first diagram, we get another epi-mono-factorization. The universal property of the limit also induces a unique morphism $\mathcal{F}(\Delta \lim D) \to \Delta \lim \mathcal{F}(D)$ through which the cone morphism $\mathcal{F}(\Delta \lim D) \to \mathcal{F}(D)$ factors. We obtain a commutative diagram

$$\begin{tikzcd}
\mathcal{F}(\Delta \lim D) \arrow[r, two heads] \arrow[d] & \mathcal{F}(D^{-\infty}) \arrow[r, hook] & \mathcal{F}(D) \\
\Delta \lim \mathcal{F}(D) \arrow[r, two heads] & \mathcal{F}(D)^{-\infty} \arrow[r, hook] & \mathcal{F}(D). \arrow[u, hook, "\mathrm{id}"]
\end{tikzcd}$$

Since epi-mono factorizations are unique up to unique isomorphism, we only need to show that the composite morphism $\mathcal{F}(\Delta \lim D) \to \mathcal{F}(D)^{-\infty}$ is epi in order to obtain our claim. So let $t_0 \in T$ and $m \in \mathcal{F}(D)^{-\infty}_{t_0}$. Because m is in the ancient part, i.e., the image of the natural map $\lim \mathcal{F}(D) \to \mathcal{F}(D)_{t_0}$, there exists a family $(m_t)_t$ with $m_t \in \mathcal{F}(D)_t$, $m_{t_0} = m$ and such that $\mathcal{F}(D)_{s,t}(m_s) = m_t$ whenever $s \leqslant t$. Now, write finite formal linear combinations $m_t = \sum_{\alpha \in A_t} \lambda_{\alpha,t} d_{\alpha,t}$ with $d_{\alpha,t} \in D_t$ and $\lambda_{\alpha,t} \neq 0$ for all $t \in T$, $\alpha \in A_t$. Because $\mathcal{F}(D)_{s,t_0}(m_s) = m_{t_0}$ holds for any $s \leqslant t_0$, we obtain that for any $\alpha \in A_{t_0}$ and $s \leqslant t_0$ there exists $\beta = \beta(\alpha, s) \in A_s$ with $(d_{\beta,s}, d_{\alpha,t_0}) \in D_{s,t_0}$. In particular, the component Q_α represented by d_{α,t_0} has support in $\mathfrak{I}^{-\infty}$ for any $\alpha \in A_{t_0}$. Thus, m is the image of $\sum_{\alpha \in A_{t_0}} \lambda_{\alpha,t_0} Q_\alpha \in \mathcal{F}(\lim D)$ under the composite morphism $\mathcal{F}(\lim D) \to \mathcal{F}(D)^{-\infty}_{t_0}$. In particular, this map is epi as we needed to show.

The claimed isomorphisms for the other lifespan functors can now be deduced from the isomorphisms we have shown already: Consider the commutative squares

$$\begin{tikzcd} \mathcal{F}(D) \arrow[r, "\eta_{\mathcal{F}(D)}"] \arrow[d, "\mathrm{id}"] & \Delta \operatorname{colim} \mathcal{F}(D) \arrow[d] \\ \mathcal{F}(D) \arrow[r, "\mathcal{F}(\eta_D)"] & \mathcal{F}(\Delta \operatorname{colim} D) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{F}(D)^{-\infty} \arrow[r, "\alpha_{\mathcal{F}(D)}"] \arrow[d] & \mathcal{F}(D) \arrow[d, "\mathrm{id}"] \\ \mathcal{F}(D^{-\infty}) \arrow[r, "\mathcal{F}(\alpha_D)"] & \mathcal{F}(D). \end{tikzcd}$$

We have just shown that the vertical maps in the square on the right are isomorphisms. The vertical maps in the square on the left are isomorphisms because $\mathcal{F}$ and colim commute by Lemma 4.5. Thus, we obtain

$$\begin{aligned} \mathcal{F}(D)^\dagger &= \ker \eta_{\mathcal{F}(D)} \cong \mathcal{F}(\ker \eta_D) = \mathcal{F}(D^\dagger) \\ \mathcal{F}(D)^\infty &= \operatorname{im} \eta_{\mathcal{F}(D)} \cong \mathcal{F}(\operatorname{im} \eta_D) = \mathcal{F}(D^\infty) \\ \mathcal{F}(D)^* &= \operatorname{coker} \alpha_{\mathcal{F}(D)} \cong \mathcal{F}(\operatorname{coker} \alpha_D) = \mathcal{F}(D^*). \end{aligned}$$

By these isomorphisms, the vertical maps in the commutative squares

$$\begin{tikzcd} \mathcal{F}(D)^\dagger \arrow[r, "\beta_{\mathcal{F}(D)}"] \arrow[d] & \mathcal{F}(D)^* \arrow[d] \\ \mathcal{F}(D^\dagger) \arrow[r, "\mathcal{F}(\beta_D)"] & \mathcal{F}(D^*) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{F}(D)^{-\infty} \arrow[r, "\gamma_{\mathcal{F}(D)}"] \arrow[d] & \mathcal{F}(D)^\infty \arrow[d] \\ \mathcal{F}(D^{-\infty}) \arrow[r, "\mathcal{F}(\gamma_D)"] & \mathcal{F}(D^\infty) \end{tikzcd}$$

are isomorphisms, too. This yields

$$\begin{aligned} \mathcal{F}(D)^{-\infty,\dagger} &= \ker \beta_{\mathcal{F}(D)} \cong \mathcal{F}(\ker \beta_D) = \mathcal{F}(D^{-\infty,\dagger}) \\ \mathcal{F}(D)^{\dagger,*} &= \operatorname{im} \beta_{\mathcal{F}(D)} \cong \mathcal{F}(\operatorname{im} \beta_D) = \mathcal{F}(D^{\dagger,*}) \\ \mathcal{F}(D)^{*,\infty} &= \operatorname{coker} \gamma_{\mathcal{F}(D)} \cong \mathcal{F}(\operatorname{coker} \gamma_D) = \mathcal{F}(D^{*,\infty}) \\ \mathcal{F}(D)^{-\infty,\infty} &= \operatorname{im} \gamma_{\mathcal{F}(D)} \cong \mathcal{F}(\operatorname{im} \gamma_D) = \mathcal{F}(D^{-\infty,\infty}). \end{aligned} \qquad \square$$

Finally, combining the fact that $\mathcal{F}$ commutes with the lifespan functors by Theorem 4.6 with the fact that passing from barcodes to persistence modules is compatible with passing from matching diagrams to persistence modules by Proposition 2.14, we can use the formulas for the effect of lifespan functors on barcodes from Theorem 4.4 to describe how the lifespan functors change the barcodes of persistence modules.

Corollary 4.7. *Let M be a persistence module. If B is a barcode of M, then*

$$B^\diamond = \{(I, a) \in B \mid I \in \mathfrak{I}^\diamond\}$$

is a barcode for $M^\diamond$, where $(-)^\diamond$ is any lifespan functor.

From Corollary 4.7 we obtain that all the short exact sequences in the lifespan diagram of an interval-decomposable persistence module are obtained up to isomorphism by applying $\mathcal{M}$ to a short exact sequence of barcodes of the form $B' \hookrightarrow B' \sqcup B'' \twoheadrightarrow B''$. The inclusion and coinclusion into and out of the disjoint union only match bars with identical underlying intervals, so they admit one-sided inverses by Lemma 2.21. Applying $\mathcal{M}$ to the sequence of barcodes above preserves these one-sided inverses, so we obtain the following corollary.

Corollary 4.8. *All short exact sequences in the lifespan diagram of an interval-decomposable persistence module split.*

For the unborn complement, the expected formula for the effect on barcodes and the compatibility with $\mathcal{F}$ hold as for the lifespan functors. We summarize the results and omit the analogous proofs.

Proposition 4.9. *The unborn complement satisfies*

$$\mathcal{B}(\mathcal{D}(B)^{\triangleleft}) \cong B^{\triangleleft} := \{(T \setminus I, a) \mid (I, a) \in B, I \neq T \text{ and } I \in \mathfrak{I}^{\infty}\}$$

for any barcode B. Moreover, the unborn complement commutes with the matching module functor up to natural transformation. In particular, if M is a persistence module with barcode B, then $B^{\triangleleft}$ is a barcode of $M^{\triangleleft}$.

For the ghost complement, however, not all of the corresponding statements hold in general: It does not commute with the matching module functor and thus does not change the barcode of a persistence module as we would like it to. This is closely related to the fact that $\mathcal{F}$ does not commute with limits as mentioned before Theorem 4.6, which can even happen for p.f.d. persistence modules indexed by the real line as the example $\bigoplus_{n\in\mathbb{N}} C((-\infty, -n))$ shows. The problems with the ghost complement disappear for classes of persistence modules where limits do commute with $\mathcal{F}$, e.g. those of *finite type*, which are persistence modules with a finite barcode. Similarly, everything works out as desired if the index set has a minimal element $t_{\min}$, because then we have $\lim \mathcal{F}(D) \cong \mathcal{F}(D_{t_{\min}}) \cong \mathcal{F}(\lim D)$.

Proposition 4.10. *Let B be a barcode satisfying* $\lim \mathcal{F}(\mathcal{D}(B)) \cong \mathcal{F}(\lim \mathcal{D}(B))$. *Then we have*

$$\mathcal{B}(\mathcal{D}(B)^{\triangleright}) \cong B^{\triangleright} := \{(T \setminus I, a) \mid (I, a) \in B, I \neq T \text{ and } I \in \mathfrak{I}^{-\infty}\}.$$

Moreover, on the full subcategory of these barcodes the ghost complement commutes with the matching module functor up to natural transformation. In particular, if M is a persistence module with barcode B, then $B^{\triangleright}$ is a barcode of $M^{\triangleright}$.

Remark 4.11. For persistence modules, some of the lifespan functors admit more explicit descriptions. In particular, the mortal part of a persistence module $M = ((M_t)_t, (m_{s,t})_{s,t})$ is the submodule given by the subspaces $M_t^{\dagger} = \bigcup_u \ker m_{t,u} \subseteq M_t$. In this form, the construction has been considered before by Höppner and Lenzing in [**25**]. They describe it as analogous to taking the submodule of all torsion elements of a module over some integral domain. Certain categories of persistence modules can be shown to be equivalent to categories of modules over some ring (cf. [**16**, **29**, **37**]), and under these equivalences, the mortal part indeed corresponds to the torsion submodule. Furthermore, the immortal part has also been considered before in applications of barcodes to symplectic geometry. For a recent example, see [**19**].

Note that, on the other hand, while the ancient part is always a submodule of the intersection of images, $M_t^{-\infty} \subseteq \bigcap_s \operatorname{im} m_{s,t}$, in general the two need not be isomorphic. The persistence module M_3 described in Remark 5.6 provides a counterexample.

4.2. Lifespan and dualization

When passing from homology to cohomology, we will see later in Section 6 that what happens on the level of persistence modules is dualization. When passing from absolute to relative persistent homology, our correspondence result Theorem 6.2 will involve the lifespan functors. So, in order to get the full picture involving all four persistence modules associated to a diagram of spaces, we now also have to analyze whether dualization is compatible with lifespan functors.

Because we dualize, we will not only consider persistence modules indexed by $(T, \leqslant)$, but also ones indexed by $(T, \geqslant)$. When interpreting lifespan in terms of barcodes, it is important to note that the index set changes the meaning of the different classes of intervals we consider, i.e., Definition 4.3 depends on whether we use the usual or the opposite order. For example, we have $\mathfrak{I}^*(T, \leqslant) = \mathfrak{I}^\dagger(T, \geqslant)$ and $\mathfrak{I}^{-\infty}(T, \leqslant) = \mathfrak{I}^\infty(T, \geqslant)$. Thus, one should expect duals of mortal parts to correspond to nascent parts of duals and so on. To avoid confusion, we introduce some notation.

Notation 4.12. In the context of the indexing category $\mathbf{T}^{\mathrm{op}}$, we will write

$$(-)_\dagger := (-)^*, \qquad (-)_\infty := (-)^{-\infty}, \qquad (-)_* := (-)^\dagger, \qquad (-)_{-\infty} := (-)^\infty,$$
$$(-)_{\dagger,*} := (-)^{\dagger,*}, \quad (-)_{-\infty,\infty} := (-)^{-\infty,\infty}, \quad (-)_{*,\infty} := (-)^{-\infty,\dagger}, \quad (-)_{-\infty,\dagger} := (-)^{*,\infty},$$
$$(-)_\triangleleft := (-)^\triangleright, \qquad (-)_\triangleright := (-)^\triangleleft.$$

The above convention now yields $\mathfrak{I}_\diamond(T, \geqslant) = \mathfrak{I}^\diamond(T, \leqslant)$ for any lifespan functor $(-)^\diamond$.

Proposition 4.13. *Let M be a persistence module. We have canonical isomorphisms*

$$(M^\dagger)^\vee \cong (M^\vee)_\dagger, \qquad (M^\infty)^\vee \cong (M^\vee)_\infty, \qquad (M^\triangleleft)^\vee \cong (M^\vee)_\triangleleft.$$

Proof. The functor $\operatorname{Hom}(-, \mathbb{F})$ takes colimits to limits, so we have a canonical isomorphism $(\Delta \operatorname{colim} M)^\vee \cong \Delta \lim M^\vee$. Together with the kernel, cokernel, and image descriptions for dual maps from Lemma 2.24, this yields the claim. □

The limit functor for persistence modules commonly exhibits less desirable properties than the colimit functor. For example, the limit functor does not preserve exactness and does not commute with the functor $\mathcal{F}$ while the colimit functor does. A similar phenomenon arises with dualization of persistence modules, preventing the previous proposition from holding for all lifespan functors: In general, we do not have an isomorphism between $(\Delta \lim M)^\vee$ and $\Delta \operatorname{colim} M^\vee$, because the vector spaces $\operatorname{Hom}(\lim M, \mathbb{F})$ and $\operatorname{colim} M^\vee$ need not be isomorphic. However, if $(T, \leqslant)$ has a minimal element $t_{\min}$, then we have $(\Delta \lim M)^\vee \cong \Delta \operatorname{Hom}(M_{t_{\min}}, \mathbb{F}) \cong \Delta \operatorname{colim} M^\vee$. Thus, we get the following.

Proposition 4.14. *Assume that $(T, \leqslant)$ has a minimal element and let M be a $\mathbf{T}$-indexed persistence module. Then we have canonical isomorphisms $(M^\diamond)^\vee \cong (M^\vee)_\diamond$ for any lifespan functor $(-)^\diamond$.*

For later use, we also record the following completely equivalent reformulation in terms of persistence modules indexed by the opposite order.

Proposition 4.15. *Assume that $(T, \leqslant)$ has a largest element and let M be a $\mathbf{T}^{\mathrm{op}}$-indexed persistence module. Then we have canonical isomorphisms $(M_\diamond)^\vee \cong (M^\vee)^\diamond$ for any lifespan functor $(-)^\diamond$.*

Note that above, we use the notation $(-)^\vee$ as a functor from $\mathbf{T}$-indexed persistence modules to $\mathbf{T}^{\mathrm{op}}$-indexed persistence modules and also vice versa.

Furthermore, in the p.f.d. case, applying any lifespan functor $(-)^\diamond$ to a persistence module M has the same effect on barcodes as the corresponding functor $(-)_\diamond$ applied to the dual persistence module $M^\vee$.

Proposition 4.16. *Let M be a p.f.d. persistence module. Then $M^\diamond$ and $(M^\vee)_\diamond$ have the same barcodes for any lifespan functor $(-)^\diamond$.*

Proof. By Lemma 2.23 we know that p.f.d. persistence modules have the same barcode as their duals, so the claim follows immediately from the explicit formula in Corollary 4.7 for the effect of lifespan functors on barcodes. □

5. Projectivity, injectivity, and lifespan

As a first application, we will use our lifespan functors to characterize projective and injective objects in the categories of barcodes, matching diagrams, and p.f.d. persistence modules.

Theorem 5.1. *A barcode B is projective if and only if $B^\dagger = 0$, and injective if and only if $B^* = 0$.*

Proof. We will only show the first statement, the second one can be shown analogously. First, assume that $B^\dagger = 0$. In order to show that B is projective, we consider some overlap matching $\sigma \colon B \to B'$ and need to show that it factors through an arbitrary epi $\tau \colon B'' \to B'$. Consider σ and τ as ordinary matchings and set $\rho = \tau^\circ \circ \sigma$, where τ° is the opposite matching of τ (see Definition 2.2). We show that ρ is in fact an overlap matching, i.e., that for any $((I, a), (I'', a'')) \in \rho$ we have that I overlaps I'' above:

Since $B^\dagger = 0$, we have $I \in \mathfrak{I}^\infty$, so that I bounds any other interval, and in particular I'', above. What is left to check is that I'' bounds I below and that the two intervals have non-empty intersection. If $((I, a), (I'', a'')) \in \rho$, then by definition of ρ there is some $(I', a') \in B'$ such that $((I, a), (I', a')) \in \sigma$ and $((I'', a''), (I', a')) \in \tau$. Since τ is epi, its cokernel vanishes and we obtain $I' \subseteq I''$ from the explicit cokernel formula in Proposition 2.20. Moreover, I overlaps I' above, so we know that I' bounds I below and that $I \cap I' \neq \emptyset$. Together with $I' \subseteq I''$, this implies that I'' bounds I below and that $I \cap I'' \neq \emptyset$. In total, I overlaps I'' above and ρ is an overlap matching.

Now, an easy calculation verifies that we have $\tau \bullet \rho = \sigma$, i.e., that when considering ρ as an overlap matching, its overlap composition with τ recovers σ. Hence, we have shown that σ factors through τ, so B is projective.

Next, assume that $B^\dagger \neq 0$. We want to show that in this case B is not projective by constructing a barcode B' and an epi $\sigma\colon B' \to B$ that does not split. To do so, choose $(I, a) \in B$ such that $I \in \mathfrak{I}^\dagger$, which is possible by our assumption $B^\dagger \neq 0$. Define

$$J = \{t \in T \mid \text{there exists } s \in I \text{ with } s \leqslant t\}.$$

Clearly, J is an interval in T and it overlaps I above. We define

$$B' = (B \setminus \{(I, a)\}) \cup \{(J, a)\}$$

and $\sigma\colon B' \to B$ by matching each element of $B \setminus \{(I, a)\}$ to itself and matching (J, a) to (I, a). This matching σ has trivial cokernel since $I \subseteq J$, so σ is epi as desired. But, we have $I \neq J$ since $I \in \mathfrak{I}^\dagger$. Thus, σ matches non-identical intervals and consequently does not split by Lemma 2.21, so B cannot be projective. □

Translating via the equivalence of barcodes and matching diagrams, we also obtain that a matching diagram is projective if and only if its mortal part vanishes and injective if and only if nascent part vanishes.

By Proposition 3.6 we know that vanishing mortal and nascent part can equivalently be described in terms of the structure maps of a diagram, given that taking limits and colimits of diagrams is exact. It is therefore interesting to check whether taking limits and colimits of matching diagrams is exact.

Proposition 5.2. *The functors* colim, lim: $\mathbf{Mch}^{\mathbf{T}} \to \mathbf{Mch}$ *are exact.*

Proof. Let $D \to D' \to D''$ be an exact sequence of matching diagrams. By Proposition 2.18 the functor $\mathcal{F}$ preserves exactness, so the sequence

$$\mathcal{F}(D) \to \mathcal{F}(D') \to \mathcal{F}(D'')$$

remains exact. It is well-known that taking colimits of persistence modules is exact. Thus, the sequence

$$\operatorname{colim} \mathcal{F}(D) \to \operatorname{colim} \mathcal{F}(D') \to \operatorname{colim} \mathcal{F}(D'')$$

is still exact. Using that $\mathcal{F}$ commutes with taking $\mathbf{T}$-indexed colimits by Lemma 4.5, we get that

$$\mathcal{F}(\operatorname{colim} D) \to \mathcal{F}(\operatorname{colim} D') \to \mathcal{F}(\operatorname{colim} D'')$$

is also exact. By Proposition 2.18 the functor $\mathcal{F}$ reflects exactness, so

$$\operatorname{colim} D \to \operatorname{colim} D' \to \operatorname{colim} D''$$

is exact, proving that taking colimits of matching diagrams is exact. Self-duality of the category $\mathbf{Mch}$ implies that taking limits then has to be exact, too. □

Knowing that taking limits and colimits of matching diagrams is exact, we can now combine the equivalent conditions for vanishing mortal and nascent parts from Proposition 3.6 and Theorem 5.1 to obtain the following.

Corollary 5.3. *A matching diagram D is projective if and only if all of its structure maps are mono, and injective if and only if all of its structure maps are epi.*

A natural question to ask is whether statements analogous to the above Theorem 5.1 and Corollary 5.3 also hold for persistence modules instead of matching diagrams: can we characterize projectivity/injectivity or structure maps being mono/epi by vanishing mortal/nascent parts? We start with some general results.

Proposition 5.4. *For any persistence module M, we have $M^\dagger = 0$ if and only if all structure maps of M are mono. Moreover, if $M^* = 0$, then all structure maps of M are epi.*

Proof. The first part of the proposition is just a special case of the first part of Proposition 3.6, noting that taking colimits of persistence modules is exact.

For the second part, we repeat parts of the dual version of the proof of Proposition 3.6: If $M^* = 0$, then $\Delta \lim M \to M$ is epi, i.e., $\lim M \to M_t$ is epi for all $t \in T$. This implies in particular that for any structure map $M_s \to M_t$, the composition $\lim M \to M_s \to M_t$ is epi since it is equal to $\lim M \to M_t$. As a consequence, $M_s \to M_t$ needs to be epi, finishing the proof. ☐

As we will see in Remark 5.6, the converse to the second part of the proposition does not hold in general. In the category $\mathbf{vec^T}$ of p.f.d. persistence modules, however, we can indeed characterize projectives and injectives in a way analogous to matching diagrams $\mathbf{Mch^T}$.

Theorem 5.5. *Let M be a p.f.d. persistence module.*

1. *The following are equivalent:*
   - *(a) All structure maps of M are mono.*
   - *(b) $M^\dagger = 0$.*
   - *(c) M is projective in $\mathbf{vec^T}$.*
2. *The following are equivalent:*
   - *(a) All structure maps of M are epi.*
   - *(b) $M^* = 0$.*
   - *(c) M is injective in $\mathbf{vec^T}$.*

Proof. Starting with the first part of the theorem, we first note that we have already shown that $M^\dagger = 0$ is equivalent to M having mono structure maps for any persistence module M in Proposition 5.4. Thus, what is left to show for the first part is that $M^\dagger = 0$ is equivalent to M being projective in the p.f.d. category. To do so, we fix a barcode decomposition $M \cong \bigoplus_\alpha C(I_\alpha)$, which is possible by Crawley-Boevey's Theorem since M is p.f.d.

Now, assume that $M^\dagger = 0$, or equivalently $M = M^\infty$. We want to show that M is projective in $\mathbf{vec^T}$. A direct sum of projectives is projective, so it suffices to check that the interval modules $C(I_\alpha)$ in the decomposition of M are projective in $\mathbf{vec^T}$. Recall that $\mathbf{vec^T}$ is abelian, so in order to show that $C(I_\alpha)$ is projective in $\mathbf{vec^T}$ we only need to show now that any epimorphism $\varphi\colon N \to C(I_\alpha)$ with N p.f.d. splits.

Choosing a barcode decomposition $N \cong \bigoplus_\beta C(J_\beta)$ induces maps

$$\varphi_\beta \colon C(J_\beta) \to C(I_\alpha)$$

for each β. Because φ is an epimorphism of p.f.d. persistence modules, there has to be some β_0 such that φ_{β_0} is epi (see also the proof of [**3**, Lemma 4.3]). This implies that $I_\alpha \subseteq J_{\beta_0}$ and that simultaneously J_{β_0} has to overlap I_α above. Since we assume $M^\dagger = 0$, $I_\alpha \in \mathfrak{I}^\infty$ holds by Corollary 4.7, so we obtain $I_\alpha = J_{\beta_0}$, which yields that φ_{β_0} is an isomorphism. We can thus define $\psi\colon C(I_\alpha) \to N$ as the composition

$$C(I_\alpha) \xrightarrow{\varphi_{\beta_0}^{-1}} C(J_{\beta_0}) \hookrightarrow \bigoplus\nolimits_\beta C(J_\beta) \cong N.$$

By construction, we have $\varphi \circ \psi = \varphi_{\beta_0}^{-1} \circ \varphi_{\beta_0}$, which is the identity on $C(I_\alpha)$, so φ splits. Thus, we have shown that $C(I_\alpha)$, and consequently M, is projective.

Next, we assume that $M^\dagger \neq 0$ and show that M is not projective in $\mathbf{vec^T}$. Because the mortal part of M does not vanish, there now has to be some α_0 with $I_{\alpha_0} \in \mathfrak{I}^\dagger$. We proceed as in the proof of Theorem 5.1 and define

$$J = \{t \in T \mid \text{there exists } s \in I_{\alpha_0} \text{ with } s \leqslant t\}.$$

Clearly, J is an interval in T and it overlaps I_{α_0} above. Since the canonical map $C(J) \to C(I_{\alpha_0})$ is an epi, we can use to construct an epi

$$\bigoplus_{\alpha \neq \alpha_0} C(I_\alpha) \oplus C(J) \to \bigoplus_{\alpha \neq \alpha_0} C(I_\alpha) \oplus C(I_{\alpha_0}) \cong M$$

in $\mathbf{vec^T}$, which is an isomorphism on all summands except for $C(J)$. If this map would split, the splitting would induce a morphism $C(I_{\alpha_0}) \to C(J)$, which cannot exist since I_{α_0} by construction does not overlap J above. Thus, the epi we constructed does not split and M is not projective in $\mathbf{vec^T}$. This finishes the proof of the first part.

For the second part, we first observe that in the p.f.d. setting, barcode decompositions can not only be interpreted as direct sums but even as biproducts. Note that a p.f.d. persistence module may have a barcode consisting of infinitely many intervals, so this assertion is not guaranteed by $\mathbf{vec^T}$ being abelian and thus having finite biproducts. However, the observation is still true due to the fact that direct sums and products of persistence modules are given pointwise, and they coincide for finite dimensional vector spaces, so they also coincide for p.f.d. persistence modules. Thus, since we have biproduct decompositions, one can now show that $M^* = 0$ is equivalent to M being injective in $\mathbf{vec^T}$ by dualizing the previous argument and exploiting the fact that products of injectives are again injectives.

That $M^* = 0$ implies M having epi structure maps has been shown for all persistence modules M before in Proposition 5.4, so what remains to be checked is that $M^* = 0$ if M is p.f.d. and its structure maps are epi. To see that this is the case, one can use the fact that the functor $\lim\colon \mathbf{vec^T} \to \mathbf{Vec}$ is exact (because derived inverse limits of p.f.d. persistence modules vanish [**28**, Proposition 1.1], [**34**, Théorème 2]) and reuse the argument in the proof of Proposition 3.6 to show that $\Delta \lim M \to M$ is epi if the structure maps of M are epi, which implies that $M^* = 0$. □

Any p.f.d. persistence module has a barcode and the lifespan functors are compatible with the passage to barcodes, so another way of phrasing the previous theorem is that a p.f.d. persistence module is projective or injective in $\mathbf{vec^T}$ if and only if its barcode has the corresponding property in $\mathbf{Barc(T)}$.

Remark 5.6. When considering persistence modules beyond the p.f.d. category, some of the equivalences established in Theorem 5.5 do not hold anymore in general. We give a few examples.

First consider the case $T = \mathbb{R}$, which is the most relevant one for persistent homology. In this case, if the structure maps of M are epi, then $M^* = 0$, providing a converse to the second part of Proposition 5.4. The assertion can be shown by a simple argument as in the proof of [36, Lemma 7]. Moreover, any injective object in $\mathbf{Vec}^{\mathbf{R}}$ has epi structure maps. The converse does not hold: the real-indexed persistence module $M_1 = C(-\infty, 0)$ satisfies $M_1^* = 0$ and has epi structure maps, but it is not injective in $\mathbf{Vec}^{\mathbf{R}}$ because the obvious mono

$$M_1 = C(-\infty, 0) \to \prod_{n \in \mathbb{N}_{>0}} C\left(-\infty, -\frac{1}{n}\right)$$

does not split. Similarly, any projective object in $\mathbf{Vec}^{\mathbf{R}}$ has vanishing mortal part and mono structure maps. Again, the converse does not hold: the real-indexed interval module $M_2 = C(0, \infty)$ satisfies $M_1^\dagger = 0$ and has mono structure maps, but it is not projective in $\mathbf{Vec}^{\mathbf{R}}$ because the obvious epi

$$\bigoplus_{n \in \mathbb{N}_{>0}} C\left(\frac{1}{n}, \infty\right) \to C(0, \infty) = M_2$$

does not split. That M_1 is not injective and M_2 is not projective in $\mathbf{Vec}^{\mathbf{R}}$ also follows from the classification of injective and projective interval modules by Bubenik and Milićević [10, Section 6].

For general totally ordered indexing sets T, we still have the implications that any injective object in $\mathbf{Vec}^{\mathbf{T}}$ has epi structure maps and that any projective object in $\mathbf{Vec}^{\mathbf{T}}$ has mono structure maps and vanishing mortal part (see [24, 25] for classification results for injectives in $\mathbf{Vec}^{\mathbf{T}}$ and [26] for a classification of projectives in $\mathbf{Vec}^{\mathbf{T}}$). However, for persistence modules, having epi structure maps does not always imply vanishing nascent part: there is a non-zero persistence module M_3 indexed by the opposite poset of the first uncountable ordinal ω_1 whose structure maps are all epi, but which satisfies $\lim M_3 = 0$ ([23, Section 3]), so that $M_3^* = M_3 \neq 0$. We are presently unable to determine whether an injective persistence module necessarily has vanishing nascent part.

6. Functorial dualities in persistent homology

In Section 6.1, we discuss functorial versions of the duality results by de Silva et al. [20]. As an application, we present some considerations in Section 6.2 on obtaining images of morphisms in persistent homology from their counterparts in relative cohomology, which is of great relevance for making their algorithmic computation more efficient.

6.1. Persistent homology dualities in terms of lifespan functors

We will now prove a generalization of the absolute-relative correspondence [20, Proposition 2.4] involving our lifespan functors. In order for this to work nicely, we only consider filtrations that satisfy the following condition.

Definition 6.1. Let X be a $\mathbf{T}$-indexed diagram of topological spaces. We say that

X is *colimit proper* if the natural maps

$$\operatorname{colim} H_d(X) \to H_d(\operatorname{colim} X) \quad \text{and} \quad H_d(\operatorname{colim} X) \to \lim H_d(\operatorname{colim} X, X)$$

are isomorphisms for all d.

Note that colimit properness is always satisfied if the diagram X is initially empty and eventually constant. In particular, if the index set has a largest element $t_{\max}$ and a minimal element $t_{\min}$ then every X with $X_{t_{\min}} = \emptyset$ is colimit proper. These properties are usually given in the computational setting for persistent homology.

Theorem 6.2. *Let X be a colimit proper filtration of topological spaces. For all d, we have the following isomorphisms, which are natural in X:*

$$\begin{aligned} H_{d-1}(X)^{\dagger} &\cong H_d(\operatorname{colim} X, X)^{*}, \\ H_d(X)^{\lhd} &\cong H_d(\operatorname{colim} X, X)^{-\infty}, \\ H_d(X)^{\infty} &\cong H_d(\operatorname{colim} X, X)^{\rhd}. \end{aligned}$$

Proof. To shorten notation, we write A for $\operatorname{colim} X$. Since X is a filtration, the natural map $C_*(X) \to C_*(\Delta A)$ is mono. Thus, we have a short exact sequence

$$0 \longrightarrow C_*(X) \longrightarrow C_*(\Delta A) \longrightarrow C_*(A, X) \longrightarrow 0.$$

This induces a long exact sequence of persistence modules

$$\cdots \longrightarrow \Delta H_d(A) \xrightarrow{\epsilon_d} H_d(A, X) \xrightarrow{\partial} H_{d-1}(X) \xrightarrow{\eta_{d-1}} \Delta H_{d-1}(A) \longrightarrow \cdots$$

Since we assume X to be colimit proper, the map ϵ_d can be identified with the counit

$$\epsilon_{H_d(A,X)} \colon \Delta \lim H_d(A, X) \to H_d(A, X)$$

of the adjunction $\Delta \dashv \lim$. Similarly, the map η_{d-1} may be identified with the unit

$$\eta_{H_{d-1}(X)} \colon H_{d-1}(X) \to \Delta \operatorname{colim} H_{d-1}(X)$$

of the adjunction $\operatorname{colim} \dashv \Delta$. Applying the definition of the lifespan functors, the claimed isomorphisms are now simply given by exactness of the above sequence :

$$\begin{aligned} H_{d-1}(X)^{\dagger} &\cong \ker \eta_{d-1} \cong \operatorname{coker} \epsilon_d \cong H_d(A, X)^{*}, \\ H_d(X)^{\lhd} &\cong \operatorname{coker} \eta_d \cong \operatorname{im} \epsilon_d \cong H_d(A, X)^{-\infty}, \\ H_d(X)^{\infty} &\cong \operatorname{im} \eta_d \cong \ker \epsilon_d \cong H_d(A, X)^{\rhd}. \end{aligned}$$

These isomorphisms are natural in X as a direct consequence of the fact that the construction of the long exact sequence is natural in X. ☐

Using the barcode formulas for lifespan functors and complements in Corollary 4.7 and Propositions 4.9 and 4.10, one can easily recover the original duality result by de Silva et al. [**20**, Proposition 2.4]. Moreover, naturality in the filtration variable implies that for a morphism $f \colon X \to Y$ between colimit proper filtrations with $\phi = \operatorname{colim} f$ we also get isomorphisms

$$H_{d-1}(f)^{\dagger} \cong H_d(\phi, f)^{*}, \qquad H_d(f)^{\lhd} \cong H_d(\phi, f)^{-\infty}, \qquad H_d(f)^{\infty} \cong H_d(\phi, f)^{\rhd}$$

in the category of morphisms of persistence modules. These also translate to isomorphisms between the corresponding images, kernels, and cokernels.

Note that the isomorphism between the mortal part of the absolute persistent homology and the nascent part of the relative persistent homology in the proof of Theorem 6.2 is induced by the boundary operator. This means that if an interval in the nascent part of the barcode of the relative persistent homology is represented by some relative cycle, the boundary of this cycle represents the same interval in the absolute persistent homology in one dimension lower, as observed in [**20**].

Remark 6.3. While the above result is stated for persistent homology of filtrations of spaces, a similar statement holds in the purely algebraic setting. Given a filtered chain complex C, we can consider the short exact sequence

$$0 \longrightarrow C \longrightarrow \Delta \operatorname{colim} C \longrightarrow C^{\triangleleft} \longrightarrow 0.$$

We can then continue as in the proof above to get natural isomorphisms

$$H_{d-1}(C)^{\dagger} \cong H_d(C^{\triangleleft})^{*} \qquad H_d(C)^{\triangleleft} \cong H_d(C^{\triangleleft})^{-\infty} \qquad H_d(C)^{\infty} \cong H_d(C^{\triangleleft})^{\triangleright}.$$

For completeness, we also record a functorial version of the correspondence between persistent homology and persistent cohomology [**20**, Proposition 2.3], which follows immediately from the universal coefficient theorem.

Proposition 6.4. *Let X be a $\mathbf{T}$-indexed diagram of topological spaces. For all d, we have the following isomorphisms, which are natural in X:*

$$H_d(X)^{\vee} \cong H^d(X),$$
$$H_d(\operatorname{colim} X, X)^{\vee} \cong H^d(\operatorname{colim} X, X).$$

While the correspondence in [**20**, Proposition 2.3] is stated on the level of barcodes, the natural isomorphism asserted in Proposition 6.4 appears in its proof, which essentially combines the previous statement with the fact that p.f.d. persistence modules have the same barcode as their duals (Lemma 2.23).

As in the absolute-relative correspondence, naturality in the variable X yields corresponding isomorphisms in the category of morphisms of persistence modules for maps $f\colon X \to Y$.

6.2. Absolute homology images from relative cohomology images

As a concrete application, we want to explain how to use our previous results for the efficient computation of barcodes for images of morphisms in persistent homology. Note that similar considerations also apply for kernels and cokernels of such morphisms.

As mentioned above, and as is explained e.g. in [**2**], one of the most efficient ways currently known to compute the barcode of the persistent homology of a filtration of finite simplicial complexes is to actually compute the barcode of the persistent relative cohomology with the so-called clearing optimization, and to then translate this to persistent homology via the two duality results from de Silva et al. [**20**].

Our generalizations of these duality results now allow us to proceed similarly for the image of a map $f\colon X \to Y$. Since we are talking about computational speed-ups, X and Y are assumed to be filtrations of finite simplicial complexes indexed by a totally ordered set $\mathbf{T}$ with a minimal element $t_{\min}$ and a largest element $t_{\max}$. We also assume that $X_{t_{\min}} = Y_{t_{\min}} = \emptyset$, so that both filtrations are colimit proper, and that $\operatorname{colim} H_d(f) = \lim H_d(\operatorname{colim} f, f) = H_d(f_{t_{\max}})$ is an isomorphism.

In order to compute the barcode for $\operatorname{im} H_d(f)$, we start with applying the (non-natural) decomposition

$$\operatorname{im} H_d(f) \cong (\operatorname{im} H_d(f))^\dagger \oplus (\operatorname{im} H_d(f))^\infty$$

from Corollary 4.8. We consider both summands separately, making use of the fact that taking barcodes is compatible with direct sums.

Starting with the first summand, we observe that because $\operatorname{colim} H_d(f) = H_d(f_{t_{\max}})$ is an isomorphism, and in particular a monomorphism, we have

$$(\operatorname{im} H_d(f))^\dagger \cong \operatorname{im}(H_d(f)^\dagger)$$

using the first part of Theorem 3.10. The natural duality Theorem 6.2, which we can apply since X and Y are colimit proper, provides an isomorphism

$$\operatorname{im}(H_d(f)^\dagger) \cong \operatorname{im}(H_{d+1}(\operatorname{colim} f, f)^*).$$

An application of the second part of Theorem 3.10 yields the isomorphism

$$\operatorname{im}(H_{d+1}(\operatorname{colim} f, f)^*) \cong (\operatorname{im} H_{d+1}(\operatorname{colim} f, f))^*$$

using that $\lim H_d(\operatorname{colim} f, f) = H_d(f_{t_{\max}})$ is epi. Finally, the duality of homology and cohomology from Proposition 6.4 yields an isomorphism

$$(\operatorname{im} H_{d+1}(\operatorname{colim} f, f))^* \cong ((\operatorname{im} H^{d+1}(\operatorname{colim} f, f))^\vee)^*,$$

where we also make use of the fact that applying dualization twice yields the identity on p.f.d persistence modules. Finally, because our index set has a largest element, Proposition 4.15 gives

$$((\operatorname{im} H^{d+1}(\operatorname{colim} f, f))^\vee)^* \cong ((\operatorname{im} H^{d+1}(\operatorname{colim} f, f))_*)^\vee.$$

In total, the above implies that $(\operatorname{im} H_d(f))^\dagger$ and $(\operatorname{im} H^{d+1}(\operatorname{colim} f, f))_*$ have the same barcode by Lemma 2.23 because we are in the p.f.d. setting, so we can obtain the mortal part of the absolute homology barcode from the one in relative cohomology.

For the second term in the mortal-immortal decomposition of $\operatorname{im} H_d(f)$, we have

$$(\operatorname{im} H_d(f))^\infty \cong H_d(X)^\infty$$

by Theorem 3.10. We proceed again with our natural absolute-relative duality from Theorem 6.2 to obtain

$$H_d(X)^\infty \cong H_d(\operatorname{colim} X, X)^\triangleright.$$

Since all modules are p.f.d., passing to cohomology with Proposition 6.4 yields

$$(H_d(\operatorname{colim} X, X))^\triangleright \cong ((H^d(\operatorname{colim} X, X))^\vee)^\triangleright.$$

Proposition 4.15 finally yields

$$((H^d(\operatorname{colim} X, X))^\vee)^\triangleright \cong (H^d(\operatorname{colim} X, X)_\triangleright)^\vee.$$

Thus, we can also obtain the immortal part of the absolute homology barcode from the one in relative cohomology. For a simplified restatement and an algorithmic application of the above argument in the computational setting of finite simplicial complexes, we refer to [**6**].

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Ulrich Bauer ulrich.bauer@tum.de

Department of Mathematics, School of Computation, Information and Technology, Technical University of Munich, Germany

Maximilian Schmahl mschmahl@mathi.uni-heidelberg.de

Mathematisches Institut, Universität Heidelberg, Germany