Homology, Homotopy and Applications, vol. 25(2), 2023, pp.275–296

THE STRUCTURING EFFECT OF A GOTTLIEB ELEMENT ON THE SULLIVAN MODEL OF A SPACE

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(*communicated by Peter Bubenik*)

Abstract

We show a Gottlieb element in the rational homotopy of a simply connected space X implies a structural result for the Sullivan minimal model, with different results depending on parity. In the even-degree case, we prove a rational Gottlieb element is a terminal homotopy element. This fact allows us to complete an argument of Dupont to prove an even-degree Gottlieb element gives a free factor in the rational cohomology of a formal space of finite type. We apply the odd-degree result to affirm a special case of the $2N$-conjecture on Gottlieb elements of a finite complex. We combine our results to make a contribution to the realization problem for the classifying space $B\mathrm{aut}_1(X)$. We prove a simply connected space X satisfying $B\mathrm{aut}_1(X_{\mathbb{Q}}) \simeq S^{2n}_{\mathbb{Q}}$ must have infinite-dimensional rational homotopy and vanishing rational Gottlieb elements above degree $2n-1$ for $n = 1, 2, 3$.

1. Introduction

Let X be simply connected and of finite CW type. A homotopy class $\alpha \in \pi_n(X)$ is a *Gottlieb element* if the map of the wedge $(\alpha \,|\, 1_X) \colon S^n \vee X \to X$ extends to a map of the product $F \colon S^n \times X \to X$. The definition directly implies the vanishing of the Whitehead product of α with any $\beta \in \pi_m(X)$. Amongst many nice results, Gottlieb proved that an even-degree Gottlieb element is in the kernel of the mod p Hurewicz homomorphism for any prime p not dividing the Euler characteristic of X [**7**, Th.4.4].

Gottlieb elements have a natural description in rational homotopy theory which we recall, briefly, here. Our general reference for rational homotopy theory is the text [**5**]. A space X, as hypothesized, has a Sullivan minimal model which is a free DG algebra $(\wedge V, d)$ over $\mathbb{Q}$ with each V^n finite-dimensional. The differential d has image in the decomposable elements of $\wedge V$. The affiliated map F for a Gottlieb class $\alpha \in \pi_n(X)$ induces a map $F^* \colon (\wedge V, d) \to (\mathbb{Q}(u), 0) \otimes (\wedge V, d)$ of DG algebras. Here u is of degree n and $(\mathbb{Q}(u), 0) \cong H^*(S^n; \mathbb{Q})$ is a non-minimal Sullivan model for S^n with trivial differential. Since F extends $(\alpha \,|\, 1_X)$ we have $F^*(v) = u$ where $v \in V^n$ is dual to α under Sullivan's isomorphism $V^n \cong \mathrm{Hom}(\pi_n(X), \mathbb{Q})$ [**5**, Th.15.11]. Further,

Received January 5, 2022, revised June 29, 2022; published on November 1, 2023.
2020 Mathematics Subject Classification: Primary 55P62, 55R35; Secondary 55R15.
Key words and phrases: Gottlieb element, Sullivan minimal model, classifying space for a fibration, derivation.
Article available at `http://dx.doi.org/10.4310/HHA.2023.v25.n2.a12`

writing $F^*(\chi) = \chi + u\theta(\chi)$ for $\chi \in \wedge V$, the linear map θ so-defined is a *derivation* of $\wedge V$, $\theta(\chi_1\chi_2) = \theta(\chi_1)\chi_2 + (-1)^{|\chi_1|n}\chi_1\theta(\chi_2)$, lowering degrees by n, of (*degree* n) and θ is a *derivation cycle*, $d\theta - (-1)^n\theta d = 0$. We say $v \in V^n$ is a *Gottlieb element* for the Sullivan minimal model $(\wedge V, d)$ if there exists a derivation cycle θ of $\wedge V$ of degree n with $\theta(v) = 1$.

In [**8**, Lem.1.1], Halperin showed that the derivation cycle θ associated to a Gottlieb element $v \in V^n$ that is a cycle, $dv = 0$, induces a change of basis for the Sullivan model $(\wedge V, d)$ and a resulting DG algebra factorization:

$$(\wedge V, d) \cong (\wedge(v), 0) \otimes (\wedge V', d').$$

Recall the space of cycles $Z(V) \subseteq V$ may be identified as the dual of the image of the rational Hurewicz homomorphism. The result thus represents a natural extension of Gottlieb's theorem, mentioned above (cf. [**13**]).

In this paper, we explore the structural consequence of a Gottlieb element $v \in V^n$ in the general case, when $dv \neq 0$. For n even, we obtain a surprising result. Say an element $v \in V^n$ is a *terminal element* if v does not appear in a differential dw for any $w \in V$.

Theorem 1.1. *An even-degree Gottlieb element $v \in V^{2n}$ in a Sullivan minimal model $(\wedge V, d)$ is a terminal element.*

We apply Theorem 1.1 to an old problem on the location of Gottlieb elements for a formal space X. We recall that a formal space X has a *bigraded* Sullivan minimal model $(\wedge V, d)$ with the generators carrying a second, or lower, grading $V = \bigoplus_{i\geqslant 0} V_i$ that extends multiplicatively to the whole of $\wedge V$. The differential satisfies $d(V_0) = 0$ and $d(V_i) \subseteq (\wedge V)_{i-1}$ for $i \geqslant 1$ (see [**9**]). The following conjecture is attributed to C. Jacobsson.

Conjecture 1.2. *If X is a formal space with bigraded Sullivan minimal model $(\wedge V, d)$, then all Gottlieb elements are contained in $V_0 \oplus V_1$.*

In [**2**], Dupont initiated work on Conjecture 1.2 and enunciated several results on this problem. However, this paper was never published and some arguments appear to be incomplete. We here reproduce Dupont's argument in the even-degree case of [**2**, Pro.4] and use Theorem 1.1 to complete the proof.

Theorem 1.3. (cf. [**2**, Pro.4]) *Let X be a formal space with finitely generated rational cohomology and suppose $y \in V^{2n}$ is an even-degree Gottlieb element in the Sullivan minimal model $(\wedge V, d)$ for X. Then $dy = 0$ and there is a DG algebra isomorphism*

$$(\wedge V, d) \cong (\wedge(y), 0) \otimes (\wedge V', d').$$

The mapping theorem for rational L.S. category ([**5**, Th.28.6]) implies strong constraints on the rational Gottlieb elements of a space X of finite L.S. category. The even-degree Gottlieb elements vanish in this case and the number of independent odd-degree Gottlieb elements is bounded above by the rational L.S. category of X [**5**, Pro.29.8]. The location of the odd-degree Gottlieb elements is the subject of the following open problem.

Conjecture 1.4. [**5**, p.518] *Let X be space with $H^*(X;\mathbb{Q})$ finite-dimensional. Let $N = \max\{n \mid H^n(X;\mathbb{Q}) \neq 0\}$. Then the rational Gottlieb elements for X are of degree $< 2N$.*

In Theorem 2.1 below, we show that a Gottlieb element $v \in V^{2n+1}$ induces a basis change for $(\wedge V, d)$ after which the appearance of v in differentials dw is explicitly constrained. We apply this result to prove:

Theorem 1.5. *Let X be space with $H^*(X;\mathbb{Q})$ finite-dimensional and with top non-trivial degree N. Let $x \in V^{2n+1}$ be a Gottlieb element in the Sullivan minimal model $(\wedge V, d)$ with dx a monomial in the generators of V. Then $2n + 1 < 2N$.*

We apply our results in both the even and odd-degree cases to another open problem in rational homotopy, concerning the classifying space $B\mathrm{aut}_1(X)$ for fibrations. We recall that X, as hypothesized, has a *universal fibration* $X \to E_X \to B\mathrm{aut}_1(X)$ such that any fibration of simply connected spaces with fibre X is equivalent to a pullback by a map into the base $B\mathrm{aut}_1(X)$, the *classifying space* (cf. [**12**]). There is an H-equivalence $\Omega B\mathrm{aut}_1(X) \simeq \mathrm{aut}_1(X)$ with $\mathrm{aut}_1(X) = \mathrm{map}(X, X; 1)$. With our hypotheses on X, the construction can be applied to the rationalization $X_{\mathbb{Q}}$ of X yielding $X_{\mathbb{Q}} \to E_{X_{\mathbb{Q}}} \to B\mathrm{aut}_1(X_{\mathbb{Q}})$, the universal fibration with fibre $X_{\mathbb{Q}}$.

The construction of algebraic models for $B\mathrm{aut}_1(X_{\mathbb{Q}})$ is the classical work of many authors (cf. [**15**, Ch.7]) and an area of continued interest. Recent advances provide models in the non simply-connected case [**1, 3**]. We make primary use here of the first and simplest description of a model, due to Sullivan. Write $\mathrm{Der}_n(\wedge V)$ for the space of degree n derivations of the Sullivan minimal model $(\wedge V, d)$ for $n \geqslant 1$ with the graded commutator bracket $[\theta, \varphi] = \theta \circ \varphi + (-1)^{|\theta||\varphi|}\varphi \circ \theta$ and differential $D(\theta) = [d, \theta]$. Sullivan's identity [**14**, Sec.11] is an isomorphism of connected, graded Lie algebras:

$$\pi_*(\Omega B\mathrm{aut}_1(X_{\mathbb{Q}})) \cong H_*(\mathrm{Der}(\wedge V), D). \tag{1}$$

The following question is due to M. Schlessinger.

Question 1.6. [**5**, p.517] *Is every simply connected space Y realized, rationally, as a classifying space, in the sense that there is some simply connected space X such that $B\mathrm{aut}_1(X_{\mathbb{Q}}) \simeq Y_{\mathbb{Q}}$?*

Question 1.6 suggests a realization problem for classifying spaces. Given a space Y, the problem is to either construct X with $B\mathrm{aut}_1(X_{\mathbb{Q}}) \simeq Y_{\mathbb{Q}}$ or, alternately, to prove no such space X exists. The identity $K(\mathbb{Q}, n) \simeq B\mathrm{aut}_1(K(\mathbb{Q}, n-1))$ shows Eilenberg–Mac Lane spaces are realized in this sense, at least for $n \geqslant 2$, Deciding whether an arbitrary product $K(\mathbb{Q}, m) \times K(\mathbb{Q}, n)$ can be so realized is already challenging (see [**10**, Th.1] for one class of examples).

In previous work, we have proved several low-dimensional rational types Y cannot be realized as $B\mathrm{aut}_1(X)$ when X is restricted to have finite-dimensional rational homotopy (X is *π-finite*). We have proved this result for $Y = S^{2n}$ for $n = 1, 2$ and for $Y = \mathbb{C}P^n$ for $n = 2, 3, 4$; namely, if $B\mathrm{aut}_1(X_{\mathbb{Q}}) \simeq Y_{\mathbb{Q}}$ for any such Y then X must be π-infinite [**10, 11**]. We extend and sharpen these results with the following.

Theorem 1.7. *Let X satisfy $B\mathrm{aut}_1(X_{\mathbb{Q}}) \simeq S^{2n}_{\mathbb{Q}}$ for $n = 1, 2, 3$. Then X is π-infinite with vanishing rational Gottlieb elements above degree $2n - 1$.*

Our proof of Theorem 1.7 streamlines the argument given in [**10**, Th.3] and extends it to include the case S^6. The advance over our previous work is that here we do not start by assuming X is π-finite with $B\mathrm{aut}_1(X_{\mathbb{Q}}) \simeq S^{2n}_{\mathbb{Q}}$. Instead, we give a constraint on the π-infinite spaces X that could satisfy this identity.

The paper is organized as follows. In Section 2, we show a rational Gottlieb element induces a basis change for a Sullivan minimal model. We deduce Theorems 1.3 and 1.5 as consequences in Section 3. In Section 4, we give a structure result for a Sullivan minimal model having Gottlieb elements of both parities in the scenario that arises in the realization problem for $Y = S^{2n}$. We apply this result to prove Theorem 1.7. In Section 5, we give a further example suggesting a negative answer to Question 1.6. We prove a space X with $B\mathrm{aut}_1(X_{\mathbb{Q}}) \simeq (S^3 \vee \cdots \vee S^3)_{\mathbb{Q}}$ must have vanishing rational Gottlieb elements.

2. Gottlieb elements and basis change for Sullivan models

In this section, we observe that a Gottlieb element $v \in V^n$ induces a *change of basis isomorphism* for the Sullivan minimal model $(\wedge V, d)$. The idea is as follows. Start with an automorphism $\phi\colon \wedge V \to \wedge V$ of graded algebras. Then ϕ induces a new differential d' on $\wedge V$ given by $d' = \phi^{-1} \circ d \circ \phi$. The map

$$\phi\colon (\wedge V, d') \to (\wedge V, d)$$

is, tautologically, a DG algebra isomorphism. Also note that, since d is decomposable, d' is decomposable as well and ϕ is an isomorphism of minimal DG algebras.

A derivation θ of $(\wedge V, d)$ induces a derivation $\theta' = \phi^{-1} \circ \theta \circ \phi$ of $\wedge V$. If θ is a derivation cycle, i.e., $[d, \theta] = 0$, then we see

$$[d', \theta'] = \big[\phi^{-1} \circ d \circ \phi, \phi^{-1} \circ \theta \circ \phi\big] = \phi^{-1} \circ [d, \theta] \circ \phi = 0.$$

We focus first on the structuring effect of an odd-degree Gottlieb element $x \in V^{2n+1}$. Let θ be of degree $2n+1$ and satisfy $[d, \theta] = 0$ and $\theta(x) = 1$. Write $V = \langle x \rangle \oplus W$ for $W \subseteq V$ a complementary subspace to $\langle x \rangle$ in V. Define a linear map $\phi\colon V \to \wedge V$ by setting $\phi(x) = x$ and, for each $v \in W$,

$$\phi(v) = v - x\theta(v). \tag{2}$$

Extend multiplicatively to a map of algebras $\phi\colon \wedge V \to \wedge V$. It is easy to see that ϕ is an automorphism. Notice further that

$$\begin{aligned} \phi(v)\phi(v') &= \big(v - x\theta(v)\big)\big(v' - x\theta(v')\big) \\ &= vv' - x\theta(v)v' - (-1)^{|v|}\, xv\theta(v') \\ &= vv' - x\theta(vv'). \end{aligned}$$

It follows that $\phi(\chi) = \chi - x\theta(\chi)$ for a general $\chi \in \wedge W$. Also note that, again for a general $\chi \in \wedge W$, we have $\phi(x\chi) = \phi(x)\phi(\chi) = x\big(\chi - x\theta(\chi)\big) = x\chi$. As above, define $d' = \phi^{-1} \circ d \circ \phi$ and $\theta' = \phi^{-1} \circ \theta \circ \phi$ so that $[d', \theta'] = 0$. We prove

Theorem 2.1. *Let $(\wedge V, d)$ be a Sullivan minimal model with a Gottlieb element $x \in V^{2n+1}$. Let $(\wedge V, d')$ be as constructed above and θ' the induced derivation cycle of degree $2n+1$ of the model $(\wedge V, d')$. Write $V = \langle x \rangle \oplus W$. Then:*

(a) $\theta'(x) = 1$ *and* $d'(x) = dx$

(b) *For* $\chi \in \wedge W$*, we have* $\theta'(\chi) = x\,\lambda(\chi)$ *for some* $\lambda \in \mathrm{Der}_{4n+2}(\wedge W)$

(c) *For* $\chi \in \wedge W$*,*

$$d'(\chi) = -\theta'(\chi)\,dx + d_0'(\chi) = -x\lambda(\chi)dx + d_0'(\chi)$$

for some $d_0' \in \mathrm{Der}_{-1}(\wedge W)$ *and* $\lambda \in \mathrm{Der}_{4n+2}(\wedge W)$ *as in (b).*

Proof. Part (a) is immediate since $\phi(x) = x$ and further ϕ is the identity on elements of degree $\leqslant 2n$. For (b) and (c), we begin with a general observation. Suppose that $\psi\colon \wedge V \to \wedge V$ is a derivation. Let $\chi \in \wedge W$. If we define linear maps on $\wedge W$ by

$$\psi(\chi) = x\psi_1(\chi) + \psi_0(\chi),$$

then both ψ_1 and ψ_0 are derivations of $\wedge W$ (of different degrees). This is easy to check, as follows. For a product of terms $\chi, \chi' \in \wedge W$, write

$$\begin{aligned}
\psi(\chi\chi') &= \psi(\chi)\chi' + (-1)^{|\psi||\chi|}\chi\psi(\chi') \\
&= \big(x\,\psi_1(\chi) + \psi_0(\chi)\big)\chi' + (-1)^{|\psi||\chi|}\chi\big(x\,\psi_1(\chi') + \psi_0(\chi')\big) \\
&= x\big(\psi_1(\chi)\chi' + (-1)^{|\psi||\chi|+|\chi|}\chi\psi_1(\chi')\big) + \big(\psi_0(\chi)\chi' + (-1)^{|\psi||\chi|}\chi\psi_0(\chi')\big) \\
&= x\psi_1(\chi\chi') + \psi_0(\chi\chi').
\end{aligned}$$

In the last line, we used that ψ_0 has the same degree as ψ, whereas the degree of ψ_1 is of opposite parity to the degree of ψ, so we have $|\psi| + 1 \equiv |\psi_0| \mod 2$.

Applying this decomposition to θ gives

$$\theta(\chi) = x\,\theta_1(\chi) + \theta_0(\chi). \tag{3}$$

As just observed, this defines derivations θ_0 and θ_1 of $\wedge W$ with θ_0 of degree $2n+1$ and θ_1 of degree $4n+2$. The isomorphism ϕ is given by

$$\phi(\chi) = \chi - x\,\theta_0(\chi). \tag{4}$$

Furthermore, as already noted, we have $\phi(x\chi) = x\chi$, and hence also $\phi^{-1}(x\chi) = x\chi$. Applying ϕ^{-1} to (4), then, yields

$$\phi^{-1}(\chi) = \chi + x\,\theta_0(\chi). \tag{5}$$

We use the notation from (3), (4), and (5) in what follows.

(b) We calculate:

$$\begin{aligned}
\theta'(\chi) = \phi^{-1}\circ\theta\circ\phi(\chi) &= \phi^{-1}\circ\theta\big(\chi - x\,\theta_0(\chi)\big) \\
&= \phi^{-1}\big(x\,\theta_1(\chi) + \theta_0(\chi) - 1\,\theta_0(\chi) + x\,\theta_0\circ\theta_0(\chi)\big) \\
&= \phi^{-1}\big(x\,\big(\theta_1(\chi) + \theta_0\circ\theta_0(\chi)\big)\big) \\
&= x\,\big(\theta_1(\chi) + \theta_0\circ\theta_0(\chi)\big),
\end{aligned}$$

with the last line following from the observation made leading into (5), that $\phi^{-1}(x\chi) = x\chi$. So θ' has the form asserted. Note that $\theta_0\circ\theta_0 = (1/2)[\theta_0, \theta_0]$ is a derivation of degree $4n+2$. We have

$$\lambda = \theta_1 + \frac{1}{2}[\theta_0, \theta_0].$$

(c) As in (3), we write

$$d(\chi) = x\, d_1(\chi) + d_0(\chi), \tag{6}$$

which defines d_1 and d_0 as derivations of $\wedge W$. We now compute:

$$\begin{aligned}
\theta \circ d(\chi) &= \theta\big(x\, d_1(\chi) + d_0(\chi)\big) \\
&= 1 \cdot d_1(\chi) - x\left(x\, \theta_1(d_1(\chi)) + \theta_0(d_1(\chi))\right) + x\, \theta_1(d_0(\chi)) + \theta_0(d_0(\chi)) \\
&= d_1(\chi) - x\, \theta_0 \circ d_1(\chi) + x\, \theta_1 \circ d_0(\chi) + \theta_0 \circ d_0(\chi)
\end{aligned}$$

and

$$\begin{aligned}
d \circ \theta(\chi) &= d\big(x\, \theta_1(\chi) + \theta_0(\chi)\big) \\
&= dx \cdot \theta_1(\chi) - x\left(x\, d_1(\theta_1(\chi)) + d_0(\theta_1(\chi))\right) + x\, d_1(\theta_0(\chi)) + d_0(\theta_0(\chi)) \\
&= dx\, \theta_1(\chi) - x\, d_0 \circ \theta_1(\chi) + x\, d_1 \circ \theta_0(\chi) + d_0 \circ \theta_0(\chi).
\end{aligned}$$

Adding these two identities, whose left-hand sides sum to zero, gives us the following two identities amongst derivations of $\wedge W$:

$$[d_0, \theta_1] = [d_1, \theta_0] \quad \text{and} \quad d_1 + [d_0, \theta_0] = -dx\, \theta_1 \tag{7}$$

with the first from collecting terms in $x \cdot \wedge W$ and the second from collecting terms in $\wedge W$, which terms are independent of each other.

Now we calculate:

$$\begin{aligned}
d'(\chi) &= \phi^{-1} \circ d \circ \phi(\chi) = \phi^{-1} \circ d\big(\chi - x\, \theta_0(\chi)\big) \\
&= \phi^{-1}\big(x\, d_1(\chi) + d_0(\chi) - dx\, \theta_0(\chi) + x\, d_0 \circ \theta_0(\chi)\big) \\
&= x\, d_1(\chi) + \big(d_0(\chi) + x\, \theta_0 \circ d_0(\chi)\big) \\
&\qquad\qquad - \big(dx + x\, \theta_0(dx)\big)\big(\theta_0(\chi) + x\, \theta_0 \circ \theta_0(\chi)\big) + x\, d_0 \circ \theta_0(\chi) \\
&= x\left(d_1(\chi) + [d_0, \theta_0](\chi) - dx\, \theta_0 \circ \theta_0(\chi)\right) + d_0(\chi) - dx\, \theta_0(\chi).
\end{aligned}$$

The last line follows using $\theta_0(dx) = 0$, which follows for degree reasons, to cancel one term from the penultimate line. Notice that we also commuted x and dx without changing the sign. Using the second identity of (7), we arrive at

$$\begin{aligned}
d'(\chi) &= -x\left(dx\, \theta_1(\chi) + dx\, \theta_0 \circ \theta_0(\chi)\right) + d_0(\chi) - dx\, \theta_0(\chi) \\
&= -x\, dx\, \lambda(\chi) + d_0'(\chi),
\end{aligned}$$

where we have written $d_0'(\chi) = d_0(\chi) - dx\, \theta_0(\chi)$ for the term not involving x. One can see this is a derivation as it is a sum of two terms, each of which acts as a derivation on $\wedge W$. Alternately, the terms here may be written in the form $d'(\chi) = x d_1'(\chi) + d_0'(\chi)$, with both d_1' and d_0' derivations of $\wedge W$ by the observation at the top of the proof. $\square$

Remark 2.2. We record some further consequences.

(1) If $dx = 0$ then, by (c), there is a DG algebra decomposition

$$(\wedge V, d') \cong (\wedge(x), 0) \otimes (\wedge W, d_0').$$

We thus recover the odd-degree case of [**8**, Lem.1].

(2) If we decompose $[\theta, \theta]$ in the form $[\theta, \theta] = x\,[\theta, \theta]_1 + [\theta, \theta]_0$, with $[\theta, \theta]_1$ and $[\theta, \theta]_0$ derivations of $\wedge W$. Then λ in part (b) is given by

$$\lambda = \frac{1}{2}\,[\theta, \theta]_0.$$

(3) Decomposing d' in this form so that $d'(\chi) = x\,d'_1(\chi) + d'_0(\chi)$ we then have the identities:

$$[d'_1, d'_0] = 0 \quad\text{and}\quad d'_0 \circ d'_0 + dx\,d'_1 = 0.$$

An even-degree Gottlieb element $y \in V^{2m}$ gives rise to a basis change, as well. In this case, the result gives Theorem 1.1, that y is a terminal element. We explain this now. Let θ be the derivation cycle of degree $2m$ with $\theta(y) = 1$ and $[d, \theta] = d\theta - \theta d = 0$. Write $V = \langle y \rangle \oplus W$ and define a linear map $\psi\colon\ \wedge V \to \wedge V$ by setting

$$\psi(y) = 0 \quad\text{and}\quad \psi(\chi) = -y\theta(\chi)$$

for $\chi \in \wedge W$. Extend multiplicatively so that the ideal of $\wedge V$ generated by y is in the kernel of ψ. Then ψ is a degree-zero derivation of $\wedge V$ and, further, ψ is *locally nilpotent* meaning that, for each element $\xi \in \wedge V$, there is some r for which $\psi^r(\xi) = 0$. We may thus exponentiate ψ to obtain a linear map

$$\phi = \exp(\psi) = \mathrm{id} + \psi + \frac{1}{2!}\,\psi^2 + \frac{1}{3!}\,\psi^3 + \cdots.$$

Then ϕ is an automorphism of $\wedge V$ and so induces a DG algebra isomorphism

$$\phi\colon (\wedge V, d') \to (\wedge V, d)$$

by setting $d' = \phi^{-1} \circ d \circ \phi$. We transfer the derivation cycle θ to one of $(\wedge V, d')$ by setting $\theta' := \phi^{-1} \circ \theta \circ \phi$.

Theorem 2.3. *Let $(\wedge V, d)$ be a Sullivan minimal model with a Gottlieb element $y \in V^{2m}$. Let $\phi\colon (\wedge V, d') \to (\wedge V, d)$ and θ' the induced derivation cycle of $(\wedge V, d')$ be as above. Write $V = \langle y \rangle \oplus W$. Then with notation as above:*

(a) $\theta'(y) = 1$ and $d'(y) = dy$.

(b) For $\chi \in \wedge W$, we have $\theta'(\chi) = 0$.

(c) The differential d' restricts to a derivation of $\wedge W$, namely, we have

$$d'(\wedge W) \subseteq \wedge W.$$

Proof. Again (a) is immediate. For (b), let $\chi \in \wedge W$ and observe

$$\phi(\chi) = \chi + \psi(\chi) + \frac{1}{2!}\,\psi^2(\chi) + \frac{1}{3!}\,\psi^3(\chi) + \cdots.$$

Notice that $\psi^2(\chi) = \psi\big(-y\theta(\chi)\big) = (-y)(-y)\theta^2(\chi)$, since we defined $\psi(y) = 0$. Generally, we have

$$\psi^k(\chi) = (-1)^k y^k \theta^k(\chi),$$

for $k \geqslant 1$. It follows that

$$\phi(\chi) = \chi + \sum_{k \geqslant 1} (-1)^k\,\frac{1}{k!}\,y^k \theta^k(\chi).$$

Recall that θ is locally-nilpotent, so the sum is finite for each χ. Applying the derivation θ, with $\theta(y) = 1$, we have

$$\begin{aligned}\theta\big(\phi(\chi)\big) &= \theta(\chi) - \theta(\chi) - y\theta^2(\chi) \\ &\qquad + \sum_{k\geqslant 2} (-1)^k \left(\frac{1}{(k-1)!}\, y^{k-1}\theta^k(\chi) + \frac{1}{k!}\, y^k\theta^{k+1}(\chi)\right) \\ &= \theta(\chi) - \theta(\chi) - y\theta^2(\chi) + y\theta^2(\chi) \\ &\qquad + \sum_{k\geqslant 3} (-1)^k \frac{1}{(k-1)!}\, y^{k-1}\theta^k(\chi) + \sum_{k\geqslant 2} (-1)^k \frac{1}{k!}\, y^k\theta^{k+1}(\chi) \\ &= \theta(\chi) - \theta(\chi) - y\theta^2(\chi) + y\theta^2(\chi) \\ &\qquad + \sum_{k\geqslant 2} \left((-1)^{k+1} \frac{1}{k!}\, y^k\theta^{k+1}(\chi) + (-1)^k \frac{1}{k!}\, y^k\theta^{k+1}(\chi)\right) \\ &= 0.\end{aligned}$$

Finally, this gives $\theta'(\chi) = \phi^{-1} \circ \theta' \circ \phi(\chi) = \phi^{-1}\big(x\lambda\big(\phi(\chi)\big)\big) = 0$, as claimed.

(c) Let $v \neq y$ be a generator of degree at least $2m+1$ and write

$$d'(v) = \chi_0 + y\chi_1 + \cdots + y^k\chi_k,$$

for some $k \geqslant 1$ and each $\chi_i \in \wedge W$, that is, not containing terms that involve y. The derivation θ' satisfies $\theta'(y) = 1$ and $\theta'(\chi_i) = 0$ for each i. Also, because θ' is a cycle with respect to $D' = \mathrm{ad}(d')$, we have

$$\theta' \circ d'(v) = d' \circ \theta'(v) = 0,$$

and hence we have

$$0 = 0 + \chi_1 + y\chi_2 + \cdots + y^{k-1}\chi_k.$$

Each of these terms must be zero independently of each other, since the χ do not involve y, an even-degree generator of $\wedge V$. Hence, each $\chi_i = 0$ for $i = 1, \ldots, k$, and we have $d'(v) = \chi_0$, which does not involve y. For generators v of degree $2m$ and lower, $d'(v)$ cannot involve y for degree reasons. ☐

Observe that the even-degree case of [**8**, Lem.1] follows from Theorem 2.3. When $dy = 0$, we have a splitting $(\wedge V, d) \cong (\wedge(y), 0) \otimes (\wedge W, d')$.

Proof of Theorem 1.1. Observe that Theorem 1.1 is a direct consequence of Theorem 2.3. We give an alternate, homotopy-theoretic proof of Theorem 1.1 in Section 5. ☐

3. Location of rational Gottlieb elements

We begin with a simple example of a non-terminal Gottlieb element, necessarily of odd degree.

Example 3.1. Consider the minimal model $(\wedge V, d)$ with $V = \langle u_2, v_2, x_3, z_3, y_6\rangle$, where subscripts denote degrees, and differential given on generators by

$$du = 0 = dv, dx = uv, dz = v^2, dy = -uvx + u^2z.$$

Set $\theta(x) = 1$ and $\theta(y) = x$, and $\theta = 0$ on all other generators. It is straightforward to check that, extended as a derivation of $\wedge V$, we have $[d, \theta] = 0$.

Observe that the even-degree element $y \in V^6$ is also a Gottlieb element in Example 3.1 as it is a terminal element. Indeed if V is finite-dimensional then we see that any $v \in V^N$ with $N = \max\{n \mid V^n \neq 0\}$ is terminal and so a Gottlieb element. The converse to this observation is relevant to the realization problem for classifying spaces, in particular, Theorem 1.7 above. Namely, we note that a space having a non-vanishing rational homotopy element in a degree that is higher than the degree of all non-vanishing Gottlieb elements must be a π-infinite space.

Suppose now that $(\wedge V, d)$ is a formal, minimal DG algebra. When V is finite-dimensional then, in the lower grading, we have $V = V_0 \oplus V_1$ by [**4**, Th.2]. Conjecture 1.2 concerns the case when V is infinite-dimensional. Recall the assertion is that a Gottlieb element $v \in V^n$ should be of lower grading $\leqslant 1$, i.e., $v \in V_0 \oplus V_1$ with $V_0 = Z(V)$ the space of cycles. The following argument is due to Dupont [**2**, Pro.4].

Proof of Theorem 1.3. Let X be a formal space with Sullivan minimal model $(\wedge V, d)$ and suppose $H^*(X; \mathbb{Q}) \cong H(\wedge V, d)$ is finitely generated. Then V_0 is finite-dimensional. Suppose $y \in V^{2m}$ is a Gottlieb element. We prove that $dy = 0$ so that $y \in V_0$.

Use the Gottlieb element y and Theorem 2.3 to obtain a change of basis $(\wedge V, d')$ so that, in $(\wedge V, d')$, the generator y does not appear in the differential d' and the derivation θ' has $\theta'(y) = 1$ and $\theta' = 0$ on all other generators. Further, we have $d'(y) = dy$.

Suppose $d'(y) \neq 0$ so that $y \notin V_0$. Write $V_0^{\text{even}} = \langle z_1, \ldots, z_k\rangle$. Let a_i be of degree $2|z_i| - 1$. Extend $(\wedge V, d')$ to a minimal DG algebra $(\wedge V \otimes \wedge(a_1, \ldots, a_k), \delta)$ by setting $\delta(a_i) = z_i^2$ and $\delta = d'$ on $\wedge V$. We prove:

Lemma 3.2. *$\dim H(\wedge V \otimes \wedge(a_1, \ldots, a_k), \delta) < \infty$.*

Proof. Write $\mathcal{H} = H(\wedge V, d')$ and $\mathcal{H}_0 = \mathcal{H}/\langle x_1^2, \ldots, x_k^2\rangle$. Since $(\wedge V, d')$ is formal we have a quasi-isomorphism $(\wedge V, d') \simeq (\mathcal{H}, 0)$ and so a corresponding quasi-isomorphism

$$(\wedge V \otimes \wedge(a_1, \ldots, a_k), \delta) \simeq (\mathcal{H} \otimes \wedge(a_1, \ldots, a_k), \delta')$$

with $\delta'(a_i) = x_i^2$ (where x_i now denotes the cohomology class represented by x_i) and $\delta' = 0$ on $\mathcal{H}$. Observe that the graded algebra $\mathcal{H} \otimes \wedge(a_1, \ldots, a_k)$ is finitely generated as a module over $\mathcal{H}$. Since $\ker \delta'$ is preserved by the action of $\mathcal{H}$, we see that the homology $H(\mathcal{H} \otimes \wedge(a_1, \ldots, a_k), \delta')$ is a module over $\mathcal{H}$ as well, and finitely generated as such. In the latter $\mathcal{H}$-module, the elements $x_i^2 \in \mathcal{H}$ act trivially. For

$$\text{if } \gamma \in \mathcal{H} \otimes \wedge(a_1, \ldots, a_k) \text{ is a } \delta'\text{-cycle then } x_i^2\gamma = \delta'(a_i\gamma) \text{ is a } \delta'\text{-boundary.}$$

We conclude that $H(\mathcal{H} \otimes \wedge(a_1, \ldots, a_k), \delta')$ is a module over $\mathcal{H}_0$ and is finitely generated as such. But $\mathcal{H}_0$ is clearly finite-dimensional. Thus

$$H(\wedge V \otimes \wedge(a_1, \ldots, a_k), \delta) \cong H(\mathcal{H} \otimes \wedge(a_1, \ldots, a_k), \delta')$$

is a finitely-generated module over a finite-dimensional algebra $\mathcal{H}_0$ and is thus finite-dimensional. □

Now consider the derivation θ' of $(\wedge V, d')$. Since y does not appear in any differential $d'(v)$ for $v \in V'$, we may extend θ' to a derivation θ'' of $(\wedge V \otimes \wedge(a_1, \ldots, a_k), \delta)$, that satisfies $[\delta, \theta''] = 0$ and $\theta''(y) = 1$ by setting $\theta''(a_i) = 0$ for each i. As a consequence, θ'' displays y as an even-degree Gottlieb element in the Sullivan minimal model $(\wedge V \otimes \wedge(a_1, \ldots, a_k), \delta)$. By Lemma 3.2, $(\wedge V \otimes \wedge(a_1, \ldots, a_k), \delta)$ has finite rational L.S. category contradicting [**5**, Pro.29.8(ii)]. We conclude that $dy = d'(y) = 0$. □

Remark 3.3. We note the need for Theorem 1.1 in the preceding. The extension of the derivation θ' to θ'' requires $y \in V^{2m}$ to be a terminal element. Otherwise it is not clear how the derivation θ'' is to be defined on the a_i.

We next apply our basis change formula to affirm Conjecture 1.4 in a restricted case. Recall our hypothesis in Theorem 1.5 is that there is an element $x \in V^{2n+1}$ that is a Gottlieb element and such that dx a monomial. We introduce notation for the latter hypothesis. Write $V = \langle u_1, \ldots, u_k \rangle$ in a basis with the u_i in (non-strictly) increasing order of degrees. Then *dx is a monomial* means we may write

$$dx = w_1 \cdots w_n$$

in which $w_j = u_{i_j}^{m_j}$ for $m_j \geqslant 1$. We assume the indices i_j are increasing and so the degrees of the u_{i_j} are increasing. Note $m_j = 1$ when u_{i_j} has odd degree.

Lemma 3.4. *With notation as above, $dw_1 = 0$.*

Proof. We have

$$0 = d^2x = d(w_1 \cdots w_n) = d(w_1 \cdots w_{n-1})(w_n) + (-1)^{|w_1|+\cdots+|w_{n-1}|}(w_1 \cdots w_{n-1})dw_n.$$

Since u_{i_n} is of maximal degree, the term $u_{i_n}^{m_n}$ does not appear in $(w_1 \cdots w_{n-1})dw_n$. We must have that the two summands above are both zero. In particular,

$$d(w_1 \cdots w_{n-1})(w_n) = 0.$$

Since u_{i_n} cannot appear in $d(w_1 \cdots w_{n-1})$ we have, in fact, $d(w_1 \cdots w_{n-1}) = 0$. The result now follows by induction. □

Proof of Theorem 1.5. Suppose we are given a Sullivan minimal model $(\wedge V, d)$ with $H^q(\wedge V, d) = 0$ for $q > N$ and $x \in V^{2n+1}$ a Gottlieb element such that dx is a monomial. We must prove $n \leqslant N - 1$. We perform the basis change of Theorem 2.1 using $x \in V^{2n+1}$ to obtain an isomorphic Sullivan minimal model $(\wedge V, d')$. We observe that $d'(x) = dx$ remains a monomial under this basis change.

Suppose now that $n \geqslant N$. Write $d'(x) = w_1 \ldots w_n$ as above so that $d'(w_1) = 0$ by Lemma 3.4. Suppose first that w_1 is of odd degree. Let $u = w_1 x$ and observe that

$$d'(u) = d'(w_1 x) = (-1)^{|w_1|} w_1^2 w_2 \cdots w_n = 0$$

since $w_1^2 = 0$. Now $|u| > N$ and so u must be a boundary of $(\wedge V, d')$. However, this contradicts Theorem 2.1 (3). If x appears in a differential $d'(\chi)$ for $\chi \in \wedge V$ then it must appear in a term of the form $x\lambda(\chi)d'(x)$.

So suppose w_1 is of even degree. In this case, the identity

$$0 = (d')^2 x = d'(w_1 \cdots w_n) = (-1)^{|w_i|} w_1 d'(w_2 \cdots w_n)$$

implies $d'(w_2 \cdots w_n) = 0$. Now, since $|d'(x)| = 2n + 2 > 2N$, either we have $|w_1| > N$

or $|w_2\cdots w_n| > N$. In the first case, w_1 is a cycle of $(\wedge V, d')$ of degree greater than N and so w_1 is a boundary. If $w_1 = d'(\eta)$ for $\eta \in \wedge V$ then $u = x - \eta w_2 \cdots w_n$ is a cycle with $|u| > N$. Note that u cannot be a boundary as this contradicts the decomposability of d'. Similarly, if $w_2 \cdots w_n$ has degree $> N$ then it must bound, say $d'(\zeta) = w_2 \cdots w_n$. Again this gives a cycle $u = x - (-1)^{|w_1|} w_1 \zeta$ with $|u| > N$ that cannot be exact. □

4. The realization problem for S^{2n}

The problem of realizing S^{2n} as a classifying space, of producing a rational space $X_{\mathbb{Q}}$ with $B\mathrm{aut}_1(X_{\mathbb{Q}}) \simeq S^{2n}_{\mathbb{Q}}$, translates, with Sullivan's identity (1), to a simple algebraic problem. Namely, a solution to the realization problem for S^{2n} is a Sullivan minimal model $(\wedge V, d)$ giving a Lie algebra isomorphism:

$$H_*(\mathrm{Der}(\wedge V), D) \cong \pi_*(\Omega S^{2n}_{\mathbb{Q}}) \cong \langle \iota_{2n-1}, [\iota_{2n-1}, \iota_{2n-1}]\rangle.$$

Here $\iota_{2n-1} \in \pi_{2n-1}(\Omega S^{2n}_{\mathbb{Q}}) \cong \pi_{2n-1}(S^{2n-1}_{\mathbb{Q}})$ corresponds to the fundamental class.

Throughout this section, we assume a solution to the realization problem is given. Specifically, we assume we have a Sullivan minimal model $(\wedge V, d)$ admitting a non-bounding derivation cycle $\theta_a \in \mathrm{Der}_{2n-1}(\wedge V)$ such that $[\theta_a, \theta_a]$ is non-bounding, as well, and that these two classes span the homology of derivations:

$$H_*(\mathrm{Der}(\wedge V), D) = \langle \theta_a, [\theta_a, \theta_a]\rangle. \tag{8}$$

Write $\theta_b = [\theta_a, \theta_a]$. We make one further hypothesis. We assume

$$\theta_b(y_0) = 1 \text{ for some } \; y_0 \in V^{4n-2}. \tag{9}$$

Note that the hypothesis (9) corresponds to the case that $(\wedge V, d)$ has a nontrivial Gottlieb element in degree $4n - 2$.

We prove that assumptions (8) and (9) lead to a contradiction when $n = 1, 2$, and 3. For the latter two cases, we will make use of a combined version of our change of basis results from Section 2. We begin with the following:

Lemma 4.1. *There is $x \in V^{2n-1}$ with $\theta_a(x) = 1$.*

Proof. By (9) we have

$$1 = \theta_b(y_0) = [\theta_a, \theta_a](y_0) = 2\theta_a(\theta_a(y_0)) = \theta_a\left(2\theta_a(y_0)\right).$$

It follows that $2\theta_a(y_0)$ has an indecomposable summand, i.e., $2\theta_a(y_0) = x + \chi$ for some $x \in V^{2n-1}$ and χ decomposable in $\wedge V$. Then $\theta_a(\chi)$ is decomposable and so, by the above equation, $\theta_a(\chi) = 0$. Thus $\theta_a(x) = 1$, as needed. □

Lemma 4.1 already leads to a contradiction in the case $n = 1$ since $V^1 = 0$. We note a stronger statement can be made in this case (see Proposition 5.1, below). We assume now that $n \geqslant 2$

Lemma 4.2. *The element $x \in V^{2n-1}$ with $\theta_a(x) = 1$ satisfies $dx \neq 0$.*

Proof. If $dx = 0$ then by Remark 2.2 (1) we have a factorization of DG algebras:

$$(\wedge V, d) \cong (\wedge(x), 0) \otimes (\wedge W, d).$$

By the above factorization, we see $(x, 1)$ is a non-bounding derivation cycle of degree $2n-1$. However, $[(x,1),(x,1)] = 0$ which is a contradiction of (8). □

We now apply our work in Section 2. First we make the change of basis

$$\phi\colon (\wedge(W, x), d') \to (\wedge V, d)$$

in (2) using the Gottlieb element $x \in V^{2n-1}$. Let $y = 2y_0$. Remark 2.2 (2) implies

$$\lambda(y) = \frac{1}{2}[\theta_a, \theta_a]_0(2y_0) = \theta_b(y_0) = 1.$$

By Theorem 2.1 (b), we have

$$\theta'(a)(y) = \lambda(y)x = x.$$

We next apply a basis change as in Theorem 2.3 except in this case we use the derivation $\lambda \in \mathrm{Der}_{4n-2}(\wedge V)$. Write $V = \langle x\rangle \oplus \langle y\rangle \oplus \widehat{V}$, so that $W = \langle y\rangle \oplus \widehat{V}$ in the notation of Theorem 2.1. First define a linear map $\psi\colon \wedge V \to \wedge V$ by setting

$$\psi(x) = 0, \quad \psi(y) = 0, \quad \text{and} \quad \psi(\chi) = -y\lambda(\chi)$$

for $\chi \in \wedge\widehat{V}$, and then extending ψ multiplicatively (so that the ideal of $\wedge V$ generated by x and y is in the kernel of ψ). Then ψ is a degree-zero, locally nilpotent derivation of $\wedge V$. We may then exponentiate ψ to an isomorphism

$$\phi' = \exp(\psi) = \mathrm{id} + \psi + \frac{1}{2!}\psi^2 + \frac{1}{3!}\psi^3 + \cdots\colon \wedge V \to \wedge V.$$

Setting $d'' = \phi'^{-1} \circ d' \circ \phi'$, we obtain a DG algebra isomorphism

$$\phi'\colon (\wedge V, d'') \to (\wedge V, d').$$

We transfer the derivation cycle θ'_a to this DG algebra by $\theta''_a = \phi'^{-1} \circ \theta'_a \circ \phi'$. Then $[d'', \theta''_a] = 0$. We prove:

Proposition 4.3. *Let $(\wedge V, d'')$ and $\theta''_a \in \mathrm{Der}_{2n-1}(\wedge V)$ be as constructed above. Write $V = \langle x\rangle \oplus W$ where $W = \langle y\rangle \oplus \widehat{V}$. Then*

(a) $d''x = d'x = dx$ and $\theta''_a(x) = 1$, $\theta''_a(y) = x$, and, $\theta''_a(\chi) = 0$ for $\chi \in \wedge\widehat{V}$.

(b) y does not appear in the differential $d''(w)$ for any $w \in V$.

(c) Given $\chi \in \wedge W$ and decompose d'' in the form $d''(\chi) = xd''_1(\chi) + d''_0(\chi)$ for derivations d''_1 and d''_0 of $\wedge W$. We then have

$$d''_1(y) = -dx \ \text{ and } \ d''_1(\chi) = 0 \ \text{ for } \chi \in \wedge\widehat{V}$$

(namely, x does not appear in any differential other than that of y, where it occurs only in the term $-xdx$).

Proof. (a) Note $d'(x) = dx$ by Theorem 2.1 (a) while $d''(x) = d'(x)$ since $\psi(x) = 0$. Since $\psi(y) = 0$, it is immediate that $\phi'(x) = \phi'^{-1}(x) = x$ and $\phi'(y) = \phi'^{-1}(y) = y$. It then follows that we have $\theta''_a(x) = 1$ and $\theta''(y) = x$. For $\chi \in \wedge\widehat{V}$, we have

$$\phi'(\chi) = \chi + \psi(\chi) + \frac{1}{2!}\,\psi^2(\chi) + \frac{1}{3!}\,\psi^3(\chi) + \cdots .$$

But notice that $\psi^2(\chi) = \psi\big(- y\lambda(\chi)\big) = (-y)(-y)\lambda^2(\chi)$, since $\psi(y) = 0$. Generally, we have

$$\psi^k(\chi) = (-1)^k y^k \lambda^k(\chi),$$

for $k \geqslant 1$. It follows that

$$\phi'(\chi) = \chi + \sum_{k\geqslant 1} (-1)^k\,\frac{1}{k!}\,y^k\lambda^k(\chi).$$

Then, applying the derivation λ, with $\lambda(y) = 1$, we have

$$\begin{aligned}
\lambda\big(\phi'(\chi)\big) &= \lambda(\chi) - \lambda(\chi) - y\lambda^2(\chi)\\
&\qquad\qquad + \sum_{k\geqslant 2} (-1)^k \left(\frac{1}{(k-1)!}\,y^{k-1}\lambda^k(\chi) + \frac{1}{k!}\,y^k\lambda^{k+1}(\chi)\right)\\
&= \lambda(\chi) - \lambda(\chi) - y\lambda^2(\chi) + y\lambda^2(\chi)\\
&\qquad\qquad + \sum_{k\geqslant 3} (-1)^k\,\frac{1}{(k-1)!}\,y^{k-1}\lambda^k(\chi) + \sum_{k\geqslant 2} (-1)^k\,\frac{1}{k!}\,y^k\lambda^{k+1}(\chi)\\
&= \lambda(\chi) - \lambda(\chi) - y\lambda^2(\chi) + y\lambda^2(\chi)\\
&\qquad\qquad + \sum_{k\geqslant 2} \left((-1)^{k+1}\,\frac{1}{k!}\,y^k\lambda^{k+1}(\chi) + (-1)^k\,\frac{1}{k!}\,y^k\lambda^{k+1}(\chi)\right)\\
&= 0.
\end{aligned}$$

Finally, this gives

$$\theta''_a(\chi) = \phi'^{-1} \circ \theta'_a \circ \phi'(\chi) = \phi'^{-1}\big(x\lambda\big(\phi'(\chi)\big)\big) = 0,$$

as claimed.

(b) Let v be a generator of degree at least $4n-3$ and write

$$d''(v) = \chi_0 + y\chi_1 + \cdots + y^k\chi_k,$$

for some $k \geqslant 1$ and each $\chi_i \in \wedge(x) \otimes \wedge(\widehat{V})$, that is, not containing terms that involve y. The derivation $[\theta''_a, \theta''_a] = 2\theta''_a \circ \theta''_a$ satisfies $[\theta''_a, \theta''_a](y) = 2$. Now $[\theta''_a, \theta''_a](v) = 0$ for $v \in \widehat{V}$ by (a). Also, because $[\theta''_a, \theta''_a]$ is a cycle with respect to $D'' = \mathrm{ad}(d'')$, we have

$$[\theta''_a, \theta''_a] \circ d''(v) = d'' \circ [\theta''_a, \theta''_a](v) = 0,$$

and hence we have

$$0 = 0 + 2\lambda\chi_1 + 4\lambda y\chi_2 + \cdots + 2k\lambda y^{k-1}\chi_k.$$

Here, we use the fact that $[\theta''_a, \theta''_a](\chi_i) = 0$ for each i. But each of these terms must be zero independently of each other, since the χ do not involve y, an even-degree generator of $\wedge V$. Hence, each $\chi_i = 0$ for $i = 1, \ldots, k$, and we have $d''(v) = \chi_0$, which does not involve y. For generators v of degree $4n-4$ and lower, $d''(v)$ cannot involve y for degree reasons. So y does not appear in d'', as claimed.

(c) Suppose given $v \in \widehat{V}^k$ with $k > 2n-2$. Write

$$d''(v) = xV_1 + V_2,$$

for $V_1, V_2 \in \wedge\widehat{V}$—recall that we just showed in (b) that y does not occur in the differential of any generator. Then we have $\theta''_a(dv) = V_1$, since $\theta''_a(\widehat{V}) = 0$. On the other hand, we have $\theta''_a(v) = 0$, for the same reason, and because θ''_a is a D''-cycle we have

$$0 = d'' \circ \theta''_a(v) = -\theta''_a \circ d''(v) = -V_1.$$

Thus $d''(v) = V_2 \in \wedge\widehat{V}$, and x does not appear in $d''(v)$. For generators v of degree $2n-2$ and lower, $d''(v)$ cannot involve x for degree reasons. Finally, from part (b) of Theorem 2.1, we have $d'(y) = -x\,dx + d'_0(y)$, and $d'_0(y)$ is a decomposable term, since the original d was decomposable. It follows that

$$\phi'^{-1}\big(-x\,dx + d'_0(y)\big) = -x\,dx + d'_0(y), \quad \text{and hence} \quad d''(y) = -x\,dx + d''_0(y).$$

□

We apply Proposition 4.3, to obtain a differential d'' on $\wedge V$. Then $y \in V^{4n-2}$ does not appear in a differential $d''(w)$ for any $w \in V$, i.e. y is terminal in $(\wedge V, d'')$. We may write

$$d''(y) = -x\,d''(x) + d''_0(y), \tag{10}$$

(here using that $d''(x) = dx$) where $d''_0(y) \in \wedge W$ (so not involving x). Furthermore, x does not appear in the differential $d''(v)$ for v in the subspace $\widehat{V}$ complementary to y in V.

Proposition 4.4. *Let $n = 2$ or 3. Let $(\wedge V, d'')$ be the Sullivan minimal model described above. Then*

$$\dim H_*(\mathrm{Der}(\wedge V), D) > 2.$$

Proof. For convenience in the proof, we omit double subscripts and write $d = d''$. We begin with the case $n = 2$ so that $x \in V^3$ and $y \in V^6$. By Lemma 4.2, we may suppose that $dx \neq 0$. Write $V^2 = \langle v_1, \ldots, v_r\rangle$ and note that, for degree reasons, V^2 consists of cycles: $d(v_j) = 0$ for each $j = 1, \ldots, r$. Now $dx \in \wedge^2 V^2$, so choose and fix elements $\alpha_j \in V^2$ for which

$$dx = v_1\alpha_1 + \cdots + v_r\alpha_r$$

(some, but not all, of the α_j may be zero). For each $v_j \in V^2$, the derivation (y, v_j) is a D-cycle since y does not appear in the differential d. As a cycle of degree 4, each (y, v_j) must be exact or the inequality is achieved. We record a consequence of this exactness in the following form.

Lemma 4.5. *Let $z \in V^t, v \in V^s$ with $s < t$ and consider a derivation of the form $\theta = (z, v) + \theta'$ with $\theta'(z) = 0$. If θ is a D-boundary, with $\theta = D(\eta)$, then there exists $v^* \in V^{t-s+1}$ such that*

$$dz = vv^* + \alpha,$$

for $\alpha \in \wedge\overline{V}$ where $V = \langle v^*\rangle \oplus \overline{V}$. Further, the derivation η satisfies $\eta(v^*) = \pm 1$.

Proof. In the expression $v = D(\eta)(z) = d\eta(z) + (-1)^{|\eta|}\eta(dz)$, the term $d\eta(z)$ is decomposable and so $d\eta(z) = 0$. Thus $v = (-1)^{|\eta|}\eta(dz)$ which implies dz must contain a quadratic term vv^* as specified with $\eta(v^*) = (-1)^{|\eta|}$. □

Returning to our case, each degree 4 derivation cycle (y, v_j) must be exact, say $(y, v_j) = D(\eta_j)$. By Lemma 4.5, there exist (independent) $v_j^* \in V^5$ such that

$$dy = -xdx + \sum_{j=1}^{r} v_j v_j^* + \beta,$$

and $\eta_j(v_j^*) = \pm 1$ where the term $-xdx$ occurring in dy is by Proposition 4.3. Here β is in $\wedge^7(\overline{V})$ with $\overline{V}$ complementary to $\langle v_1^*, \ldots, v_r^*\rangle$ in V.

Now define a degree 3 derivation

$$\gamma = (y, x) - \alpha_1 \cdot \eta_1 - \cdots - \alpha_r \cdot \eta_r.$$

Notice that, since $d(\alpha_j) = 0$, we have

$$D(\alpha_j \cdot \eta_j) = \alpha_j D(\eta_j) = \alpha_j (y, v_j) = (y, v_j\alpha_j).$$

Also, since y does not appear in the differential d, we have $D(y, x) = (y, dx)$. Hence,

$$D(\gamma) = (y, dx) - (y, v_1\alpha_1) - \cdots - (y, v_r\alpha_r) = 0.$$

Thus, γ is a D-cycle. Since $\theta_a(x) = 1$, γ is not a multiple of θ_a. Thus γ must be exact. Applying Lemma 4.5 with $z = y$ and $v = x$, we conclude that dy contains a term of the form xx^* for an indecomposable x^* of degree 3 with $\eta(x^*) = \pm 1$. We have arrived at a contradiction: other than in the term xdx, the generator x does not occur in dy.

For the case $n = 3$, we have $x \in V^5$ and $y \in V^{10}$. By Lemma 4.2, we again assume $dx \neq 0$. If $V^2 = 0$, then $dx \in \wedge^2 V^3$ and we may define the derivation γ and proceed to a contradiction exactly as above.

Suppose then that there is some $v \in V^2$. We define a degree 3 derivation

$$\zeta = (x, v) + (y, vx).$$

We claim ζ is a D-cycle. Since $dv = 0$, and since y is terminal, we see $D(y, vx) = (y, vdx)$. Since x only appears in the differential dy and there as the term $-xdx$, we see $D(x, v) = (x, v) \circ d = (y, -vdx)$. Thus $D(\zeta) = 0$ and so, for degree reasons, ζ is a D-boundary. Write $\zeta = D(\rho)$ for $\rho \in \mathrm{Der}^4(\wedge V)$. By Lemma 4.5, dx contains a term vv^* for $v^* \in V^4$ with $\rho(v^*) = \pm 1$. Further, we have that $D(\rho)(y) = vx$ with $D(\rho)$ vanishing on a complementary subspace of $\langle x, y\rangle$ in V.

If $V^2 = \langle v\rangle$ is one-dimensional, write $d(v^*) = vw$ for $w \in V^3$. Note that $dw = 0$. The D-cycle (y, v) must be a boundary, so write $(y, v) = D(\eta)$ for η of degree 9. Then, set

$$\beta = (y, v^*) - w \cdot \eta.$$

We see that β is D-cycle of degree 6 that must be a D-boundary. By Lemma 4.5 again, this implies a term v^*v^{**} appears in dy with $v^{**} \in V^7$. But $\rho(v^*) = \pm 1$ and so $D(\rho)(y) = \rho(dy)$ has a summand v^{**} which contradicts $D(\rho)(y) = vx$.

It remains to handle the case $V^2 = \langle v_1, v_2, \ldots, v_n\rangle$ for $n \geqslant 2$. Here we begin by defining D-cycles $\zeta_j = (x, v_j) + (y, v_jx)$ of degree 3 as above. These must each bound which implies we have derivations ρ_j of degree 4 with $D(\rho_j) = \zeta_j$. Using Lemma 4.5, we may write

$$dx = v_1 v_1^* + \cdots + v_n v_n^* + \chi$$

for $v_j^* \in V^4$ where $\chi \in \wedge \overline{V}$ and $\overline{V}$ is complementary to $\langle v_1^*, \cdots, v_n^* \rangle$ in V. We have $\rho_j(v_j^*) = \pm 1$. Consider the derivation

$$\alpha = v_1 \cdot \rho_2 - v_2 \cdot \rho_1$$

of degree 2. We see

$$D(\alpha)(x) = v_2 v_1 - v_1 v_2 = 0 \text{ and } D(\alpha)(y) = v_1 v_2 x - v_2 v_1 x = 0.$$

Since $D(\rho_1) = \zeta_1$ and $D(\rho_2) = \zeta_2$ and the latter vanish on a complementary subspace to $\langle x, y \rangle$ in V so does $D(\alpha)$. Thus α is a D-cycle which, for degree reasons, must be a D-boundary. Write $\alpha = D(\sigma)$ for $\sigma \in \mathrm{Der}_3(\wedge V)$. Then $D(\sigma)(v_1^*) = \alpha(v_1^*) = -v_2$ and $D(\sigma)(v_2^*) = \alpha(v_2^*) = v_1$. From this we deduce there is $w \in V^3$ with $\sigma(w) = 1$ and the term $-wv_2$ appears in $d(v_1^*)$ while the term wv_1 appears in $d(v_2^*)$.

Finally, consider the D-cycles of degree 8 given by (y, v_j) for $j = 1, \ldots, r$. Since these must be D-boundaries we obtain degree 9 derivations η_j satisfying $D(\eta_j) = (y, v_j)$. Since $w \in V^3$ we may write $dw = \alpha_1 v_1 + \cdots + \alpha_r v_r$ for choices of elements $\alpha_j \in V^2$. Define a degree 7 derivation

$$\gamma = (y, w) - \sum_{j=1}^{r} \alpha_j \eta_j.$$

We directly check that γ is a D-cycle and so a D-boundary. Then, by Lemma 4.5 applied with $z = y$ and $v = w$, we must have a generator $w^* \in V^8$ such that dy contains the term ww^*. Since $\sigma(w) = 1$ we see that w^* occurs in $D(\sigma)(y)$ as an indecomposable summand. On the other hand, we have

$$D(\sigma)(y) = \alpha(y) = v_1 \cdot \rho_2(y) - v_2 \cdot \rho_1(y)$$

is decomposable since $\rho_j(y)$ is of degree 6. This contradiction completes the proof. □

Proof of Theorem 1.7. Suppose given a simply connected space X satisfying

$$B\mathrm{aut}_1(X_{\mathbb{Q}}) \simeq S^{2n}_{\mathbb{Q}}$$

for $n = 1, 2, 3$. The Sullivan minimal model $(\wedge V, d)$ for X then satisfies (8) and so we have $\theta_a \in \mathrm{Der}_{2n-1}(\wedge V)$ and $\theta_b = [\theta_a, \theta_a] \in \mathrm{Der}_{4n-2}(\wedge V)$ which are non-bounding derivation cycles. If X has a non-vanishing rational Gottlieb element in degree at least $2n$, then there is a derivation cycle $\theta \in \mathrm{Der}_q(\wedge V)$ with $\theta(v) = 1$ for some $v \in V^q$. The minimality condition for d implies θ cannot be a D-boundary and so, since $q > 2n - 1$, we must have $q = 4n - 2$ and $\theta = c\,\theta_b$ for some $c \neq 0$. Thus, taking $y_0 = cv \in V^{4n-2}$ we have $\theta_b(y_0) = 1$. When $n = 1$, we have the contradiction $V^1 = 0$. Proposition 4.4 gives the result for $n = 2$ and 3. □

5. The universal fibration and the realization problem

Question 1.6 naturally leads to consideration of the universal fibration. For observe that $Y_{\mathbb{Q}} \simeq B\mathrm{aut}_1(X_{\mathbb{Q}})$ for some space $X_{\mathbb{Q}}$ implies the existence of a fibration

$$X_{\mathbb{Q}} \to E_{X_{\mathbb{Q}}} \to B\mathrm{aut}_1(X_{\mathbb{Q}}) \simeq Y_{\mathbb{Q}}$$

that is universal for fibrations with fibre $X_{\mathbb{Q}}$. We consider this expanded view here. Write $\partial_\infty \colon \Omega B\mathrm{aut}_1(X_{\mathbb{Q}}) \to X_{\mathbb{Q}}$ for the connecting homomorphism. We recall that $\alpha \in \pi_n(X_{\mathbb{Q}})$ is a Gottlieb element if and only if

$$\alpha = (\partial_\infty)_\sharp(\beta) \text{ for some } \beta \in \pi_{n+1}(B\mathrm{aut}_1(X_{\mathbb{Q}})) \quad [\mathbf{7}, Th.2.6].$$

Suppose now we have such universal fibration for $Y = S^2$. That is, suppose we have a simply connected space X with $B\mathrm{aut}_1(X_{\mathbb{Q}}) \simeq S^2_{\mathbb{Q}}$. By Theorem 1.7, X has vanishing rational Gottlieb elements. By Gottlieb's result, mentioned above, it follows that $p_{X_{\mathbb{Q}}} \colon E_{X_{\mathbb{Q}}} \to B\mathrm{aut}_1(X_{\mathbb{Q}})$ induces a surjection on rational homotopy groups. We can also see this directly from the long-exact homotopy sequence of the universal fibration. As a consequence, we deduce:

Proposition 5.1. *Suppose X is a simply connected and satisfies $B\mathrm{aut}_1(X_{\mathbb{Q}}) \simeq S^2_{\mathbb{Q}}$. Then the universal fibration $X_{\mathbb{Q}} \to E_{X_{\mathbb{Q}}} \to B\mathrm{aut}_1(X_{\mathbb{Q}})$ with fibre $X_{\mathbb{Q}}$ has a section.*

Proof. We obtain a section of $p_{X_{\mathbb{Q}}} \colon E_{X_{\mathbb{Q}}} \to B\mathrm{aut}_1(X_{\mathbb{Q}})$ as the lift of the identity homotopy class $S^2_{\mathbb{Q}} \simeq B\mathrm{aut}_1(X_{\mathbb{Q}}) \to B\mathrm{aut}_1(X_{\mathbb{Q}})$ to $E_{X_{\mathbb{Q}}}$. □

Proposition 5.1 gives a strong restriction on a rational space $X_{\mathbb{Q}}$ which satisfies $B\mathrm{aut}_1(X_{\mathbb{Q}}) \simeq S^2_{\mathbb{Q}}$. Namely, $X_{\mathbb{Q}}$ must have the property that every rational fibration $X_{\mathbb{Q}} \to E_{\mathbb{Q}} \to B_{\mathbb{Q}}$ has a section. We give an example to show that it is possible to have a non-trivial, sectioned fibration with base S^2 and fibre X such that X has no rational Gottlieb elements.

Example 5.2. Let $p \colon S^2 \vee S^2 \to S^2$ denote the projection onto the first summand. Convert p into a fibration, to obtain a fibre sequence

$$X \xrightarrow{j} S^2 \vee S^2 \xrightarrow{p} S^2.$$

Then p admits the obvious section $i_1 \colon S^2 \to S^2 \vee S^2$ which is just inclusion into the first summand. Also, the fibration is not trivial, even after rationalization, since $H^*(S^2 \vee S^2; \mathbb{Q})$ does not split as a tensor product $H^*(S^2;\mathbb{Q}) \otimes H^*(X;\mathbb{Q})$. Next, note that X has infinitely many non-zero rational homotopy groups since, from the long exact sequence in rational homotopy, we have $\pi_i(X_{\mathbb{Q}}) \cong \pi_i((S^2 \vee S^2)_{\mathbb{Q}})$ for $i \geqslant 4$, and $S^2 \vee S^2$ has infinitely many non-zero rational homotopy groups. To see that X has no rational Gottlieb elements, we apply the mapping theorem for rational L.S. category ([**5**, Th.28.6]). Note that that the fibre inclusion $j \colon X \to S^2 \vee S^2$ induces an injection of rational homotopy groups and $\mathrm{cat}_0(X) \leqslant \mathrm{cat}_0(S^2 \vee S^2) = 1$ Now X is π-infinite and so it follows that X has the rational homotopy type of a wedge of at least two spheres. Thus, the homotopy Lie algebra $\pi_*(\Omega X_{\mathbb{Q}})$ is a free Lie algebra on at least two generators. It follows directly that $X_{\mathbb{Q}}$ has no nontrivial Gottlieb elements; for recall a Gottlieb element $\alpha \in \pi_n(X_{\mathbb{Q}})$ has vanishing Whitehead products $[\alpha, \beta] = 0$ for all $\beta \in \pi_q(X_{\mathbb{Q}})$.

The rationalization of the fibration of Example 5.2 cannot be a universal fibration. We argue as follows. First by a Serre spectral sequence argument, $H^i(X;\mathbb{Q}) \neq 0$ for $i \geqslant 2$. Since $\mathrm{cat}_0(X) = 1$, $X_{\mathbb{Q}}$ is an infinite wedge of spheres. We prove:

Proposition 5.3. *Suppose that X has the rational homotopy type of a wedge of infinitely many spheres. Then $B\mathrm{aut}_1(X_{\mathbb{Q}})$ is π-infinite.*

Proof. We note that Gatsinzi has proved essentially this fact for $X_{\mathbb{Q}}$ a finite wedge (cf. [**6**]). We use the model for $B\mathrm{aut}_1(X_{\mathbb{Q}})$ described in [**15**, VII.2] that derives from a Quillen model of X. Since X is a wedge of spheres, it has Quillen model a free Lie algebra $\mathbb{L}(V)$ with zero differential and V a graded vector space of generators isomorphic to the (de-suspension of the) reduced homology of X. We assume this homology is infinite-dimensional, and so write $V = \{v_i\}_{i\in\mathbb{N}}$ with the generators v_i written in order of increasing degree: thus $i < j \implies |v_i| \leqslant |v_j|$. Now the model for $B\mathrm{aut}_1(X_{\mathbb{Q}})$ is a DG Lie algebra written as

$$(s\mathbb{L}(V) \oplus \mathrm{Der}\,\mathbb{L}(V), \delta),$$

where $s\mathbb{L}(V)$ denotes the abelian Lie algebra on the vector space $\mathbb{L}(V)$ with degrees shifted up by one, $\mathrm{Der}\,\mathbb{L}(V)$ denotes the usual Lie algebra of derivations that increase degree, except that in degree 1 we restrict to just the cycles. The differential δ restricts to the usual differential on $\mathrm{Der}\,\mathbb{L}(V)$ which, since we assume X is a wedge of spheres, is zero here. The only non-trivial differentials, therefore, are of elements from $s\mathbb{L}(V)$, where we have $\delta(sx) = \mathrm{ad}(x) \in \mathrm{Der}\,\mathbb{L}(V)$, for $x \in \mathbb{L}(V)$. See [**15**, VII.2(6)] for details. Suppose that $v_n \in V$ is the first generator of degree strictly greater than that of v_0. Then we define an infinite sequence of non-zero derivations $\theta_i \in \mathrm{Der}\,\mathbb{L}(V)$ for $i \geqslant n$ by setting $\theta_i(v_0) = v_i$ and $\theta_i = 0$ on all other generators, and then extending θ_i as a derivation to an element of $\mathrm{Der}\,\mathbb{L}(V)$. Then each θ_i is a δ-cycle in $s\mathbb{L}(V) \oplus \mathrm{Der}\,\mathbb{L}(V)$ that cannot be exact, since any boundary $\mathrm{ad}(x)$ will map v_0 to elements of bracket length at least two in $\mathbb{L}(V)$. The homology of the Quillen model $(s\mathbb{L}(V) \oplus \mathrm{Der}\mathbb{L}(V), \delta)$ gives the homotopy of $B\mathrm{aut}_1(X_{\mathbb{Q}})$, which therefore is non-zero in infinitely many degrees. $\square$

Our main result in this section is a further example with the same conclusion as Proposition 5.1.

Theorem 5.4. *Suppose X is a simply connected space with*

$$B\mathrm{aut}_1(X_{\mathbb{Q}}) \simeq \left(S^3 \vee \cdots \vee S^3\right)_{\mathbb{Q}} \quad \text{(a wedge of two or more copies of } S^3\text{)}.$$

Then the universal fibration $X_{\mathbb{Q}} \to E_{X_{\mathbb{Q}}} \to B\mathrm{aut}_1(X_{\mathbb{Q}})$ has a section.

We prove a preliminary result on the homology of derivations of a Sullivan minimal model having a free factor generated by an element of even degree. We say an element x in a graded Lie algebra L is a *zero-divisor* of L if there exists $y \in L$ with $y \notin \langle x\rangle$ such that $[x, y] = 0$.

Lemma 5.5. *Let $(\wedge V, d)$ be a Sullivan minimal model admitting a DG algebra factorization $(\wedge V, d) \cong (\wedge(y), 0) \otimes (\wedge W, d)$ for some $y \in V^{2m}$. Let $\theta \in \mathrm{Der}_{2m}(\wedge V)$ be dual to y. If the homology class $[\theta]$ is not a zero divisor in $H_*(\mathrm{Der}(\wedge V), D)$, then*

$$H_{\geqslant 2m}(\mathrm{Der}(\wedge V), D) = \langle[\theta]\rangle.$$

Proof. Suppose that we have a derivation cycle $\alpha \in \mathrm{Der}_q(\wedge V)$ with $q \geqslant 2m$ representing a class $[\alpha] \in H_{\geqslant 2m}(\mathrm{Der}(\wedge V), D)$ in the complement of $\langle[\theta]\rangle$. We will show that if $[\theta]$ is not a zero divisor then α is a D-boundary.

Without loss of generality, we assume that $\alpha(y) = 0$, for otherwise we may replace α with $\alpha - \theta$. Given $\chi \in \wedge W$, write

$$\alpha(\chi) = \alpha_0(\chi) + y\alpha_1(\chi) + \cdots + y^r\alpha_r(\chi) + \cdots ,$$

which defines derivations α_i of $\wedge W$ for $i \geqslant 0$. Since α is a D-cycle, and y is a d-cycle, and the differential d restricts to one of $\wedge W$, it follows that each α_i is a D-cycle. Notice that the sum here is locally finite, namely only finitely many terms are non-zero for a fixed χ. We will show inductively that we have a sequence of derivations $\{\eta_i\}_{i\geqslant 0}$ of $\wedge W$, for which $D(\eta_i) = \alpha_i$, and it follows that the (locally finite) sum

$$\eta(\chi) = \eta_0(\chi) + y\eta_1(\chi) + \cdots + y^r\eta_r(\chi) + \cdots$$

has $D(\eta) = \alpha$. Induction starts with $i = -1$, where there is nothing to prove. For some $r \geqslant -1$, assume that we have derivations $\{\eta_i\}_{0\leqslant i\leqslant r}$ with $D(\eta_i) = \alpha_i$. Then $D(y^i\eta_i) = y^i\alpha$, and

$$\alpha - D\left(\sum_{i=0}^{r} y^i\eta_i\right)(\chi) = y^{r+1}\alpha_{r+1}(\chi) + y^{r+2}\alpha_{r+2}(\chi) + \cdots .$$

Write $\beta(\chi) = \alpha_{r+1}(\chi) + y\alpha_{r+2}(\chi) + y^2\alpha_{r+3}(\chi) + \cdots$, so that

$$\alpha - D\left(\sum_{i=0}^{r} y^i\eta_i\right) = y^{r+1}\beta.$$

Since the left-hand side here is a D-cycle, it follows that so too is β a D-cycle. Now use β to construct the derivation

$$\widehat{\beta} = \beta - y\,[\theta,\beta] + \frac{y^2}{2!}\,\big[\theta,[\theta,\beta]\big] + \cdots + \frac{y^n}{n!}\,\mathrm{ad}^n(\theta)(\beta) + \cdots .$$

Since y is a d-cycle, and θ and β are both D-cycles, each term in this (locally finite) sum is a derivation that is a D-cycle. Hence, $\widehat{\beta}$ is a D-cycle. But observe that we have

$$[\theta,\widehat{\beta}] = ([\theta,\beta] - 1\cdot[\theta,\beta]) + (-y\,\big[\theta,[\theta,\beta]\big] + y\,\big[\theta,[\theta,\beta]\big]) + \cdots = 0.$$

Then our assumption that θ is not a zero divisor implies that $\widehat{\beta} = D(\eta)$ for some $\eta \in \mathrm{Der}_{q+1}(\wedge V)$. However, we have $\widehat{\beta} = \beta_0$. This follows by writing out terms in $\widehat{\beta}(\chi)$, as

$$\begin{aligned}
\widehat{\beta}(\chi) &= \beta(\chi) - y\,[\theta,\beta](\chi) + \cdots \\
&= \beta(\chi) - y\,\theta\circ\beta(\chi) + \cdots + \frac{y^n}{n!}\,\theta^n\circ\beta(\chi) + \cdots \\
&= (\alpha_{r+1}(\chi) + y\alpha_{r+2}(\chi) + y^2\alpha_{r+3}(\chi) + \cdots) \\
&\qquad\qquad - y\,\theta(\alpha_{r+1}(\chi) + y\alpha_{r+2}(\chi) + y^2\alpha_{r+3}(\chi) + \cdots) \\
&\qquad\qquad + \cdots + \frac{y^n}{n!}\,\theta^n(\alpha_{r+1}(\chi) + y\alpha_{r+2}(\chi) + y^2\alpha_{r+3}(\chi) + \cdots) + \cdots
\end{aligned}$$

$$\begin{aligned}&= (\alpha_{r+1}(\chi) + y\alpha_{r+2}(\chi) + y^2\alpha_{r+3}(\chi) + \cdots) \\ &\qquad - (y\alpha_{r+2}(\chi) + 2y^2\alpha_{r+3}(\chi) + 3y^3\alpha_{r+4}(\chi) + \cdots) \\ &\qquad + \cdots + (y^n\alpha_{r+1+n}(\chi) + \binom{n+1}{n} y^{n+1}\,\alpha_{r+1+n+1}(\chi) + \cdots) + \cdots .\end{aligned}$$

Notice above, in passing from the first to the second lines, we replaced $\mathrm{ad}^n(\theta)(\beta)(\chi)$ with $\theta^n \circ \beta(\chi)$ since $\theta(\chi) = 0$. A careful tallying of the terms in this last expression reveals that it consists of a sum of terms $y^n\alpha_{r+1+n}(\chi)$ for $n \geqslant 0$ and, for $n \geqslant 1$, the coefficient of $y^n\alpha_{r+1+n}(\chi)$ is the alternating sum

$$1 - n + \binom{n}{2} + \cdots + (-1)^t \binom{n}{t} + \cdots + (-1)^n.$$

But this is zero, since it is the value of $(1-x)^n$ when $x = 1$. That is, we have

$$D(\eta)(\chi) = \widehat{\beta}(\chi) = \alpha_{r+1}(\chi).$$

But this means that we have $D(\eta) = \alpha_{r+1}$, as derivations of $\wedge W$, and setting $\eta = \eta_r$ completes the inductive step. Notice that α_r is of degree $|\alpha_r| = |\alpha| - 2mr$, and $|\eta_r| = |\alpha_r| + 1$, so for a fixed χ only finitely many $\eta_r(\chi)$ can be non-zero, for degree reasons. It follows from the induction that, with

$$\eta = \sum_{i \geqslant 0} y^i\eta_i,$$

we have $D(\eta) = \alpha$. □

Proof of Theorem 5.4. Suppose X with Sullivan minimal model $(\wedge V, d)$ also satisfies $B\mathrm{aut}_1(X_{\mathbb{Q}}) \simeq (S^3 \vee \cdots \vee S^3)_{\mathbb{Q}}$. Let $p_{X_{\mathbb{Q}}}\colon E_{X_{\mathbb{Q}}} \to B\mathrm{aut}_1(X_{\mathbb{Q}})$ denote the universal fibre map for $X_{\mathbb{Q}}$. Suppose $(p_{X_{\mathbb{Q}}})_\sharp\colon \pi_3(E_{X_{\mathbb{Q}}}) \to \pi_3(B\mathrm{aut}_1(X_{\mathbb{Q}}))$ is not surjective. Then there is a Gottlieb element $y \in V^2$. Since $(\wedge V, d)$ is simply connected, $dy = 0$. Thus we have a factorization $(\wedge V, d) \cong (\wedge(y), 0) \otimes (\wedge W, d)$ for W complementary to $\langle y \rangle$ in V. We may apply Lemma 5.5. However, here we have

$$H_*(\mathrm{Der}(\wedge V), D) \cong \pi_*\big(\Omega B\mathrm{aut}_1(X_{\mathbb{Q}})\big) \cong \pi_*\big(\Omega(S^3 \vee \cdots \vee S^3)_{\mathbb{Q}}\big)$$

and the latter is the free graded Lie algebra generated in degree 2 by at least two elements. No element of degree 2 (or any other degree) can be a zero divisor and yet neither can we have $H_{\geqslant 2}(\mathrm{Der}(\wedge V), D)$ one-dimensional (indeed, $H_{\geqslant 2}(\mathrm{Der}(\wedge V), D)$ is infinite-dimensional). We have a contradiction.

We may assume $(p_{X_{\mathbb{Q}}})_\sharp\colon \pi_3(E_{X_{\mathbb{Q}}}) \to \pi_3(B\mathrm{aut}_1(X_{\mathbb{Q}}))$ is surjective. Then, for each summand $S^3_{\mathbb{Q}}$ involved in the wedge, we may lift the inclusion $S^3_{\mathbb{Q}} \to (S^3 \vee \cdots \vee S^3)_{\mathbb{Q}}$ through $p_{X_{\mathbb{Q}}}$ to a map $S^3_{\mathbb{Q}} \to E_{X_{\mathbb{Q}}}$. Assembling these liftings gives a lifting of the identity map of $B\mathrm{aut}_1(X_{\mathbb{Q}}) \simeq (S^3 \vee \cdots \vee S^3)_{\mathbb{Q}}$ through $p_{X_{\mathbb{Q}}}$ and so a section of the universal fibration. □

We conclude with a homotopy-theoretic proof of Theorem 1.1 using the universal fibration for rational spaces.

Alternate Proof of Theorem 1.1. Given a space X with a Gottlieb element $\alpha \in \pi_{2n}(X)$ we show that there is a corresponding terminal element $y \in V^{2n}$ in a Sullivan minimal model $(\wedge V, d)$ for X. Let $F\colon S^{2n} \times X \to X$ be the affiliated map for α. The rationalization of F is a map $F_0\colon S^{2n}_{\mathbb{Q}} \times X_{\mathbb{Q}} \to X_{\mathbb{Q}}$ which has adjoint $\beta_0\colon S^{2n}_{\mathbb{Q}} \to \mathrm{aut}_1(X_{\mathbb{Q}})$. From the isomorphisms

$$\pi_{2n+1}(B\mathrm{aut}_1(X_{\mathbb{Q}})) \cong \pi_{2n}(\Omega B\mathrm{aut}_1(X_{\mathbb{Q}})) \cong \pi_{2n}(\mathrm{aut}_1(X_{\mathbb{Q}}))$$

we may view β_0 as a class $\beta_0\colon S^{2n+1}_{\mathbb{Q}} \to B\mathrm{aut}_1(X_{\mathbb{Q}})$. The pullback of the universal fibration $X_{\mathbb{Q}} \to E_{X_{\mathbb{Q}}} \to B\mathrm{aut}_1(X_{\mathbb{Q}})$ by β_0 is a fibration of the form

$$X_{\mathbb{Q}} \to E_{\mathbb{Q}} \to S^{2n+1}_{\mathbb{Q}} \quad \text{with connecting homomorphism satisfying} \quad \partial_\sharp(\iota_{2n+1}) = \alpha_0,$$

where $\iota_{2n+1} \in \pi_{2n+1}(S^{2n+1}_{\mathbb{Q}})$ is the fundamental class and $\alpha_0 \in \pi_{2n}(X_{\mathbb{Q}})$ the image of α under rationalization. Stepping back one stage in the Puppe sequence gives a fibration $\Omega S^{2n+1}_{\mathbb{Q}} \to X_{\mathbb{Q}} \to E_{\mathbb{Q}}$. Now observe $\Omega S^{2n+1}_{\mathbb{Q}} = K(\mathbb{Q}, 2n)$. The relative Sullivan model for this fibration is thus of the form

$$(\wedge V', d') \to (\wedge(V' \oplus \langle y \rangle), D) \to (\wedge(y), 0).$$

Since $y \in V^{2n}$ is a Gottlieb element, using [5, Pro.15.13] we can see that the middle term is a minimal Sullivan model for $X_{\mathbb{Q}}$ having terminal homotopy element y corresponding to the class α. $\square$

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