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HAEFLIGER'S APPROACH FOR SPHERICAL KNOTS MODULO IMMERSIONS

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(*communicated by Dev P. Sinha*)

Abstract

We show that for the spaces of spherical embeddings modulo immersions $\overline{Emb}(S^n, S^{n+q})$ and long embeddings modulo immersions $\overline{Emb}_\partial(D^n, D^{n+q})$, the set of connected components is isomorphic to $\pi_{n+1}(SG, SG_q)$ for $q \geqslant 3$. As a consequence, we show that all the terms of the long exact sequence of the triad $(SG; SO, SG_q)$ have a geometric meaning relating to spherical embeddings and immersions.

1. Introduction

The spaces of embeddings modulo immersions in recent years attracted a lot of attention [**1, 2, 3, 4, 5, 6, 7, 8, 10, 20, 24**]. In particular, it was shown in [**5, 7, 20**] that there is a natural little disks operad action on the spaces of (framed) disk embeddings modulo immersions. Also, for codimension at least 3, there are several delooping results on such spaces by means of smoothing theory [**20**] and the Goodwillie–Weiss functor calculus on manifolds [**4, 16**]. Furthermore, the rational homotopy and homology of such spaces were studied in [**1, 2, 3, 9, 10**], and in case of spherical embeddings, the explicit computations were done in [**24**]. The advantage of these spaces over the usual embedding spaces is mainly the easier description of their rational homology and homotopy as well as deloopings. The main objective of this paper is to study the set of isotopy classes of spherical and disk embeddings modulo immersions by extending Haefliger's work [**13**] on isotopy classes of (framed) spherical embeddings of codimension at least 3. Also, we establish a natural connection between our and Haefliger's results in terms of long exact sequences.

We let $Emb(S^n, S^{n+q})$ be the space of smooth embeddings $S^n \hookrightarrow S^{n+q}$ and let $Imm(S^n, S^{n+q})$ be the space of smooth immersions $S^n \looparrowright S^{n+q}$. We define the space of *spherical embeddings modulo immersions* $\overline{Emb}(S^n, S^{n+q})$ as the homotopy fiber of $Emb(S^n, S^{n+q}) \hookrightarrow Imm(S^n, S^{n+q})$ over the trivial inclusion $id\colon S^n \subset S^{n+q}$. An element in this space is represented by a pair (f, α), where $f\colon S^n \hookrightarrow S^{n+q}$ is a smooth

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embedding together with a regular homotopy $\alpha\colon [0,1] \to Imm(S^n, S^{n+q})$, between f and the trivial inclusion $S^n \subset S^{n+q}$. Moreover, $Emb_\partial(D^n, D^{n+q})$ denotes the space of disk embeddings $D^n \hookrightarrow D^{n+q}$ with the fixed behavior near the boundary, and we similarly define $\overline{Emb}_\partial(D^n, D^{n+q})$ as the space of disk embeddings modulo immersions. For framed spherical/disk embeddings, we consider the spaces

$$Emb^{fr}(S^n, S^{n+q}),\ \overline{Emb}^{fr}(S^n, S^{n+q}),\ Emb^{fr}_\partial(D^n, D^{n+q}) \ \text{ and } \ \overline{Emb}^{fr}_\partial(D^n, D^{n+q})$$

in the same manner. Throughout the paper for spherical embeddings we assume that the framing respects the natural orientation: if one takes the orientation of S^n and completes it with the orientation of the normal bundle induced by the framing, one obtains the standard orientation of the ambient sphere S^{n+q}. For disk embeddings the framing is standard near the boundary.

In [**13**, Theorems 3.4 and 5.7], Haefliger has shown that for $q \geqslant 3$, the group of isotopy classes of (framed) spherical embeddings of S^n in S^{n+q} can be represented in terms of the homotopy group of a triad i.e.,

$$\begin{aligned} C^q_n &:= \pi_0 Emb(S^n, S^{n+q}) = \pi_{n+1}(SG; SO, SG_q) \ \text{ and} \\ FC^q_n &:= \pi_0 Emb^{fr}(S^n, S^{n+q}) = \tilde{\pi}_{n+1}(SG; SO, SG_q). \end{aligned}$$

We recall these homotopy groups and isomorphisms later in Section 2.

1.1. Main results

Let $\overline{FC}^q_n$ denote the group of isotopy classes of "framed disked embeddings", which we discuss in more detail in Section 3.

Theorem 1.1. *For $q \geqslant 3$,*

$$\begin{aligned} \overline{FC}^q_n &= \pi_0 \overline{Emb}(S^n, S^{n+q}) = \pi_0 \overline{Emb}_\partial(D^n, D^{n+q}) \\ &= \pi_0 \overline{Emb}^{fr}(S^n, S^{n+q}) = \pi_0 \overline{Emb}^{fr}_\partial(D^n, D^{n+q}) = \pi_{n+1}(SG, SG_q). \end{aligned}$$

The result is an immediate corollary of Theorems 3.1 and 4.3, and Lemma 4.1.

Alternatively, this result can be obtained using smoothing theory, as a consequence of [**14**, Section 6] and [**20**, Theorem 1.1]. There is even a stronger result:

$$\pi_i \overline{Emb}_\partial(D^n, D^{n+q}) = \pi_{n+i+1}(SG_{n+q}, SG_q) \text{ for } i \leqslant 2q-5,$$

which follows from the work of Lashof [**17**], Millett [**19**, Theorem 2.3] and Sakai [**20**, Theorem 1.1 and Remark 2.3]. Moreover, for $i \leqslant q-3$,

$$\pi_{n+i+1}(SG_{n+q}, SG_q) = \pi_{n+i+1}(SG, SG_q),$$

see Lemma 6.3. However, our goal is to review Haefliger's construction and give a geometric meaning to $\pi_{n+1}(SG, SG_q)$ i.e., $\pi_{n+1}(SG, SG_q) = \overline{FC}^q_n$.

Another main result that does not immediately follow from smoothing theory is to geometrically interpret both of the long exact sequences associated with the triad $(SG; SO, SG_q)$ considered by Haefliger [**13**, Sections 4.4 and 5.9]. Let Im^q_n and FIm^q_n denote the group of regular homotopy classes of immersions $S^n \looparrowright S^{n+q}$ and framed immersions $S^n \looparrowright S^{n+q}$, respectively. It is natural to ask which (framed) spherical immersions can be realized as (framed) embeddings, or when two (framed) spherical embeddings are equivalent as (framed) immersions. Answers to these questions are encoded by the lower exact sequences in (1) and (2) of Theorem 1.2, in which $\overline{FC}^q_n$ naturally fits.

Theorem 1.2. *For $q \geqslant 3$, the two long exact sequences of the triad $(SG; SO, SG_q)$ are isomorphic to the corresponding geometric long exact sequences:*

$$\begin{array}{ccccccccc} \to \pi_{n+1}(SG, SG_q) & \to & \pi_{n+1}(SG; SO, SG_q) & \to & \pi_n(SO, SO_q) & \to & \pi_n(SG, SG_q) \to \\ \| & & \| & & \| & & \| \\ \longrightarrow \overline{FC}^q_n & \longrightarrow & C^q_n & \longrightarrow & Im^q_n & \longrightarrow & \overline{FC}^q_{n-1} \longrightarrow \end{array} \qquad (1)$$

and

$$\begin{array}{ccccccccc} \to \pi_{n+1}(SG, SG_q) & \to & \tilde{\pi}_{n+1}(SG; SO, SG_q) & \to & \pi_n(SO) & \to & \pi_n(SG, SG_q) \to \\ \| & & \| & & \| & & \| \\ \longrightarrow \overline{FC}^q_n & \longrightarrow & FC^q_n & \longrightarrow & FIm^q_n & \longrightarrow & \overline{FC}^q_{n-1} \longrightarrow . \end{array} \qquad (2)$$

Note that the upper sequences in (1) and (2) are the long exact sequences of the homotopy groups of pairs $(SG/SG_q, SO/SO_q)$ and $(SG/SG_q, SO)$, respectively, see Remarks 2.3 and 2.5.

1.2. Outline of the paper

The paper is organized as follows: we give a quick review of Haefliger's result [**13**] for (framed) spherical embeddings $S^n \hookrightarrow S^{n+q}$ in Section 2. In Section 3, we define the group $\overline{FC}^q_n$ and show that $\overline{FC}^q_n = \pi_{n+1}(SG, SG_q)$. We prove Theorems 1.1 and 1.2 in Section 4 and Section 5, respectively. We recall some computations and prove a few applications of Theorem 1.1 in Section 6. Throughout the paper we work in the smooth category and assume $q \geqslant 3$.

1.3. Terminology

Let D^n be the standard unit disk in $\mathbb{R}^n$, and $\{e_1 \ldots, e_n\}$ denote the natural basis of $\mathbb{R}^n$. Let $S^n = \partial D^{n+1}$ be the unit sphere such that

$$S^n = D^n_- \cup D^n_+ \text{ with } D^n_- = \{x \in S^n | x_1 \leqslant 0\} \text{ and } D^n_+ = \{x \in S^n | x_1 \geqslant 0\}.$$

According to Haefliger [**13**], the *suspension* of a map $f\colon D^n \to D^n$ is given by the map $S(f)\colon D^{n+1} \to D^{n+1}$ sending the arc of circle going from e_{n+1}, by $x \in D^n$, to $-e_{n+1}$ on the arc of circle from e_{n+1}, by $f(x)$, to $-e_{n+1}$. The suspension $S^{n+1} \to S^{n+1}$ of a map $S^n \to S^n$ is defined in the same way.

Abusing terminology, the suspension of an embedding $S^n \overset{f}{\hookrightarrow} S^{n+q}$ is the composition $S^n \overset{f}{\hookrightarrow} S^{n+q} \subset S^{n+q+1}$. For the suspension of a framed embedding $S^n \hookrightarrow S^{n+q}$, the framing is completed by adding the standard vector e_{n+q+2} as the last vector. We often say suspension for an iterated suspension defined inductively. For example, when we say $S^n \hookrightarrow S^{n+N}$ is the suspension of a framed embedding $S^n \overset{f}{\hookrightarrow} S^{n+q}$ for $N > q$, we mean that it is defined as the composition $S^n \overset{f}{\hookrightarrow} S^{n+q} \subset S^{n+N}$ and the framing is obtained by adding vectors $\{e_{n+q+2}, \ldots, e_{n+N+1}\}$ to the initial framing. We define the suspension of a (framed) disk embedding similarly.

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2. Embeddings of S^n in S^{n+q}

Haefliger [13] proved that the group of concordance classes of embeddings of S^n in S^{n+q} is isomorphic to $\pi_{n+1}(SG; SO, SG_q)$ for $q \geqslant 3$.

2.1. The group C_n^q

$$C_n^q := \{\text{concordance classes of smooth embeddings } S^n \hookrightarrow S^{n+q}\}.$$

Theorem 2.1 ([13, Theorem 1.2]). *Two concordant embeddings of S^n in S^{n+q} are isotopic when $q \geqslant 3$, i.e. $C_n^q = \pi_0 Emb(S^n, S^{n+q})$, the set of connected components of the space of embeddings of S^n in S^{n+q}.*

Furthermore, the equality $\pi_0 Emb(S^n, S^{n+q}) = \pi_0 Emb_\partial(D^n, D^{n+q})$ equips C_n^q with an additive multiplication, and the existence of inverses is guaranteed, as we consider concordance classes. Hence, C_n^q is an abelian group.

Lemma 2.2 ([13, Section 1]). *An embedding $S^n \hookrightarrow S^{n+q}$ is concordant to the trivial one if and only if it is slice, in other words, if it can be extended to an embedding $D^{n+1} \hookrightarrow D^{n+q+1}$.*

2.2. The group $\pi_{n+1}(SG; SO, SG_q)$

Let SG_q be the space of degree one maps $S^{q-1} \to S^{q-1}$, $SG = \cup SG_q$ under suspension, and $SO = \cup SO_q$, where SO_q is the special orthogonal group.

An element in $\pi_{n+1}(SG; SO, SG_q)$ is represented by a continuous based map $\phi\colon D^{n+1} \to SG$ i.e., for $x \in D^{n+1}$, $\phi(x)\colon S^{N-1} \to S^{N-1}$, for some large N, such that $\phi(D_-^n) \subset SO_N$ and $\phi(D_+^n) \subset SG_q$. Note that the equator $S^{n-1} = \partial D_-^n = \partial D_+^n$ goes to $SO \cap SG_q = SO_q$, and $\phi(*) = id$ for the base-point $* = e_2 \in S^{n-1}$.[1] Abusing notation, we also view ϕ as a map $\phi\colon D^{n+1} \times S^{N-1} \to S^{N-1}$, and sometimes for $\phi(x)$ we write $\phi_x = \phi(x, -)\colon S^{N-1} \to S^{N-1}$.

Two such maps $\phi\colon D^{n+1} \times S^{N-1} \to S^{N-1}$ and $\phi'\colon D^{n+1} \times S^{N'-1} \to S^{N'-1}$ represent the same element in $\pi_{n+1}(SG; SO, SG_q)$ if there exists a choice of homotopy $\phi_t\colon D^{n+1} \times S^{M-1} \to S^{M-1}$ for some $M \geqslant N, N'$ and $t \in [0,1]$, satisfying the above conditions and such that for any $x \in D^{n+1}$, the maps

$$\phi_0(x, -), \phi_1(x, -)\colon S^{M-1} \to S^{M-1}$$

are suspensions of

$$\phi(x, -) \text{ and } \phi'(x, -),$$

respectively. The product operation of any two elements in $\pi_{n+1}(SG; SO, SG_q)$ is defined point-wise.

[1] Haefliger in [13] does not consider the base-point condition, but it is immediate that adding it yields the same homotopy group, since $SO_q = SO \cap SG_q$ is connected.

Remark 2.3. Recall that the upper long exact sequence in (1) is the long exact sequence of the pair $(SG/SG_q, SO/SO_q)$. Indeed, Milgram [**18**, Section 1] interpreted the group $\pi_{n+1}(SG; SO, SG_q)$ as

$$\pi_n\Big(\operatorname{hofib}(SO/SO_q \to SG/SG_q) \simeq \operatorname{hofib}(SG_q/SO_q \to SG/SO)\Big).^2$$

One way to see this interpretation is that the group $\pi_{n+1}(SG; SO, SG_q)$ is obviously isomorphic to $\pi_n(SO \times^h_{SG} SG_q, SO_q)$, where $SO \times^h_{SG} SG_q$ is the homotopy pullback of $SO \to SG \leftarrow SG_q$.

2.3. The isomorphism $\psi\colon C^q_n \to \pi_{n+1}(SG; SO, SG_q)$

Although these two groups look completely different, there is a natural map between them. To see the relation, Haefliger considers representatives in C^q_n to be framed embeddings of D^{n+1} with different boundary conditions on $D^n_- \subset S^n = \partial D^{n+1}$ and $D^n_+ \subset S^n = \partial D^{n+1}$. Framing will be crucial to relate such embeddings to our target $\pi_{n+1}(SG; SO, SG_q)$ by means of Pontryagin–Thom type construction [**13**, Section 3].

An embedding $f\colon S^n \hookrightarrow S^{n+q}$ is called a ***special embedding*** if $f|_{D^n_-} = id$ and $f(\text{int } D^n_+) \subset \text{int } D^{n+q}_+$. We can always extend $f\colon S^n \hookrightarrow S^{n+q}$ to a disk embedding $\bar f\colon D^{n+1} \hookrightarrow D^{n+N+1}$, for some N large enough (in fact $N > n+2$). We refer the obtained pair $(f, \bar f)\colon (S^n, D^{n+1}) \hookrightarrow (S^{n+q}, D^{n+N+1})$ as a ***disked embedding***. Any element in C^q_n can be represented by a special disked embedding $(f, \bar f)$ together with some framing on $\bar f$ defined as follows:

- Fix the base-point $* = e_2 \in S^{n-1} = D^n_- \cap D^n_+$ and endow it with the framing $\{e_{n+2}, \ldots, e_{n+q+1}\}$.
- Extend the framing from $* = e_2$ to D^n_+ inside D^{n+q}_+. Since $* \hookrightarrow D^n_+$ is a homotopy equivalence, this extension is unique up to homotopy. Take the suspension of this framing in D^{n+N+1} by adding $\{e_{n+q+2}, \ldots, e_{n+N+1}\}$ as last vectors.
- Extend the obtained framing from D^n_+ to the entire disk D^{n+1} inside D^{n+N+1}. Again this framing is defined uniquely up to homotopy.

Note that even though the knot f is trivial on D^n_-, the extended framing can be non-trivial. Moreover, the framing on $\bar f|_{D^n_-}$ inside D^{n+N+1} might not be a suspension, while the framing on $\bar f|_{D^n_+}$ inside D^{n+N+1} is the suspension of a framing inside D^{n+q}_+. We refer this boundary condition on the framing defined on $\bar f$ as ***Type I*** (in Sections 2.4 and 3, we will also consider framing with Type II and Type III boundary conditions). Hence, any embedding $f\colon S^n \hookrightarrow S^{n+q}$ representing an element in C^q_n can be considered as a ***special disked embedding*** $(f, \bar f)\colon (S^n, D^{n+1}) \hookrightarrow (S^{n+q}, D^{n+N+1})$ ***with Type I framing***, i.e., $\bar f|_{S^n = \partial D^{n+1}} = f$ is a special knot, and the framing on $\bar f$ has boundary condition defined as above.

[2] The spaces are equivalent because they describe the total homotopy fiber of the square

$$\begin{array}{ccc} BSO_q & \longrightarrow & BSG_q \\ \big\downarrow & & \big\downarrow \\ BSO & \longrightarrow & BSG\,. \end{array}$$

Any two special disked embeddings $(f_0, \bar{f}_0)\colon (S^n, D^{n+1}) \hookrightarrow (S^{n+q}, D^{n+N_0+1})$ and $(f_1, \bar{f}_1)\colon (S^n, D^{n+1}) \hookrightarrow (S^{n+q}, D^{n+N_1+1})$ with Type I framing are *concordant* if there exists an embedding $F\colon D^{n+1} \times [0,1] \hookrightarrow D^{n+N+1} \times [0,1]$ for $N \geqslant max\{N_0, N_1\}$, such that $F|_{D^{n+1}\times i} = \bar{f}_i$ for $i = 0, 1$, $F|_{D^n_- \times [0,1]} = id$ and $F|_{D^n_+ \times [0,1]} \subset D^{n+q}_+ \times [0,1]$. Furthermore, the framing on $F|_{D^n_+ \times [0,1]}$ and $F|_{D^{n+1}\times i}$, $i = 0, 1$, is given by suspension of a framing inside $D^{n+q}_+ \times [0,1]$ and $D^{n+N_i+1} \times i$, respectively. Similarly, we define the isotopy relation to be a level-preserving concordance. All the following groups are isomorphic for $q \geqslant 3$:

$$\begin{gathered}
C^q_n = \{\text{concordance/isotopy classes of embeddings } S^n \hookrightarrow S^{n+q}\} \\
\updownarrow \\
\{\text{concordance/isotopy classes of special embeddings } S^n \hookrightarrow S^{n+q}\} \\
\updownarrow \\
\{\text{concordance/isotopy classes of special disked embeddings} \\
(S^n, D^{n+1}) \hookrightarrow (S^{n+q}, D^{n+N+1})\} \\
\updownarrow \\
\{\text{concordance/isotopy classes of special disked embeddings} \\
(S^n, D^{n+1}) \hookrightarrow (S^{n+q}, D^{n+N+1}) \text{ with Type I framing}\}.
\end{gathered}$$

Furthermore, as a consequence of the tubular neighborhood theorem, one can choose a representative f in C^q_n such that $f(S^n)$ is contained in a subspace of S^{n+q} which can be identified with $S^n \times D^q$. Thus, we can consider a special knot to be $f\colon S^n \hookrightarrow S^n \times D^q$ such that $f|_{D^n_-}$ is the natural inclusion $D^n_- \hookrightarrow D^n_- \times 0$ and $f(\text{int}\, D^n_+) \subset \text{int}(D^n_+ \times D^q)$, together with a disk extension $\bar{f}\colon D^{n+1} \hookrightarrow D^{n+1} \times\ D^N$ with a similarly defined framing of Type I. The homomorphism

$$\psi\colon C^q_n \to \pi_{n+1}(SG; SO, SG_q)$$

is then defined as follows.

Theorem 2.4. *Given an element $\alpha \in C^q_n$ represented by a special disked embedding $(f, \bar{f})\colon (S^n, D^{n+1}) \hookrightarrow (S^n \times D^q, D^{n+1} \times D^N)$ with Type I framing, a specified map $\phi\colon D^{n+1} \times S^{N-1} \to S^{N-1}$ represents $\psi(\alpha) \in \pi_{n+1}(SG; SO, SG_q)$ if there exists an extension $\bar{\phi}\colon D^{n+1} \times D^N \to D^N$ i.e., $\bar{\phi}|_{D^{n+1}\times S^{N-1}} = \phi$ such that:*

(i) $\bar{\phi}$ is regular on $0 \in D^N$ and $\bar{\phi}^{-1}(0) = \bar{f}(D^{n+1})$ as framed submanifolds,

(ii) $\bar{\phi}_x \in SO_N$ for $x \in D^n_-$,

(iii) $\bar{\phi}_x$ is the suspension of a map $D^q \to D^q$ for $x \in D^n_+$.

*The homomorphism $\psi\colon C^q_n \to \pi_{n+1}(SG; SO, SG_q)$ is well defined [**13**, Theorem 2.3] and is an isomorphism for $q \geqslant 3$ [**13**, Theorem 3.4].*

In the proof of well-definedness of ψ [**13**, Theorem 2.3], Haefliger shows the existence of such a map $\bar{\phi}$ as follows. Define $\bar{\phi}_-\colon D^n_- \times D^N \to D^N$ uniquely as a linear map such that $(\bar{\phi}_-)_x \in SO_N$ for $x \in D^n_-$ and $\bar{\phi}^{-1}_-(0) = f(D^n_-)$, as framed submanifolds. Using obstruction theory [**13**, Lemma 2.4] the restriction $\bar{\phi}_-|_{S^{n-1}\times D^q}$ can be extended to $\bar{\phi}_-|_{D^n_+\times D^q}$ with the given framing on $f(D^n_+)$. Then we define $\bar{\phi}_+\colon D^n_+ \times D^N \to D^N$ to be the $(N-q)$-suspension of $\bar{\phi}_-\colon D^n_+ \times D^q \to D^q$. By using

[**13**, Lemma 2.4] again, we extend

$$\bar{\phi}_- \cup \bar{\phi}_+ \colon S^n \times D^N \to D^N \quad \text{to a map} \quad \bar{\phi} \colon D^{n+1} \times D^N \to D^N$$

verifying $(i) - (iii)$ above. To show ψ is well defined, he uses the same argument invoking [**13**, Lemma 2.4] twice in order to construct a homotopy between two maps $\phi_0, \phi_1 \colon D^{n+1} \times S^{N-1} \to S^{N-1}$ corresponding to two concordant embeddings

$$(f_0, \bar{f}_0), (f_1, \bar{f}_1) \colon (S^n, D^{n+1}) \hookrightarrow (S^n \times D^q, D^{n+1} \times D^N).$$

To prove the isomorphism [**13**, Theorem 3.4], Haefliger interprets the target group $\pi_{n+1}(SG; SO, SG_q)$ in terms of cobordisms (we refer to this as a Pontryagin–Thom type construction). An element of $\pi_{n+1}(SG; SO, SG_q)$ represented by a map as in Subsection 2.2, $\phi \colon D^{n+1} \times S^{N-1} \to S^{N-1}$, which is regular on e_1, corresponds to a framed $(n+1)$-submanifold $V = \phi^{-1}(e_1) \subset D^{n+1} \times S^{N-1}$ with two parts of boundary:

- $V \cap (D^n_- \times S^{N-1})$ is the graph of some map $g \colon D^n_- \to S^{N-1}$ with the framing at points $(x, g(x))$ lying inside $x \times S^{N-1}$ and orthonormal. Indeed, for $x \in D^n_-$, the map $\phi_x \colon S^{N-1} \to S^{N-1}$ is linear and therefore the preimage of e_1 is just a point.
- $V \cap (D^n_+ \times S^{N-1})$ is the suspension of a framed submanifold in $D^n_+ \times S^{q-1}$, since for any $x \in D^n_+$, the map $\phi_x \colon S^{N-1} \to S^{N-1}$ is the suspension of a map $S^{q-1} \to S^{q-1}$.

Thus, $\pi_{n+1}(SG; SO, SG_q)$ can be described as the group of cobordisms of framed $(n+1)$-manifolds with such boundary conditions.

He then considers $\bar{\phi}$ which exists by [**13**, Theorem 2.3]. Note that one can always slightly change $\bar{\phi} \colon D^{n+1} \times D^N \to D^N$ so that $\bar{\phi}^{-1}(\partial D^N) \subset D^{n+1} \times \partial D^N$. The preimage $\bar{\phi}^{-1}(I) \subset D^{n+1} \times D^N$ of the segment I joining 0 and e_1 within D^N is a framed $(n+2)$-manifold W with corners ($\bar{\phi}$ is chosen to be transversal to I). In particular, ∂W has the following strata:

- a free face given by the framed disk $\bar{f}(D^{n+1}) = \bar{\phi}^{-1}(0)$,
- $\partial W \cap (D^{n+1} \times S^{N-1}) = \bar{\phi}^{-1}(e_1) = V$,
- $\partial W \cap (D^n_- \times D^N)$ is the radial extension of $V \cap (D^n_- \times S^{N-1})$,
- $\partial W \cap (D^n_+ \times D^N)$ is the $(N-q)$-fold suspension of a framed submanifold in $D^n_+ \times D^q$.

As a result, he restates the homomorphism defined in Theorem 2.4 as follows. Given an element $\alpha \in C^q_n$ represented by a special disked embedding

$$(f, \bar{f}) \colon (S^n, D^{n+1}) \hookrightarrow (S^n \times D^q, D^{n+1} \times D^N)$$

with Type I framing, a framed submanifold $V \subset D^{n+1} \times\ S^{N-1}$ as defined above represents $\psi(\alpha) \in \pi_{n+1}(SG; SO, SG_q)$ if there exists a choice of framed submanifold $W \subset D^{n+1} \times D^N$ with the boundary strata as given above.

According to [**13**, Argument 3.5], to show surjectivity he applies surgery to construct W satisfying $\partial W \cap (D^{n+1} \times S^{N-1}) = V$ for a given V. For injectivity, he shows if $[(f, \bar{f})]$ maps to the trivial element $[V]$ of $\pi_{n+1}(SG; SO, SG_q)$, then the corresponding W can be modified using surgery so that it is embedded in $D^{n+1} \times D^q$. In particular, the free face $\bar{f}(D^{n+1})$ of W is inside $D^{n+1} \times D^q \cong D^{n+q+1}$, and therefore the corresponding $f = \bar{f}|_{\partial D^{n+1}}$ is slice i.e., concordant to the trivial embedding of S^n in S^{n+q}, by Lemma 2.2.

2.4. Framed embeddings of S^n in S^{n+q}

Let us recall that we always consider framed embeddings with a framing preserving the natural orientation. For $q \geqslant 3$, Haefliger expressed the group FC_n^q of concordance classes of framed embeddings of S^n in S^{n+q} as $\tilde{\pi}_{n+1}(SG; SO, SG_q)$. An element in $\tilde{\pi}_{n+1}(SG; SO, SG_q)$ is represented by a continuous map $\phi\colon D^{n+1} \to SG$ i.e., for $x \in D^{n+1}$, $\phi(x)\colon S^{N-1} \to S^{N-1}$, for some large N, such that $\phi(D^n_-) \subset SO$, $\phi(D^n_+) \subset SG_q$ and $\phi(\partial D^n_- = \partial D^n_+) = id$. Again, abusing notation we also view ϕ as a map $\phi\colon D^{n+1} \times S^{N-1} \to S^{N-1}$ and sometimes write ϕ_x for $\phi(x)$.

Remark 2.5. It is easy to see that the group $\tilde{\pi}_{n+1}(SG; SO, SG_q)$ is isomorphic to $\pi_n\Big((SO \times^h_{SG} SG_q) \simeq \mathrm{hofib}(SO \to SG/SG_q)\Big)$. Moreover, the upper long exact sequence in (2) is the long exact sequence of the pair $(SG/SG_q, SO)$.

Remark 2.6 ([**13**, Section 5.1]). Two concordant framed embeddings of S^n in S^{n+q} are isotopic when $q \geqslant 3$ and therefore, $FC_n^q = \pi_0 Emb^{fr}(S^n, S^{n+q}) = \pi_0 Emb^{fr}_\partial(D^n, D^{n+q})$.

Lemma 2.7 ([**13**, Section 5])**.** *A framed embedding $S^n \hookrightarrow S^{n+q}$ is concordant to the trivial one if and only if it is slice i.e., if it can be extended along with the framing to an embedding $D^{n+1} \hookrightarrow D^{n+q+1}$.*

2.4.1. The isomorphism $\tilde{\psi}\colon FC_n^q \to \tilde{\pi}_{n+1}(SG; SO, SG_q)$

The natural map $\tilde{\psi}$ between the two groups is defined as in the "non-framed" case. Firstly, an element in FC_n^q can be represented by a special framed knot $f\colon S^n \hookrightarrow S^{n+q}$ which is the natural inclusion on D^n_- with trivial framing $\{e_{n+2}, \ldots, e_{n+q+1}\}$, and $f(\mathrm{int}\, D^n_+) \subset \mathrm{int}(D^{n+q}_+)$ with some non-trivial framing. Such a framed knot is assigned a special disked embedding $(f, \bar{f})\colon (S^n, D^{n+1}) \hookrightarrow (S^{n+q}, D^{n+N+1})$ along with a framing as follows. We extend $f\colon S^n \hookrightarrow S^{n+q}$ to a disk embedding $\bar{f}\colon D^{n+1} \hookrightarrow D^{n+N+1}$ for N large enough. For the framing on $\bar{f}(D^{n+1})$, which is defined uniquely up to homotopy, we first suspend the framing on D^n_+ inside D^{n+q}_+ to a framing inside D^{n+N+1} by adding vectors $\{e_{n+q+2}, \ldots, e_{n+N+1}\}$. Then we extend the obtained framing to the entire disk D^{n+1} inside D^{n+N+1}. Note that the framing on $\bar{f}|_{D^n_-}$ may now be non-trivial (and does not have to be a suspension), while the framing on $\bar{f}|_{D^n_+}$ is the suspension of the framing on D^n_+ inside D^{n+q}_+. But we still obtain a trivial framing on the equator $S^{n-1} = D^n_- \cap D^n_+$. Such boundary condition on the framing defined on $\bar{f}$ is referred as ***Type II***. Therefore, a representative in FC_n^q can be considered to be a ***special disked embedding*** $(f, \bar{f})$ ***with Type II framing***. For $q \geqslant 3$, the following groups are isomorphic:

$$
\begin{gathered}
FC_n^q = \{\text{concordance/isotopy classes of framed embeddings } S^n \hookrightarrow S^{n+q}\} \\
\updownarrow \\
\{\text{concordance/isotopy classes of special framed embeddings } S^n \hookrightarrow S^{n+q}\} \\
\updownarrow \\
\{\text{concordance/isotopy classes of special disked embeddings} \\
(S^n, D^{n+1}) \hookrightarrow (S^{n+q}, D^{n+N+1}) \text{ with Type II framing}\}.
\end{gathered}
$$

Using the tubular neighborhood theorem, we can transform any special framed knot $f\colon S^n \hookrightarrow S^{n+q}$ into $f\colon S^n \hookrightarrow S^n \times D^q$, equipped with a framed disk extension $\bar{f}\colon D^{n+1} \hookrightarrow D^{n+1} \times D^N$. Thus, an element in FC_n^q can be represented by a pair $(f, \bar{f})\colon (S^n, D^{n+1}) \hookrightarrow (S^n \times D^q, D^{n+1} \times D^N)$ with a framing of Type II. We define the homomorphism $\tilde{\psi}\colon FC_n^q \to \tilde{\pi}_{n+1}(SG; SO, SG_q)$ exactly as in Theorem 2.4 by adding to condition (ii) that $\bar{\phi}_x = id$ when $x \in S^{n-1}$. For $q \geqslant 3$, $\tilde{\psi}$ is an isomorphism [**13**, Theorem 5.7]. This result is stated without proof because the argument follows the same lines as in the "non-framed" case. Note that Lemma 2.7 is used in the proof of injectivity of $\tilde{\psi}$ in the same way as Lemma 2.2 is necessary for injectivity of ψ.

3. Framed disked embeddings

We now define a new group of concordance classes of special disked embeddings with a ***Type III*** framing. Namely, this time we require the framing to be trivial along D^n_-. To be precise, we consider special disked embeddings of the form $(f, \bar{f})\colon (S^n, D^{n+1}) \hookrightarrow (S^{n+q}, D^{n+N+1})$ where the framing on $\bar{f}$ comes with the following boundary condition: $\bar{f}|_{D^n_-}$ has trivial framing, while the framing on $\bar{f}|_{D^n_+}$ inside D^{n+N+1} is obtained as the suspension of a framing inside D^{n+q}_+.

$$\overline{FC}_n^q := \{\text{concordance classes of special disked embeddings}$$
$$(f, \bar{f})\colon (S^n, D^{n+1}) \hookrightarrow (S^{n+q}, D^{n+N+1}) \text{ with Type III framing}\}.$$

Note that since the codimension condition $q \geqslant 3$ is satisfied, concordance and isotopy relations coincide for special disked embeddings with all three boundary restrictions on framing.

3.1. The group $\pi_{n+1}(SG, SG_q)$

An element in $\pi_{n+1}(SG, SG_q)$ is represented by a continuous map $\phi\colon D^{n+1} \to SG$ such that $\phi|_{D^n_-} = id$ and $\phi(D^n_+) \subset SG_q$.

This representation is equivalent to the usual definition of a relative homotopy group i.e., $\pi_{n+1}(SG; *, SG_q) = \pi_{n+1}(SG, SG_q)$, since D^n_- can be collapsed to get the base-point in the relative group.

3.2. The isomorphism $\xi\colon \overline{FC}_n^q \to \pi_{n+1}(SG, SG_q)$

Following the same argument as in Subsection 2.3, when an element in $\overline{FC}_n^q$ is represented by a special disked embedding $(f, \bar{f})\colon (S^n, D^{n+1}) \hookrightarrow (S^n \times D^q, D^{n+1} \times D^N)$ with Type III framing, there is a natural homomorphism $\xi\colon \overline{FC}_n^q \to \pi_{n+1}(SG, SG_q)$ defined as in Theorem 2.4 by replacing condition *(ii)* with $\bar{\phi}_x = id$ for $x \in D^n_-$. By Haefliger's surgery construction [**13**, Argument 3.5] that proves [**13**, Theorem 3.4], we conclude:

Theorem 3.1. *The homomorphism $\xi\colon \overline{FC}_n^q \to \pi_{n+1}(SG, SG_q)$ is an isomorphism for $q \geqslant 3$.*

The sliceness Lemma 3.4 is used to prove injectivity of ξ, similarly to the cases of ψ and $\tilde{\psi}$. Note that Theorem 3.1 can be deduced from the proof of Theorem 1.2 given in Section 5. In particular, with ψ and η as isomorphisms in (5), ξ is also an isomorphism by the five lemma.

As a review, the following tables point to the main difference among all the groups we discussed in the three cases. In terms of special disked embeddings with different boundary conditions on framing of $\bar{f}$:

C_n^q	trivial framing at the base-point $*$ (Type I)
FC_n^q	trivial framing at the equator S^{n-1} (Type II)
$\overline{FC}_n^q$	trivial framing at D_-^n (Type III)

The corresponding homotopy groups differ as follows:

$\pi_{n+1}(SG;SO,SG_q)$	$\phi(*)=id$
$\tilde{\pi}_{n+1}(SG;SO,SG_q)$	$\phi(S^{n-1})=id$
$\pi_{n+1}(SG,SG_q)$	$\phi(D_-^n)=id$

Remark-Definition 3.2. Consider a disked embedding

$$(f,\bar{f})\colon (S^n,D^{n+1})\hookrightarrow (S^{n+q},D^{n+N+1})$$

which is not necessarily special, i.e., without a fixed behavior at D_-^n. Assume both f and $\bar{f}$ are framed embeddings such that framing on $\bar{f}(D^{n+1})$ inside D^{n+N+1} is defined by extending the suspension of the framing of $f(S^n)\subset S^{n+q}$. We call such a pair $(f,\bar{f})$ a ***framed disked embedding***. The concordance classes of such embeddings are the same as those of special ones with Type III framing representing elements in $\overline{FC}_n^q$. It is because given any framed disked embedding, we can always isotope it, so that near the base-point $*\in\partial D_-^n=S^{n-1}$ it is the identity inclusion with the trivial framing. Then we can reparametrize the sphere so that the small neighborhood of $*$ is D_-^n and the rest is D_+^n. As a result, we get a special disked embedding $(f,\bar{f})$ with Type III framing. Therefore, we can describe $\overline{FC}_n^q$ as the group of concordance classes of framed disked embeddings $(f,\bar{f})$.

Thus, all the groups C_n^q, FC_n^q and $\overline{FC}_n^q$ can be described as groups of concordance classes of "non-special" embeddings:

C_n^q	embeddings $S^n\hookrightarrow S^{n+q}$
FC_n^q	framed embeddings $S^n\hookrightarrow S^{n+q}$
$\overline{FC}_n^q$	framed disked embeddings $(S^n,D^{n+1})\hookrightarrow(S^{n+q},D^{n+N+1})$

Note that special disked embeddings with framing of Type I or Type II are not framed disked embeddings because for latter we require the framing on $\bar{f}$ to be the suspension on entire boundary $\partial D^{n+1}=S^n$, see the definition above.

3.3. Sliceness

In this subsection, we study an interesting property of sliceness for framed disked embeddings representing elements in the group $\overline{FC}_n^q$.

Definition 3.3. A framed disked embedding $(f,\alpha)\colon (S^n,D^{n+1})\hookrightarrow(S^{n+q},D^{n+N+1})$ is ***slice*** if there exists a framed embedding $H\colon D^{n+2}\hookrightarrow D^{n+N'+2}$ where $N'\geqslant N$, such that $H|_{(\partial_- D^{n+2}=D_-^{n+1})}=\alpha$ and $H|_{\partial_+ D^{n+2}}$ is the suspension of a framed embedding inside D^{n+q+1} i.e., $H(\partial_+D^{n+2})\subset D^{n+q+1}\subset\partial_+D^{n+N'+2}=D^{n+N+1}$.

The trivial element in $\overline{FC}_n^q$ is given by the equivalence class of the trivial framed disked embedding $(id, id)\colon (S^n, D^{n+1}) \subset (S^{n+q}, D^{n+N+1})$ i.e., the trivial pair with the trivial framing.

Lemma 3.4. *A framed disked embedding $(f, \alpha)\colon (S^n, D^{n+1}) \hookrightarrow (S^{n+q}, D^{n+N+1})$ representing an element in $\overline{FC}_n^q$ is concordant to the trivial element*

$$(id, id)\colon (S^n, D^{n+1}) \subset (S^{n+q}, D^{n+N+1}),$$

if and only if (f, α) is slice.

Proof. Let $F\colon D^{n+1} \times [0,1] \hookrightarrow D^{n+N'+1} \times [0,1]$, where $N' \geqslant N$ be a concordance between (f, α) and (id, id). Since at $t = 1$ we have a trivial framing, we attach a half disk $\frac{1}{2}D^{n+N'+2}$ along the trivial embedding such that it extends D^{n+1} to the disk D^{n+2}, see Figure 1. Since F takes the boundary inside $S^{n+q} \times [0,1]$, therefore attaching this half disk gives the sliceness of the framed knot $S^n \hookrightarrow S^{n+q}$ i.e., a framed extension $D^{n+1} \hookrightarrow D^{n+q+1}$. As a consequence, we get a framed embedding $H\colon D^{n+2} \hookrightarrow D^{n+N'+2}$, which on one part of ∂D^{n+2} gives α and on the other, an embedding to D^{n+q+1}. Therefore, (f, α) is slice.

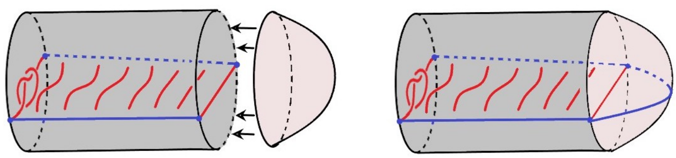

Figure 1: Attaching a half disk $D^{n+N'+2}$ to $D^{n+N'+1}$ at $t = 1$.

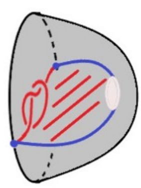

Figure 2: Removing a small half disk $D^{n+N'+2}$ going inside.

The converse is easy to prove by reversing the above argument. We remove a small half disk $\frac{1}{2}D^{n+N'+2}$ around a point in $D^{n+q+1} \subset D^{n+N'+2}$, see Figure 2, such that the resulting space acts as a concordance between (f, α) and (id, id). $\square$

4. Embeddings modulo immersions as special disked embeddings

Let $D^{n+\infty} := \cup_N D^{n+N}$. By a smooth embedding $D^n \hookrightarrow D^{n+\infty}$, we understand $D^n \hookrightarrow D^{n+N}$ for some N large enough. Set

$$Emb_\partial^{fr}(D^n, D^{n+\infty}) := \bigcup_N Emb_\partial^{fr}(D^n, D^{n+N}),$$

$$Imm^{fr}_\partial(D^n, D^{n+\infty}) := \bigcup_N Imm^{fr}_\partial(D^n, D^{n+N}).$$

Similarly, we consider SDE^q_n to be the space of special disked embeddings $(f,\alpha)\colon (S^n, D^{n+1}) \hookrightarrow (S^{n+q}, D^{n+1+\infty})$ with Type III framing. By construction, $\pi_0 SDE^q_n = \overline{FC}^q_n$.

We claim that the space SDE^q_n has the same set of connected components as the space of embeddings modulo immersions i.e.,$\overline{FC}^q_n = \pi_0\overline{Emb}_\partial(D^n, D^{n+q})$. First we prove the following lemma which gives different geometric interpretations of the group $\pi_0\overline{Emb}_\partial(D^n, D^{n+q})$.

Lemma 4.1. *For $q \geqslant 3$,*

$$\begin{aligned}\pi_0\overline{Emb}(S^n, S^{n+q}) &= \pi_0\overline{Emb}_\partial(D^n, D^{n+q}) = \pi_0\overline{Emb}(S^n, \mathbb{R}^{n+q}) \\ &= \pi_0\overline{Emb}^{fr}(S^n, S^{n+q}) = \pi_0\overline{Emb}^{fr}_\partial(D^n, D^{n+q}) = \pi_0\overline{Emb}^{fr}(S^n, \mathbb{R}^{n+q}).\end{aligned} \tag{3}$$

Proof. Let us first prove the "non-framed" case

$$\pi_0\overline{Emb}(S^n, S^{n+q}) = \pi_0\overline{Emb}(S^n, \mathbb{R}^{n+q}).$$

Consider the following diagram, where the horizontal lines are fiber sequences:

$$\begin{array}{ccccc}\overline{Emb}(S^n,\mathbb{R}^{n+q}) & \longrightarrow & Emb(S^n,\mathbb{R}^{n+q}) & \longrightarrow & Imm(S^n,\mathbb{R}^{n+q}) \\ \downarrow & & \downarrow & & \downarrow \\ \overline{Emb}(S^n,S^{n+q}) & \longrightarrow & Emb(S^n,S^{n+q}) & \longrightarrow & Imm(S^n,S^{n+q}).\end{array}$$

We view $\mathbb{R}^{n+q} = S^{n+q} - \infty$, where the "infinity" point is of codimension $n+q$. Given an embedding (resp. immersion) from S^n to S^{n+q}, we can perturb it slightly in a way that it misses the "infinity" point as $q \geqslant 3$, so that we get an embedding (resp. immersion) from S^n to $\mathbb{R}^{n+q}$. Therefore, the second (resp. third) vertical map is surjective on the level of π_0. For injectivity, note that an isotopy (resp. regular homotopy) of an embedding (resp. immersion) of S^n in S^{n+q} is $(n+1)$-dimensional, while the "infinity" point has codimension $n+q$, so it can still miss the point given $q \geqslant 3$, and therefore the second (resp. third) vertical map is bijective on π_0. The same argument holds for π_1 because $q \geqslant 3$. Therefore, the second and third vertical maps induce isomorphisms on π_0 and π_1 when $q \geqslant 3$. By five lemma, we get

$$\pi_0\overline{Emb}(S^n, S^{n+q}) = \pi_0\overline{Emb}(S^n, \mathbb{R}^{n+q}).$$

It is proved in [**24**, Theorem 1.1] that

$$\pi_0\overline{Emb}(S^n, \mathbb{R}^{n+q}) = \pi_0\overline{Emb}_\partial(D^n, D^{n+q}).$$

Using the argument from [**23**, Proposition 1.2], we have that the following natural projections are weak equivalences:

$$\overline{Emb}^{fr}_\partial(D^n, D^{n+q}) \to \overline{Emb}_\partial(D^n, D^{n+q}),$$

$$\overline{Emb}^{fr}(S^n, \mathbb{R}^{n+q}) \to \overline{Emb}(S^n, \mathbb{R}^{n+q}).$$

Similarly, one can show that $\overline{Emb}^{fr}(S^n, S^{n+q}) \to \overline{Emb}(S^n, S^{n+q})$ is a weak equivalence. Thus, we get different representations for $\pi_0\overline{Emb}_\partial(D^n, D^{n+q})$ as in (3). □

By Smale–Hirsch theory [**15**, **22**], we have $Imm_\partial^{fr}(D^n, D^{n+q}) \simeq \Omega^n SO(n+q)$, and since we consider the ambient dimension to tend to infinity, we obtain $Imm_\partial^{fr}(D^n, D^{n+\infty}) \simeq \Omega^n SO$. Note that $Emb_\partial^{fr}(D^n, D^{n+N})$ is an open, dense subset of $Imm_\partial^{fr}(D^n, D^{n+N})$ of codimension $N-n$. As a consequence, as N gets large, the inclusion $Emb_\partial^{fr}(D^n, D^{n+N}) \hookrightarrow Imm_\partial^{fr}(D^n, D^{n+N})$ becomes highly connected, and we get $Emb_\partial^{fr}(D^n, D^{n+\infty}) \simeq Imm_\partial^{fr}(D^n, D^{n+\infty}) \simeq \Omega^n SO$.

Lemma 4.2. *For $q \geqslant 3$,*

$$\pi_0 \overline{Emb}_\partial^{fr}(D^n, D^{n+q}) = \pi_0 hofib\Big(Emb_\partial^{fr}(D^n, D^{n+q}) \to Emb_\partial^{fr}(D^n, D^{n+\infty})\Big).$$

Proof. By definition, $\pi_0 \overline{Emb}_\partial^{fr}(D^n, D^{n+q})$ is equal to

$$\pi_0 \operatorname{hofib}(Emb_\partial^{fr}(D^n, D^{n+q}) \to Imm_\partial^{fr}(D^n, D^{n+q}) \simeq \Omega^n SO(n+q)),$$

which is isomorphic to $\pi_0 \operatorname{hofib}(Emb_\partial^{fr}(D^n, D^{n+q}) \to \Omega^n SO)$ using the stability of the homotopy groups of SO:

$$\pi_i SO(n+q) = \pi_i SO, \quad \text{ if } i \leqslant n+q-2.$$

Therefore, for $q \geqslant 3$, we have that

$$\pi_0 \Omega^n SO(n+q) = \pi_n SO(n+q) = \pi_n SO = \pi_0 \Omega^n SO$$

and similarly $\pi_1 \Omega^n SO(n+q) = \pi_1 \Omega^n SO$. Since $\Omega^n SO \simeq Emb_\partial^{fr}(D^n, D^{n+\infty})$, we get the result as a consequence of five lemma. □

Thus, for any element $[(f,\alpha)]$ in $\pi_0 \overline{Emb}_\partial^{fr}(D^n, D^{n+q})$ there corresponds an equivalence class of a pair $(\tilde{f}, \tilde{\alpha})$ where $\tilde{f} \colon D^n \hookrightarrow D^{n+q}$ and $\tilde{\alpha} \colon [0,1] \to Emb_\partial^{fr}(D^n, D^{n+\infty})$ i.e, $\tilde{\alpha} \colon D^n \times [0,1] \hookrightarrow D^{n+N} \times [0,1]$ such that $\tilde{\alpha}|_{D^n \times 0} = id$ and $\tilde{\alpha}|_{D^n \times 1} = \tilde{f}$, together with framing.

We consider $D^{n+1} \cong D^n \times [0,1]$ obtained by identifying D^n_+ to $D^n \times \{1\}$ and D^n_- to $D^n \times \{0\} \cup S^{n-1} \times [0,1]$, and then smoothening the corners. We similarly identify $D^{n+N+1} \cong D^{n+N} \times [0,1]$, for some large N. Therefore, each pair $(\tilde{f}, \tilde{\alpha})$ can be thought of as a special disked embedding with Type III framing i.e., a pair $(id \cup \tilde{f}, \tilde{\alpha}) \colon (S^n, D^{n+1}) \hookrightarrow (S^{n+q}, D^{n+N+1})$ such that $\tilde{\alpha}|_{D^n_-}$ is the trivial inclusion $id \colon D^n_- \hookrightarrow D^{n+q}_-$ with trivial framing, and $\tilde{\alpha}|_{D^n_+}$ is the framed embedding $\tilde{f} \colon D^n_+ \hookrightarrow D^{n+q}_+$. In other words, one has a natural map

$$\mu \colon \operatorname{hofib}\Big(Emb_\partial^{fr}(D^n, D^{n+q}) \to Emb_\partial^{fr}(D^n, D^{n+\infty})\Big) \longrightarrow SDE^q_n. \tag{4}$$

On the level of π_0, we obtain

$$\mu_* \colon \pi_0 \overline{Emb}_\partial^{fr}(D^n, D^{n+q}) \to \overline{FC}^q_n,$$

$$[(f,\alpha)] \mapsto [(id \cup \tilde{f}, \tilde{\alpha})].$$

Theorem 4.3. *For $q \geqslant 3$, μ_* is an isomorphism, and therefore*

$$\pi_0 \overline{Emb}_\partial^{fr}(D^n, D^{n+q}) = \overline{FC}^q_n.$$

Proof. To show that μ_* is bijective, it suffices to show that

$$\text{hofib}\left(Emb_\partial^{fr}(D^n, D^{n+q}) \to Emb_\partial^{fr}(D^n, D^{n+\infty})\right) \quad \text{and} \quad SDE_n^q$$

are weakly homotopy equivalent, and then Lemma 4.2 concludes the result.

Consider the following diagram, where the vertical lines are fiber sequences, and the map μ is defined above (4).

$$\begin{tikzcd}
\Omega Emb_\partial^{fr}(D^n, D^{n+\infty}) \arrow[r] \arrow[d] & Emb_\partial^{fr}(D^{n+1}, D^{n+1+\infty}) \arrow[d] \\
\text{hofib}\left(Emb_\partial^{fr}(D^n, D^{n+q}) \to Emb_\partial^{fr}(D^n, D^{n+\infty})\right) \arrow[r, "\mu"] \arrow[d] & SDE_n^q \arrow[d] \\
Emb_\partial^{fr}(D^n, D^{n+q}) \arrow[r, equal] & Emb_\partial^{fr}(D^n, D^{n+q}).
\end{tikzcd}$$

Note that the top map is just restriction on the fibers. Moreover, it is a homotopy equivalence since

$$\begin{aligned}
Emb_\partial^{fr}(D^{n+1}, D^{n+1+\infty}) &\simeq Imm_\partial^{fr}(D^{n+1}, D^{n+1+\infty}) \simeq \Omega^{n+1} SO \\
&\simeq \Omega\Omega^n SO \simeq \Omega Emb_\partial^{fr}(D^n, D^{n+\infty}).
\end{aligned}$$

Thus, the map in the middle μ is also a weak homotopy equivalence. By Lemma 4.2, we get $\pi_0 \overline{Emb}_\partial^{fr}(D^n, D^{n+q}) = \overline{FC}_n^q$. $\square$

Theorem 1.1 is immediate by combining Lemma 4.1 and Theorems 3.1 and 4.3.

5. Geometric interpretation of long exact sequences associated with the triad $(SG; SO, SG_q)$

In this section, we prove Theorem 1.2 for the "non-framed" case i.e., we show the isomorphism between the sequences in (1). The proof for the framed case is similar.

Recall that Im_n^q is the group of concordance (or equivalently regular homotopy) classes of immersions of S^n in S^{n+q}. According to Haefliger [**14**, Section 4], any representative in Im_n^q is regular homotopic to a special immersion i.e., an immersion $f\colon S^n \looparrowright S^{n+q}$ such that $f|_{D_-^n}$ is the natural inclusion in D_-^{n+q} and $f|_{D_+^n}$ is an immersion in D_+^{n+q}. We can extend this immersion as a disk immersion $\bar{f}\colon D^{n+1} \looparrowright D^{n+N+1}$ for N large enough. Furthermore, we add framing on $\bar{f}$ by first extending the framing from the base-point $* = e_2$ to D_+^n inside D_+^{n+q}, and then we extend this framing to D^{n+1} inside D^{n+N+1} after taking the suspension. In other words, we add disk structure and Type I framing in the same way as we did for special embeddings representing elements in C_n^q. Thus, any element in Im_n^q can be represented by a special disked immersion $(f, \bar{f})\colon (S^n, D^{n+1}) \looparrowright (S^{n+q}, D^{n+N+1})$ with Type I framing.

Haefliger [**13**, Section 4.2] has shown that Im_n^q is isomorphic to the homotopy group $\pi_n(SO, SO_q)$ where his map $\eta\colon Im_n^q \to \pi_n(SO, SO_q)$ is defined as follows: given a special disked immersion $(f, \bar{f})\colon (S^n, D^{n+1}) \looparrowright (S^{n+q}, D^{n+N+1})$ with Type I framing, one considers the trivialization of the normal bundle induced by the framing of $\bar{f}$. To each $x \in S^n$, one associates the $(N-q)$ frame $e_{n+q+2}, \dots, e_{N+n+1}$ with respect to this trivialization. This $(N-q)$ frame defines a map $h_f \colon S^n \to V_{N,N-q}$ that represents a homotopy class $[h_f]$ in $\pi_n(V_{N,N-q}) = \pi_n(SO, SO_q)$, where $V_{N,N-q} = SO_N/SO_{N-q}$ is the Stiefel manifold.

Remark 5.1. Note that the restriction $h_f|_{D^n_+}$ is constantly equal to the identity inclusion $\mathbb{R}^{N-q} \subset \mathbb{R}^N$ (viewed as the base-point of $V_{N,N-q}$) because the framing on D^n_+ is given by suspension and the last $(N-q)$ vectors are $e_{n+q+2}, \dots, e_{N+n+1}$. Hence, the class $[h_f]$ depends only on the framing at D^n_-.

Let us now describe the map θ appearing in the geometric long exact sequence:

$$\cdots \longrightarrow \overline{FC}^q_n \longrightarrow C^q_n \longrightarrow Im^q_n \xrightarrow{\theta} \overline{FC}^q_{n-1} \longrightarrow \cdots$$

Note that

$$Im^q_n = \pi_0 Imm_\partial(D^n, D^{n+q}) = \pi_n V_{n+q,n} = \pi_1 Imm_\partial(D^{n-1}, D^{n+q-1}).$$

The natural map $\Omega Imm_\partial(D^{n-1}, D^{n+q-1}) \to \overline{Emb}_\partial(D^{n-1}, D^{n+q-1})$ induces a map $Im^q_n = \pi_1 Imm_\partial(D^{n-1}, D^{n+q-1}) \to \pi_0 \overline{Emb}_\partial(D^{n-1}, D^{n+q-1}) = \overline{FC}^q_{n-1}$.

We can also interpret $\theta \colon Im^q_n \to \overline{FC}^q_{n-1}$ in terms of disked embeddings/immersions as follows: given a special disked immersion $(f, \bar{f}) \colon (S^n, D^{n+1}) \looparrowright (S^{n+q}, D^{n+N+1})$ with Type I framing representing an element in Im^q_n, we consider the restriction $f|_{S^{n-1} = D^n_- \cap D^n_+} = g = id \colon S^{n-1} \hookrightarrow S^{n+q-1}$, which is the natural inclusion. Moreover, we get the disk immersion $f|_{D^n_+} \colon D^n_+ \looparrowright D^{n+q}_+$, which can be immersed inside a bigger disk D^{n+N}_+ by allowing more dimensions. As a result, we obtain a disk immersion $\bar{g} := id \circ f|_{D^n_+} \colon D^n_+ \looparrowright D^{n+N}_+$ with the restricted framing from $\bar{f}|_{D^n_+}$. Since N is large enough, we can change the framed immersion $\bar{g}$ into a framed embedding $\bar{g}' \colon D^n \hookrightarrow D^{n+N}$. The obtained pair $(g, \bar{g}') \colon (S^{n-1}, D^n) \hookrightarrow (S^{n+q-1}, D^{n+N})$ is a disked embedding where the framing on $\bar{g}'|_{S^{n-1}}$ is given by suspension of a framing inside S^{n+q-1} i.e., $(g, \bar{g}')$ is a framed disked embedding. Therefore, given a special disked immersion $(f, \bar{f})$ with Type I framing, we can assign a framed disked embedding $(g, \bar{g}')$ to it. Thus, we get a well defined map from Im^q_n to $\overline{FC}^q_{n-1}$.

The commutativity of the following diagram is given by a similar argument as in the proof of Theorem 4.3.

$$\begin{array}{ccc} Im^q_n & \xrightarrow{\quad\quad\theta\quad\quad} & \overline{FC}^q_{n-1} \\ \simeq\uparrow & & \simeq\uparrow \\ \pi_1 Imm(D^{n-1}, D^{n+q-1}) & \longrightarrow & \pi_0 \overline{Emb}_\partial(D^{n-1}, D^{n+q-1}). \end{array}$$

Remark 5.2. Note that while defining θ, instead of

$$(g, \bar{g}) = (id, id \circ f|_{D^n_+}) \colon (S^{n-1}, D^n) \looparrowright (S^{n+q-1}, D^{n+N}),$$

we can consider the pair $(id, id \circ f|_{D^n_-})$ which is same as (id, id) since $f|_{D^n_-} = id$ with framing restricted from $\bar{f}$ (such framing may not be a suspension). In $\overline{FC}^q_{n-1}$, the representative $(g, \bar{g}')$ corresponding to $(g, \bar{g}) = (id, id \circ f|_{D^n_+})$ is equivalent to the framed trivial disked embedding $(id, id) = (id, id \circ f|_{D^n_-})$ with framing as on $\bar{f}|_{D^n_-}$, since $\bar{f}$ acts as a concordance between $id \circ f|_{D^n_+}$ and $id \circ f|_{D^n_-}$. To be precise, we take a perturbation $\bar{f}'$ of $\bar{f}$ which acts as a concordance. We will use this in the following proof to show commutativity of the third square in (5).

Proof of Theorem 1.2. To prove the result, we need to show that each square in the following diagram commutes:

$$\begin{tikzcd}
\pi_{n+1}(SG, SG_q) \arrow[r] & \pi_{n+1}(SG; SO, SG_q) \arrow[r] & \pi_n(SO, SO_q) \arrow[r] & \pi_n(SG, SG_q) \\
\overline{FC}^q_n \arrow[r] \arrow[u, "\xi", "\simeq"'] & C^q_n \arrow[r] \arrow[u, "\psi", "\simeq"'] & Im^q_n \arrow[r, "\theta"] \arrow[u, "\eta", "\simeq"'] & \overline{FC}^q_{n-1}. \arrow[u, "\xi", "\simeq"']
\end{tikzcd} \tag{5}$$

For the first square, the map $\overline{FC}^q_n \to C^q_n$ is an inclusion on the level of representatives i.e., a framed disked embedding representing an element in $\overline{FC}^q_n$ clearly represents an element in C^q_n. Therefore, the commutativity of this square is straightforward from the construction.

The commutativity of the second square is given by Haefliger [**13**, Section 4.4] and is easy to see. The map $C^q_n \to Im^q_n$ is obvious since an embedding is also an immersion. We have seen that the vertical map η on a given representative in Im^q_n depends only on the behavior of the representative on D^n_-, see Remark 5.1. Similarly, the top horizontal map $\pi_{n+1}(SG; SO, SG_q) \to \pi_n(SO, SO_q)$ is defined by restricting the representatives in $\pi_{n+1}(SG; SO, SG_q)$ to the half-disk D^n_-.

We now check the commutativity of the third square. Given an element $\alpha \in Im^q_n$ represented by a special disked immersion

$$(f, \bar{f})\colon (S^n, D^{n+1}) \looparrowright (S^{n+q}, D^{n+N+1})$$

with Type I framing, by Remark 5.2, the corresponding element $\theta(\alpha)$ in $\overline{FC}^q_{n-1}$ can be represented by the framed trivial disked embedding

$$(id, id) = (id, id \circ \ f|_{D^n_-})\colon (S^{n-1}, D^n) \hookrightarrow (S^{n-1} \times D^q, D^n \times D^N),$$

with the framing obtained as a restriction $\bar{f}|_{D^n_-}$. Recall that on $S^{n-1} = D^n_- \cap D^n_+$, the framing is given by suspension of a framing inside S^{n+q-1}. We can homotope the obtained framing on S^{n-1} so that it becomes trivial on D^{n-1}_-. Now, under the vertical map ξ, the image of $\theta(\alpha)$ is represented by a map

$$\phi\colon D^n \times S^{N-1} \to S^{N-1} \ \text{ with an extension } \ \bar{\phi}\colon D^n \times D^N \to D^N$$

defined linearly by $(x, y) \mapsto r(x)(y)$, for some rotation r given by the framing on $\bar{f}|_{D^n_-}$. More precisely, $r\colon D^n \to SO(N)$ is such that $r|_{\partial D^n = S^{n-1}}$ is a suspension of rotation in $SO(q)$ with $r = id$ on D^{n-1}_-, by construction. The map $\bar{\phi}$ satisfies the definition of ξ (see Subsection 3.2), since

$$\bar{\phi}^{-1}(0) = \bar{f}(D^n_-) = D^n \times 0, \quad \text{with} \quad \bar{\phi}|_{S^{n-1}} \in SO(q)$$

such that $\bar{\phi}_x = id$ for $x \in D^{n-1}_-$ and $\bar{\phi}_x$ is the suspension of a map $D^q \to D^q$ for any $x \in D^{n-1}_+$. Moreover, $\bar{\phi}$ also represents an element in $\pi_n(SO, SO_q)$ and is precisely the representative that we get for $\eta(\alpha)$, as η also depends only on the non-trivial framing on D^n_- (see Remark 5.1). Therefore, the square commutes. ☐

6. Applications

6.1. Known computations

For C_n^q, the well-known computations were done by Haefliger [**11, 12, 13**] in the 1960s and later by Milgram [**18**] in the early 1970s. To the best of our knowledge, no computations were done ever since. The Manifold Atlas webpage [**25**] describes all the known groups C_n^q. Haefliger [**11**] has shown that $C_n^q = 0$ for $n < 2q-3$. Furthermore, he proved that for $q \geqslant 3$ (see [**13**, Corollary 8.14]),

$$C_{2q-3}^q = \begin{cases} \mathbb{Z} & q \textit{ odd}, \\ \mathbb{Z}_2 & q \textit{ even}. \end{cases}$$

For odd q, the generator is given by the Haefliger trefoil knot [**12**]. It is an interesting question whether the Haefliger trefoil is a generator for the even case.

There are only a few computations for FC_n^q in the literature. For example, Haefliger [**13**, Theorem 5.17] has shown that $FC_3^3 = \mathbb{Z} \oplus \mathbb{Z}$. Moreover, it is easy to see that for $n < 2q-3$, we get $FC_n^q = \pi_n(SO_q)$, see Proposition 6.2.

The groups $\overline{FC}_n^q = \pi_{n+1}(SG, SG_q)$ are related to the homotopy groups of spheres. Some of these groups are known, in particular,

$$\overline{FC}_2^3 = \pi_3(SG, SG_3) = \mathbb{Z}_2 \quad \text{and} \quad \overline{FC}_3^3 = \pi_4(SG, SG_3) = \mathbb{Z},$$

found in [**13**, Section 5.16] and [**21**, Proof of Lemma 3.1].

The rational computations of $\overline{FC}_n^q$ are known [**3**, Corollary 20], [**10**, Section 5.7], and can also be computed directly as follows:

Proposition 6.1. *For $q \geqslant 3$,*

$$\overline{FC}_n^q \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & n = q-1, \quad q \textit{ even}, \\ \mathbb{Q} & n = 2q-3, \quad q \textit{ odd}, \\ 0 & \textit{otherwise}. \end{cases}$$

Proof. Since $\overline{FC}_n^q = \pi_{n+1}(SG, SG_q)$, we consider the long exact sequence of the pair (SG, SG_q):

$$\cdots \longrightarrow \pi_{n+1}(SG) \longrightarrow \pi_{n+1}(SG, SG_q) \longrightarrow \pi_n(SG_q) \longrightarrow \pi_n(SG) \longrightarrow \cdots \tag{6}$$

The rational homotopy groups $\pi_n^{\mathbb{Q}}(SG_q)$ can be easily computed by considering the long exact sequence associated with the fibration $\Omega_*^{q-1}S^{q-1} \to SG_q \to S^{q-1}$, where $\Omega_*^{q-1}S^{q-1}$ is the component of loops of degree one. The connecting homomorphism $\pi_{n+1}(S^{q-1}) \to \pi_n(\Omega_*^{q-1}S^{q-1}) = \pi_{n+q-1}(S^{q-1})$ is given by the Whitehead bracket $[id_{S^{q-1}}, -]$. Note that all $\pi_n(SG)$ are torsions being the stable homotopy groups of spheres, hence, $\pi_n^{\mathbb{Q}}(SG) = 0$. Using the rational homotopy groups of spheres, we get

$$\pi_n^{\mathbb{Q}}(SG_q) = \begin{cases} \mathbb{Q} & n = q-1, \quad q \textit{ even}, \\ \mathbb{Q} & n = 2q-3, \quad q \textit{ odd}, \\ 0 & \textit{otherwise}. \end{cases}$$

Thus, from the long exact sequence (6) we get $\pi_{n+1}^{\mathbb{Q}}(SG, SG_q) = \pi_n^{\mathbb{Q}}(SG_q)$ and that concludes the result. ☐

6.2. Metastable range

From [**13**, Section 4.4], one can deduce the following stability result for the groups $\overline{FC}^q_n$ and FC^q_n:

Proposition 6.2. *For $n < 2q-3$, $\overline{FC}^q_n = \pi_{n+1}(SO, SO_q)$ and $FC^q_n = \pi_n SO_q$.*

Proof. Consider the long exact sequence associated with the triad $(SG; SO, SG_q)$ given in [**13**, Section 4.4]:

$$\to \pi_{n+1}(SG, SG_q) \to \pi_{n+1}(SG; SO, SG_q) \to \pi_n(SO, SO_q) \to \pi_n(SG, SG_q) \to \quad (7)$$

By [**13**, Corollary 6.6], the groups $\pi_{n+1}(SG; SO, SG_q) = C^q_n = 0$ for $n < 2q-3$. Therefore, from the above sequence, we get

$$\pi_{n+1}(SO, SO_q) = \pi_{n+1}(SG, SG_q) = \overline{FC}^q_n \quad \text{for} \quad n < 2q-4.$$

Moreover, any element of C^q_n is trivial as immersion for $n < 2q-1$, see [**13**, Corollary 6.10]. Thus, the homomorphism $\pi_{2q-2}(SG; SO, SG_q) \to \pi_{2q-3}(SO, SO_q)$ in (7) is trivial, and we get $\pi_{2q-3}(SO, SO_q) = \pi_{2q-3}(SG, SG_q) = \overline{FC}^q_{2q-4}$.

For the second equality, we consider the geometric long exact sequence given by Haefliger [**13**, Section 5.9]:

$$\cdots \longrightarrow \pi_n SO_q \longrightarrow FC^q_n \longrightarrow C^q_n \longrightarrow \pi_{n-1} SO_q \longrightarrow \cdots \quad (8)$$

The result easily follows for $n < 2q-4$ since $C^q_n = 0$ for $n < 2q-3$. When $n = 2q-4$, the homomorphism $C^q_{2q-3} \to \pi_{2q-4} SO_q$ in (8) is the composition

$$C^q_{2q-3} = \pi_{2q-2}(SG; SO, SG_q) \xrightarrow{0} \pi_{2q-3}(SO, SO_q) \to \pi_{2q-4}(SO_q),$$

and therefore is also trivial. □

Lemma 6.3. *For $i \leqslant q-2$, $\pi_i(SG_q) = \pi_i(SG)$.*

Proof. When $i \leqslant q-1$, we have $\pi_i(SG, SG_q) = \pi_i(SO, SO_q) = 0$, where we get the first equality from Proposition 6.2, and the second one using the fact that $\pi_i(SO, SO_q) = 0$ for $i < q$. Therefore, from the long exact sequence (6), we conclude that for $i \leqslant q-2$, $\pi_i(SG_q) = \pi_i(SG)$. □

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