Homology, Homotopy and Applications, vol. 25(2), 2023, pp.21–36

SELF-CLOSENESS NUMBERS OF NON-SIMPLY-CONNECTED SPACES

YICHEN TONG

(*communicated by Natàlia Castellana*)

Abstract

The self-closeness number of a connected CW complex is the least integer n such that any of its self-maps inducing an isomorphism in π_* for $* \leqslant n$ is a homotopy equivalence. We prove that under a mild condition, the self-closeness number of a non-simply-connected finite complex coincides with that of its universal cover whenever the universal cover is a finite H_0-space or a finite co-H_0-space. We give several interesting examples to which the result applies.

1. Introduction

Let X be a (pointed) connected CW complex. The subset $\mathcal{E}(X)$ of a pointed homotopy set $[X, X]$ consisting of homotopy equivalences is a group, which is called the *group of self-homotopy equivalences* of X. The group $\mathcal{E}(X)$ has been broadly studied for a long time. See [**2**] for its basics. Recently, the following approach to the groups of self-homotopy equivalences was proposed by Choi and Lee [**5**]. For $n \geqslant 0$, let $\mathcal{A}^n_\sharp(X)$ denote the subset of $[X, X]$ consisting of self-maps inducing isomorphisms in the homotopy groups π_* for $* \leqslant n$. Then $\mathcal{A}^n_\sharp(X)$ is a monoid such that there is a sequence of submonoids

$$\mathcal{A}^1_\sharp(X) \supset \cdots \supset \mathcal{A}^n_\sharp(X) \supset \mathcal{A}^{n+1}_\sharp(X) \supset \cdots \supset \mathcal{A}^\infty_\sharp(X) = \mathcal{E}(X)$$

which captures properties of $\mathcal{E}(X)$ that classical results cannot do, where the last equality follows from the J.H.C. Whitehead theorem. In this approach, it is of particular importance to find whether or not there is an integer n satisfying

$$\mathcal{A}^n_\sharp(X) = \mathcal{E}(X). \tag{1}$$

For instance, if X is a connected n-dimensional CW complex, then $\mathcal{A}^n_\sharp(X) = \mathcal{E}(X)$ as in [**5**, Theorem 2]. Moreover, $\mathcal{A}^n_\sharp(X)$ can coincide with $\mathcal{E}(X)$ for n much smaller than $\dim X$ as we can see from complex projective spaces. So finding the smallest such integer is of particular interest, and so we are led to the following definition.

Definition 1.1. The *self-closeness number* $\mathrm{N}\mathcal{E}(X)$ of a connected CW complex X is defined as the least integer n such that the equality (1) holds.

Received January 20, 2022, revised May 26, 2022; published on September 27, 2023.
2020 Mathematics Subject Classification: 55P10, 55S37.
Key words and phrases: self-closeness number, group of self-equivalences, H_0-space, co-H_0-space, p-universal space.
Article available at `http://dx.doi.org/10.4310/HHA.2023.v25.n2.a2`

We can alternatively say that $N\mathcal{E}(X)$ is the least integer n such that any self-map of X is a homotopy equivalence whenever it is an isomorphism in π_* for $* \leqslant n$. For a simply-connected CW complex, we can use (co)homology for evaluating its self-closeness number as we did above, and the self-closeness numbers of simply-connected CW complexes were intensely studied in [**5, 11, 13, 14, 15, 16**]. However, the self-closeness numbers of non-trivial non-simply connected CW complexes were only studied for real projective spaces and lens spaces, by the ways that only apply to these cases. Then at the moment, there is no general result on the self-closeness numbers of non-simply-connected CW complexes.

In this paper, we study the self-closeness number of a non-simply-connected CW complex by assuming conditions on its universal cover. To state the main theorem, we set notation and terminology. A simply-connected space is called an *H_0-space* if its rationalization is an H-space. Note that an H_0-space needs not be an H-space. For example, every odd sphere of dimension $\neq 1, 3, 7$ is not an H-space but it is an H_0-space (see Proposition 4.4). For a graded algebra A, let $\mathrm{d}(A)$ denote the maximal degree of generators of A. Now we are ready to state the main theorem, which is the first general result on the self-closeness numbers of non-simply-connected CW complexes.

Theorem 1.2. *Let X be a finite complex such that the universal cover $\widetilde{X}$ is a finite H_0-space. If $\mathrm{d}(H^*(\widetilde{X};\mathbb{Z})) = \mathrm{d}(H^*(\widetilde{X};\mathbb{Q}))$ and $\pi_1(X)$ acts trivially on $H^*(\widetilde{X};\mathbb{Q})$, then*

$$N\mathcal{E}(X) = N\mathcal{E}(\widetilde{X}) = \mathrm{d}(H^*(\widetilde{X};\mathbb{Q})).$$

We will give in Section 2 several interesting examples to which Theorem 1.2 applies. We will also prove the dual version of Theorem 1.2. To state it, we set notation and terminology. Dually to an H_0-space, a simply-connected space is called a *co-H_0-space* if its rationalization is a co-H-space. Recall that the *cohomological dimension* of X is defined by

$$\mathrm{cd}(X) = \sup\{n \mid H^n(X;M) \neq 0 \text{ for some } \pi_1(X)\text{-module } M\}.$$

Now we state the dual version of Theorem 1.2.

Theorem 1.3. *Let X be a finite complex such that the universal cover $\widetilde{X}$ is a finite co-H_0-space. If $\mathrm{cd}(X) = \mathrm{d}(H^*(\widetilde{X};\mathbb{Q}))$ and $\pi_1(X)$ acts trivially on $H^*(\widetilde{X};\mathbb{Q})$, then*

$$N\mathcal{E}(X) = N\mathcal{E}(\widetilde{X}) = \mathrm{d}(H^*(\widetilde{X};\mathbb{Q})).$$

The paper is structured as follows. Section 2 gives examples to which Theorem 1.2 applies. Section 3 gives some upper bounds for self-closeness numbers that will be needed. Section 4 recalls the p-universality introduced by Mimura, O'Neill and Toda [**12**] and shows properties of (co-)H_0-spaces that we will use. Sections 5 and 6 prove Theorems 1.2 and 1.3.

Acknowledgments

I would like to thank Daisuke Kishimoto for suggesting to me to study the self-closeness number of non-simply-connected spaces. I would also like to thank Nobuyuki Oda, Toshihiro Yamaguchi and Masahiro Takeda for useful comments. This work was supported by JST SPRING, Grant Number JPMJSP2110.

2. Example

We give interesting examples of non-simply-connected CW complexes to which Theorem 1.2 applies.

2.1. Topological spherical space form

Recall that a topological spherical space form is the quotient space of a sphere by a free action of a finite group. Simplest examples are real projective spaces and lens spaces.

Proposition 2.1. *Let X be a topological spherical space form of dimension $2n-1$. Then*

$$\mathrm{N}\mathcal{E}(X) = 2n-1.$$

Proof. The case $n=1$ is trivial, and so we assume $n \geqslant 2$. Let G be a finite group acting freely on S^{2n-1} such that $X = S^{2n-1}/G$. By the Lefschetz fixed point theorem, every orientation reversing map of S^{2n-1} must have a fixed point. Then the action of G on $H^*(S^{2n-1};\mathbb{Q})$ is trivial. Clearly, we have

$$\mathrm{d}(H^*(S^{2n-1};\mathbb{Z})) = \mathrm{d}(H^*(S^{2n-1};\mathbb{Q})) = 2n-1.$$

Thus since S^{2n-1} is a finite H_0-space by Proposition 4.4 below, the proof is done by Theorem 1.2. □

If a topological spherical space form X is of dimension $2n$, then the action of $\pi_1(X)$ on the universal cover S^{2n} is orientation reversing, so that Theorem 1.2 does not apply. However, it is known that X is homotopy equivalent to $\mathbb{R}P^{2n}$, so that $\mathrm{N}\mathcal{E}(X) = \mathrm{N}\mathcal{E}(\mathbb{R}P^{2n}) = 2n$ as in [**15**]. Kishimoto and Oda [**10**] determined the monoid $[X,X]$ for every topological spherical space form X, which also proves Proposition 2.1 and $\mathrm{N}\mathcal{E}(X) = 2n$ for $\dim X = 2n$.

2.2. Product of spheres

First, we consider an explicit free action of the symmetric group Σ_3 on a product of two spheres given by Blaszczyk [**4**, Proposition 5.1]. Let S^{2m+1} be the unit sphere of $\mathbb{C}^{m+1}$. Since there is a presentation

$$\Sigma_3 = \langle a, b \mid a^2 = b^3 = (ab)^2 = 1\rangle,$$

we get an action of Σ_3 on $S^{2m+1} \times S^n$ by

$$a(x,y) = (\bar{x}, -y), \quad b(x,y) = (e^{2\pi\sqrt{-1}/3}x, y)$$

for $(x,y) \in S^{2m+1} \times S^n$. It is easy to see that this action is free.

Proposition 2.2. *Consider the above free action of Σ_3 on $S^{2m+1} \times S^n$. If m is even with $m > 0$ and n is odd with $n \geqslant 3$, then*

$$\mathrm{N}\mathcal{E}((S^{2m+1} \times S^n)/\Sigma_3) = \max\{2m+1, n\}.$$

Proof. Since $m > 0$ and n is odd with $n \geqslant 3$, $S^{2m+1} \times S^n$ is a finite H_0-space by Proposition 4.4 below. By definition, the action of Σ_3 is trivial in cohomology if m is even and n is odd. Clearly,

$$\mathrm{d}(H^*(S^{2m+1}\times S^n;\mathbb{Z}))=\mathrm{d}(H^*(S^{2m+1}\times S^n;\mathbb{Q}))=\max\{2m+1,n\}.$$

Thus the proof is finished by Theorem 1.2. □

We can easily generalize the above free action of Σ_3 on $S^{2m+1}\times S^n$ to a free action of the dihedral group

$$D_{2q}=\langle a,b\mid a^q=b^2=(ab)^2=1\rangle$$

on $S^{2m+1}\times S^n$. Then quite similarly to Proposition 2.2, we can prove:

Proposition 2.3. *Consider the above free action of D_{2q} on $S^{2m+1}\times S^n$. If m is even with $m>0$ and n is odd with $n\geqslant 3$, then*

$$\mathrm{N}\mathcal{E}((S^{2m+1}\times S^n)/D_{2q})=\max\{2m+1,n\}.$$

Next, we consider an abstract free action of a finite group on a product of odd spheres.

Proposition 2.4. *Suppose that the canonical action of a finite subgroup G of $U(n)$ on $U(n)/U(n-m)$ is free and the order of G is prime to $(n-1)!$. Then for $m\geqslant 1$, G acts freely on $X=\prod_{i=n-m+1}^{n}S^{2i-1}$ such that*

$$\mathrm{N}\mathcal{E}(X/G)=2n-1.$$

Proof. Since $m\geqslant 1$, X is a finite H_0-space. Clearly,

$$\mathrm{d}(H^*(X;\mathbb{Z}))=\mathrm{d}(H^*(X;\mathbb{Q}))=2n-1.$$

Adem, Davis and Ünlü [1] proved that there is a free action of G on X, which is trivial in rational cohomology. Thus the proof is done by Theorem 1.2. □

Yet more free actions of finite groups on products of spheres are given in [6].

2.3. Lie group

Let K be a compact simply-connected Lie group. Then every element of K is connected to the unit by a path, implying that the canonical action of every subgroup of K is trivial in rational cohomology. If K has no torsion in homology, then

$$H^*(K;\mathbb{Z})=\Lambda(x_1,\dots,x_n).$$

In particular, $\mathrm{d}(H^*(K;\mathbb{Z}))=\mathrm{d}(H^*(K;\mathbb{Q}))=\max\{|x_1|,\dots,|x_n|\}$. Thus as a consequence of Theorem 1.2, we get:

Proposition 2.5. *Let K be a compact simply-connected Lie group having no torsion in homology, and let G be a finite subgroup of K. Then*

$$\mathrm{N}\mathcal{E}(K/G)=\mathrm{N}\mathcal{E}(K)=\mathrm{d}(H^*(K;\mathbb{Q})).$$

This proposition can be generalized to certain homogeneous spaces, including complex Stiefel manifolds and $U(2n)/Sp(n)$, as follows.

Proposition 2.6. *Let K be a compact simply-connected Lie group having no torsion in homology, and let H be a closed subgroup of K such that the projection $K \to K/H$ is injective in cohomology. If the action of a finite subgroup G on K induces a free action on K/H, then*

$$\mathrm{N}\mathcal{E}(G\backslash K/H) = \mathrm{N}\mathcal{E}(K/H) = \mathrm{d}(H^*(K/H;\mathbb{Q})).$$

Proof. By assumption, the cohomology of K/H is an exterior algebra generated by elements of odd degrees. Then K/H is a finite H_0-space by Proposition 4.4 such that $\mathrm{d}(H^*(K/H;\mathbb{Z})) = \mathrm{d}(H^*(K/H;\mathbb{Q}))$. Since the action of G on the cohomology of K is trivial and the projection $K \to K/H$ is compactible with the action of G and injective in cohomology, the action of G on the cohomology of K/H is trivial too. Thus by Theorem 1.2, the proof is complete. □

3. Upper bound

This section considers some upper bounds for self-closeness numbers, which will be used later. First, we consider a relation among self-closeness numbers of a total space and a base space of a covering.

Lemma 3.1. *Let $E \to B$ be a covering, where E and B are connected CW complexes. Then*

$$\mathrm{N}\mathcal{E}(B) \leqslant \max\{\mathrm{N}\mathcal{E}(E), 1\}.$$

Proof. Let $\mathrm{N}\mathcal{E}(E) = n$. If $n = 0$, then $\pi_*(E) = 0$ for each $*$, implying $B = K(\pi_1(B), 1)$. Thus $\mathrm{N}\mathcal{E}(B) \leqslant 1$, so that the inequality in the statement holds. Now we suppose $n \geqslant 1$. Let $f\colon B \to B$ be a map which is an isomorphism in π_* for $* \leqslant n$. Then it lifts to a self-map $\tilde{f}\colon E \to E$, and by the homotopy exact sequence, $\tilde{f}$ is an isomorphism in π_* for $* \leqslant n$. Hence $\tilde{f}$ is a homotopy equivalence, implying that f is an isomorphism in π_* for each $*$ by the homotopy exact sequence. Thus by the J.H.C. Whitehead theorem, f is a homotopy equivalence, and so $\mathrm{N}\mathcal{E}(B) \leqslant n$. Therefore, the inequality in the statement is proved. □

We will use the following inequality later.

Proposition 3.2. *Let X be a connected CW complex of finite dimension, and let $\widetilde{X}$ denote its universal cover. If $\pi_1(X)$ is finite, then*

$$\mathrm{N}\mathcal{E}(X) \leqslant \mathrm{N}\mathcal{E}(\widetilde{X}).$$

Proof. By Lemma 3.1, $\mathrm{N}\mathcal{E}(X) \leqslant \max\{\mathrm{N}\mathcal{E}(\widetilde{X}), 1\}$. It remains to show $\mathrm{N}\mathcal{E}(\widetilde{X}) \geqslant 1$. If $\mathrm{N}\mathcal{E}(\widetilde{X}) = 0$, then $X = K(\pi_1(X), 1)$ as in the proof of Lemma 3.1. So since $\pi_1(X)$ is finite, X is of infinite dimension, which is a contradiction. Thus we obtain $\mathrm{N}\mathcal{E}(\widetilde{X}) \geqslant 1$, completing the proof. □

Next, we consider a cohomological upper bound for the self-closeness number of a simply-connected CW complex. Oda and Yamaguchi [**15**] defined the *homological self-closeness number* of a path-connected space X, denoted $\mathrm{N}_*\mathcal{E}(X)$, as the least integer n such that a self-map of X is an isomorphism in the integral homology of all dimensions whenever it is an isomorphism in the integral homology of dimensions

$\leqslant n$. If X is a simply-connected CW complex of finite type, then by the Hurewicz theorem and the J.H.C. Whitehead theorem, we have

$$\mathrm{N}\mathcal{E}(X) = \mathrm{N}_*\mathcal{E}(X).$$

However, this equality fails, in general, if X is non-simply-connected; non-trivial acyclic spaces are typical examples. As in [**15**], we can define the *cohomological self-closeness number* of a path-connected space X, denoted $\mathrm{N}^*\mathcal{E}(X)$, quite similarly to the homological self-closeness number $\mathrm{N}_*\mathcal{E}(X)$. Cohomological self-closeness numbers are more useful practically than homological ones because cohomology has products. We prove a basic property of cohomological self-closeness numbers.

Proposition 3.3. *Let X be a connected CW complex of finite type. Then*

$$\mathrm{N}\mathcal{E}(X) \leqslant \mathrm{N}^*\mathcal{E}(X).$$

Proof. Suppose that a self-map $f\colon X \to X$ is an isomorphism in π_* for $* \leqslant n$. Then since X is a connected CW complex of finite type, it follows from [**15**, Proposition 40] that f is an isomorphism in H_* for $* \leqslant n$. By the naturality of the universal coefficient theorem, there is a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ext}(H_{*-1}(X;\mathbb{Z}),\mathbb{Z}) & \longrightarrow & H^*(X;\mathbb{Z}) & \longrightarrow & \mathrm{Hom}(H_*(X;\mathbb{Z}),\mathbb{Z}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle (f_*)^*} & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle (f_*)^*} & & \\
0 & \longrightarrow & \mathrm{Ext}(H_{*-1}(X;\mathbb{Z}),\mathbb{Z}) & \longrightarrow & H^*(X;\mathbb{Z}) & \longrightarrow & \mathrm{Hom}(H_*(X;\mathbb{Z}),\mathbb{Z}) & \longrightarrow & 0
\end{array}$$

where rows are split exact. Since f is an isomorphism in H_* for $* \leqslant n$, the left and the right vertical maps are isomorphisms for $* \leqslant n$. Then by the five lemma, f is an isomorphism in H^* for $* \leqslant n$, completing the proof. □

The inequality in Proposition 3.3 can be strict. For example,

$$\mathrm{N}\mathcal{E}(S^2 \cup_2 e^3) = 2 < 3 = \mathrm{N}^*\mathcal{E}(S^2 \cup_2 e^3).$$

We give a computable upper bound for cohomological self-closeness numbers.

Proposition 3.4. *Let X be a connected CW complex of finite type. Then*

$$\mathrm{N}^*\mathcal{E}(X) \leqslant \mathrm{d}(H^*(X;\mathbb{Z})).$$

Proof. If $f\colon X \to X$ is an isomorphism in H^* for $* \leqslant \mathrm{d}(H^*(X;\mathbb{Z}))$, then it is surjective in H^* for each $*$ because f induces an algebra homomorphism in cohomology. Since X is of finite type, $H^*(X;\mathbb{Z})$ is finitely generated for each $*$. Then since a surjective endomorphism of a finitely generated abelian group is an isomorphism, the proof is finished. □

4. p-Universality

This section recalls the p-universality introduced by Mimura, O'Neill and Toda [**12**] and shows properties of (co-)H_0-spaces that we are going to use. Throughout this section, let p denote a prime or 0. Following [**12**], we say that a map of spaces $X \to Y$ is a *p-equivalence* if it is an isomorphism in mod p homology for p odd and in rational homology for $p = 0$. A 0-equivalence is often called a *rational equivalence*, alternatively. By [**9**, Chapter 2, Theorem 1.14], a map between simply-connected CW complexes of finite type is a p-equivalence if and only if it is a p-local homotopy equivalence. Note that a p-equivalence needs not have an inverse, or even a reverse p-equivalence. Then the following notion was introduced in [**12**].

Definition 4.1. A simply-connected finite complex K is called *p-universal* if for any p-equivalence $f\colon X \to Y$ and a map $g\colon K \to Y$, there is a homotopy commutative diagram

$$\begin{array}{ccc} K & \xrightarrow{h} & K \\ \downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array}$$

where h is a p-equivalence.

Let $Y = K$ and $g = 1$ in the above definition. Then we can understand a p-universal space as a space having a "partial right inverse" for any p-equivalence into it. Then it is natural to ask whether or not a p-universal space has a "partial left inverse" for any p-equivalence out of it. To answer this question, the following is proved in [**12**, Theorem 2.1]

Proposition 4.2. *Let K be a p-universal space. For any p-equivalence $f\colon X \to Y$ and a map $g\colon X \to K$, there is a homotopy commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle g}\downarrow & & \downarrow \\ K & \xrightarrow{h} & K \end{array}$$

where h is a p-equivalence.

A simply-connected space is called a *mod p H-space* if its p-localization is itself an H-space. Note that a mod p H-space itself is not an H-space in general. For instance, an odd sphere of dimension $\neq 1, 3, 7$ is a mod p H-space for any odd prime p and $p = 0$ while the sphere itself is not an H-space. Dually, a simply-connected space is called a *mod p co-H-space* if its p-localization is a co-H-space. We say that a mod p (co-)H-space is finite if it is a finite complex. Mimura, O'Neill and Toda [**12**, Theorem 4.2] proved:

Proposition 4.3. *Every mod p finite (co-)H-space is p-universal.*

A mod 0 (co-)H-space is called a *(co-)H_0-space*. The following characterization given in [**17**] of a finite H_0-space is quite useful.

Proposition 4.4. *Let X be a simply-connected finite complex. Then the following conditions are equivalent:*

1. *X is an H_0-space;*
2. *there is a rational equivalence*
$$S^{2n_1+1} \times \cdots \times S^{2n_r+1} \to X.$$

A similar characterization of a finite co-H_0-space is also proved by Arkowitz and Curjel [**3**, Theorem 2.5].

Proposition 4.5. *Let X be a simply-connected finite complex. Then the following conditions are equivalent:*

1. *X is a co-H_0-space;*
2. *there is a rational equivalence*

$$X \to S^{n_1} \vee \cdots \vee S^{n_r}.$$

Now we prove a key lemma on finite (co-)H_0-spaces.

Lemma 4.6. *Let X be a finite (co-)H_0-space. If $H^*(X;\mathbb{Q})$ has an indecomposable element of degree n, then there are maps*

$$f\colon S^n \to X \quad \textit{and} \quad g\colon X \to S^n$$

such that $g \circ f\colon S^n \to S^n$ is a rational equivalence.

Proof. We only prove the case that X is a finite H_0-space since the co-H_0-case is proved quite similarly. By Proposition 4.4, there is a rational equivalence

$$\bar{f}\colon Y \to X$$

for $Y = S^{2n_1+1} \times \cdots \times S^{2n_r+1}$, and by Proposition 4.3, there is also a rational equivalence

$$\bar{g}\colon X \to Y$$

such that $\bar{g} \circ \bar{f}$ is a rational equivalence. Since $H^*(X;\mathbb{Q})$ has an indecomposable element of degree n, we may assume $n = 2n_1 + 1$. Let $F, G \in \mathrm{GL}(r, \mathbb{Q})$ represent the induced maps $\bar{f}^*\colon QH^*(X;\mathbb{Q}) \to QH^*(Y;\mathbb{Q})$ and $\bar{g}^*\colon QH^*(Y;\mathbb{Q}) \to QH^*(X;\mathbb{Q})$, where QA denotes the module of indecomposables of an augmented algebra A and a basis of $QH^*(Y;\mathbb{Q})$ is a collection of the Kronecker duals of the Hurewicz images of inclusions $S^{2n_i+1} \to Y$. Clearly, there is an integer $c \neq 0$ such that $c(FG)^{-1}$ is an integer matrix. Then we can easily construct a self-map $h\colon Y \to Y$ inducing $c(FG)^{-1}$ in cohomology. So the composite $h \circ \bar{g} \circ \bar{f}\colon Y \to Y$ induces the multiplication by c in $QH^*(Y;\mathbb{Q})$. Let f and g be the composites

$$S^n \xrightarrow{\text{incl}} Y \xrightarrow{\bar{f}} X \quad \text{and} \quad X \xrightarrow{h \circ \bar{g}} Y \xrightarrow{\text{proj}} S^n.$$

Then $g \circ f\colon S^n \to S^n$ is of degree $c \neq 0$, completing the proof. □

5. Proof of Theorem 1.2

First, we characterize the triviality of an action of a finite group in rational cohomology.

Lemma 5.1. *Let X be a finite G-complex, where G is a finite group. The following statements are equivalent:*

1. *the quotient map $X \to X/G$ is an isomorphism in rational cohomology;*
2. *the action of G on $H^*(X;\mathbb{Q})$ is trivial.*

Proof. By [7, Proposition 3G.1], there is an isomorphism $H^*(X/G;\mathbb{Q}) \cong H^*(X;\mathbb{Q})^G$ such that the map $H^*(X/G;\mathbb{Q}) \to H^*(X;\mathbb{Q})$ induced from the natural quotient map $X \to X/G$ is identified with the inclusion $H^*(X;\mathbb{Q})^G \to H^*(X;\mathbb{Q})$. Then the two statements are equivalent. □

For the rest of this section, let X be a finite H_0-space, and let G be a finite group acting freely and cellularly on X. Then Theorem 1.2 is equivalent to the following theorem, and so we aim to prove it.

Theorem 5.2. *If* $\mathrm{d}(H^*(X;\mathbb{Z})) = \mathrm{d}(H^*(X;\mathbb{Q}))$ *and* G *acts trivially on* $H^*(X;\mathbb{Q})$*, then*

$$\mathrm{N}\mathcal{E}(X/G) = \mathrm{N}\mathcal{E}(X) = \mathrm{d}(H^*(X;\mathbb{Q}))$$

For the rest of this section, we assume that $\mathrm{d}(H^*(X;\mathbb{Z})) = \mathrm{d}(H^*(X;\mathbb{Q}))$ and G acts trivially on $H^*(X;\mathbb{Q})$. By Proposition 4.4, $\mathrm{d}(H^*(X;\mathbb{Q}))$ is an odd integer, which we set $2n-1$ for the rest of this section. By Lemma 4.6, there are maps

$$\epsilon\colon S^{2n-1} \to X \quad \text{and} \quad \rho\colon X \to S^{2n-1}$$

such that $\rho \circ \epsilon\colon S^{2n-1} \to S^{2n-1}$ is of degree $M \neq 0$. By composing a degree -1 map with ϵ if necessary, we may assume $M > 0$.

We determine $\mathrm{N}\mathcal{E}(X)$. Let $(Y,B)^k$ denote the k-skeleton of a relative CW complex (Y,B).

Lemma 5.3. *For some positive integer* N*, there is a homotopy commutative diagram*

$$\begin{array}{ccc} X \vee S^{2n-1} & \xrightarrow{1 \vee N\epsilon} & X \\ {\scriptstyle \mathrm{incl}}\downarrow & & \Vert \\ X \times S^{2n-1} & \xrightarrow{\mu} & X. \end{array}$$

Proof. The first obstruction for extending $1 \vee k\epsilon\colon X \vee S^{2n-1} \to X$ with $k \in \mathbb{Z}$ over $X \times S^{2n-1}$ is given by

$$\mathfrak{o}(k) \in H^{m+1}(X \times S^{2n-1}, X \vee S^{2n-1}; \pi_m(X)).$$

Since there is a natural isomorphism

$$H^{m+1}(X \times S^{2n-1}, X \vee S^{2n-1}; \pi_m(X)) \cong H^{m+1}(X \wedge S^{2n-1}; \pi_m(X)),$$

the naturality of obstruction classes implies

$$\mathfrak{o}(k) = k\mathfrak{o}(1).$$

Since X is an H_0-space, the obstruction $\mathfrak{o}(1)$ becomes trivial after rationalization, so that $N_1\mathfrak{o}(1) = 0$ for some positive integer N_1. Then $1 \vee N_1\epsilon\colon X \vee S^{2n-1} \to X$ extends over $(X \times S^{2n-1}, X \vee S^{2n-1})^{m+1}$, so that the first obstruction for extending $1 \vee N_1\epsilon\colon X \vee S^{2n-1} \to X$ over $X \times S^{2n-1}$ belongs to

$$H^{l+1}(X \times S^{2n-1}, X \vee S^{2n-1}; \pi_l(X)).$$

for $l > m$. Arguing as above, we can see that $1 \vee N_1N_2\epsilon\colon X \vee S^{2n-1} \to X$ extends over $(X \times S^{2n-1}, X \vee S^{2n-1})^{l+1}$ for some positive integer N_2. Then since X is finite dimensional, the induction shows that there are positive integers $N_1, \ldots, N_r$ such that $1 \vee N_1 \cdots N_r\epsilon\colon X \vee S^{2n-1} \to X$ extends over $X \times S^{2n-1}$, completing the proof. □

Proposition 5.4. $\mathrm{N}\mathcal{E}(X) = \mathrm{d}(H^*(X;\mathbb{Q}))$.

Proof. Let $f\colon X \to X$ denote the composition of maps

$$X \xrightarrow{\Delta_X} X \times X \xrightarrow{1\times\rho} X \times S^{2n-1} \xrightarrow{\mu} X,$$

where Δ_X denotes the diagonal map of X and μ is as in Lemma 5.3. Then f is the identity map in π_* for $* < 2n-1$. We also have

$$\begin{aligned} f \circ \epsilon &= \mu \circ (1 \times \rho) \circ \Delta_X \circ \epsilon \\ &= \mu \circ (1 \times \rho) \circ (\epsilon \times \epsilon) \circ \Delta_{S^{2n-1}} \\ &= \mu \circ (\epsilon \times M) \circ \Delta_{S^{2n-1}} \\ &= \mu \circ (\epsilon \vee M) \circ \nabla \\ &= (1 + MN)\epsilon, \end{aligned}$$

where ∇ is the comultiplication of S^{2n-1} and N is as in Lemma 5.3. Since ϵ is of infinite order in $\pi_{2n-1}(X)$, f is not an isomorphism in π_{2n-1}, implying $\mathrm{N}\mathcal{E}(X) \geqslant 2n-1$. On the other hand, by Propositions 3.3 and 3.4,

$$\mathrm{N}\mathcal{E}(X) \leqslant \mathrm{N}^*\mathcal{E}(X) \leqslant \mathrm{d}(H^*(X;\mathbb{Z})) = \mathrm{d}(H^*(X;\mathbb{Q})) = 2n-1.$$

Thus we obtain $\mathrm{N}\mathcal{E}(X) = 2n-1$. ☐

We prove Theorem 1.2 by constructing a self-map of X/G similar to the self-map f of X in the proof of Proposition 5.4. First, we construct a map $\bar{\rho}\colon X/G \to S^{2n-1}$ which is compatible with ρ.

Lemma 5.5. *For any $\alpha \in \pi_i(S^{2n-1})$ with $i > 2n-1$, there is a positive integer k such that $k \circ \alpha = 0$.*

Proof. By assumption, α is of finite order. By [8, Theorem 6.7], $m \circ \alpha = m\alpha$ for $m \equiv 0 \mod 4$. Then for $k = 4|\alpha|$, we have $k \circ \alpha = k\alpha = 0$, where $|\alpha|$ denotes the order of α. ☐

Let $q\colon X \to X/G$ denote the projection.

Lemma 5.6. *For some positive integer K, there is a homotopy commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{\rho} & S^{2n-1} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle K} \\ X/G & \xrightarrow{\bar{\rho}} & S^{2n-1}. \end{array}$$

Proof. Let I_q denote the mapping cylinder of $q\colon X \to X/G$. Then it suffices to show that for some positive integer k, the map $k \circ \rho\colon X \to S^{2n-1}$ extends over I_q. A possible first non-trivial obstruction for such an extension is given by

$$\mathfrak{o}(k) \in H^{2n}(I_q, X; \pi_{2n-1}(S^{2n-1})).$$

By the naturality of obstruction classes,

$$\mathfrak{o}_1(k) = k\mathfrak{o}_1(1).$$

By Lemma 5.1, we have $H^{2n}(I_q, X; \pi_{2n-1}(S^{2n-1})) \otimes \mathbb{Q} \cong H^{2n}(I_q, X; \mathbb{Q}) = 0$, implying $\mathfrak{o}(K_1) = K_1\mathfrak{o}(1) = 0$ for some positive integer K_1. Then the first obstruction for extending $kK_1 \circ \rho\colon X \to S^{2n-1}$ extends over I_q is

$$\mathfrak{o}_2(k) \in H^m(I_q, X; \pi_{m-1}(S^{2n-1}))$$

for $m > 2n$. By the naturality of obstruction classes,

$$\mathfrak{o}_2(k) = k\mathfrak{o}_2(1).$$

Then since $\pi_{m-1}(S^{2n-1})$ is of finite order, it follows from Lemma 5.5 that there is a positive integer K_2 such that $\mathfrak{o}_2(K_2) = K_2\mathfrak{o}_2(1) = 0$. Thus the induction on the skeleton of a relative CW complex (I_q, X) works, and so since I_q is finite dimensional, there are positive integers $K_1, \dots, K_r$ such that the map $K_1 \cdots K_r \circ \rho\colon X \to S^{2n-1}$ extends over I_q. Therefore, the proof is complete. $\square$

Next, we construct a map

$$\bar{\mu}\colon X/G \times S^{2n-1} \to X/G$$

which is an analog of

$$\mu\colon X \times S^{2n-1} \to X.$$

To this end, we will use the following lemma. Let $\underline{\pi_*(X/G)}$ denote the local coefficient system defined by the natural action of $G = \pi_1(X/G)$ on $\pi_*(X/G)$.

Lemma 5.7. *There is an isomorphism*

$$H^{*+1}(X/G \times S^{2n-1}, X/G \vee S^{2n-1}; \underline{\pi_*(X/G)}) \cong H^{*-2n+2}(X/G, \bar{x}_0; \underline{\pi_*(X/G)}) \otimes H^{2n-1}(S^{2n-1}; \mathbb{Z})$$

which is natural with respect to a self-map of S^{2n-1}, where $\bar{x}_0$ is the basepoint of X/G.

Proof. Let $W = X/G \vee S^{2n-1}$, and let $\widetilde{W}$ denote its universal cover. Since G acts freely on $C_*(X)$, $C_*(X \times S^{2n-1})$ and $C_*(\widetilde{W})$ are free G-modules, where $C_*(-)$ denotes a cellular chain complex. As in [**18**, (4.1.4)],

$$H^{*+1}(X/G \times S^{2n-1}, X/G \vee S^{2n-1}; \underline{\pi_*(X/G)})$$

is the cohomology of

$$\mathrm{Hom}_G(C_{*+1}(X \times S^{2n-1})/C_{*+1}(\widetilde{W}), \pi_*(X/G)),$$

where $\mathrm{Hom}_G(-,-)$ denotes the abelian group of G-homomorphisms between the specified G-modules. Then we identify a G-module $C_{*+1}(X \times S^{2n-1})/C_{*+1}(\widetilde{W})$. Note that $\widetilde{W}$ is obtained by attaching S^{2n-1} to X at each $x_0 g \in X$ for $g \in G$, where x_0 is the basepoint of X. Then

$$C_{*+1}(\widetilde{W}) = C_{*+1}(X) \oplus (C_0(x_0 G) \otimes C_{*+1}(S^{2n-1}))$$

so that there is an isomorphism of G-modules

$$C_{*+1}(X \times S^{2n-1})/C_{*+1}(\widetilde{W}) \cong C_{*-2n+2}(X, x_0 G) \otimes C_{2n-1}(S^{2n-1}).$$

Thus the proof is done. $\square$

Now we prove:

Lemma 5.8. *For some positive integer L, there is a homotopy commutative diagram*

$$\begin{array}{ccc} X/G \vee S^{2n-1} & \xrightarrow{1\vee L(q\circ\epsilon)} & X/G \\ {\scriptstyle \mathrm{incl}}\downarrow & & \Vert \\ X/G \times S^{2n-1} & \xrightarrow{\bar{\mu}} & X/G. \end{array}$$

Proof. Since $X/G \times S^{2n-1}$ is obtained from $X/G \vee S^{2n-1}$ by attaching cells of dimensions $\geqslant 2n$, as in [18, Theorem 4.3.7], the first obstruction for extending

$$1 \vee l(q \circ \epsilon)\colon X/G \vee S^{2n-1} \to X/G$$

over $X/G \times S^{2n-1}$ is given by

$$\mathfrak{o}(l) \in H^{*+1}(X/G \times S^{2n-1}, X/G \vee S^{2n-1}; \pi_*(X/G))$$

for $* > 2n-1$. By Lemma 5.7,

$$\mathfrak{o}(l) = l\mathfrak{o}(1).$$

Then since $\pi_*(X/G)$ is of finite order for $* > 2n-1$, there is a positive integer L_1 such that $\mathfrak{o}(L_1) = L_1\mathfrak{o}(1) = 0$. Thus by arguing as in the proof of Lemma 5.3, the proof is finished. □

We are ready to prove Theorem 5.2.

Proof of Theorem 5.2. Define a map $\bar{f}\colon X/G \to X/G$ by the composite

$$X/G \xrightarrow{\Delta_{X/G}} X/G \times X/G \xrightarrow{1\times\bar{\rho}} X/G \times S^{2n-1} \xrightarrow{\bar{\mu}} X/G,$$

where $\bar{\rho}$ and $\bar{\mu}$ are as in Lemmas 5.6 and 5.8. Then $\bar{f}$ is the identity map in π_* for $* < 2n-1$. We also have

$$\begin{aligned} \bar{f} \circ q \circ \epsilon &= \bar{\mu} \circ (1 \times \bar{\rho}) \circ \Delta_{X/G} \circ q \circ \epsilon \\ &= \bar{\mu} \circ (1 \times \bar{\rho}) \circ (q \circ \epsilon \times q \circ \epsilon) \circ \Delta_{S^{2n-1}} \\ &= \bar{\mu} \circ (q \circ \epsilon \times KM) \circ \Delta_{S^{2n-1}} \\ &= \bar{\mu} \circ (q \circ \epsilon \vee KM) \circ \nabla \\ &= (1 + KLM)(q \circ \epsilon), \end{aligned}$$

where K and L are as in Lemmas 5.6 and 5.8. Thus since $q \circ \epsilon$ is of infinite order in $\pi_{2n-1}(X/G)$, $\bar{f}$ is not an isomorphism in π_{2n-1}, implying $\mathrm{N}\mathcal{E}(X/G) \geqslant 2n-1$. Therefore, by Propositions 3.2 and 5.4, we obtain

$$2n-1 \leqslant \mathrm{N}\mathcal{E}(X/G) \leqslant \mathrm{N}\mathcal{E}(X) = 2n-1 = \mathrm{d}(H^*(X;\mathbb{Q})),$$

completing the proof. □

6. Proof of Theorem 1.3

Throughout this section, let X denote a finite co-H_0-space, and let G be a finite group acting on X freely and cellularly. Then Theorem 1.3 is equivalent to the following theorem, so that we will prove it.

Theorem 6.1. *If* $\mathrm{cd}(X/G) = \mathrm{d}(H^*(X;\mathbb{Q}))$ *and* G *acts trivially on* $H^*(X;\mathbb{Q})$*, then*

$$\mathrm{N}\mathcal{E}(X/G) = \mathrm{N}\mathcal{E}(X) = \mathrm{d}(H^*(X;\mathbb{Q})).$$

Theorem 6.1 is proved by dualizing the proof of Theorem 5.2. First, we show an easy implication of the condition in Theorem 6.1.

Lemma 6.2. *If* $\mathrm{cd}(X/G) = \mathrm{d}(H^*(X;\mathbb{Q}))$ *and* G *acts trivially on* $H^*(X;\mathbb{Q})$*, then*

$$\mathrm{cd}(X) = \mathrm{d}(H^*(X;\mathbb{Z})).$$

Proof. Since

$$H^*(X/G;\underline{\mathbb{Z}G}) \cong H^*(X;\mathbb{Z}), \quad \text{we have} \quad \mathrm{cd}(X/G) \geqslant \mathrm{cd}(X),$$

where $\underline{\mathbb{Z}G}$ denotes the local coefficient system. Clearly,

$$\mathrm{cd}(X) \geqslant \mathrm{d}(H^*(X;\mathbb{Z})) \geqslant \mathrm{d}(H^*(X;\mathbb{Q})).$$

Thus the condition $\mathrm{cd}(X/G) = \mathrm{d}(H^*(X;\mathbb{Q}))$ implies $\mathrm{cd}(X) = \mathrm{d}(H^*(X;\mathbb{Z}))$. □

Hereafter, we will always assume that $\mathrm{cd}(X/G) = \mathrm{d}(H^*(X;\mathbb{Q}))$ and the induced action of G on $H^*(X;\mathbb{Q})$ is trivial. Next, we compute $\mathrm{N}\mathcal{E}(X)$. Let $\mathrm{d}(H^*(X;\mathbb{Q})) = n$. Then by Lemma 4.6, there are maps

$$\epsilon\colon S^n \to X \quad \text{and} \quad \rho\colon X \to S^n$$

such that $\rho \circ \epsilon\colon S^n \to S^n$ is of degree $M > 0$.

Lemma 6.3. *There is a homotopy commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{\ \phi\ } & X \vee S^n \\ \Vert & & \downarrow{\scriptstyle \mathrm{incl}} \\ X & \xrightarrow{1\times\rho} & X \times S^n. \end{array}$$

Proof. We aim to lift a map $1 \times \rho\colon X \to X \times S^n$ to $X \vee S^n$ through the inclusion $X \vee S^n \to X \times S^n$, up to homotopy. The first obstruction for this lift belongs to

$$H^{*+1}(X;\pi_*(X \times S^n, X \vee S^n))$$

for some $* \geqslant 0$. By Lemma 6.2, $H^{*+1}(X;\pi_*(X \times S^n, X \vee S^n)) = 0$ for $* \geqslant n$. On the other hand, since $X \times S^n$ and $X \vee S^n$ have a common n-skeleton,

$$\pi_*(X \times S^n, X \vee S^n) = 0 \ \ \text{for} \ \ * < n,$$

implying

$$H^{*+1}(X;\pi_*(X \times S^n, X \vee S^n)) = 0 \ \ \text{for} \ \ * < n.$$

Then we obtain the desired lift, completing the proof. □

Proposition 6.4. $\mathrm{N}\mathcal{E}(X) = n$*.*

Proof. Define a self-map $f\colon X \to X$ by the composite

$$X \xrightarrow{\phi} X \vee S^n \xrightarrow{1\vee\epsilon} X \vee X \xrightarrow{\psi} X,$$

where ϕ is as in Lemma 6.3 and ψ denotes the folding map. Then f is the identity map in π_* for $* \leqslant n-1$. For a generator $u \in H_n(S^n;\mathbb{Z})$, we have

$$\begin{aligned}(f\circ\epsilon)_*(u) &= (\psi\circ(1\vee\epsilon)\circ\phi\circ\epsilon)_*(u)\\ &= \psi_*(\epsilon_*(u)\times 1 + 1\times(\epsilon\circ\rho\circ\epsilon)_*(u))\\ &= \psi_*(\epsilon_*(u)\times 1 + 1\times M\epsilon_*(u))\\ &= (1+M)\epsilon_*(u).\end{aligned}$$

Then since $\epsilon_*(u)$ is of infinite order and in the Hurewicz image, we get $\mathrm{N}\mathcal{E}(X/G) \geqslant n$. On the other hand, it follows from Propositions 3.3 and 3.4 that

$$\mathrm{N}\mathcal{E}(X) \leqslant \mathrm{N}^*\mathcal{E}(X) \leqslant \mathrm{d}(H^*(X;\mathbb{Z})) = \mathrm{d}(H^*(X;\mathbb{Q})) = n.$$

Therefore, we obtain $\mathrm{N}\mathcal{E}(X) = n$, completing the proof. □

Next, we construct maps $\bar\rho\colon X/G \to S^n$ and $\bar\phi\colon X/G \to X/G \vee S^n$ having properties analogous to ρ and ϕ.

Lemma 6.5. *For some positive integer K, there is a homotopy commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{\rho} & S^n \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle K} \\ X/G & \xrightarrow{\bar\rho} & S^n. \end{array}$$

Proof. We extend $k\circ\rho\colon X \to S^n$ over the mapping cylinder I_q of $q\colon X \to X/G$. The first possible obstruction for such an extension is given by

$$\mathfrak{o}(k) \in H^{n+1}(I_q, X; \pi_n(S^n)).$$

Since $H^{n+1}(I_q, X;\pi_n(S^n))\otimes\mathbb{Q} \cong H^{n+1}(I_q, X;\mathbb{Q}) = 0$, $\mathfrak{o}(1)$ is of finite order. By the naturality of obstruction classes,

$$\mathfrak{o}(k) = k\mathfrak{o}(1).$$

Then $\mathfrak{o}(K) = K\mathfrak{o}(1) = 0$ for some positive integer K, and so the first obstruction for extending $K\circ\rho\colon X \to S^n$ over I_q belongs to

$$H^{*+1}(I_q, X; \pi_*(S^n))$$

for $* > n$. Since $\mathrm{cd}(X/G) = n$ and $\mathrm{cd}(X) = n$ by Lemma 6.2,

$$H^{*+1}(I_q;\pi_*(S^n)) = 0 \quad\text{and}\quad H^{*+1}(X;\pi_*(S^n)) = 0$$

for $* > n$. Then it follows from the cohomology exact sequence that

$$H^{*+1}(I_q, X;\pi_*(S^n)) = 0 \ \text{ for } \ * > n.$$

Thus $K\circ\rho\colon X \to S^n$ extends over I_q, completing the proof. □

Lemma 6.6. *There is a homotopy commutative diagram*

$$\begin{array}{ccc} X/G & \xrightarrow{\bar{\phi}} & X/G \vee S^n \\ \Vert & & \downarrow{\scriptstyle \mathrm{incl}} \\ X/G & \xrightarrow{1\times\bar{\rho}} & X/G \times S^n. \end{array}$$

Proof. The first obstruction for lifting

$$1 \times \bar{\rho}\colon X/G \to X/G \times S^n$$

to $X/G \vee S^n$ through the inclusion $X/G \vee S^n \to X/G \times S^n$, up to homotopy, belongs to

$$H^{*+1}(X/G; \mathcal{R}_*),$$

where $\mathcal{R}_*$ denotes the local coefficient system defined by the canonical G-action on $\pi_*(X/G \times S^n, X/G \vee S^n)$. Since $X/G \times S^n$ and $X/G \vee S^n$ have an n-skeleton in common, $\pi_*(X/G \times S^n, X/G \vee S^n) = 0$ for $* \leqslant n$, implying $H^{*+1}(X/G; \mathcal{R}_*) = 0$ for $* \leqslant n$. Since $\mathrm{cd}(X/G) = n$, we also have $H^{*+1}(X/G; \mathcal{R}_*) = 0$ for $* > n$. Then there is no obstruction for lifting $1 \times \bar{\rho}\colon X/G \to X/G \times S^n$, completing the proof. □

Now we are ready to prove Theorem 6.1.

Proof of Theorem 6.1. Define a self-map $\bar{f}\colon X/G \to X/G$ by the composite

$$X/G \xrightarrow{\bar{\phi}} X/G \vee S^n \xrightarrow{1\vee(q\circ\epsilon)} X/G \vee X/G \xrightarrow{\psi} X/G.$$

Then $\bar{f}$ is the identity map in π_* for $* < n$. Let u denote a generator of $H_n(S^n; \mathbb{Z}) \cong \mathbb{Z}$. Then

$$\begin{aligned}(\bar{f} \circ q \circ \epsilon)_*(u) &= (\psi \circ (1 \vee (q \circ \epsilon)) \circ \phi \circ q \circ \epsilon)_*(u) \\ &= \psi_*((q \circ \epsilon)_*(u) \times 1 + 1 \times (q \circ \epsilon \circ \bar{\rho} \circ q \circ \epsilon)_*(u)) \\ &= \psi_*((q \circ \epsilon)_*(u) \times 1 + 1 \times K(q \circ \epsilon \circ \rho \circ \epsilon)_*(u)) \\ &= \psi_*((q \circ \epsilon)_*(u) \times 1 + 1 \times KM(q \circ \epsilon)_*(u)) \\ &= (1 + KM)(q \circ \epsilon)_*(u).\end{aligned}$$

Since $(q \circ \epsilon)_*(u)$ is of infinite order and in the Hurewicz image, $\bar{f}$ is not an isomorphism in π_n, implying $\mathrm{N}\mathcal{E}(X/G) \geqslant n$. Therefore, by Propositions 3.2 and 6.4, we obtain

$$n \leqslant \mathrm{N}\mathcal{E}(X/G) \leqslant \mathrm{N}\mathcal{E}(X) = n = \mathrm{d}(H^*(X; \mathbb{Q})),$$

completing the proof. □

References

[1] A. Adem, J.F. Davis, and Ö. Ünlü, Fixity and free group actions on products of spheres, *Comment. Math. Helv.* **79** (2004), no. 4, 758–778.

[2] M. Arkowitz, *The group of self-homotopy equivalences – a survey, in Groups of Self-Equivalences and Related Topics* (Montreal, PQ, 1988), Lecture Notes in Math. **1425**, pp. 170–203, Springer, Berlin-Heidelberg, 1990.

[3] M. Arkowitz and C.R. Curjel, The Hurewicz homomorphism and finite homotopy invariants, *Trans. Amer. Math. Soc.* **110** (1964), no. 3, 538–551.

[4] Z. Blaszczyk, Free $\mathcal{A}_4$-actions on products of spheres, *Trans. Inst. Math. NAS Ukraine* **10** (2013), no. 6, 86–96.

[5] H.W. Choi and K.Y. Lee, Certain numbers on the groups of self-homotopy equivalences, *Topology Appl.* **181** (2015), 104–111.

[6] J.F. Davis and R.J. Milgram, Free actions on products of spheres: The rational case, *Math. Z.* **207** (1991), no. 3, 379–390.

[7] A. Hatcher, *Algebraic Topology*, Cambridge University Press, Cambridge, 2002.

[8] P. Hilton, On the homotopy groups of the union of spheres, *J. Lond. Math. Soc.* **30** (1955), no. 2, 154–172.

[9] P. Hilton, G. Mislin, and J. Roitberg, *Localization of Homotopy Types, Localization of Nilpotent Groups and Spaces*, North-Holland Mathematics Studies **15**, North-Holland, 1975.

[10] D. Kishimoto and N. Oda, Monoids of self-maps of topological spherical space forms, *Homol. Homotopy Appl.* **23** (2020), no. 2, 141–149.

[11] P. Li, (Co)homology self-closeness numbers of simply-connected spaces, *Homol. Homotopy Appl.* **23** (2021), no. 1, 1–16.

[12] M. Mimura, R.C. O'Neill, and H. Toda, On p-equivalence in the sense of Serre, *Jpn. J. Math.* **40** (1971), 1–10.

[13] N. Oda and T. Yamaguchi, Self-homotopy equivalences and cofibrations, *Topology Appl.* **228** (2017), 341–354.

[14] N. Oda and T. Yamaguchi, Self-maps of spaces in fibrations, *Homol. Homotopy Appl.* **20** (2018), no. 2, 289–313.

[15] N. Oda and T. Yamaguchi, Self-closeness numbers of finite cell complexes, *Topology Appl.* **272** (2020), 107062.

[16] T. Yamaguchi, Relative self-closeness numbers, *Bull. Korean Math. Soc.* **58** (2021), no. 2, 445–449.

[17] M. Mimura and H. Toda, On p-equivalences and p-universal spaces, *Comment. Math. Helv.* **46** (1971), 87–97.

[18] H. J. Baues, *Obstruction Theory on Homotopy Classification of Maps*, Lecture Notes in Mathematics **628**, Springer-Verlag, Berlin-New York, 1977.

Yichen Tong tong.yichen.25m@st.kyoto-u.ac.jp

Department of Mathematics, Kyoto University, Kyoto, 606-8502, Japan