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TWO THEOREMS ON COHOMOLOGICAL PAIRINGS

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(*communicated by Daniel C. Isaksen*)

Abstract

We give a new, elegant description of the Tate duality pairing as a Brauer–Manin pairing for associated embedding problems and prove a new theorem on the cup product.

1. Introduction

The aim of this paper to prove two theorems on cohomological pairings which are crucial ingredients in the companion paper [**6**], but we think are quite interesting on their own, too. The first such result is fairly simple to state.

Definition 1.1. Let M be a finite abelian group and let

$$1 \longrightarrow M \longrightarrow \Omega \longrightarrow \Pi \longrightarrow 1 \tag{1.1.1}$$

be an exact sequence in the category of prodiscrete groups. Suppose A is a discrete Π-module and let $H^2(\Omega, A)_0$ denote the kernel of the restriction map:

$$H^2(\Omega, A) \longrightarrow H^2(M, A).$$

Moreover let

$$\delta\colon H^2(\Omega, A)_0 \longrightarrow H^1(\Pi, \mathrm{Hom}(M, A)) = H^1(\Pi, H^1(M, A))$$

be the homomorphism furnished by the Hochschild–Serre spectral sequence:

$$H^p(\Pi, H^q(M, A)) \Rightarrow H^{p+q}(\Omega, A),$$

where we equip M with its Π-module structure induced by the exact sequence (1.1.1). For every pair of sections $s_1, s_2\colon \Pi \to \Omega$ of the exact sequence (1.1.1) the 1-cochain in $C^1(\Pi, M)$ given by the rule $g \mapsto s_1(g)s_2(g)^{-1}$ is actually a cocycle. Now suppose $[s_1 - s_2] \in H^1(\Pi, M)$ is the cohomology class represented by this cocycle. Finally let

$$\cup\colon H^1(\Pi, M) \times H^1(\Pi, \mathrm{Hom}(M, A)) \longrightarrow H^2(\Pi, A)$$

be the cup product induced by the evaluation map $M \otimes \mathrm{Hom}(M, A) \to A$.

As a consequence to our first main result (Theorem 2.3) we will deduce the following

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Corollary 1.2. *For every $c \in H^2(\Omega, A)_0$ and for every pair of sections s_1, s_2 of the exact sequence* (6.6.1) *we have:*

$$s_1^*(c) - s_2^*(c) = [s_1 - s_2] \cup \delta(c) \in H^2(\Pi, A).$$

Now let F be an arbitrary field. Fix a separable closure $\overline{F}$ of F and denote by $\Gamma = \mathrm{Gal}(\overline{F}|F)$ the absolute Galois group of F. An embedding problem $\mathbf{E}$ over F is a diagram:

$$\begin{array}{ccc} & & \Gamma \\ & & \downarrow{\scriptstyle \psi} \\ G_1 & \xrightarrow{\phi} & G_2, \end{array} \tag{1.2.1}$$

where G_1, G_2 are finite groups, ϕ and ψ are group homomorphisms, the map ϕ is surjective, and ψ is assumed to be continuous with respect to the Krull topology on Γ and the discrete topology on G_2. We say that the embedding problem $\mathbf{E}$ is solvable if there is a continuous homomorphism $\widetilde{\psi}\colon \Gamma \to G_1$ which makes the diagram above commutative. We will call such a homomorphism $\widetilde{\psi}$ a solution of $\mathbf{E}$. Then we let $\mathrm{Ker}(\mathbf{E}) = \mathrm{Ker}(\phi)$. We will say that two solutions of $\mathbf{E}$ are conjugate if they are conjugate by an element of $\mathrm{Ker}(\mathbf{E})$. Conjugacy is clearly an equivalence relation. Let $\mathrm{Sol}(\mathbf{E})$ denote the set of equivalence classes of this relation. Note that in the subject of field arithmetic it is common to define a solution to an embedding problem to be a surjective continuous homomorphism $\widetilde{\psi}\colon \Gamma \to G_1$, and to refer to our notion of solution as a weak solution. In this paper we will not impose the surjectivity condition.

Assume now that F is a global field. In this case there is an obvious family of obstructions to the solvability of $\mathbf{E}$ which we will call local obstructions. Let $|F|$ denote the set of all places of F and for every $x \in |F|$ let F_x denote the completion of F with respect to x. Fix a separable closure $\overline{F}_x$ of F_x and let $\Gamma_x = \mathrm{Gal}(\overline{F}|F)$ denote the absolute Galois group of F_x. The choice of an F-embedding $\eta_x\colon \overline{F} \to \overline{F}_x$ induces an injective homomorphism $\iota_x\colon \Gamma_x \to \Gamma$ whose conjugacy class is actually independent of these choices. Let $\mathbf{E}$ be an embedding problem over F. Then for every $x \in |F|$ we define the embedding problem $\mathbf{E}_x$ over F_x associated to $(\mathbf{E}, x)$ to be the diagram:

$$\begin{array}{ccc} & & \Gamma_x \\ & & \downarrow{\scriptstyle \iota_x \circ \psi} \\ G_1 & \xrightarrow{\phi} & G_2. \end{array}$$

Clearly the embedding problem $\mathbf{E}_x$ is solvable if the problem $\mathbf{E}$ is; this is the local obstruction we mentioned above. For every non-archimedean $x \in |F|$ let $u_x\colon \Gamma_x \to \widehat{\mathbb{Z}}$ denote the homomorphism onto the Galois group of the maximal unramified extension of F_x in $\overline{F}_x$. We say that a continuous homomorphism $h\colon \Gamma_x \to G$ is unramified if x is non-archimedean and h factors through u_x.

Clearly a solution conjugate to an unramified solution is also unramified. Let

$\mathrm{Sol}_{un}(\mathbf{E}_x)$ be the set of conjugacy classes of all solutions of $\mathbf{E}_x$ which are unramified in the sense above. Let $\mathrm{Sol}_{\mathbb{A}}(\mathbf{E})$ denote the set:

$$\mathrm{Sol}_{\mathbb{A}}(\mathbf{E}) = \Big\{ \prod_{x\in|F|} h_x | h_x \in \mathrm{Sol}_{un}(\mathbf{E}_x) \text{ for almost all } x \in |F| \Big\} \subseteq \prod_{x\in|F|} \mathrm{Sol}(\mathbf{E}_x).$$

Similarly by an adèlic solution of the embedding problem $\mathbf{E}$ we mean an expression $\prod_{x\in|F|} h_x$ such that h_x is a solution of the embedding problem $\mathbf{E}_x$ for every $x \in |F|$ which is unramified for almost all x. For each $x \in |F|$ let $r_x \colon \mathrm{Sol}(\mathbf{E}) \to \mathrm{Sol}(\mathbf{E}_x)$ denote the map furnished by the rule $\widetilde{\psi} \mapsto \iota_x \circ \widetilde{\psi}$. Then image of the map

$$r = \prod_{x\in|F|} r_x \colon \mathrm{Sol}(\mathbf{E}) \to \prod_{x\in|F|} \mathrm{Sol}(\mathbf{E}_x)$$

lies in $\mathrm{Sol}_{\mathbb{A}}(\mathbf{E})$. In order to study the image of $\mathrm{Sol}(\mathbf{E})$ in $\mathrm{Sol}_{\mathbb{A}}(\mathbf{E})$ under the map r we will define the Brauer group $\mathrm{Br}(\mathbf{E})$ of the embedding problem $\mathbf{E}$ (see Definition 2.2) and a pairing:

$$\langle \cdot, \cdot \rangle \colon \mathrm{Sol}_{\mathbb{A}}(\mathbf{E}) \times \mathrm{Br}(\mathbf{E}) \to \mathbb{Q}/\mathbb{Z}$$

(see Definition 3.6) such that $r(\mathrm{Sol}(\mathbf{E}))$ is annihilated by this pairing (see Lemma 3.5). (This idea can be found already in the paper [**10**].) Moreover we will also define two subgroups $\mathrm{Br}_1(\mathbf{E})$ and $\text{Б}(\mathbf{E})$ of $\mathrm{Br}(\mathbf{E})$ (see Definition 3.2) analogous to the algebraic Brauer group and the Brauer group of locally constant elements of algebraic varieties, respectively, playing an important role in our main results. Let $\mathrm{Ker}(\mathbf{E})_{ab}$ denote the abelianization of $\mathrm{Ker}(\mathbf{E})$. Note that the natural action of G_1 on $\mathrm{Ker}(\mathbf{E})_{ab}$ via conjugation factors through G_2 hence $\mathrm{Ker}(\mathbf{E})_{ab}$ is naturally equipped with a Γ-action. Let $\mathrm{Ker}(\mathbf{E})_{ab}^\vee$ denote its dual as a Γ-module. We will show that the group $\mathrm{Br}(\mathbf{E})$ sits in an exact sequence:

$$0 \to H^1(\Gamma, \mathrm{Ker}(\mathbf{E})_{ab}^\vee) \xrightarrow{j_{\mathbf{E}}} \mathrm{Br}(\mathbf{E}) \to H^2\Big(\mathrm{Ker}(\mathbf{E}), \bigoplus_{p\neq \mathrm{char}(F)} \mathbb{Q}_p/\mathbb{Z}_p\Big)^\Gamma \to H^2(\Gamma, \mathrm{Ker}(\mathbf{E})_{ab}^\vee)$$

(see Lemma 3.1) and we will identify the subgroups $\mathrm{Br}_1(\mathbf{E})$ and $\text{Б}(\mathbf{E})$ with the image of the group $H^1(\Gamma, \mathrm{Ker}(\mathbf{E})_{ab}^\vee)$ and its Tate–Shafarevich subgroup $Ш^1(F, \mathrm{Ker}(\mathbf{E})_{ab}^\vee)$ under the map $j_{\mathbf{E}}$, respectively (see Definition 3.2 and Proposition 3.4).

Definition 1.3. Let F be again an arbitrary field with absolute Galois group Γ and let $\mathbf{E}$ be an embedding problem given by the diagram (1.2.1). Let $\Gamma(\mathbf{E})$ denote the fibre product group:

$$\Gamma(\mathbf{E}) = \{(a,b) \in G_1 \times \Gamma | \phi(a) = \psi(b)\} \leqslant G_1 \times \Gamma.$$

Then $\Gamma(\mathbf{E})$ sits in the exact sequence:

$$1 \longrightarrow \mathrm{Ker}(\mathbf{E}) \xrightarrow{i_{\mathbf{E}}} \Gamma(\mathbf{E}) \xrightarrow{\pi_{\mathbf{E}}} \Gamma \longrightarrow 1, \tag{1.3.1}$$

where the map $i_{\mathbf{E}}$ is given by the rule $a \mapsto (a,1)$, and the homomorphism $\pi_{\mathbf{E}}$ is the restriction onto $\Gamma(\mathbf{E})$ of the projection of $G_1 \times \Gamma$ onto the second factor. The group $\Gamma(\mathbf{E})$ also inherits a topology from the product topology on $G_1 \times \Gamma$ which makes $\Gamma(\mathbf{E})$ a profinite group and (1.3.1) an exact sequence in the category of Hausdorff topological groups in the sense that $\mathrm{Ker}(\mathbf{E})$ is a closed normal subgroup, it is the kernel of the continuous surjective homomorphism $\pi_{\mathbf{E}}$, and Γ is equipped with the quotient topology.

Note that discrete Γ-modules are actually the abelian sheaves on a site, so they form an abelian category. Let M be a discrete finite abelian Γ-module. For every embedding problem $\mathbf{E}$ over F such that $\mathrm{Ker}(\mathbf{E})=M$ and the set $\mathrm{Sol}_{\mathbb{A}}(\mathbf{E})$ is non-empty let $c_{\mathbf{E}}\in H^2(F,M)=H^2(F,\mathrm{Ker}(\mathbf{E}))$ denote the class of the extension (1.3.1). Note that $c_{\mathbf{E}}\in Ш^2(F,M)$ since we assumed that $\mathrm{Sol}_{\mathbb{A}}(\mathbf{E})$ is non-empty. Conversely for every $c\in Ш^2(F,M)$ there is an embedding problem $\mathbf{E}$ as above such that $c=c_{\mathbf{E}}$. Let

$$б\colon Ш^1(F,M^\vee)\times Ш^2(F,M)\to\mathbb{Q}/\mathbb{Z} \tag{1.3.2}$$

be the unique pairing such that $б(b,c_{\mathbf{E}})=\langle h,b\rangle$ for every $b\in Ш^1(F,M^\vee)$, for every $h\in\mathrm{Sol}_{\mathbb{A}}(\mathbf{E})$, and embedding problem $\mathbf{E}$ as above. The pairing $б$ is well-defined. Assume now that $\mathrm{char}(F)$ does not divide the order of M and let

$$\tau\colon Ш^1(F,M^\vee)\times Ш^2(F,M)\longrightarrow\mathbb{Q}/\mathbb{Z}$$

denote the Tate duality pairing. Our second main theorem gives an elegant description of the Tate duality pairing in terms of the Brauer–Manin pairing:

Theorem 1.4. *We have* $б=-\tau$.

In the paper [2] Harari and Szamuely gave a geometric interpretation of the duality pairing. Our result offers another geometric interpretation by relating it to the Brauer–Manin pairing.

Contents 1.5. In the second chapter we prove a theorem on the cup product in topology which we use to deduce a similar statement (Corollary 1.2) in group cohomology. We will define the Brauer group of embedding problems and study its structure and introduce the analogue of the Manin pairing in the third chapter. We compare the Tate duality pairing with the Brauer–Manin paring in the fourth chapter.

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2. A topological theorem on the cup product

Definition 2.1. In this chapter all topological spaces are Hausdorff and locally contractible. For every topological space T and abelian group A let A_T denote the constant sheaf A on T. Let $p\colon X\to Y$ be a fibre bundle with a connected fibre F. Let $r\colon Y\to X$ and $s\colon Y\to X$ be sections of the fibration p. Let $p_\natural$ denote the derived left adjoint of the pull-back functor p^* from the category of complexes of sheaves on Y to category of complexes of sheaves on X. (The adjoint $p_\natural$ exists because p^* commutes with arbitrary limits, since we assumed p to be a fibre bundle, and so we may apply Freyd's adjoint functor theorem.) By functoriality both r and s induces maps $r^h,s^h\in[\mathbb{Z}_Y,p_\natural(\mathbb{Z}_X)]$ in the derived category of complexes of sheaves on Y. Let $\deg \in [p_\natural(\mathbb{Z}_X), \mathbb{Z}_Y]$ denote the map from $p_\natural(\mathbb{Z}_X)$ onto its 0-th homology. (The latter is $\mathbb{Z}_Y$ because we assumed that the fibres are connected.) Since both r and s are sections, the compositions of r^h, s^h with deg are both the identity map in $[\mathbb{Z}_Y, \mathbb{Z}_Y]$. Let $\tau_{>0}(p_\natural(\mathbb{Z}_X))$ be the fibre of the map deg in the derived category and let

$$\tau_{>0}(p_\natural(\mathbb{Z}_X)) \xrightarrow{f_0} p_\natural(\mathbb{Z}_X) \xrightarrow{\deg} \mathbb{Z}_Y \longrightarrow \tau_{>0}(p_\natural(\mathbb{Z}_X))[1]$$

be the corresponding distinguished triangle. Therefore their difference

$$r^h - s^h \in [\mathbb{Z}_Y, p_\natural(\mathbb{Z}_X)]$$

is the image of a map $[r-s] \in [\mathbb{Z}_Y, \tau_{>0}(p_\natural(\mathbb{Z}_X))]$ such that $r^h - s^h = f_0 \circ [r-s]$. This map is unique since $[\mathbb{Z}_Y, \mathbb{Z}_Y[-1]]$ is zero. Let $\mathcal{B}$ be the first homology of the complex $\tau_{>0}(p_\natural(\mathbb{Z}_X))$, let $h_1 \colon \tau_{>0}(p_\natural(\mathbb{Z}_X)) \to \mathcal{B}[1]$ be the Postnikov truncation, and consider the distinguished triangle:

$$\tau_{>1}(p_\natural(\mathbb{Z}_X)) \xrightarrow{f_1} \tau_{>0}(p_\natural(\mathbb{Z}_X)) \xrightarrow{h_1} \mathcal{B}[1] \longrightarrow \tau_{>1}(p_\natural(\mathbb{Z}_X))[1].$$

Let $\Delta(r,s) \in [\mathbb{Z}_Y, \mathcal{B}[1]] = H^1(Y, \mathcal{B})$ denote $h_1 \circ [r-s]$.

Definition 2.2. For every $y \in Y$ let $X_y = p^{-1}(y)$ be the fibre of p over y and let $i_y \colon X_y \to X$ denote the inclusion map. Let $\mathcal{A}$ be a locally constant sheaf of abelian groups on the base Y and let $H^2(X, p^*(\mathcal{A}))_0$ denote the intersection of the kernels of the maps:

$$i_x^* \colon H^2(X, p^*(\mathcal{A})) \longrightarrow H^2(X_y, p^*(\mathcal{A})|_{X_y})$$

for every $y \in Y$. Note that

$$H^2(X, p^*(\mathcal{A}))_0 = \bigcap_{y \in S} \ker(i_y^*),$$

where $S \subseteq Y$ is any set such that for every connected component $C \subseteq Y$ there is a $y \in C \cap S$. Note that $H^2(X, p^*(\mathcal{A}))_0$ is the kernel of the edge homomorphism:

$$\epsilon \colon H^2(X, p^*(\mathcal{A})) \longrightarrow H^0(Y, R^2 p_*(p^*(\mathcal{A})))$$

furnished by the Leray spectral sequence:

$$H^p(Y, R^q p_*(p^*(\mathcal{A}))) \Rightarrow H^{p+q}(X, p^*(\mathcal{A})).$$

So the higher edge homomorphism of this spectral sequence is a homomorphism:

$$\delta \colon H^2(X, p^*(\mathcal{A}))_0 \longrightarrow H^1(Y, R^1 p_*(p^*(\mathcal{A}))) = H^1(Y, \mathrm{Hom}(\mathcal{B}, \mathcal{A})),$$

where we used that there is a natural isomorphism:

$$R^1 p_*(p^*(\mathcal{A})) = \mathrm{Hom}(\mathcal{B}, \mathcal{A}).$$

Let

$$\cup \colon H^1(Y, \mathcal{B}) \times H^1(Y, \mathrm{Hom}(\mathcal{B}, \mathcal{A})) \longrightarrow H^2(Y, \mathcal{A})$$

be the cup product furnished by the evaluation map:

$$\mathcal{B} \otimes \mathrm{Hom}(\mathcal{B}, \mathcal{A}) \longrightarrow \mathcal{A}.$$

Note that every section s of p induces a homomorphism:

$$s^*\colon H^i(X, p^*(\mathcal{A})) \longrightarrow H^i(Y, s^*(p^*(\mathcal{A}))) = H^i(Y, \mathcal{A}) \quad (\forall i \in \mathbb{N}).$$

Theorem 2.3. *For every $c \in H^2(X, p^*(\mathcal{A}))_0$ and for every pair of sections r, s of the fibration p we have:*

$$r^*(c) - s^*(c) = \Delta(r, s) \cup \delta(c) \in H^2(Y, \mathcal{A}).$$

Proof. For every topological space T and complexes of sheaves $\mathcal{C}, \mathcal{D}$ and $\mathcal{E}$ on T let

$$m(\mathcal{C}, \mathcal{D}, \mathcal{E})\colon [\mathcal{C}, \mathcal{D}] \times [\mathcal{D}, \mathcal{E}] \longrightarrow [\mathcal{C}, \mathcal{E}]$$

denote the map given by the rule $(f, g) \mapsto f \circ g$. Moreover for every $f \in [\mathcal{C}, \mathcal{D}]$ let

$$f\circ\colon [\mathcal{D}, \mathcal{E}] \longrightarrow [\mathcal{C}, \mathcal{E}]$$

denote the map given by the rule $g \mapsto f \circ g$, and similarly for every $g \in [\mathcal{D}, \mathcal{E}]$ let

$$\circ g\colon [\mathcal{C}, \mathcal{D}] \longrightarrow [\mathcal{C}, \mathcal{E}]$$

denote the map given by the rule $f \mapsto f \circ g$. Then we have the following commutative diagram:

$$\begin{array}{ccc}
[\mathbb{Z}_Y, \mathcal{B}[1]] \times [\mathcal{B}[1], \mathcal{A}[2]] & \xrightarrow{m(\mathbb{Z}_Y, \mathcal{B}[1], \mathcal{A}[2])} & \\
{\scriptstyle \circ h_1}\uparrow \qquad \downarrow{\scriptstyle h_1 \circ} & & \\
[\mathbb{Z}_Y, \tau_{>0}(p_\natural(\mathbb{Z}_X))] \times [\tau_{>0}(p_\natural(\mathbb{Z}_X)), \mathcal{A}[2]] & \xrightarrow{m(\mathbb{Z}_Y, \tau_{>0}(p_\natural(\mathbb{Z}_X)), \mathcal{A}[2])} & [\mathbb{Z}_Y, \mathcal{A}[2]]. \\
{\scriptstyle \circ f_0}\downarrow \qquad \uparrow{\scriptstyle f_0 \circ} & & \\
[\mathbb{Z}_Y, p_\natural(\mathbb{Z}_X)] \times [p_\natural(\mathbb{Z}_X), \mathcal{A}[2]] & \xrightarrow{m(\mathbb{Z}_Y, p_\natural(\mathbb{Z}_X), \mathcal{A}[2])} &
\end{array}$$

For every section $t\colon Y \to X$ of p let $t^h \in [\mathbb{Z}_Y, p_\natural(\mathbb{Z}_X)]$ be the map induced by t, similarly to the notation we introduced in Definition 2.1. Note that under the identification $[p_\natural(\mathbb{Z}_X), \mathcal{A}[2]] = H^2(X, \mathcal{A})$, for every section $t\colon Y \to X$ as above and each $c \in H^2(X, \mathcal{A})$ we have $t^h \circ c = t^*(c)$. Therefore

$$r^*(c) - s^*(c) = r^h \circ c - s^h \circ c = (r^h - s^h) \circ c = [r - s] \circ (f_0 \circ c)$$

by the commutativity of the diagram above. Note that

$$[\mathcal{B}[1], \mathcal{A}[2]] = [\mathcal{B}, \mathcal{A}[1]] = H^1(Y, \mathrm{Hom}(\mathcal{B}, \mathcal{A})),$$

and under this identification $\delta(c) \in [\mathcal{B}[1], \mathcal{A}[2]]$ is such that $h_1 \circ \delta(c) = f_0 \circ c$. Indeed

$$[\tau_{>1}(p_\natural(\mathbb{Z}_X)), \mathcal{A}[2])] = H^0(Y, R^2 p_*(p^*(\mathcal{A}))$$

and under this identification

$$(f_1 \circ f_0)\circ\colon [p_\natural(\mathbb{Z}_X), \mathcal{A}[2])] \longrightarrow [\tau_{>1}(p_\natural(\mathbb{Z}_X)), \mathcal{A}[2])]$$

is the edge homomorphism ϵ in Definition 2.2. In particular the kernel of $(f_1 \circ f_0)\circ$ is $H^2(X, \mathcal{A})_0$. The second distinguished triangle in Definition 2.1 induces a long exact sequence:

$$\begin{array}{ccccc} [\tau_{>1}(p_\natural(\mathbb{Z}_X))[1],\mathcal{A}[2])] & & & & \\ \downarrow & & & & \\ [\mathcal{B}[1],\mathcal{A}[2]] & \xrightarrow{h_1\circ} & [\tau_{>0}(p_\natural(\mathbb{Z}_X)),\mathcal{A}[2])] & \xrightarrow{f_1\circ} & [\tau_{>1}(p_\natural(\mathbb{Z}_X)),\mathcal{A}[2])]. \end{array}$$

Since

$$[\tau_{>1}(p_\natural(\mathbb{Z}_X))[1],\mathcal{A}[2])] = [\tau_{>1}(p_\natural(\mathbb{Z}_X)),\mathcal{A}[1])] = 0,$$

there is a unique homomorphism:

$$\partial\colon H^2(X,\mathcal{A})_0 \longrightarrow [\mathcal{B}[1],\mathcal{A}[2]] = H^1(Y,\mathrm{Hom}(\mathcal{B},\mathcal{A}))$$

such that the diagram

$$\begin{array}{ccc} H^2(X,\mathcal{A})_0 & & \\ \downarrow{\scriptstyle\partial} & \searrow^{f_0\circ|_{H^2(X,\mathcal{A})_0}} & \\ [\mathcal{B}[1],\mathcal{A}[2]] & \xrightarrow{h_1\circ} & [\tau_{>0}(p_\natural(\mathbb{Z}_X)),\mathcal{A}[2])] \end{array}$$

is commutative. The map ∂ is actually the higher edge homomorphism δ in Definition 2.2. Now the relation $h_1\circ\delta(c) = f_0\circ c$ is clear. Now $[\mathbb{Z}_Y,\mathcal{B}[1]] = H^1(Y,\mathcal{B})$ and $[\mathbb{Z}_Y,\mathcal{B}[2]] = H^2(Y,\mathcal{B})$, and under these identifications $m(\mathbb{Z}_Y,\mathcal{B}[1],\mathcal{A}[2])$ is the cup product:

$$\cup\colon H^1(Y,\mathcal{B})\times H^1(Y,\mathrm{Hom}(\mathcal{B},\mathcal{A})) \longrightarrow H^2(Y,\mathcal{A})$$

in Definition 2.2 above. So by using the commutativity of the diagram above again we get that

$$[r-s]\circ(f_0\circ c) = \Delta(r,s)\circ\delta(c) = \Delta(r,s)\cup\delta(c),$$

and the theorem follows. □

Proof of Corollary 1.2. We may assume that Π is actually finite by applying the usual limit argument. The proof will be based on giving topological interpretation to both sides of the equation. The homomorphism $\Omega\to\Pi$ furnishes a Serre-fibration of classifying spaces $p\colon B\Omega\to B\Pi$ with fibre BM. The sections s_1, s_2 induce sections of the fibration p which we will denote by the same symbols by abuse of notation. Let $\mathcal{B}$ denote the locally constant sheaf on $B\Pi$ corresponding to the Π-module A. Then $H^1(\Omega,M) = H^1(B\Omega,\mathcal{B})$ and the cohomology classes denoted by $[s_1-s_2]$ in Definitions 2.1 and 1.1 correspond to each other. There is a locally constant sheaf $\mathcal{A}$ on $B\Pi$ corresponding to the Π-module A. Note that $H^2(\Omega,A) = H^2(B\Omega,\mathcal{A})$, and also $H^2(\Omega,A)_0 = H^2(B\Omega,\mathcal{A})_0$, where we use the notation of Definition 2.2 for the fibration p. Moreover $H^1(\Omega,A) = H^2(B\Omega,\mathcal{A})$, and the edge homomorphisms

$$H^2(\Omega,A)_0 \longrightarrow H^1(\Pi,\mathrm{Hom}(M,A)) \text{ and } H^2(B\Omega,\mathcal{A})_0 \longrightarrow H^1(B\Pi,\mathrm{Hom}(\mathcal{B},\mathcal{A}))$$

correspond to each other under these identifications. Consequently the cohomology classes denoted by $\delta(c)$ in Definitions 2.2 and 1.1 also correspond to each other. The claim now follows immediately from Theorem 2.3. □

3. The Brauer group and the Manin pairing of embedding problems

Definition 3.1. Via the homomorphism $\pi_{\mathbf{E}}$ we may consider every discrete Γ-module M a discrete $\Gamma(\mathbf{E})$-module, too, which will be denoted also by M by slight abuse of notation. For every abelian group M let M^{ct} denote the quotient of M by its torsion. Let $\mathrm{Br}(\mathbf{E})$ denote the cokernel of the homomorphism

$$\pi_{\mathbf{E}}^*\colon H^2(\Gamma, \overline{F}^*) \to H^2(\Gamma(\mathbf{E}), \overline{F}^*).$$

For every finite discrete abelian Γ-module M let $M^\vee$ denote the dual of M:

$$M^\vee = \mathrm{Hom}(M, \overline{F}^*).$$

Assume now that F is either a local or a global field and continue to use the notation which we introduced above.

Lemma 3.1. *We have the following short exact sequence:*

$$0 \to H^1(\Gamma, \mathrm{Ker}(\mathbf{E})_{ab}^\vee) \xrightarrow{j_{\mathbf{E}}} \mathrm{Br}(\mathbf{E}) \to H^2(\mathrm{Ker}(\mathbf{E}), \bigoplus_{p \neq \mathrm{char}(F)} \mathbb{Q}_p/\mathbb{Z}_p)^\Gamma \to H^2(\Gamma, \mathrm{Ker}(\mathbf{E})_{ab}^\vee),$$

where we equip $\bigoplus_{p \neq \mathrm{char}(F)} \mathbb{Q}_p/\mathbb{Z}_p$ *with the trivial* $\mathrm{Ker}(\mathbf{E})$*-action.*

Proof. This is Lemma 2.3 in [6]. □

Assume now that F is a global field. Note that for every $x \in |F|$ the following diagram:

$$\begin{array}{ccccc}
H^2(\Gamma, \overline{F}^*) & \xrightarrow{\iota_x^*} & H^2(\Gamma_x, \overline{F}^*) & \xrightarrow{(\eta_x)_*} & H^2(\Gamma_x, \overline{F}_x^*) \\
\downarrow{\scriptstyle \pi_{\mathbf{E}}^*} & & \downarrow{\scriptstyle \pi_{\mathbf{E}_x}^*} & & \downarrow{\scriptstyle \pi_{\mathbf{E}_x}^*} \\
H^2(\Gamma(\mathbf{E}), \overline{F}^*) & \xrightarrow{(\mathrm{id}_{G_1} \times \iota_x)^*} & H^2(\Gamma_x(\mathbf{E}_x), \overline{F}^*) & \xrightarrow{(\eta_x)_*} & H^2(\Gamma_x(\mathbf{E}_x), \overline{F}_x^*)
\end{array}$$

is commutative, where the maps on the left are restriction maps in group cohomology, and hence it gives rise to a map:

$$j_x\colon \mathrm{Br}(\mathbf{E}) \longrightarrow \mathrm{Br}(\mathbf{E}_x).$$

Definition 3.2. Let Б$(\mathbf{E})$ denote the intersection:

$$\text{Б}(\mathbf{E}) = \bigcap_{x \in |F|} \mathrm{Ker}(j_x) \leqslant \mathrm{Br}(\mathbf{E}).$$

Let $\mathrm{Br}_1(\mathbf{E})$ denote the kernel of the homomorphism:

$$\mathrm{Br}(\mathbf{E}) \longrightarrow H^2(\mathrm{Ker}(\mathbf{E}), \bigoplus_{p \neq \mathrm{char}(F)} \mathbb{Q}_p/\mathbb{Z}_p)^\Gamma.$$

Lemma 3.2. *The group* Б$(\mathbf{E})$ *is a subgroup of* $\mathrm{Br}_1(\mathbf{E})$*.*

Proof. This is Lemma 2.5 of [6]. □

Notation 3.3. For every $k \in \mathbb{N}$ and for finite discrete Γ-module M let $Ш^k(F, M)$ denote the subgroup:

$$Ш^k(F, M) = \operatorname{Ker}\left(\prod_{x \in |F|} \iota_x^* \colon H^k(\Gamma, M) \to H^k(\Gamma_x, M)\right) \leqslant H^k(\Gamma, M).$$

Proposition 3.4. *We have:*

$$Б(\mathbf{E}) = Ш^1(F, \operatorname{Ker}(\mathbf{E})_{ab}^\vee).$$

Proof. This is Proposition 2.8 of [6]. □

Notation 3.5. Let F be again an arbitrary field with absolute Galois group Γ and let $\mathbf{E}$ be an embedding problem given by the diagram (1.2.1). Note that the map which assigns to every solution h of $\mathbf{E}$ the homomorphism

$$s(h) \colon \Gamma \to \Gamma(\mathbf{E}) \subseteq G_1 \times \Gamma$$

given by the rule $g \mapsto (h(g), g)$ is a bijection between the solutions of $\mathbf{E}$ and continuous sections of the exact sequence (1.3.1). This bijection induces a bijection between $\operatorname{Sol}(\mathbf{E})$ and the conjugacy classes of sections of (1.3.1). We will always identify these two pairs of sets under these bijections.

Assume now that F is a global field and let $\mathbf{E}$ be as above. For every $x \in |F|$ let

$$\operatorname{inv}_x \colon \operatorname{Br}(F_x) = H^2(\Gamma_x, \overline{F}_x^*) \to \mathbb{Q}/\mathbb{Z}$$

denote the canonical invariant of the Brauer group $\operatorname{Br}(F_x)$ of the local field F_x. Let Inf denote the inflation map in group cohomology, as usual.

Lemma 3.3. *For every $c \in H^2(\Gamma(\mathbf{E}), \overline{F}^*)$ and for every adèlic solution $\prod_{x \in |F|} h_x$ of $\mathbf{E}$ the image of c under the composition:*

$$\begin{array}{l} H^2(\Gamma(\mathbf{E}), \overline{F}^*) \\ \quad \downarrow (\mathrm{id}_{G_1} \times \iota_x)^* \\ H^2(\Gamma_x(\mathbf{E}_x), \overline{F}^*) \xrightarrow{(\eta_x)_*} H^2(\Gamma_x(\mathbf{E}_x), \overline{F}_x^*) \xrightarrow{s(h_x)^*} H^2(\Gamma_x, \overline{F}_x^*) \xrightarrow{\operatorname{inv}_x} \mathbb{Q}/\mathbb{Z} \end{array}$$

is zero for almost all $x \in |F|$.

Proof. This is Lemma 3.3 of [6]. □

Note that for every $x \in |F|$ and h_x as above the map $s(h_x)^*$ only depends on the conjugacy class of h_x, therefore the pairing:

$$(\cdot, \cdot) \colon \operatorname{Sol}_{\mathbb{A}}(\mathbf{E}) \times H^2(\Gamma(\mathbf{E}), \overline{F}^*) \to \mathbb{Q}/\mathbb{Z}$$

given by the rule

$$\left(\prod_{x \in |F|} h_x, c\right) = \sum_{x \in |F|} \operatorname{inv}_x(s(h_x)^*((\eta_x)_*((\mathrm{id}_{G_1} \times \iota_x)^*(c))))$$

is well-defined, because all but finitely many of the summands are zero by the lemma above.

Lemma 3.4. *The image of group $H^2(\Gamma,\overline{F}^*)$ with respect to the homomorphism $\pi^*_{\mathbf{E}}$ is annihilated by the pairing $(\cdot,\cdot)$.*

Proof. This is Lemma 3.4 of [6]. □

Definition 3.6. By the lemma above we have a pairing:

$$\langle\cdot,\cdot\rangle\colon \mathrm{Sol}_{\mathbb{A}}(\mathbf{E})\times\mathrm{Br}(\mathbf{E})\to\mathbb{Q}/\mathbb{Z}$$

such that for every $h\in\mathrm{Sol}_{\mathbb{A}}(\mathbf{E})$ and $c\in H^2(\Gamma(\mathbf{E}),\overline{F}^*)$ we have:

$$(h,c)=\langle h,\sigma_{\mathbf{E}}(c)\rangle,$$

where $\sigma_{\mathbf{E}}\colon H^2(\Gamma(\mathbf{E}),\overline{F}^*)\to\mathrm{Br}(\mathbf{E})$ is the tautological surjection. For every subset $X\subseteq\mathrm{Br}(\mathbf{E})$ let $\mathrm{Sol}^X_{\mathbb{A}}(\mathbf{E})$ denote the set:

$$\mathrm{Sol}^X_{\mathbb{A}}(\mathbf{E})=\{h\in\mathrm{Sol}_{\mathbb{A}}(\mathbf{E})|\langle h,c\rangle=0\ (\forall c\in X)\}.$$

In the special case when $X=\mathrm{Br}(\mathbf{E}),\mathrm{Br}_1(\mathbf{E})$ or $\text{Б}(\mathbf{E})$ we will use the shorter superscripts $\mathrm{Br},\mathrm{Br}_1$ and Б, respectively. Clearly we have the inclusions:

$$\mathrm{Sol}^{\mathrm{Br}}_{\mathbb{A}}(\mathbf{E})\subseteq\mathrm{Sol}^{\mathrm{Br}_1}_{\mathbb{A}}(\mathbf{E})\subseteq\mathrm{Sol}^{\text{Б}}_{\mathbb{A}}(\mathbf{E})\subseteq\mathrm{Sol}_{\mathbb{A}}(\mathbf{E}).$$

Lemma 3.5. *We have $r(\mathrm{Sol}(\mathbf{E}))\subseteq\mathrm{Sol}^{\mathrm{Br}}_{\mathbb{A}}(\mathbf{E})$.*

Proof. This is Lemma 3.6 of [6]. □

Definition 3.7. Let $\mathbf{E}$ be an embedding problem over F such that the set $\mathrm{Sol}_{\mathbb{A}}(\mathbf{E})$ is non-empty and let b be an element of $\text{Б}(\mathbf{E})$. Choose a $b'\in H^2(\Gamma(\mathbf{E}),\overline{F}^*)$ such that $\sigma_{\mathbf{E}}(b')=b$. By definition for every $x\in|F|$ there is a $b_x\in H^2(\Gamma_x,\overline{F}^*_x)$ such that $(\eta_x)_*(\mathrm{id}_{G_1}\times\iota_x)^*(b')=\pi^*_{\mathbf{E}_x}(b_x)$. Therefore for every solution h_x of $\mathbf{E}_x$ we have:

$$s(h_x)^*((\eta_x)_*((\mathrm{id}_{G_1}\times\iota_x)^*(b')))=s(h_x)^*(\pi^*_{\mathbf{E}_x}(b_x))=b_x,$$

and hence the value of

$$\text{б}_{\mathbf{E}}(b)=\langle h,b\rangle=\sum_{x\in|F|}\mathrm{inv}_x(b_x)$$

does not depend on the choice of the adèlic solution $h=\prod_{x\in|F|}h_x$ of $\mathbf{E}$. Let

$$\text{б}_{\mathbf{E}}\colon\text{Б}(\mathbf{E})=\text{Ш}^1(F,\mathrm{Ker}(\mathbf{E})^\vee_{ab})\to\mathbb{Q}/\mathbb{Z}$$

denote the function defined by the formula above.

4. The Tate duality pairing and the Brauer–Manin pairing

Definition 4.1. By a continuous (or discrete) module over a pro-finite group Δ we mean a Δ-module M such that the action of Δ is continuous with respect to the discrete topology on M. For every pro-finite group Δ let $\mathcal{M}(\Delta),\mathcal{C}(\Delta),\mathcal{C}^+(\Delta),\mathcal{C}^-(\Delta)$, and $\mathcal{C}^\pm(\Delta)$ denote the category of continuous Δ-modules, the category of complexes of continuous Δ-modules, the category of complexes in $\mathcal{C}(\Delta)$ bounded from above, the category of complexes in $\mathcal{C}(\Delta)$ bounded from below, and the category of complexes in $\mathcal{C}(\Delta)$ which are either bounded from above or below, respectively. For every object

C of $\mathcal{C}(\Delta)$ let $H_n(C)$ denote the n-th homology group of C. When Δ is the absolute Galois group $\Gamma = \mathrm{Gal}(\overline{F}|F)$ of a field F, for every complex C

$$\cdots \longleftarrow C_{-1} \longleftarrow C_{-1} \longleftarrow C_0 \longleftarrow C_1 \longleftarrow \cdots$$

in $\mathcal{C}(\Delta)$ let $C^\vee$ denote the dual complex:

$$\cdots \longleftarrow \mathrm{Hom}(C_1, \overline{F}^*) \longleftarrow \mathrm{Hom}(C_0, \overline{F}^*) \longleftarrow \mathrm{Hom}(C_{-1}, \overline{F}^*) \longleftarrow \cdots,$$

where Hom denotes the group of continuous group homomorphisms (and we equip $\overline{F}^*$ with the discrete topology).

Definition 4.2. Note that for every pro-finite group Δ the category $\mathcal{M}(\Delta)$ has enough injectives, so right exact functors from $\mathcal{M}(\Delta)$ has derived functors. For every complex C in $\mathcal{C}^{\pm}(\Delta)$ let $\mathbb{H}^i(\Delta, C)$ denote its hypercohomology with respect to the functor of Δ-invariants. Similarly for any object C in $\mathcal{C}^{\pm}(\Delta)$ let $\mathrm{Ext}^n_\Delta(C, \cdot)$ denote the n-th derived functor of $\mathrm{Hom}_{\mathcal{C}(\Delta)}(C, \cdot)$. When $\Delta = \Gamma = \mathrm{Gal}(\overline{F}|F)$, as above, we will use the notation $\mathbb{H}^i(F, C)$ for $\mathbb{H}^i(\Gamma, C)$. When F is a global field let

$$\mathbb{H}^i_\Pi(F, C) = \prod_{x \in |F|} \mathbb{H}^i(F_x, C),$$

where for every $x \in |F|$ we consider C as an object of $\mathcal{C}(\Gamma_x)$ via the embedding $\iota_x \colon \Gamma_x \to \Gamma$, and we interpret $\mathbb{H}^i(F_x, C)$ accordingly. For every such x there is a pull-back map $i^*_x \colon \mathbb{H}^i(F, C) \to \mathbb{H}^i(F_x, C)$. Let

$$Ш^i(F, C) = \mathrm{Ker}\Big(\prod_{x \in |F|} \iota^*_x \colon \mathbb{H}^i(F, C) \to \mathbb{H}^i_\Pi(F, C)\Big).$$

Theorem 4.3. *Let F be a global field and let C be a complex in $\mathcal{C}^{\pm}(\Gamma)$ such that $H_n(C)$ is finite for every n and not divisible by the characteristic of F. Then there is a perfect pairing*

$$\langle \cdot, \cdot \rangle \colon Ш^i(F, C) \times Ш^{3-i}(F, C^\vee) \longrightarrow \mathbb{Q}/\mathbb{Z}.$$

Proof. This is exactly Theorem 3.5.9 from [4], when F is a number field. The function field case can be proved exactly the same way. It is also important to note that this pairing specializes to the usual Poitou–Tate pairing for Γ-modules, i.e. for complexes concentrated in degree zero. □

Definition 4.4. Let $\mathbf{E}$ be an embedding problem over an arbitrary field F given by the diagram (1.2.1). Let $\mathbf{E}_*$ denote the contractible simplicial set freely generated by G_1. At the level of sets $\mathbf{E}_i = G_1^{i+1}$. The diagonal right-action of G_1 on each $\mathbf{E}_i$ induces a free right action of G_1 on $\mathbf{E}_*$, and therefore a free right action of $\mathrm{Ker}(\mathbf{E})$ on $\mathbf{E}_*$, too. Then we have a left action of G_2 on $\mathbf{E}_*/\mathrm{Ker}(\mathbf{E})$ and thus by pulling back with respect to ϕ a left action of Γ on $\mathbf{E}_*/\mathrm{Ker}(\mathbf{E})$. Let $B(\mathbf{E})_*$ denote this simplicial object in the category of Γ-sets. Let $\mathbb{Z}B(\mathbf{E})_*$ denote the complex where $\mathbb{Z}B(\mathbf{E})_n$ is the free abelian group generated by $B(\mathbf{E})_n$ and the differential is the usual alternating sum. Equipped with the induced Γ-action this complex is an object of $\mathcal{C}^+(\Gamma)$.

Definition 4.5. As a simplicial set $B(\mathbf{E})_*$ is weakly equivalent to the Eilenberg–MacLane space $B\mathrm{Ker}(\mathbf{E})$, and hence

$$H_n(\mathbb{Z}B(\mathbf{E})_*) \cong H_n(\mathrm{Ker}(\mathbf{E}), \mathbb{Z}).$$

Let $\deg \in [\mathbb{Z}B(\mathbf{E})_*, \mathbb{Z}]$ denote the map from $\mathbb{Z}B(\mathbf{E})_*$ onto its 0-th homology, let $\tau_{>0}(\mathbb{Z}B(\mathbf{E})_*)$ be the fibre of the map deg in the derived category, and let

$$\tau_{>0}(\mathbb{Z}B(\mathbf{E})_*) \xrightarrow{f_0} \mathbb{Z}B(\mathbf{E})_* \xrightarrow{\deg} \mathbb{Z} \longrightarrow \tau_{>0}(\mathbb{Z}B(\mathbf{E})_*)[1]$$

be the corresponding distinguished triangle. By construction

$$H_n(\tau_{>0}(\mathbb{Z}B(\mathbf{E})_*)) \cong \begin{cases} H_n(\mathrm{Ker}(\mathbf{E}), \mathbb{Z}), & \text{if } n \neq 0, \\ 0, & \text{if } n = 0. \end{cases}$$

In particular when $\mathrm{Ker}(\mathbf{E})$ is abelian we have

$$H_1(\tau_{>0}(\mathbb{Z}B(\mathbf{E})_*)) \cong H_1(\mathrm{Ker}(\mathbf{E}), \mathbb{Z}) \cong \mathrm{Ker}(\mathbf{E}).$$

Let $h_1 \colon \tau_{>0}(\mathbb{Z}B(\mathbf{E})_*) \to \mathrm{Ker}(\mathbf{E})[1]$ be the Postnikov truncation in this case.

Definition 4.6. Let

$$\pi^*_{\mathbf{E}} \colon \mathcal{C}(\Gamma) \to \mathcal{C}(\Gamma(\mathbf{E}))$$

denote the functor which we get by pulling back with respect to the surjective homomorphism $\pi_{\mathbf{E}} \colon \Gamma(\mathbf{E}) \to \Gamma$. For any object M of $\mathcal{M}(\Gamma(\mathbf{E}))$ let $\pi_{\mathbf{E}!}(M)$ denote the $\mathrm{Ker}(\mathbf{E})$-coinvariants of M, that is, the quotient of M by the subgroup generated by the set:

$$\{x - \gamma(x) | x \in M, \gamma \in \mathrm{Ker}(\mathbf{E})\}.$$

Since the latter is a $\Gamma(\mathbf{E})$-submodule, there is a natural action of Γ on $\pi_{\mathbf{E}!}(M)$, and hence we get a functor $\pi_{\mathbf{E}} \colon \mathcal{M}(\Gamma(\mathbf{E})) \to \mathcal{M}(\Gamma)$ which in turn induces a functor:

$$\pi_{\mathbf{E}!} \colon \mathcal{C}(\Gamma(\mathbf{E})) \longrightarrow \mathcal{C}(\Gamma).$$

It can be easily seen that this functor is the left adjoint of $\pi^*_{\mathbf{E}}$.

Definition 4.7. Let $\mathbb{Z}\mathbf{E}_*$ denote the chain complex of the contractible simplicial set $\mathbf{E}_*$. For every object C of $\mathcal{C}(\Gamma(\mathbf{E}))$ we may take (the total complex of) the tensor product $C \otimes \mathbb{Z}\mathbf{E}_*$ in the category of complexes of $\mathbb{Z}$-modules and equip it with the diagonal $\Gamma(\mathbf{E})$-action; this makes $C \otimes \mathbb{Z}\mathbf{E}_*$ an object of $\mathcal{C}(\Gamma(\mathbf{E}))$. Let $\mathbb{L}\pi_{\mathbf{E}!}(C)$ denote $\pi_{\mathbf{E}!}(C \otimes \mathbb{Z}\mathbf{E}_*)$. As we will shortly see, the functor $\mathbb{L}\pi_{\mathbf{E}!}$ is the left derived functor of $\pi_{\mathbf{E}!}$ in a suitable interpretation, although the latter is not defined in the sense of classical homological algebra, as $\mathcal{M}(\Gamma(\mathbf{E}))$ does not have enough projectives.

Lemma 4.1. *There is an isomorphism:*

$$\mathbb{L}\pi_{\mathbf{E}!}(\mathbb{Z}) \cong \mathbb{Z}B(\mathbf{E})_*.$$

Proof. Clearly $\mathbb{Z} \otimes \mathbb{Z}\mathbf{E}_* \cong \mathbb{Z}\mathbf{E}_*$ and $\pi_{\mathbf{E}!}(\mathbb{Z}\mathbf{E}_*) \cong \mathbb{Z}B(\mathbf{E})_*$. □

Definition 4.8. Let Δ be any pro-finite group, as above, and for any pair M, N of continuous Δ-modules let $\mathrm{Hom}_\Delta(M, N)$ denote the group of Δ-module homomorphisms from M to N. Now let $A = \{A_n\}_{n\in\mathbb{Z}}, B = \{B_n\}_{n\in\mathbb{Z}}$ be two complexes in $\mathcal{C}(\Delta)$. Let $\underline{\mathrm{Hom}}_\Delta(A,B) = \{\underline{\mathrm{Hom}}^n_\Delta(A,B)\}_{n\in\mathbb{Z}}$ be the equivariant mapping complex from A to B, where

$$\underline{\mathrm{Hom}}^n_\Delta(A,B) = \prod_{i\in\mathbb{Z}} \mathrm{Hom}_\Delta(A_i, B_{i-n}),$$

and the differential

$$d\colon \underline{\mathrm{Hom}}^n_\Delta(A,B) \longrightarrow \underline{\mathrm{Hom}}^{n+1}_\Delta(A,B)$$

for any $f = \prod_{i\in\mathbb{Z}} f_i \in \prod_{i\in\mathbb{Z}} \mathrm{Hom}_\Delta(A_i, B_{i-n})$ is given by

$$d(f)_i = f_{i-1} \circ d^A_i + (-1)^n d^B_{i-n} \circ f_i,$$

where $d^A_i\colon A_i \to A_{i-1}$ and $d^B_i\colon B_i \to B_{i-1}$ are differentials of A and B, respectively. We denote the kernel of d by $\mathcal{Z}^n(A,B) \subseteq \underline{\mathrm{Hom}}^n_\Delta(A,B)$. Note that $\mathcal{Z}^n(A,B)$ consists of exactly those elements of $\underline{\mathrm{Hom}}^n_\Delta(A,B)$ which are maps of complexes of degree n from A to B.

Lemma 4.2. *There are natural isomorphisms:*

$$\mathrm{Ext}^n_\Gamma(\mathbb{L}\pi_{\mathbf{E}!}(C), D) \cong \mathrm{Ext}^n_{\Gamma(\mathbf{E})}(C, \pi^*_{\mathbf{E}}(D)) \quad (\forall n \in \mathbb{N}),$$

for every C in $\mathcal{C}(\Gamma(\mathbf{E}))$ and D in $\mathcal{C}(\Gamma)$.

Proof. This isomorphism can be explained as an instance of Quillen adjunctions between model categories, or ∞-adjunctions between $(\infty,1)$-categories. However we will give a simple direct proof. Let $\widetilde{D}$ be an resolution of D by injective Γ-modules. The groups $\mathrm{Ext}^n_\Gamma(\mathbb{L}\pi_{\mathbf{E}!}(C), D)$ are the homologies of $\underline{\mathrm{Hom}}_\Gamma(\mathbb{L}\pi_{\mathbf{E}!}(C), \widetilde{D})$. Then we have:

$$\underline{\mathrm{Hom}}_\Gamma(\mathbb{L}\pi_{\mathbf{E}!}(C), \widetilde{D}) \cong \underline{\mathrm{Hom}}_\Gamma(\pi_{\mathbf{E}!}(C \otimes \mathbb{Z}\mathbf{E}_*), \widetilde{D}) \cong \underline{\mathrm{Hom}}_{\Gamma(\mathbf{E})}(C \otimes \mathbb{Z}\mathbf{E}_*, \pi^*_{\mathbf{E}}(\widetilde{D})),$$

where we used the definition of $\mathbb{L}\pi_{\mathbf{E}!}$ in the first isomorphism, and the fact that $\pi_{\mathbf{E}!}$ is the left adjoint of $\pi^*_{\mathbf{E}}$ in the second. Moreover

$$\underline{\mathrm{Hom}}_{\Gamma(\mathbf{E})}(C \otimes \mathbb{Z}\mathbf{E}_*, \pi^*_{\mathbf{E}}(\widetilde{D})) \cong \underline{\mathrm{Hom}}_{\Gamma(\mathbf{E})}(C, \underline{\underline{\mathrm{Hom}}}(\mathbb{Z}\mathbf{E}_*, \pi^*_{\mathbf{E}}(\widetilde{D}))),$$

where $\underline{\underline{\mathrm{Hom}}}(\cdot,\cdot)$ denotes the internal Hom in the category of $\Gamma(\mathbf{E})$-complexes. (Explicitly $\underline{\underline{\mathrm{Hom}}}(\cdot,\cdot)$ is the mapping complex of the underlying $\mathbb{Z}$-complexes which we equip with a continuous Γ-action via conjugation.) In order to conclude it is enough to note that $\underline{\underline{\mathrm{Hom}}}(\mathbb{Z}\mathbf{E}_*, \pi^*_{\mathbf{E}}(\widetilde{D}))$ is an injective resolution of $\pi^*_{\mathbf{E}}(D)$. □

Lemma 4.3. *There are natural isomorphisms:*

$$\mathbb{H}^n(\Gamma(\mathbf{E}), \overline{F}^*) \cong \mathbb{H}^n(\Gamma, \mathbb{Z}B(\mathbf{E})^\vee_*) \quad (\forall n \in \mathbb{N}).$$

Proof. By the uniqueness of n-th derived functors we have:

$$\mathbb{H}^n(\Gamma(\mathbf{E}), \overline{F}^*) \stackrel{\text{def}}{=} \mathbb{H}^n(\Gamma(\mathbf{E}), \pi^*_{\mathbf{E}}(\overline{F}^*)) \cong \mathrm{Ext}^n_{\Gamma(\mathbf{E})}(\mathbb{Z}, \pi^*_{\mathbf{E}}(\overline{F}^*)),$$

as there is a natural isomorphism $\mathbb{H}^n(\Delta, C) \cong \mathrm{Ext}^n_\Delta(\mathbb{Z}, C)$ (where C is an object of

$\mathcal{C}(\Delta)$ and Δ is any pro-finite group). By Lemma 4.2 we have:

$$\mathrm{Ext}^n_{\Gamma(\mathbf{E})}(\mathbb{Z}, \pi^*_{\mathbf{E}}(\overline{F}^*)) \cong \mathrm{Ext}^n_{\Gamma}(\mathbb{L}\pi_{\mathbf{E}!}(\mathbb{Z}), \overline{F}^*).$$

Note that there is a spectral sequence:

$$\mathrm{Ext}^p_{\Gamma}(\mathbb{Z}, \underline{\underline{\mathrm{Ext}}}^q(\mathbb{L}\pi_{\mathbf{E}!}(\mathbb{Z}), \overline{F}^*)) \Rightarrow \mathrm{Ext}^{p+q}_{\Gamma}(\mathbb{L}\pi_{\mathbf{E}!}(\mathbb{Z}), \overline{F}^*),$$

where $\underline{\underline{\mathrm{Ext}}}^*(\mathbb{L}\pi_{\mathbf{E}!}(\mathbb{Z}), \cdot)$ is the derived functor of $\underline{\underline{\mathrm{Hom}}}(\mathbb{L}\pi_{\mathbf{E}!}(\mathbb{Z}), \cdot)$. Since $\overline{F}^*$ is divisible, this sequence degenerates, and hence we have an isomorphism:

$$\mathrm{Ext}^n_{\Gamma}(\mathbb{L}\pi_{\mathbf{E}!}(\mathbb{Z}), \overline{F}^*) \cong \mathrm{Ext}^n_{\Gamma}(\mathbb{Z}, \mathbb{L}\pi_{\mathbf{E}!}(\mathbb{Z})^\vee),$$

while by Lemma 4.1 and by the uniqueness of n-th derived functors we have:

$$\mathrm{Ext}^n_{\Gamma}(\mathbb{Z}, \mathbb{L}\pi_{\mathbf{E}!}(\mathbb{Z})^\vee) \cong \mathrm{Ext}^n_{\Gamma}(\mathbb{Z}, \mathbb{Z}B(\mathbf{E})^\vee_*) \cong \mathbb{H}^n(\Gamma, \mathbb{Z}B(\mathbf{E})^\vee_*).$$ □

Lemma 4.4. *Assume that F is either a global or a local field. Then we have:*

$$\mathrm{Br}(\mathbf{E}) \cong \mathbb{H}^2(\Gamma, \tau_{>0}(\mathbb{Z}B(\mathbf{E})_*)^\vee).$$

Proof. The distinguished triangle:

$$\tau_{>0}(\mathbb{Z}B(\mathbf{E})_*) \xrightarrow{f_0} \mathbb{Z}B(\mathbf{E})_* \xrightarrow{\deg} \mathbb{Z} \longrightarrow \tau_{>0}(\mathbb{Z}B(\mathbf{E})_*)[1]$$

gives rise to another distinguished triangle:

$$\overline{F}^* \longrightarrow \mathbb{Z}B(\mathbf{E})^\vee_* \longrightarrow \tau_{>0}(\mathbb{Z}B(\mathbf{E})_*)^\vee \longrightarrow \overline{F}^*[1]$$

by taking duals. Since $H^3(\Gamma, \overline{F}^*) = 0$ (see Proposition 15 of [9] on page 93 when F is a local field, and see Corollary 4.21 of [5], page 80 when F is a global field), the associated cohomological long exact sequence looks like:

$$\mathbb{H}^2(\Gamma, \overline{F}^*) \to \mathbb{H}^2(\Gamma, \mathbb{Z}B(\mathbf{E})^\vee_*) \to \mathbb{H}^2(\Gamma, \tau_{>0}(\mathbb{Z}B(\mathbf{E})_*)^\vee) \to 0.$$

The first map is the composition:

$$\mathbb{H}^2(\Gamma, \overline{F}^*) \cong \mathrm{Ext}^2_{\Gamma}(\mathbb{Z}, \overline{F}^*) \to \mathrm{Ext}^2_{\Gamma}(\mathbb{Z}B(\mathbf{E})_*, \overline{F}^*) \cong \mathbb{H}^2(\Gamma, \mathbb{Z}B(\mathbf{E})^\vee_*),$$

where the middle map is induced by the degree map deg: $\mathbb{Z}B(\mathbf{E})_* \to \mathbb{Z}$. The derived adjunction $\mathbb{L}\pi_{\mathbf{E}!} \dashv \pi^*$ give rise to a co-unit map: $\mathbb{L}\pi_{\mathbf{E}!}(\pi^*(\mathbb{Z})) \to \mathbb{Z}$ which, under the identification in Lemma 4.1, is deg. Using Lemma 4.3 the first map of the sequence above can be viewed as a homomorphism:

$$H^2(\Gamma, \overline{F}^*) = \mathbb{H}^2(\Gamma, \overline{F}^*) \to \mathbb{H}^2(\Gamma, \mathbb{Z}B(\mathbf{E})^\vee_*) = H^2(\Gamma(\mathbf{E}), \overline{F}^*).$$

As this map is induced by the co-unit, it is the pull-back map (with respect to the surjection $\Gamma(\mathbf{E}) \to \Gamma$). The cokernel of the latter is $\mathrm{Br}(\mathbf{E})$ by definition, so the claim follows. □

Corollary 4.9. *Assume that F is a global field. Then we have:*

$$\text{Б}(\mathbf{E}) \cong \text{Ш}^2(F, \tau_{>0}(\mathbb{Z}B(\mathbf{E})_*)^\vee).$$

Proof. This follows from Lemma 4.4 applied to F and all its completions. □

Remark 4.10. The Postnikov truncation

$$h_1 : \tau_{>0}(\mathbb{Z}B(\mathbf{E})_*) \to \mathrm{Ker}(\mathbf{E})[1]$$

furnishes an isomorphism:

$$Ш^1(F, \tau_{>0}(\mathbb{Z}B(\mathbf{E})_*)) \longrightarrow Ш^1(F, \mathrm{Ker}(\mathbf{E})[1]) \cong Ш^2(F, \mathrm{Ker}(\mathbf{E})).$$

Definition 4.11. Let $s\colon \Gamma \to \Gamma(\mathbf{E})$ be a continuous section of the homomorphism $\pi_{\mathbf{E}}\colon \Gamma(\mathbf{E}) \to \Gamma$. The identity map $\mathrm{id} \in \mathrm{Ext}^0_\Gamma(\mathbb{L}\pi_{\mathbf{E}_!}(\mathbb{Z}), \mathbb{L}\pi_{\mathbf{E}_!}(\mathbb{Z}))$ via the isomorphism:

$$\mathrm{Ext}^0_\Gamma(\mathbb{L}\pi_{\mathbf{E}_!}(\mathbb{Z}), \mathbb{L}\pi_{\mathbf{E}_!}(\mathbb{Z})) \cong \mathrm{Ext}^0_{\Gamma(\mathbf{E})}(\mathbb{Z}, \pi^*_{\mathbf{E}}(\mathbb{L}\pi_{\mathbf{E}_!}(\mathbb{Z})))$$

furnished by Lemma 4.2 furnishes an element $\mathrm{id}_{\mathbf{E}} \in \mathrm{Ext}^0_{\Gamma(\mathbf{E})}(\mathbb{Z}, \pi^*_{\mathbf{E}}(\mathbb{L}\pi_{\mathbf{E}_!}(\mathbb{Z})))$. By pulling back with respect to s we get an element:

$$s^*(\mathrm{id}_{\mathbf{E}}) \in \mathrm{Ext}^0_\Gamma(s^*(\mathbb{Z}), s^*(\pi^*_{\mathbf{E}}(\mathbb{L}\pi_{\mathbf{E}!}(\mathbb{Z})))).$$

Since $s^* \circ \pi^*_{\mathbf{E}} = \mathrm{id}^*_\Gamma = \mathrm{id}$ and $s^*(\mathbb{Z}) \cong \mathbb{Z}$, we get an element:

$$[s] \in \mathrm{Ext}^0_\Gamma(\mathbb{Z}, \mathbb{L}\pi_{\mathbf{E}!}(\mathbb{Z})) \cong \mathbb{H}^0(\Gamma, \mathbb{Z}B(\mathbf{E})_*)$$

(using Lemma 4.1), which we will call the classifying element of the section s. Note that the map

$$\mathbb{H}^0(\Gamma, \mathbb{Z}B(\mathbf{E})_*) \longrightarrow \mathbb{H}^0(\Gamma, \mathbb{Z}) \cong \mathbb{Z}$$

induced by deg sends $[s]$ to 1.

Note that each element in $b \in \mathbb{H}^2(\Gamma(\mathbf{E}), \overline{F}^*)$ can be considered as an element of $\mathbb{H}^2(\Gamma, \mathbb{Z}B(\mathbf{E})^\vee_*)$ via the isomorphism in Lemma 4.3. Let

$$\cup\colon \mathbb{H}^2(\Gamma, \mathbb{Z}B(\mathbf{E})^\vee_*) \times \mathbb{H}^0(\Gamma, \mathbb{Z}B(\mathbf{E})_*) \to \mathbb{H}^2(\Gamma, \overline{F}^*)$$

be the cup product induced by the natural bilinear pairing:

$$\mathbb{Z}B(\mathbf{E})^\vee_* \times \mathbb{Z}B(\mathbf{E})_* \longrightarrow \overline{F}^*$$

of complexes.

Lemma 4.5. *We have:*

$$b \cup [s] = s^*(b) \in \mathbb{H}^2(\Gamma, \overline{F}^*)$$

for every $b \in \mathbb{H}^2(\Gamma(\mathbf{E}), \overline{F}^*)$ *and continuous section* s *of* $\pi_{\mathbf{E}}$.

Proof. This claim follows at once from comparing the cup product above with the Yoneda pairing:

$$\mathrm{Ext}^0_\Gamma(\mathbb{Z}, \mathbb{Z}B(\mathbf{E})_*) \times \mathrm{Ext}^2_\Gamma(\mathbb{Z}B(\mathbf{E})_*, \overline{F}^*) \to \mathrm{Ext}^2_\Gamma(\mathbb{Z}, \overline{F}^*)$$

via the isomorphisms

$$\mathbb{H}^0(\Gamma, \mathbb{Z}B(\mathbf{E})_*) \cong \mathrm{Ext}^0_\Gamma(\mathbb{Z}, \mathbb{Z}B(\mathbf{E})_*), \quad \mathbb{H}^2(\Gamma, \mathbb{Z}B(\mathbf{E})^\vee_*) \cong \mathrm{Ext}^2_\Gamma(\mathbb{Z}B(\mathbf{E})_*, \overline{F}^*),$$

$$\mathbb{H}^2(\Gamma, \overline{F}^*) \cong \mathrm{Ext}^0_\Gamma(\mathbb{Z}, \overline{F}^*).$$

We leave the details to the reader. □

Definition 4.12. Let M be a discrete finite abelian Γ-module. For every embedding problem $\mathbf{E}$ over F such that $\mathrm{Ker}(\mathbf{E}) = M$ and the set $\mathrm{Sol}_{\mathbb{A}}(\mathbf{E})$ is non-empty, let $c_{\mathbf{E}} \in H^2(F, M) = H^2(F, \mathrm{Ker}(\mathbf{E}))$ denote the class of the extension (1.3.1). Note that $c_{\mathbf{E}} \in Ш^2(F, M)$ since we assumed that $\mathrm{Sol}_{\mathbb{A}}(\mathbf{E})$ is non-empty. Conversely for every $c \in Ш^2(F, M)$ there is an embedding problem $\mathbf{E}$ as above such that $c = c_{\mathbf{E}}$. Let

$$б\colon Ш^1(F, M^\vee) \times Ш^2(F, M) \to \mathbb{Q}/\mathbb{Z} \tag{4.12.1}$$

be the unique pairing such that $б(b, c_{\mathbf{E}}) = б_{\mathbf{E}}(b)$ for every $b \in Ш^1(F, M^\vee)$ and embedding problem $\mathbf{E}$ as above. Since for every $b \in Ш^1(F, M^\vee)$ the value of $б_{\mathbf{E}}(b)$ only depends on the isomorphism class of the embedding problem $\mathbf{E}$, the pairing $б$ is well-defined. Assume now that $\mathrm{char}(F)$ does not divide the order of M and let

$$\tau\colon Ш^1(F, M^\vee) \times Ш^2(F, M) \longrightarrow \mathbb{Q}/\mathbb{Z}$$

denote the Tate duality pairing.

Theorem 4.13. *We have* $б = -\tau$.

We think that this theorem is very interesting on its own, since it gives an elegant description of the Tate duality pairing. It will proved in the rest of this section. We will continue to use the notation which we have introduced so far. Let $\mathbf{E}$ an embedding problem of the type considered in Definition 4.12. Consider the cohomological long exact sequence:

$$\mathbb{H}^0(F, \mathbb{Z}B(\mathbf{E})_*) \xrightarrow{\deg} \mathbb{H}^0(F, \mathbb{Z}) \xrightarrow{\partial} \mathbb{H}^1(F, \tau_{>0}(\mathbb{Z}B(\mathbf{E})_*))$$

corresponding to the distinguished triangle:

$$\tau_{>0}(\mathbb{Z}B(\mathbf{E})_*) \xrightarrow{f_0} \mathbb{Z}B(\mathbf{E})_* \xrightarrow{\deg} \mathbb{Z} \longrightarrow \tau_{>0}(\mathbb{Z}B(\mathbf{E})_*)[1],$$

and set $\delta = \partial(1) \in Ш^1(F, \tau_{>0}(\mathbb{Z}B(\mathbf{E})_*))$.

Lemma 4.6. *The image of the classifying element* $c_{\mathbf{E}} \in Ш^2(F, \mathrm{Ker}(\mathbf{E}))$ *under the isomorphism*

$$Ш^2(F, \mathrm{Ker}(\mathbf{E})) \cong Ш^1(F, \tau_{>0}(\mathbb{Z}B(\mathbf{E})_*))$$

in Remark 4.10 is the δ *above.*

Proof. This is just a direct computation involving the representing cocycles. The details are left to the reader. □

Let

$$\langle\cdot,\cdot\rangle\colon Ш^1(F, \tau_{>0}(\mathbb{Z}B(\mathbf{E})_*)) \times Ш^2(F, \tau_{>0}(\mathbb{Z}B(\mathbf{E})_*)^\vee) \longrightarrow \mathbb{Q}/\mathbb{Z}$$

be the perfect pairing in Theorem 4.3. Now let $b \in Ш^2(F, \tau_{>0}(\mathbb{Z}B(\mathbf{E})_*)^\vee)$ be arbitrary. Since $\mathbb{H}^3(F, \overline{F}^*) = 0$, the map

$$\mathbb{H}^2(F, \mathbb{Z}B(\mathbf{E})_*^\vee) \to \mathbb{H}^2(F, \tau_{>0}(\mathbb{Z}B(\mathbf{E})_*)^\vee)$$

is onto, and thus b can be lifted to an element $\overline{b} \in \mathbb{H}^2(F, \mathbb{Z}B(\mathbf{E})_*^\vee)$, and we can take the localisation map to obtain

$$\overline{b}_x \stackrel{\mathrm{def}}{=} \iota_x^*(\overline{b}) \in \mathbb{H}^2(F_x, \mathbb{Z}B(\mathbf{E})_*^\vee).$$

Recall that we assumed that $\mathrm{Sol}_{\mathbb{A}}(\mathbf{E})$ is non-empty, so for every $x \in |F|$ let h_x be a solution of $\mathbf{E}_x$ such that h_x is unramified for almost all x. Let s_x denote the section $s(h_x)$ corresponding to h_x for every $x \in |F|$. Finally let $\cup$ denote the cup product introduced after Definition 4.11 (over any field, including all completions of F).

Proposition 4.14. *We have the equality:*

$$\langle \delta, b \rangle = -\sum_{x \in |F|} \mathrm{inv}_x([s_x] \cup \bar{b}_x) \in \mathbb{Q}/\mathbb{Z}.$$

By Lemma 4.5 the right hand side is $-б_{\mathbf{E}}(b)$. On the other hand the isomorphisms:

$$Ш^1(F, \mathrm{Ker}(\mathbf{E})^\vee) \cong Ш^2(F, \tau_{>0}(\mathbb{Z}B(\mathbf{E})_*)^\vee), Ш^2(F, \mathrm{Ker}(\mathbf{E})) \cong Ш^1(F, \tau_{>0}(\mathbb{Z}B(\mathbf{E})_*))$$

in Corollary 4.9 and Remark 4.10 respect the pairing in the sense that the resulting diagram:

$$\begin{array}{ccc} Ш^1(F, \mathrm{Ker}(\mathbf{E})^\vee) \times Ш^2(F, \mathrm{Ker}(\mathbf{E})) & \xrightarrow{\langle \cdot, \cdot \rangle} & \mathbb{Q}/\mathbb{Z} \\ \downarrow \qquad\qquad \downarrow & & \| \\ Ш^2(F, \tau_{>0}(\mathbb{Z}B(\mathbf{E})_*^\vee) \times Ш^1(F, \tau_{>0}(\mathbb{Z}B(\mathbf{E})_*)) & \xrightarrow{\langle \cdot, \cdot \rangle} & \mathbb{Q}/\mathbb{Z} \end{array}$$

is commutative. Therefore by Lemma 4.6 the left hand side is $\tau(c_{\mathbf{E}}, b)$. So Theorem 4.13 follows from Proposition 4.14, and hence we only have to prove the latter.

Definition 4.15. For every pro-finite group Δ and open normal subgroup $U \leqslant \Delta$ let $E(\Delta/U)$ denote the standard (bar) resolution complex by free $\mathbb{Z}[\Delta/U]$-modules of $\mathbb{Z}$, equipped with the tautological Δ-action. Note that since $E(\Delta/U)$ is quasi-isomorphic to $\mathbb{Z}$, any map of degree i between $E(\Delta/U)$ and another complex C in $\mathcal{C}(\Delta)$ gives rise to a hypercohomology class in $\mathbb{H}^i(\Delta, C)$. For every $g \in \mathcal{Z}^i(E(\Delta/U), C)$ let $[g]$ denote the class in $\mathbb{H}^i(\Delta, C)$ represented by g.

Notation 4.16. Now let Δ' be another pro-finite group, let $U' \leqslant \Delta'$ be an open normal subgroup, and let $\phi\colon \Delta' \to \Delta$ be a continuous homomorphism such that $\phi(U') \subseteq U$. Then ϕ induces a homomorphism $\Delta'/U' \to \Delta/U$, which induces a map $E(\Delta'/U') \to E(\Delta/U)$ of complexes, which furnishes a homomorphism

$$\phi^*\colon \underline{\mathrm{Hom}}^i_\Delta(E(\Delta/U), C) \to \underline{\mathrm{Hom}}^i_{\Delta'}(E(\Delta'/U'), C)$$

compatible with the pull-back map on cohomology. We will drop ϕ^* from the notation when ϕ is the identity map on Δ.

Definition 4.17. Now let $A = \{A_n\}_{n\in\mathbb{Z}}, B = \{B_n\}_{n\in\mathbb{Z}}$ and $C = \{C_n\}_{n\in\mathbb{Z}}$ be three complexes in $\mathcal{C}(\Delta)$ such that there is a pairing:

$$m\colon A \otimes B \longrightarrow C.$$

Let U and $E(\Delta/U)$ be as above, and let

$$c\colon E(\Delta/U) \longrightarrow E(\Delta/U) \otimes E(\Delta/U)$$

denote the Alexander–Whitney map (see formula (1.4) of [1] on page 108). Now consider the composition:

$$\begin{array}{c}\underline{\mathrm{Hom}}^*_\Delta(E(\Delta/U),A)\times\underline{\mathrm{Hom}}^*_\Delta(E(\Delta/U),B)\\ \big\downarrow\\ \underline{\mathrm{Hom}}^*_\Delta(E(\Delta/U)\otimes E(\Delta/U),A\otimes B)\\ \big\downarrow\\ \underline{\mathrm{Hom}}^*_\Delta(E(\Delta/U),A\otimes B)\\ \big\downarrow\\ \underline{\mathrm{Hom}}^*_\Delta(E(\Delta/U),C),\end{array}$$

where the first map is furnished by the functorial property of tensor products, the second is induced by the co-multiplication c, and the third map is induced by the multiplication m. Let $\cup$ denote the resulting pairing of complexes:

$$\underline{\mathrm{Hom}}^*_\Delta(E(\Delta/U),A)\times\underline{\mathrm{Hom}}^*_\Delta(E(\Delta/U),B)\longrightarrow\underline{\mathrm{Hom}}^*_\Delta(E(\Delta/U),C).$$

Note the induced map on the cohomology is the exterior cup product.

Proof of Proposition 4.14. In order to prove the statement we shall use an explicit description of the pairing

$$\langle\cdot,\cdot\rangle\colon Ш^1(F,\tau_{>0}(\mathbb{Z}B(\mathbf{E})_*))\times Ш^2(F,\tau_{>0}(\mathbb{Z}B(\mathbf{E})_*)^\vee)\longrightarrow\mathbb{Q}/\mathbb{Z}$$

similar to the one given by Milne in §I.4 of [5]. Let $c\in\mathbb{Z}B(\mathbf{E})_0$ be such that $\deg(c)=1$. Denote by $U\lhd\Gamma$ the stabiliser of c. Let $g\in\underline{\mathrm{Hom}}^0_\Gamma(E(\Gamma/U),\mathbb{Z}B(\mathbf{E})_*)$ be such that $g_0(\sigma)=\sigma c$ for $\sigma\in\Gamma/U$ and $g_i=0$ for $i\neq 0$. Note that $[\deg\circ g]$ represents $1\in\mathbb{H}^0(F,\mathbb{Z})$ and so

$$\alpha=dg\in\mathcal{Z}^1(E(\Gamma/U),\tau_{>0}(\mathbb{Z}B(\mathbf{E})_*))$$

represents

$$\delta=\partial(1)\in\mathbb{H}^1(F,\tau_{>0}(\mathbb{Z}B(\mathbf{E})_*)).$$

Shrink U enough so that one can represent b by a map

$$\beta\in\mathcal{Z}^2\left(E(\Gamma/U),\tau_{>0}(\mathbb{Z}B(\mathbf{E})_*)^\vee\right),$$

and $\overline{b}$ by

$$\overline{\beta}\in\mathcal{Z}^2(E(\Gamma/U),\mathbb{Z}B(\mathbf{E})_*^\vee).$$

Now set

$$\epsilon=g\cup\overline{\beta}\in\underline{\mathrm{Hom}}^2_\Gamma(E(\Gamma/U),\overline{F}^*).$$

Note that $d\epsilon=dg\cup\overline{\beta}=\alpha\cup\overline{\beta}$. Set $g_x=\iota_x^*(g)$ for every $x\in|F|$. For every place x we can take a small enough $U_x\lhd\Gamma_x$ such that we can represent g_x in the group $\underline{\mathrm{Hom}}^0_\Gamma(E(\Gamma_x/U_x),\mathbb{Z}B(\mathbf{E})_*)$ (that is, we have $U_x\subseteq\Gamma_x\cap U$), and we can represent $[s_x]\in\mathbb{H}^0(F_x,\mathbb{Z}B(\mathbf{E})_*)$ by an $f_x\in\mathcal{Z}^0(E(\Gamma_x/U_x),\mathbb{Z}B(\mathbf{E})_*)$. Set $h_x=g_x-f_x$. Then

$$dh_x=dg_x-df_x=\alpha_x,$$

where $\alpha_x=\iota_x^*(\alpha)$ (for every $x\in|F|$). Since $\deg(h_x)=0$ we see that h_x actually lies in $\underline{\mathrm{Hom}}^0_\Gamma(E(\Gamma_x/U_x), \tau_{>0}(\mathbb{Z}B(\mathbf{E})_*))$. Hence we can cup it with

$$\beta_x = \iota_x^*(\beta) \in \mathcal{Z}^1(E(\Gamma_x/U_x), \tau_{>0}(\mathbb{Z}B(\mathbf{E})_*)^\vee)$$

and get an element in $\underline{\mathrm{Hom}}^1_\Gamma(E(\Gamma_x/U_x), \overline{F}^*_x)$. We then observe that

$$d(h_x \cup \beta_x) = dh_x \cup \beta_x = \alpha_x \cup \beta_x = d\epsilon_x,$$

where $\epsilon_x = \iota_x^*(\epsilon)$ (for every $x \in |F|$), and so we can define

$$c_x = [h_x \cup \beta_x - \epsilon_x] \in \mathbb{H}^2(F_x, \overline{F}^*_x).$$

Our generalisation for Milne's formula is the following expression for the pairing:

$$\langle \delta, b \rangle = \sum_{x \in |F|} \mathrm{inv}_x(c_x) \in \mathbb{Q}/\mathbb{Z}.$$

Now by naturality

$$h_x \cup \beta_x = h_x \cup \overline{\beta}_x,$$

where the first cup is computed in $\tau_{>0}(\mathbb{Z}B(\mathbf{E})_*), \tau_{>0}(\mathbb{Z}B(\mathbf{E})_*)^\vee$ and the second in $\mathbb{Z}B(\mathbf{E})_*, \mathbb{Z}B(\mathbf{E})_*^\vee$. We then get

$$\begin{aligned} c_x = [h_x \cup \beta_x - \epsilon_x] &= [h_x \cup \beta_x - g_x \cup \overline{\beta}_x] = [h_x \cup \overline{\beta}_x - g_x \cup \overline{\beta}_x] \\ &= [-f_x \cup \overline{\beta}_x] = -[s_x] \cup \overline{b}_x, \end{aligned}$$

because $\overline{b}_x = [\overline{\beta}_x]$ (for every $x \in |F|$). □

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