

A DEGREE FORMULA FOR EQUIVARIANT COHOMOLOGY RINGS

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Abstract

This paper generalizes a result of Lynn on the “degree” of an equivariant cohomology ring $H_G^*(X)$. The degree of a graded module is a certain coefficient of its Poincaré series, and is closely related to multiplicity. In the present paper, we study these commutative algebraic invariants for equivariant cohomology rings. The main theorem is an additivity formula for degree:

$$\deg(H_G^*(X)) = \sum_{[A,c] \in \mathcal{Q}'_{max}(G,X)} \frac{1}{|W_G(A,c)|} \deg(H_{C_G(A,c)}^*(c)).$$

We also show how this formula relates to the additivity formula from commutative algebra, demonstrating both the algebraic and geometric character of the degree invariant.

1. Introduction

In the 1950’s and 60’s equivariant cohomology developed in many arenas: Cartan’s theory of equivariant differential forms, representation theory, and Atiyah’s K -theory, to name a few. However, the commutative algebraic properties of the Borel equivariant cohomology ring $H_G^*(X)$ associated to the action of a group G on a space X , weren’t well understood until 1971 when Quillen published *The Spectrum of an Equivariant Cohomology Ring: I/II* [16, 17]. Quillen explored the Krull dimension, prime spectrum, and localization of the equivariant cohomology ring $H_G^*(X)$, where G is a compact Lie group, X is a G -space satisfying certain hypotheses, and cohomology coefficients are taken in a field k of characteristic p . He showed that tucked away in each of these commutative algebraic constructs is a great deal of information about the topology and geometry of the G -action on X . Building on Quillen’s insights, many authors have researched commutative algebraic invariants of equivariant cohomology rings. For example, more recent research includes [6, 7], where the concepts of localization and primary decomposition are explored. Symonds [19] studied Castelnuovo–Mumford regularity and the article of Lynn [11] is a direct precursor for this paper. For torus or elementary abelian group actions, the Cohen–Macaulay property from commutative algebra was studied in [9, 1, 2]. This is by no means an

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exhaustive list of references, but shows how the study of commutative-algebraic concepts and invariants in algebraic topology, begun by Quillen, continues to the present day.

The present paper answers a problem posed by Lynn [11] about a certain algebraic invariant called the *degree*. The degree of a finitely generated graded module M over a positively graded ring S , finitely generated as an algebra over S_0 , with S_0 an Artinian ring, is defined in terms of the Poincaré series of M ($PS(M)$):

$$\text{deg}(M) \doteq \lim_{t \rightarrow 1} (1 - t)^{\dim_S(M)} PS(M);$$

here “ $\dim_S(M)$ ” is the Krull dimension of the S -module M . The study of degree is ubiquitous in commutative algebra and algebraic geometry; indeed, for a “standard” graded algebra R —in other words, if R is a finitely generated graded algebra over a field, generated by elements of graded degree 1—the degree of a finitely generated graded R -module M is equal to its Samuel multiplicity. Now, equivariant cohomology rings are usually not generated by elements in a single graded degree, and the degree of M is not always an integer, but there is still a relationship between the degree and an analog of the Samuel multiplicity for graded modules over graded rings: for an exposition of the precise relationship between degree and this analog of the Samuel multiplicity in graded rings, see the preprint of the first author [4].

As far as the authors know, the first study of degree in algebraic topology was done by Maiorana [12, 13] (Maiorana called the invariant “ $c(M)$ ”, rather than “degree”; here, we follow the nomenclature of [3]). In particular, the paper [13] contains some interesting calculations of degree in equivariant cohomology, presenting applications of the degree in algebraic topology which, again as far as the authors know, haven’t yet been fully explored, and will be the subject of future research.

Equivariant cohomology rings are complicated graded rings, in general. In the case where $X = \{pt\}$ is a one point space, and G is a finite group, the equivariant cohomology is the cohomology ring of the group G , which may be defined in a purely algebraic way, and many texts and journal papers have been written on computational examples. Quillen’s theorems in [16, 17] tells us that the complexity of this cohomology ring (recall coefficients are taken in a field k of characteristic p) is directly linked to the poset of elementary abelian p -subgroups of G , and much has been made of this since then. On the other hand, if the group G is very nice; for example, if G is a torus or an elementary abelian p -group, but X is not a point, Quillen’s theorems tells us that the complexity of the equivariant cohomology ring is linked to the cohomology structure of fixed-point sets X^B of the subgroups B of G . Again, much has been made of this by many different authors.

The degree formula proven in this paper is (without explaining now the meaning of the various terms, which come directly from Quillen’s work)

Theorem 1.1. $\text{deg}(H_G^*(X)) = \sum_{[A,c] \in \mathcal{Q}'_{max}(G,X)} \frac{1}{|W_G(A,c)|} \text{deg}(H_{C_G(A,c)}^*(c)).$

This degree formula falls in line with Quillen’s type of results, presenting a way to “compute” the degree from subgroups of the group G associated to elementary p -abelian subgroups of G and fixed point sets of these elementary abelian subgroups, reducing the calculation of degree for an arbitrary group G to calculations to hopefully simpler types of groups acting on spaces in a simpler way: central extensions of an

elementary abelian group A , with A acting trivially on a connected space. This paper does not actually present the study of any sample computations for these sorts of actions; we hope to return to this study in future work. However, see the paper of Lynn [11] for an example when X is a point.

Returning to our introduction of the ideas presented in this paper, given any compact Lie group G , one may smoothly embed it into a unitary group U . For a given prime p , define the space S to be the space of all diagonal matrices of order p in U . The resulting quotient $F \doteq U/S$ is a smooth G -manifold (known in some circles as Quillen’s magic space!). Lynn derives her additivity formula, which is the additivity formula (1) above, in the case $X = \{pt\}$, by cleverly using geometric properties of the orbits of G on certain sub-manifolds of F . These orbits carve up the space in just the right way (Theorems 5.1 and 7.1 of [11]) so that on the algebraic side, the degree invariant adds over the components. For this reason, we refer to Lynn’s additivity formula as “geometric”.

On the other hand, if one appeals only to commutative algebra, there is a different additivity formula for the degree of a graded S -module M , which we will call the “algebraic” additivity formula. Specifically,

$$\deg(M) = \sum_{\mathfrak{p} \in \mathcal{D}(M)} \ell_{S[\mathfrak{p}]}(M_{[\mathfrak{p}]}) \cdot \deg(S/\mathfrak{p}), \tag{1}$$

where $\mathcal{D}(M)$ is the set of minimal primes \mathfrak{p} such that $\dim_S(S/\mathfrak{p}) = \dim_S(M)$. Here, $\dim_A(X)$ is the Krull dimension of the A -module X , $\ell_A(X)$ is the length, in the algebraic sense, of the A -module X , and $Y_{[\mathfrak{p}]}$ is the graded localization of Y at a graded prime ideal \mathfrak{p} .

Lynn concludes her paper asking how to relate this “algebraic” formula (2) to her version of the “geometric” formula (1). The main theorem (5.5) – additivity formula (1) – of the present paper makes this connection precise, while also significantly generalizing Lynn’s result. This theorem is proven for a compact Lie group G acting on a topological space X which is either compact or has finite mod p cohomological dimension.

While Quillen’s theorems say that the sums on the right side of (1) and (2) have essentially the same index sets, we show that the geometric additivity formula (1) is term-by-term equal with the algebraic additivity formula (2) stated above. In fact, the proof of our additivity formula (1) uses the algebraic additivity formula (2) twice. Thus, the connection between commutative algebra and geometry is built into our methodology.

By contrast, the key tool in Lynn’s proof of the special case of (1), where X is a point, is the Gysin sequence of differential topology, which depends fundamentally on the existence of a Riemannian metric on a smooth manifold. Thus, Lynn’s results leading to her proof of the special case of (1) are stated only in the category of smooth G -manifolds. Because our methods of proof rely mainly on commutative algebra, we were able to get away from some of the more complicated geometric/topological arguments Lynn resorts to. Consequently, the hypotheses of our results are far less restrictive (no smoothness assumptions) and agree with the hypotheses of Quillen’s theorems.

2. Summary of Quillen’s results

Most of the results of this section can be found in Quillen [16]. For any compact Lie group G , there exists a “universal principal bundle” for G , denoted $EG \rightarrow BG$. (See Milnor for a construction [15].) Recall that G acts freely on EG and $EG/G \cong BG$.

If X is a topological space on which G acts continuously (a “ G -space”), G acts diagonally on $EG \times X$, and one can consider the associated fiber bundle: $EG \times_G X \doteq (EG \times X)/G$. Then, the *equivariant cohomology ring* is defined by

$$H_G^*(X) \doteq H^*(EG \times_G X).$$

The coefficient ring isn’t noted always, but generally we’ll consider coefficients in a field whose characteristic divides the order of G , if G is finite.

For our purposes, H^* represents singular cohomology, and the product is cup product. In Quillen’s formulation [16] he uses sheaf cohomology, but we don’t lose any of the fundamental properties by switching to singular cohomology (this is explained on pg. 1163 in [19].)

If G and G' are compact Lie groups with X a G -space and X' a G' -space, consider a pair of maps $(u, f) : (G, X) \rightarrow (G', X')$ such that u is a Lie group homomorphism, and f a continuous map that is u -equivariant, i.e. $f(gx) = u(g)f(x)$ for all $g \in G$ and all $x \in X$. These are the morphisms in the category of pairs (X, G) , with X a G -space, that we are considering. With respect to such morphisms, equivariant cohomology is contravariantly functorial [16]. In particular, if $Y \subseteq X$, H is a subgroup of G and u and f are inclusion mappings, then the induced map $H_G^*(X) \rightarrow H_H^*(Y)$ is called the restriction map and usually denoted res_H^G . Also, if $h \in G$, consider the automorphism $(c_h, \mu_h) : (G, X) \rightarrow (G, X)$, where c_h is defined for each $g \in G$ by $c_h(g) \doteq hgh^{-1}$ and μ_h is defined for each $x \in X$ by $\mu_h(x) = hx$. We will call the map on equivariant cohomology induced by such a map an “inner automorphism”. In [16], one finds the proof of

Lemma 2.1. *Inner automorphisms act trivially on equivariant cohomology: If X is a G -space, then for every $h \in G$, $(c_h, \mu_h)^* : H_G^*(X) \rightarrow H_G^*(X)$ is the identity map.*

2.1. Commutative algebra of equivariant cohomology rings

Quillen laid the foundation for the study of the commutative algebra of equivariant cohomology rings. The theorems of Quillen’s papers [16, 17] that we need here are presented below. The two main theorems relate the prime spectrum of $H_G^*(X)$ to the elementary abelian groups which appear as subgroups of G , and have fixed points when acting on X .

Standard properties of cup product say that, in general, the graded ring $H_G^*(X) \doteq \bigoplus_{i \geq 0} H_G^i(X)$ will not be a commutative ring – although when the characteristic of the coefficient ring is 2, we do get commutativity of the product. In order to directly apply results from commutative algebra, we make the following definition for any commutative coefficient ring k : If $char(k)$ is odd or 0, $H_G(X, k) \doteq \bigoplus_{i \geq 0} H_G^{2i}(X, k)$ is the even degree part of $H_G^*(X)$. If $char(k) = 2$ then $H_G(X, k) \doteq H_G^*(X, k)$. Of course, $H_G(X)$ is always a commutative graded ring.

The following results allow for the study of geometric and commutative algebraic properties of equivariant cohomology rings; coefficients will always be taken in a field k .

Theorem 2.2 ([16] Corollary 2.2). *Let G be a compact Lie group, X be a G -space and suppose that $H^*(X)$ is a finitely generated k -vector space. Then, $H_G^*(X)$ is a finitely generated graded k -algebra.*

It's not hard to see that this implies

Corollary 2.3. *With the hypotheses of Theorem 2.2, $H_G(X)$ is a finitely generated commutative graded k -algebra. In particular, $H_G^0(X)$ is a finite dimensional algebra over k and is thus an Artinian ring.*

Another “finite generation” result we shall use is

Theorem 2.4 ([16] Corollary 2.3). *If $(u, f) : (G, X) \rightarrow (G', X')$ is a morphism such that u is injective and $H^*(X)$ is a finitely generated k -module, then the ring homomorphism $(u, f)^* : H_{G'}^*(X') \rightarrow H_G^*(X)$ is finite; i.e. with respect to this homomorphism, $H_G^*(X)$ is a finitely generated $H_{G'}^*(X')$ -module.*

Also, $H_G^(X)$ and $H_G(X)$ are finitely generated graded modules over the commutative ring $H_{G'}(X')$, with respect to the same ring homomorphism $(u, f)^*$, restricted to $H_{G'}(X')$.*

When relating the equivariant cohomology ring of a subgroup G to that of a larger group G' , the theorem above allows us to take advantage of the theory of *integral extensions* from commutative algebra. A more thorough explanation of the theory of integral extensions can be found in [8]. For a subring R of S , an element $s \in S$ is said to be *integral* over R if it is the root of a monic polynomial with coefficients in R . We say that S is integral over R if every element of S is integral over R . When $f : R \rightarrow S$ is a ring homomorphism, we say that S is integral over R (with respect to f) when S is integral over its subring $f(R)$.

Given a ring homomorphism $f : R \rightarrow S$ where S is a finitely generated R -module with respect to f , a standard result from commutative algebra says that S must also be integral over R . In the case of equivariant cohomology rings, if a morphism $(u, f) : (G, X) \rightarrow (G', X')$ is such that u is injective and $H^*(X)$ is a finitely generated k -module (where k is the coefficient field), we see that $H_G(X)$ is integral over $H_{G'}(X')$ (with respect to $(u, f)^*$). For this paper, the fact we need about integral extensions is:

- “Lying over”: For an integral extension $f : R \rightarrow S$, for every $\mathfrak{p} \in \text{Spec}(R)$ containing $\ker f$, there exists a $\mathfrak{q} \in \text{Spec}(S)$ such that $f^{-1}(\mathfrak{q}) = \mathfrak{p}$.

2.2. Main theorems of Quillen

Quillen’s main theorems hold in the following setting, which we assume for the remainder of the paper: G is a compact Lie group which acts continuously on a space X . We also require that X is Hausdorff, and that X is either compact, or is paracompact with finite mod- p cohomological dimension (see [16] for a definition.) Finally, all cohomology is taken with coefficients in the prime field $k \doteq \mathbb{F}_p$ for p a prime number, which is fixed throughout the following discussion, and $H^*(X)$ is a finite dimensional k -vector space.

Recall that an *elementary abelian p -subgroup* of G is a subgroup A of G such that A is isomorphic to a direct product of a finite number r of cyclic groups $\mathbb{Z}/p\mathbb{Z}$. The

number of factors r is called the rank of A . Since p will be taken as a fixed prime number throughout we often omit the reference to p .

Given a pair (G, X) which satisfy the conditions above, we denote *Quillen's category of pairs* by $\mathcal{Q}(G, X)$. Objects of this category are pairs (A, c) where A is an elementary abelian p -subgroup of G , $X^A \neq \emptyset$, and c is a connected component of X^A . If (A, c) and (A', c') are objects in $\mathcal{Q}(G, X)$, then there is a morphism between them if there exists an element $g \in G$ such that $gAg^{-1} \leq A'$ and $c' \subseteq gc$.

If there exists a morphism $(A, c) \rightarrow (A', c')$ in $\mathcal{Q}(G, X)$, we say that (A, c) is *subconjugate* to (A', c') or $(A, c) \lesssim (A', c')$. We define *conjugate* objects, denoted $(A, c) \sim (A', c')$, to be two objects that are isomorphic in $\mathcal{Q}(G, X)$; isomorphism is an equivalence relation on $\mathcal{Q}(G, X)$ and the equivalence class of (A, c) is denoted by $[A, c]$.

By definition, an element $(A, c) \in \mathcal{Q}(G, X)$ is a *maximal* element of $\mathcal{Q}(G, X)$ if and only if any morphism with source (A, c) is an isomorphism. As Quillen [17] notes, (A, c) is a maximal element of $\mathcal{Q}(G, X)$ if and only if A is a maximal elementary abelian subgroup of G_x for every $x \in c$.

Definition 2.5. For a G -space X ,

$$\mathcal{Q}'(G, X) \doteq \{[A, c] : (A, c) \in \mathcal{Q}(G, X), (A, c) \text{ is a maximal element of } \mathcal{Q}(G, X)\}.$$

Within $\mathcal{Q}'(G, X)$, we are often interested in objects corresponding to elementary abelian subgroups whose rank is largest, so we define

$$\mathcal{Q}'_{max}(G, X) \doteq \{[A, c] \in \mathcal{Q}'(G, X) : rk(A) \geq rk(B), \forall [B, d] \in \mathcal{Q}'(G, X)\}.$$

We refer the reader to [14] for an example where $\mathcal{Q}'_{max}(G, X)$ is distinct from $\mathcal{Q}'(G, X)$; in this case, $G = GL_n(\mathbb{Z}/p)$ for $n \geq 4$.

Definition 2.6. Let $(A, c) \in \mathcal{Q}(G, X)$, and pick a point $x_0 \in c$. We define $\mathfrak{p}_{A,c}$ as the kernel of the following composition

$$H_G(X) \xrightarrow{res_A^G} H_A(x_0) \rightarrow H_A(x_0)/\sqrt{0}.$$

Since $H_A/\sqrt{0}$ is a polynomial ring, using the well-known computation of H_A^* , we see that $\mathfrak{p}_{A,c} \in Spec(H_G(X))$.

Theorem 2.7 ([17] Proposition 11.2). *For a G -space X ,*

- $\mathfrak{p}_{(A,c)} \supseteq \mathfrak{p}_{(A',c')}$ if and only if $(A, c) \lesssim (A', c')$. Also, $\mathfrak{p}_{(A,c)} = \mathfrak{p}_{(A',c')}$ if and only if $(A, c) \sim (A', c')$.
- The correspondence $[A, c] \mapsto \mathfrak{p}_{(A,c)}$ defines a bijective function between the elements of $\mathcal{Q}'(G, X)$ and the set of minimal prime ideals in $H_G(X)$.

As a corollary,

Theorem 2.8 ([16]). *For a G -space X , the Krull dimension of $H_G(X)$ equals the maximal rank of an elementary abelian p -subgroup such that $X^A \neq \emptyset$.*

For example, in the case where X is a point, Quillen's theorem says that Krull dimension of H_G equals the maximal rank of an elementary abelian p -group in G .

2.3. Quillen’s magic space F

For any compact Lie group G , there exists a (not unique) unitary group U and a closed embedding of G into U . Given a compact Lie group G , we often begin by fixing such an embedding. Then, define S to be the set of diagonal matrices of order p in U , where p is a fixed prime. Observe that U is equipped with both a left G -action and a right S -action.

The topological space $F \doteq U/S$ then has a natural G -action: if $g \in G$ and $uS \in F$, then the action of G on F is defined by $g \cdot (uS) \doteq guS$. Quillen showed how the product space $X \times F$ can be used to encode information about how G acts on X . In fact, Quillen used the following result to prove the two main theorems of the last section (2.8, 2.7), and we make use of this result in the next section. Recall

that a sequence of modules $A \xrightarrow{h} B \xrightleftharpoons[g]{f} C$ is called an *equalizer sequence* when $0 \rightarrow A \xrightarrow{h} B \xrightarrow{f-g} C$ is an exact sequence.

Lemma 2.9 ([16], Lemma 6.5 and preceding discussion). *The following is an equalizer sequence of $H_G(X)$ -modules:*

$$H_G^*(X) \rightarrow H_G^*(X \times F) \rightrightarrows H_G^*(X \times F \times F);$$

defined by applying the equivariant cohomology functor to the sequence

$$X \times F \times F \xrightleftharpoons[\pi_{13}]{\pi_{12}} X \times F \xrightarrow{\pi_1} X,$$

where π is the projection map onto the indicated components.

Quillen then shows there is an isomorphism of $H_G(X)$ -modules

$$H_G^*(X \times F) \cong H_G^*(X) \otimes_k H^*(F).$$

Here, if $r \in H_G(X)$, $x \otimes y \in H_G^(X) \otimes_k H^*(F)$, then the $H_G(X)$ -module structure is defined by $r(x \otimes y) \doteq (rx) \otimes y$.*

3. Summary of Duflot’s localization results

The material developed in this section is leveraged to prove the main theorem of this paper (Theorem 5.5). We do not present the details of proofs here, but we do give some discussion.

Definition 3.1. Let G be a group, X a G -space, and let $(A, c) \in \mathcal{Q}(G, X)$.

- $N_G(A, c) \doteq \{g \in G : gAg^{-1} = A\} \cap \{g \in G : gc = c\}$
- $C_G(A, c) \doteq \{g \in G : ga = ag, \forall a \in A\} \cap \{g \in G : gc = c\}$
- $W_G(A, c) \doteq N_G(A, c)/C_G(A, c)$

In the case that X is just a point, we omit the c ’s from the notation, e.g. $N_G(A)$ instead of $N_G(A, c)$.

If Z is any G -space and $(A, c) \in \mathcal{Q}(G, Z)$, then $N_G(A, c)$ acts on Z^A , because $N_G(A, c) \subseteq N_G(A)$; of course, $N_G(A, c)$ acts on c as well. We will also need the following notation: if $Y \subseteq Z$, then $G \cdot Y \doteq \{g \cdot y \mid y \in Y\}$ is the indicated G -subspace of Z .

Recall some definitions from graded commutative algebra: if L is a \mathbb{Z} -graded commutative ring, M is a graded A -module and S is a multiplicatively closed subset consisting entirely of homogeneous elements of L , then the abelian group $S^{-1}M$ has a grading in the usual way: the homogeneous elements of $S^{-1}M$ of degree j are the elements m/s where $m \in M$ is homogeneous and $j = \text{deg}(m) - \text{deg}(s)$. With this definition, $S^{-1}L$ is a graded ring, and $S^{-1}M$ is a graded $S^{-1}L$ -module.

Definition 3.2. Let \mathfrak{p} be a graded prime ideal in L , and M a graded L -module.

- $M_{[\mathfrak{p}]}$ is the localization $S^{-1}M$ where S is the set of homogeneous elements of L not in \mathfrak{p} . We will say that $M_{[\mathfrak{p}]}$ is the *graded localization* of M at \mathfrak{p} .
- $M_{(\mathfrak{p})}$ is the ungraded (or “concentrated in degree zero”) module consisting of the elements of $M_{[\mathfrak{p}]}$ of degree zero.
- $M_{\mathfrak{p}}$ is the ungraded module $\tilde{S}^{-1}M$ where \tilde{S} is the set of all elements of L not in \mathfrak{p} .

We won’t use the second notion of localization in the list above, which is that used in algebraic geometry [10], in this paper. We’ll say that the third type of localization above is “ordinary” localization.

The main result of [6] is:

Theorem 3.3 ([6] Theorem 3.2). *Suppose that $[A, c] \in \mathcal{Q}'(G, X)$. Then there is an isomorphism of $H_G(X)_{[\mathfrak{p}(A, c)]}$ -modules*

$$H_G^*(X)_{[\mathfrak{p}(A, c)]} \xrightarrow{(\text{res}_{C_G(A, c)}^G)_{[\mathfrak{p}(A, c)]}} H_{C_G(A, c)}^*(c)_{[\mathfrak{p}(A, c)]}^{W_G(A, c)}.$$

Remark 3.4. In the original paper [6], it was not clear whether the localization was ordinary localization or graded localization. In [7], Dufлот clarifies this, and we’ve added the brackets here to indicate graded localization.

Let’s discuss this result a bit, giving an outline of the proof. Our outline focuses on results we will need in the final section of the paper, and we do not give proofs. Suppose that an embedding of G into a unitary group U is fixed, and F is the G -space U/S as before.

First, $N_G(A, c)$ acts on c and hence on $c \times F^i$ and on $c \times (F^A)^i$ diagonally. Furthermore, these actions induce actions of $W_G(A, c)$ on $H_{C_G(A, c)}^*(Z)$, whether Z is $c, c \times F^i$ or $c \times (F^A)^i$: for any $n \in N_G(A, c)$, we have the equivariant conjugation map on the pair $(C_G(A, c), Z) \rightarrow (C_G(A, c), Z)$ which is the pair $g \mapsto ngn^{-1}$ and $z \mapsto n \cdot z$. Refer to this map as n , and the induced ring automorphism on equivariant cohomology as $n^* : H_{C_G(A, c)}^*(Z) \rightarrow H_{C_G(A, c)}^*(Z)$. Recall that inner automorphisms act trivially on equivariant cohomology, therefore if $n \in C_G(A, c)$ then $n^* = \text{id}$ on $H_{C_G(A, c)}^*(Z)$. Therefore, $W_G(A, c)$ has a well-defined action on $H_{C_G(A, c)}^*(Z)$.

Now, for each of the spaces Z in the paragraph above, there are equivariant maps $\theta : (C_G(A, c), Z) \rightarrow (G, X)$: for the groups involved, we use inclusions, and for the spaces, if $Z = c$, we use inclusion $c \rightarrow X$, while for $Z = c \times F^i$ or $Z = c \times (F^A)^i$, θ is the inclusion $Z \rightarrow X \times -$, followed by projection onto the first factor. We point out that all of the induced maps on equivariant cohomology $\theta^* : H_G^*(X) \rightarrow H_{C_G(A, c)}^*(Z)$ have the property that, for every (homogeneous) $x \in H_G^*(X), y \in H_{C_G(A, c)}^*(Z)$ and

every $w \in W_G(A, c)$, $\theta^*(x)(w \cdot y) = w \cdot (\theta^*(x)y)$; in other words, the automorphisms induced by the elements of $W_G(A, c)$ on $H_{C_G(A,c)}^*(Z)$ are $H_G(X)$ -module maps. Note that this implies the image of θ^* lands in the invariant subring $H_{C_G(A,c)}^*(Z)^W$ since every $w \in W$ induces a ring automorphism preserving the multiplicative identity 1 of the ring:

$$\theta^*(x) = \theta^*(x)(w \cdot 1) = w \cdot \theta^*(x).$$

To see this, look at the commutative diagrams below; n is always the map sending $g \mapsto ngn^{-1}$, $z \mapsto nz$, for every $n \in N_G(A, c)$:

$$\begin{CD} (G, X) @>n>> (G, X) \\ @V\theta VV @VV\theta V \\ (C_G(a, c), Z) @>n>> (C_G(A, c), Z). \end{CD}$$

Thus there is a commutative diagram

$$\begin{CD} H_G^*(X) @<n^*<< H_G^*(X) \\ @V\theta^*VV @VV\theta^*V \\ H_{C_G(A,c)}^*(Z) @<n^*<< H_{C_G(A,c)}^*(Z). \end{CD}$$

Now, Lemma 2.1 implies that the upper ring homomorphism n^* on $H_G^*(X)$ is the identity map, so commutativity implies that

$$n^*(\theta^*(x)y) = n^*(\theta^*(x))n^*(y) = \theta^*(n^*(x))n^*(y) = \theta^*(x)n^*(y),$$

which is our point: by definition, if w is represented by $n \in N_G(A, c)$, $w \cdot ? \doteq n^*(?)$.

Dufлот passes a problem about $H_G^*(X)$ to the space $H_G^*(X \times F)$ using Quillen’s equalizer sequence 2.9; first, the following two results are proved:

Lemma 3.5 ([6] pg. 98, 99). *Suppose that X is a G -space and $(A, c) \in \mathcal{Q}(G, X)$; note that the subgroup $N_G(A, c)$ of G acts on $c \times F^A$ (diagonally).*

- i. *The set of components of $c \times (F^A)^i$, $\pi_0(c \times (F^A)^i)$, is a finite set. The action of $N_G(A, c)$ on $c \times (F^A)^i$ induces an action of $N_G(A, c)$ on $\pi_0(c \times (F^A)^i)$ such that if d is a component of $c \times (F^A)^i$, then for every $g \in C_G(A, c)$, $g \cdot d = d$. Therefore, $W_G(A, c)$ acts on the set of components of $c \times (F^A)^i$.*
- ii. *$W_G(A, c)$ acts freely on the set of components of $c \times (F^A)^i$, $i = 1, 2$.*

Although the corollary below is stated in [6], we’ll provide details about why it follows from 3.5 later in this paper, in 5.3.

Corollary 3.6 ([6] Lemma 3.5). *If $(A, c) \in \mathcal{Q}(G, X)$, then $H_{C_G(A,c)}^q(c \times (F^A)^i)$ is a free $k[W_G(A, c)]$ -module for all $i \geq 1$ and all $q \geq 0$.*

Now, Dufлот’s proof of her localization theorem uses results about the following sequence of homomorphisms of graded rings, for a pair $[A, c] \in \mathcal{Q}'(G, X)$:

$$H_G^*(X) \xrightarrow{\textcircled{1}} H_G^*(G \cdot (c \times F^A)) \xrightarrow{\textcircled{2}} H_{N_G(A,c)}^*(c \times F^A) \xrightarrow{\textcircled{3}} H_{C_G(A,c)}^*(c \times F^A)^{W_G(A,c)} \tag{*}$$

The sequence of facts from [6] that we will need concerning these maps are as follows.

Lemma 3.7 ([6] Lemma 3.4). *For $[A, c] \in \mathcal{Q}'(G, X)$, there is an isomorphism (the map ② in (\star) above)*

$$H_G^*(G \cdot (c \times F^A)) \cong H_{N_G(A,c)}^*(c \times F^A)$$

Lemma 3.8 ([6] Lemma 3.6). *For $(A, c) \in \mathcal{Q}(G, X)$, there is an isomorphism (the map ③ in (\star) above)*

$$H_{N_G(A,c)}^*(c \times F^A) \cong H_{C_G(A,c)}^*(c \times F^A)^{W_G(A,c)}.$$

Finally, one takes graded localizations at minimal primes and obtains

Theorem 3.9 ([6] pg. 100). *Fix an $[A, c] \in \mathcal{Q}'(G, X)$ with corresponding minimal prime $\mathfrak{p}_{(A,c)} \doteq \mathfrak{p}$. The following diagram is commutative, each of the vertical arrows are graded isomorphisms of $R_{[\mathfrak{p}]} = H_G(X)_{[\mathfrak{p}]}$ -modules, and the first and last rows are equalizer sequences.*

$$\begin{array}{ccccc}
 H_G^*(X)_{[\mathfrak{p}]} & \longrightarrow & H_G^*(X \times F)_{[\mathfrak{p}]} & \xlongequal{\quad} & H_G^*(X \times F^2)_{[\mathfrak{p}]} \\
 & & \downarrow \textcircled{1} & & \downarrow \\
 & & H_G^*(G \cdot (c \times F^A))_{[\mathfrak{p}]} & \xlongequal{\quad} & H_G^*(G \cdot (c \times (F^A)^2))_{[\mathfrak{p}]} \\
 & & \downarrow \textcircled{2} & & \downarrow \\
 & & H_{N_G(A,c)}^*(c \times F^A)_{[\mathfrak{p}]} & \xlongequal{\quad} & H_{N_G(A,c)}^*(G \cdot (c \times (F^A)^2))_{[\mathfrak{p}]} \\
 & & \downarrow \textcircled{3} & & \downarrow \\
 & & H_{C_G(A,c)}^*(c \times F^A)_{[\mathfrak{p}]}^{W_G(A,c)} & \xrightarrow{\quad} & H_{C_G(A,c)}^*(G \cdot (c \times (F^A)^2))_{[\mathfrak{p}]}^{W_G(A,c)} \\
 & & \uparrow \textcircled{4} & & \uparrow \\
 H_{C_G(A,c)}^*(c)_{[\mathfrak{p}]}^{W_G(A,c)} & \longrightarrow & H_{C_G(A,c)}^*(c \times F)_{[\mathfrak{p}]}^{W_G(A,c)} & \xlongequal{\quad} & H_{C_G(A,c)}^*(G \cdot (c \times F^2))_{[\mathfrak{p}]}^{W_G(A,c)}
 \end{array}$$

Note that, before localizing, all modules in the diagram are $R \doteq H_G(X)$ -modules, using restriction maps. Taking the claims of this theorem for granted, Duflot’s localization result is obtained from this diagram since each square is commutative and all of the vertical arrows are isomorphisms. The crucial vertical arrow ① comes from a restriction map, but is only an isomorphism after graded (or ungraded) localization at $\mathfrak{p} \subseteq H_G(X)$; the details are explained in Lemma 3.3 of [6] (which we have not included here). Similarly, the bijectivity of the arrow ④, which comes from a restriction map, is also an application of Lemma 3.3 of [6] in a different case.

The exactness of the first and final rows of Duflot’s diagram 3.9 come from Lemma 2.9. Specifically, apply the result to the pairs (G, X) and $(C_G(A, c), c)$ respectively, to get the equalizer sequences:

$$H_G^*(X) \rightarrow H_G^*(X \times F) \rightrightarrows H_G^*(X \times F^2),$$

and

$$H_{C_G(A,c)}^*(c) \rightarrow H_{C_G(A,c)}^*(c \times F) \rightrightarrows H_{C_G(A,c)}^*(c \times F^2).$$

Since localization is exact, and taking invariants is left exact, we get exactness of

the top and bottom rows of Dufлот’s diagram 3.9, and conclude that $H_G^*(X)_{[p]} \cong H_{C_G(A,c)}^*(c)_{[p]}^{W_G(A,c)}$.

4. The degree of an equivariant cohomology ring

In this section we state the definition of the degree of a graded module, and state relevant results. This means that there will be two different usages of “degree”: the degree of a homogeneous element in a graded ring, and the degree of the ring itself. The usages are clearly distinguished by context.

Suppose L is a \mathbb{Z} -graded Noetherian ring; we denote the category of finitely-generated graded L -modules by $\mathfrak{grmod}(L)$.

In this paper, all of our graded rings will be of one of two types:

- I. A positively graded Noetherian ring R ($R_i = 0$ for $i < 0$), with R_0 an Artinian ring that is a finitely generated k -algebra: thus R_0 must be finite dimensional as a vector space over k .
- II. A graded localization of a positively graded ring of type I at a graded prime ideal.

We use an asterisk to denote a usual definition in non-graded algebra, adapted for the graded category; this is fairly standard notation – e.g., it is used in [5]. For example, if $M \in \mathfrak{grmod}(L)$ is a graded L -module, we denote its graded Krull dimension by $*dim_L(M)$, i.e. this is the longest chain of graded prime ideals in L containing the L -annihilator of M . For the type of rings that interest us, it turns out that graded and ungraded measures are directly comparable; for some measures this is well-known, in other cases we will refer to [4] for discussion and proofs. We choose to use the $*$ -notation in any case, as it is often practical to work only with homogeneous elements.

From now on, we will use R to denote a general ring of type I.

Definition 4.1. For $M \in \mathfrak{grmod}(R)$, the Poincaré series of M may be defined by

$$P_M(t) = \sum_i \text{vdim}_k(M_i)t^i,$$

where $\text{vdim}_k(V)$ is the vector space dimension of the vector space V over k .

The Poincaré series may also be computed using length, rather than vector space dimension. To see this, if $M \in \mathfrak{grmod}(R)$, since R_0 is an Artinian ring and M_j is a finitely generated R_0 -module, M_j is an Artinian R_0 -module. Thus the length of M_j as an R_0 -module, with the usual definition of length, is defined. Now, the claim is that $\ell_{R_0}(M_j) = \text{vdim}_k(M_j)$. This follows since, for every maximal ideal \mathfrak{m}_0 of R_0 , R_0/\mathfrak{m}_0 is isomorphic to the field k . Therefore,

$$P_M(t) = \sum_i \text{vdim}_k(M_i)t^i = \sum_i \ell_{R_0}(M_i)t^i.$$

Recall that, since R is of type I, $*dim_R(M) = dim_R(M)$ and $*dim_R(M)$ is the order of the pole of $P_M(t)$ at $t = 1$ ([4] contains an account of this, but these facts are well-known and hold for any positively graded Noetherian ring whose elements in degree zero form an Artinian ring).

Definition 4.2. If R is of type I, $M \in \mathbf{grmod}(R)$, $M \neq 0$ and $D(M) = {}^*\dim_R(M)$, then

$$\deg_R(M) \doteq \lim_{t \rightarrow 1} (1 - t)^{D(M)} P_M(t)$$

is a well-defined, strictly positive, rational number. For convenience, let $\deg_R(0) = 0$.

Given $M \in \mathbf{grmod}(R)$, $M \neq 0$, we can read off the degree of a module directly from the Poincaré series if we expand it as a Laurent series about $t = 1$:

$$P_R(t) = \frac{\deg(M)}{(1 - t)^{D(M)}} + \text{“higher order terms”}.$$

The degree of M can be regarded as a species of non-integer multiplicity associated to M . As noted in the Introduction, this rational number is called $c(M)$ in [12]; as far as we know, this is the first reference that discusses this numerical invariant for equivariant cohomology. However, we use the terminology “degree”, as used in [3]. The reference [20] uses “degree” to refer to the integer invariant of local algebra also called “multiplicity”. The first author’s preprint [4] exposits the concept of a “Samuel multiplicity” associated to certain ideals in a graded ring, and makes precise the relationship between the degree and this Samuel multiplicity. We don’t make this more precise here, because making all the definitions necessary to state the relationship would add to the length of this paper and is not used anywhere here; instead we refer the reader to [4].

We may make the following definitions:

Definition 4.3. Let L be a \mathbb{Z} -graded ring.

- A nonzero graded L -module M is a $*$ simple L -module if and only if the only graded L -submodules of M are 0 and M .
- A $*$ composition series for a graded L -module M is a sequence of graded submodules of M of the form

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

such that for each i , M_{i+1}/M_i is a $*$ simple L -module. The $*$ length of this $*$ composition series is defined to be n .

The expected properties of $*$ composition series hold and we may talk about the $*$ length of an L -module M (denoted $*\ell_L(M)$). One can see that $*\ell_L$ adds over short exact sequences of graded L -modules; another straightforward lemma is:

Lemma 4.4. *Let L be a \mathbb{Z} -graded ring with L_0 is a finite dimensional vector space over a field k . (This implies that L_0 is an Artinian ring.) Suppose that M is a graded L -module and V is a graded finite dimensional vector space over k , consider $M \otimes_k V$ as a graded L -module defined by $r \cdot (m \otimes v) \doteq (r \cdot m) \otimes v$ for $r \in L$ (with the usual grading on the tensor product). If $*\ell_L(M)$ is finite, so is $*\ell_L(M \otimes_k V)$; in fact, $*\ell_L(M \otimes_k V) = *\ell_L(M)t$, where t is the (total) dimension of V as a vector space over k .*

For comparisons to ungraded concepts and further discussion, we refer to [4]; to give some context to results proved here, a sample fact from [4] is

Lemma 4.5 (Theorem 4.10 of [4]). *Let R be of type I, and let \mathfrak{p} be a minimal prime for $M \in \mathbf{grmod}(R)$. Then, \mathfrak{p} is graded and $*\ell_{R[\mathfrak{p}]}(M_{[\mathfrak{p}]}) = \ell_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) < \infty$.*

The following theorem gives an expected additivity formula for degree, where the sum is indexed by the minimal primes of maximal dimension. This theorem just states the behavior expected of some multiplicity-type numerical invariants, and its proof is similar to corresponding statements for other sorts of multiplicities. For example, see [18] for what Serre calls Samuel multiplicity for local rings; [20] gives an account of some other types of multiplicity. The main theorem of this paper (5.5) places this “algebraic” additivity formula in a geometric context when applied to equivariant cohomology rings.

Theorem 4.6 (see, e.g., [4] Theorem 6.7). *Suppose R is of type I. Let $M \in \mathbf{grmod}(R)$, and $\mathcal{D}(M)$ be defined as the set of prime ideals \mathfrak{p} in R , necessarily minimal primes for M and graded, such that $*\dim_R(R/\mathfrak{p}) = *\dim_R(M)$. Then,*

$$\deg(M) = \sum_{\mathfrak{p} \in \mathcal{D}(M)} *\ell_{R[\mathfrak{p}]}(M_{[\mathfrak{p}]}) \cdot \deg(R/\mathfrak{p}).$$

4.1. Degree decomposition of Lynn

This short section serves simply to give a snapshot of Lynn’s results [11]. Her main result is a geometric additivity formula for the degree of an equivariant cohomology ring in the case where X is taken to be a point. This work was the inspiration for our more general additivity formula (Theorem 5.5); however, we do not use any of Lynn’s results in this paper.

As noted in the introduction, Lynn’s methods rely on the smoothness of the G -space X . In order to prove her main theorem, she uses Gysin-type sequences to prove

Theorem 4.7 ([11] Theorem 4.21). *Let G be a compact Lie group, and let X be a smooth, compact G -manifold. Let $Z = \cup_{i=1}^n Z_i$, where the Z_i ’s are closed, G -invariant, disjoint submanifolds of X such that ν_{Z_i} (the normal bundle) is orientable for all i . Assume that $\dim(H_G^*(X)) = \dim(H_G^*(Z_i))$ for all i , and if $z \notin Z$, then we require $\dim(H_{G_z}^*) < \dim(H_G^*(Z_i))$ for all i . Then,*

$$\deg(H_G^*(X)) = \sum_{i=1}^n \deg(H_G^*(Z_i)).$$

Embedding G in a unitary group U , and taking $F = U/S$ as usual, Lynn shows that for any $[A, c] \in \mathcal{Q}'_{max}(G, X)$ the sub-space $G \cdot (c \times F^A)$ satisfies the hypotheses required of Z_i in the above theorem, so deduces:

Corollary 4.8. *For X a compact, smooth manifold with G a compact Lie group acting smoothly on X , $\deg(H_G^*(X)) = \sum_{[A,c] \in \mathcal{Q}'_{max}(G,X)} \deg(H_G^*(G \cdot (c \times F^A)))$*

Lynn argues by descent, which simply means using 2.9, and taking X to be a point, to derive the following additivity formula for degree.

Theorem 4.9 ([11], Theorem 8.5). *Let G be a compact Lie group, and let $\mathcal{Q}'_{max}(G)$ be the set of conjugacy classes of maximal rank elementary abelian p -groups of G .*

Then,

$$\deg(H_G^*) = \sum_{[A] \in \mathcal{Q}'_{max}(G)} \frac{1}{|W_G(A)|} \deg(H_{C_G(A)}^*).$$

As noted in the introduction, she doesn't prove such a formula for equivariant cohomology $H_G^*(X)$ where X is any G -space, and in any case, one would only expect her methods to generalize in the case of a smooth G -space X .

5. Main theorem on degree

As usual, we assume that G is compact Lie, and it acts continuously on the Hausdorff space X , which is compact or paracompact with finite mod- p cohomological dimension. Fix a prime p , and take $k = \mathbb{F}_p$ to be the field of coefficients for cohomology.

Before we start, we make the simple observation that the set of primes for $H_G^*(X)$ as an $H_G(X)$ -module is the set of prime ideals in $H_G(X)$, since $\text{Ann}_{H_G(X)} H_G^*(X) = \{0\}$.

We will make use of Dufлот's localization result 3.3, so let's look a little more closely at the ring $H_{C_G(A,c)}^*(c)$ for a given pair $(A, c) \in \mathcal{Q}(G, X)$. A first observation is that $H_{C_G(A,c)}^*(c)$ may be thought of as a module in the category $\mathbf{grmod}(H_{C_G(A,c)}(c))$, or as a module in $\mathbf{grmod}(H_G(X))$. Its structure as a finitely generated, graded module over $H_G(X)$ comes from the restriction map $\text{res}_C^G : H_G^*(X) \rightarrow H_{C_G(A,c)}^*(c)$ - an application of Theorem 2.4. The following lemma relates the minimal primes of $H_{C_G(A,c)}^*(c)$ as a module over these two rings.

Lemma 5.1. *Let $[A, c] \in \mathcal{Q}'(G, X)$. Let $R = H_G(X)$, $S = H_{C_G(A,c)}(c)$, and make S -modules into R -modules via the restriction map $\text{res}_C^G : R \rightarrow S$, so that $H_{C_G(A,c)}^*(c)$ is an element of $\mathbf{grmod}(R)$ and $\mathbf{grmod}(S)$. Then,*

- i. $\mathcal{Q}'(C_G(A, c), c) = \{[A, c]\}$.
- ii. $\mathfrak{p}^C \doteq \ker(H_{C_G(A,c)}(c) \rightarrow H_A/\sqrt{0})$ is the unique minimal prime for $H_{C_G(A,c)}^*(c)$ as an S -module.
- iii. $(\text{res}_C^G)^{-1}(\mathfrak{p}^C) = \mathfrak{p}$, where $\mathfrak{p} \doteq \ker(H_G(X) \rightarrow H_A/\sqrt{0})$.
- iv. \mathfrak{p} is the unique minimal prime for $H_{C_G(A,c)}^*(c)$ as an R -module.
- v. $*\ell_{S[\mathfrak{p}^C]}(H_{C_G(A,c)}^*(c)_{[\mathfrak{p}^C]}) < \infty$ and $*\ell_{R[\mathfrak{p}]}(H_{C_G(A,c)}^*(c)_{[\mathfrak{p}]}) < \infty$; also,

$$\begin{aligned} \deg(H_{C_G(A,c)}^*(c)) &= *\ell_{S[\mathfrak{p}^C]}(H_{C_G(A,c)}^*(c)_{[\mathfrak{p}^C]}) \deg(S/\mathfrak{p}^C) \\ &= *\ell_{R[\mathfrak{p}]}(H_{C_G(A,c)}^*(c)_{[\mathfrak{p}]}) \deg(R/\mathfrak{p}). \end{aligned}$$

Proof. Write $C \doteq C_G(A, c)$. Of course $A \subseteq C$ since A is abelian. Since A acts trivially on c , we see that, for every $x \in c$, $A \subseteq C_x \subseteq G_x$. We know that A is a maximal elementary abelian subgroup of G_x , so A is a maximal elementary abelian subgroup of C_x as well. Thus $[A, c] \in \mathcal{Q}'(C, c)$. On the other hand, suppose that $[B, d] \in \mathcal{Q}'(C, c)$. Then, B is a maximal elementary abelian subgroup of C_x for every $x \in d$, which is a component of c^B . Let D be the subgroup of C generated by A and B . Then, D is an elementary abelian subgroup of C , by definition of C . Suppose that $x \in d \subseteq c^B$.

Since A fixes x and B fixes x , so does D , so that $D \subseteq C_x$. By maximality of B in C_x , $B = D$, so that $A \subseteq B$, and by maximality of A in G_x , $A = B$, so $c^B = c$ and $d = c$. Thus, $(A, c) = (B, d)$ as pairs. This proves i), and ii) follows from 2.7.

To see iii), note that the following diagram is commutative, where pt is any point in c :

$$\begin{array}{ccc} (G, X) & \longleftarrow & (C_G(A, c), c) \\ \uparrow & \nearrow & \\ (A, pt) & & \end{array}$$

By functoriality, there is a commutative diagram:

$$\begin{array}{ccc} H_G(X) & \xrightarrow{res_C^G} & H_{C_G(A, c)}(c) \\ \pi \circ res_A^G \downarrow & \nearrow \pi \circ res_A^C & \\ H_A(pt)/\sqrt{0} & & \end{array}$$

Now, \mathfrak{p} is a minimal prime in $H_G(X)$ by 2.7, so by commutativity of the above diagram, $\mathfrak{p} = res_C^{G^{-1}}(\mathfrak{p}^C)$.

To see iv), as noted above, \mathfrak{p} is a minimal prime for $H_G(X)$ and thus for $H_G^*(X)$ as an R -module. Let's show that \mathfrak{p} is a minimal prime for $H_{C_G(A, c)}^*(c)$ as an R -module. We need to show that \mathfrak{p} is minimal over

$$Ann_R(H_{C_G(A, c)}^*(c)) \doteq \left\{ r \in R : res_C^G(r) \cdot x = 0, \text{ for all } x \in H_{C_G(A, c)}^*(c) \right\}.$$

But $H_{C_G(A, c)}^*(c)$ is a unital ring, so $Ann_R(H_{C_G(A, c)}^*(c)) = \ker(res_C^G)$, and by commutativity of the diagram $\ker(res_C^G) \subseteq \mathfrak{p}$. Since \mathfrak{p} is minimal in R , it must be minimal over $Ann_R(H_{C_G(A, c)}^*(c))$ and thus is a minimal prime for $H_{C_G(A, c)}^*(c)$ as an R -module.

Finally, we need to show that \mathfrak{p} is the unique minimal prime for $H_{C_G(A, c)}^*(c)$ as an R -module. Since we know that $res_C^G : R \rightarrow S$ is an integral extension, we may use "lying over". Let $\mathfrak{q} \subseteq R$ be another minimal prime for $H_{C_G(A, c)}^*(c)$ as an R -module; "lying over" implies there exists a $\tilde{\mathfrak{q}} \in Spec(S)$ such that $(res_C^G)^{-1}(\tilde{\mathfrak{q}}) = \mathfrak{q}$. But, \mathfrak{p}^C is the only minimal prime in $Spec(S)$, so $\mathfrak{p}^C \subseteq \tilde{\mathfrak{q}}$. Thus, $(res_C^G)^{-1}(\mathfrak{p}^C) \subseteq (res_C^G)^{-1}(\tilde{\mathfrak{q}})$ which implies that $\mathfrak{p} \subseteq \mathfrak{q}$, but by assumption \mathfrak{q} is minimal, and therefore $\mathfrak{p} = \mathfrak{q}$.

For the last proof, set $N \doteq H_{C_G(A, c)}^*(c)$. Parts ii) and iv) of this lemma show that \mathfrak{p} is a minimal prime for N as an R -module and \mathfrak{p}^C is a minimal prime for N as an S -module. Using 4.5, $N_{[\mathfrak{p}]}$ is a $*$ Artinian $R_{[\mathfrak{p}]}$ -module, and $N_{[\mathfrak{p}^C]}$ is a $*$ Artinian $S_{[\mathfrak{p}^C]}$ -module, which proves the claim on finite $*$ length.

To prove the claim on degree, we use the algebraic additivity formula for degree, Theorem 4.6. Since we've shown in parts ii) and iv), that \mathfrak{p} is the unique minimal prime for N as an R -module, and \mathfrak{p}^C is the unique minimal prime for N as an S -module, there is only one summand in the algebraic additivity formula for degree, whether we consider N as an R -module or an S -module. Therefore, thinking of N as an R -module, $\deg(N) = *l_{R_{[\mathfrak{p}]}}(N_{[\mathfrak{p}]}) \deg(R/\mathfrak{p})$; similarly the algebraic additivity formula applied to N as an S -module gives $\deg(N) = *l_{S_{[\mathfrak{p}^C]}}(N_{[\mathfrak{p}^C]}) \deg(S/\mathfrak{p}^C)$. \square

We state the following theorem in a general algebraic setting, and will demonstrate an application (Theorem 5.3) to the cohomology ring $H_{C_G(A,c)}^*(c)^{W_G(A,c)}$.

Theorem 5.2. *Let L be a \mathbb{Z} -graded k -algebra for a field k (concentrated in degree 0) and let P be a graded L -module. Suppose the following:*

- P is a direct sum of graded L -submodules indexed by a finite set π_0 : that is, $P = \bigoplus_{c \in \pi_0} P_c$.
- W is a finite group which acts as a group of graded L -module automorphisms on P .
- W acts freely on the set π_0 and this action is compatible with the W -action on P and the direct sum decomposition: to be precise, for any $x \in P_c$, $w \cdot x \in P_{w \cdot c}$.

Then,

- i. Let $c_1, \dots, c_t \in \pi_0$ be a set of representatives for the orbits of W on π_0 . Note that $t = |\pi_0|/|W|$. For each j from 1 to t , define $P[j] = \bigoplus_{w \in W} P_{wc_j}$. Then, $P[j]$ is a graded submodule of P with respect to both L and $k[W]$. Also, $P = \bigoplus_{j=1}^t P[j]$. Here, $k[W]$ is regarded as a graded ring concentrated in degree 0.
- ii. P is a free $k[W]$ -module.
- iii. P^W is isomorphic as a graded L -module to $\bigoplus_{j=1}^t P_{c_j}$.
- iv. If P is a $*$ Artinian L -module, then so is P^W and,

$$*\ell_L(P) = |W| * \ell_L(P^W).$$

Proof. For ease of notation, we assume that the direct sum decomposition is an internal direct sum decomposition, except in item iii).

Property i) is more or less by definition, using the hypotheses: since the action of W on π_0 is free, $P_{wc_j} \cap P_{\tilde{w}c_j} = \{0\}$ if $w \neq \tilde{w}$, so $P[j]$ is a direct sum of graded L -submodules, as desired. $P[j]$ is a $k[W]$ -module by hypothesis, and its definition. Using the given decomposition $P = \bigoplus_c P_c$ along with the fact that we've picked c_1, \dots, c_t as representatives for the orbits under W (and hence $P[j] \cap P[i] = 0$ for $i \neq j$) gives us the decomposition by orbits: $P = \bigoplus_{j=1}^t P[j]$.

Property ii) is straightforward, but we give details: let E_d be a k -vector space basis for the degree d homogeneous component of P_{c_j} . We claim that E_d is a $k[W]$ -basis for the degree d component of $P[j]$. By picking a k -basis for every homogeneous component, and every j , we get a $k[W]$ -basis consisting of homogeneous elements for $P = \bigoplus_{j=1}^t P[j]$ (note that the basis may not be finite).

Writing out the details to see how the hypotheses are used, let $x \in P[j]_d$, and write $x = \sum_{w \in W} x_w$, $x_w \in P_{w \cdot c_j}$. By hypothesis, $w^{-1}x_w \in P_{w^{-1}w \cdot c_j} = P_{c_j}$ for each w . Use the vector space basis of P_{c_j} to write $w^{-1}x_w = \sum_{e \in E_d} \alpha_e(w)e$ where $\alpha_e(w) \in k$ for every $e \in E_d$ and equal to 0 for almost all e . Thus, $x = \sum_w x_w = \sum_w \sum_e \alpha_e(w)we$, which is in the $k[W]$ -span of the E_d . For linear independence, suppose $\sum_e \xi_e e = 0$ for $\xi_e \in k[W]$; again $\xi_e = 0$ for almost all e . For each e , write ξ_e as a linear combination $\sum_{w \in W} \alpha_e(w)w$, $\alpha_e(w) \in k$ for every e . Then,

$$\sum_e \xi_e e = \sum_e \sum_w \alpha_e(w)we = \sum_w \left(\sum_e \alpha_e(w)we \right).$$

Now for each $w \in W$, $w \cdot E \doteq \{w \cdot e \mid e \in E_d\}$ is contained in P_{wc_j} , and $w \cdot E$ is a

vector space basis for the degree d homogeneous component of P_{wc_j} because w is a graded automorphism of P . Therefore $\sum_e \alpha_e(w)w \cdot e = 0$, for every w , using the direct sum decomposition; in turn, $\alpha_e(w) = 0$, for every w and every e , so $\xi_e = 0$ for every e .

For property iii), define $\theta_j : P_{c_j} \rightarrow P[j]$ by $x \mapsto \sum_{w \in W} w \cdot x$. Since each w acts as a graded L -module homomorphism, θ_j is a graded L -module homomorphism. Further, θ_j is injective since $P[j]$ is direct sum.

We claim that the image of θ_j is $P[j]^W$; in this case, the result is obtained since invariants distribute over direct sums of $k[W]$ -modules, i.e. $\bigoplus_j \theta_j$ is an isomorphism from $\bigoplus_{j=1}^t P_{c_j}$ to P^W . It's straightforward to see that the image of θ_j is contained in $P[j]^W$. On the other hand, let $x \in P[j]^W$, and write x uniquely as $\sum_{w \in W} x_w$, where $x_w \in P_{wc_j}$ for every w . Now, for every $g \in W$, $gx = \sum_{w \in W} gx_w = \sum_{w \in W} x_w = x$. Since $gx_w \in P_{gwc_j}$, using the direct sum property, if 1 is the identity in the group W , $wx_1 = x_w$ for every w and $\theta_j(x_1) = \sum_{w \in W} wx_1 = x$.

For iv), as expected, $*\ell$ is additive over short exact sequences of L -modules and thus over direct sums. Of course, for each w and each c_j , $*\ell_L(P_{c_j}) = *\ell_L(P_{wc_j})$ since W acts as a group of L -automorphisms. Thus,

$$*\ell_L(P) = \sum_{j=1}^t \sum_{w \in W} *\ell_L(P_{wc_j}) = |W| \sum_{j=1}^t *\ell_L(P_{c_j}) = |W| *\ell_L(P^W). \quad \square$$

In the following theorem, we've fixed an embedding of G in a unitary group U , and $F \doteq U/S$ is the G -space described previously; if $(A, c) \in \mathcal{Q}(G, X)$, we've also already explained how we are considering $H_{C_G(A,c)}^*(c \times F^A)$ as a graded $H_G(X)$ -module.

Theorem 5.3. *Let $[A, c] \in \mathcal{Q}'(G, X)$ and $\mathfrak{p} \doteq \mathfrak{p}_{(A,c)}$.*

- i. Let π_0 be the set of connected components of $c \times F^A$. Then,*
 - (a) $H_{C_G(A,c)}^*(c \times F^A)$ is a free $k[W_G(A, c)]$ -module.*
 - (b) If $c_i, 1 \leq i \leq t$, are the representatives for the orbits of $W_G(A, c)$ acting on the set of components of $c \times F^A$, then*

$$H_{C_G(A,c)}^*(c \times F^A)^{W_G(A,c)} \cong \bigoplus_{i=1}^t H_{C_G(A,c)}^*(c_i),$$

as graded $H_G(X)$ -modules.

- ii. In addition,*

$$\begin{aligned} & *\ell_{H_G(X)_{[\mathfrak{p}]}} \left(H_{C_G(A,c)}^*(c \times F^A)_{[\mathfrak{p}]}^{W_G(A,c)} \right) \\ &= \frac{1}{|W_G(A, c)|} *\ell_{H_G(X)_{[\mathfrak{p}]}} \left(H_{C_G(A,c)}^*(c \times F^A)_{[\mathfrak{p}]} \right). \end{aligned}$$

Proof. Set $R \doteq H_G(X)$, $W \doteq W_G(A, c)$. Lemma 3.5 tells us that W acts freely on the set π_0 of connected components of $c \times F^A$, and that this set of components is a finite set. Quillen shows that W is a finite group [17].

Using Lemma 3.5, each $d \in \pi_0$ is a $C_G(A, c)$ -space, so that $H_{C_G(A,c)}^*(c \times F^A) = \bigoplus_{d \in \pi_0} H_{C_G(A,c)}^*(d)$; this is an isomorphism of graded R -modules. Now, the action of W on $H_{C_G(A,c)}^q(c \times F^A)$, for any q , comes from the natural geometric action of $N_G(A, c)$ on $c \times F^A$ (see the discussion on page 4) and thus takes components to components, so

for every $x \in H_{C_G(A,c)}^q(d)$, and every $n \in N_G(A, c)$, $n^*(x) \in H_{C_G(A,c)}^q(n \cdot d)$. Thus, the same is true for the W -action on cohomology; the discussion on pages 4 and 5 tells us that this W -action on cohomology gives compatibility with the R -module structure on $H_{C_G(A,c)}^q(c \times F^A)$. Putting this together, we see that the hypotheses required for conclusions i), ii), and iii) of Theorem 5.2 are true, for $L = R$, $W = W_G(A, c)$, $P = H_{C_G(A,c)}^*(c \times F^A)$ and π_0 as in this theorem. Therefore, i.a) and i.b) are true.

Finally, using 2.7, \mathfrak{p} is a minimal prime in R , so, using 4.5, the graded localization of $H_{C_G(A,c)}^*(c \times F^A)$ at the minimal prime \mathfrak{p} is a $*$ Artinian $R_{[\mathfrak{p}]}$ -module. Now, note that W acts on $H_{C_G(A,c)}^*(c \times F^A)_{[\mathfrak{p}]}$ via the formula $w \cdot (a/b) = (w \cdot a)/b$. This is well-defined since W acts as a group of R -module isomorphisms (see the discussion on pages 4 and 5) and the denominators all come from R . Since we've already verified the hypotheses of Theorem 5.2 for $L = R$, $P = H_{C_G(A,c)}^*(c \times F^A)$, and these verifications are well-behaved with respect to localization, the hypotheses of Theorem 5.2 are also true for $L = R_{[\mathfrak{p}]}$ and $P = H_{C_G(A,c)}^*(c \times F^A)_{[\mathfrak{p}]}$; applying part iv) of 5.2 to the localized set-up, we are done. \square

Theorem 5.4. *Let $R = H_G(X)$. Suppose that $[A, c] \in \mathcal{Q}'(G, X)$ and $\mathfrak{p} \doteq \mathfrak{p}_{(A,c)}$. Then,*

$$*\ell_{R_{[\mathfrak{p}]}}(H_G^*(X)_{[\mathfrak{p}]}) = \frac{1}{|W_G(A, c)|} * \ell_{R_{[\mathfrak{p}]}}(H_{C_G(A,c)}^*(c)_{[\mathfrak{p}]}).$$

Proof. Using (2.7) we know that \mathfrak{p} is a minimal prime ideal in R .

In this computation abbreviate $C_G(A, c)$ to C , $W_G(A, c)$ to W . Note the following facts.

- 1) We use Lemma 2.9, computing $H_G^*(X \times F) \cong H_G^*(X) \otimes_k H^*(F)$, where the R -module structure is given by $r(a \otimes v) \doteq ra \otimes v$. Therefore, when we localize,

$$H_G^*(X \times F)_{[\mathfrak{p}]} \cong H_G^*(X)_{[\mathfrak{p}]} \otimes_k H^*(F)$$

as $R_{[\mathfrak{p}]}$ -modules.

Now, F is a finite dimensional compact manifold, so $H^*(F)$ is a finite dimensional graded vector space over k , let's say it has dimension m ; thus, using Lemma 4.4,

$$*\ell_{R_{[\mathfrak{p}]}}(H_G^*(X \times F)_{[\mathfrak{p}]}) = *\ell_{R_{[\mathfrak{p}]}}(H_G^*(X)_{[\mathfrak{p}]}) \cdot m.$$

- 2) Refer back to the diagram of Theorem 3.9; that theorem asserts that

$$H_G^*(X \times F)_{[\mathfrak{p}]} \cong H_C^*(c \times F^A)_{[\mathfrak{p}]}^W$$

as $R_{[\mathfrak{p}]}$ -modules.

- 3) Using Theorem 5.3,

$$*\ell_{R_{[\mathfrak{p}]}} \left(H_C^*(c \times F^A)_{[\mathfrak{p}]}^W \right) = \frac{1}{|W|} * \ell_{R_{[\mathfrak{p}]}} \left(H_C^*(c \times F^A)_{[\mathfrak{p}]} \right).$$

- 4) By referring again to the diagram of Theorem 3.9, the fourth vertical arrow gives an isomorphism:

$$H_C^*(c \times F^A)_{[\mathfrak{p}]} \cong H_C^*(c \times F)_{[\mathfrak{p}]}$$

as $R_{[\mathfrak{p}]}$ -modules.

5) Again using Lemma 2.9 and the properties of localization for the case of the pair (C, c) , $H_C^*(c \times F)_{[p]} \cong H_C^*(c)_{[p]} \otimes_k H^*(F)$ as graded $R_{[p]}$ -modules.

Putting these five results together gives us the following computation:

$$\begin{aligned}
 * \ell_{R_{[p]}}(H_G^*(X)_{[p]}) &\stackrel{(1)}{=} \frac{1}{m} * \ell_{R_{[p]}}(H_G^*(X \times F)_{[p]}) \\
 &\stackrel{(2)}{=} \frac{1}{m} * \ell_{R_{[p]}}(H_C^*(c \times F^A)_{[p]}^W) \\
 &\stackrel{(3)}{=} \frac{1}{m} \frac{1}{|W|} * \ell_{R_{[p]}}(H_C^*(c \times F^A)_{[p]}) \\
 &\stackrel{(4)}{=} \frac{1}{m} \frac{1}{|W|} * \ell_{R_{[p]}}(H_C^*(c \times F)_{[p]}) \\
 &\stackrel{(5)}{=} \frac{1}{m} \frac{1}{|W|} \cdot m \cdot * \ell_{R_{[p]}}(H_C^*(c)_{[p]}) \\
 &= \frac{1}{|W|} * \ell_{R_{[p]}}(H_C^*(c)_{[p]}). \quad \square
 \end{aligned}$$

We are now in position to prove our main result of the paper.

Theorem 5.5. *Suppose X is a G -space. Then,*

$$\deg(H_G^*(X)) = \sum_{[A,c] \in \mathcal{Q}'_{max}(G,X)} \frac{1}{|W_G(A,c)|} \deg(H_{C_G(A,c)}^*(c)).$$

Furthermore, recall the algebraic additivity formula for degree (Theorem 4.6) for a graded module $M \in \mathbf{grmod}(R)$:

$$\deg(M) = \sum_{\mathfrak{p} \in \mathcal{D}(M)} * \ell_{R_{[\mathfrak{p}]}}(M_{[\mathfrak{p}]}) \cdot \deg(R/\mathfrak{p}).$$

Then, for $M = H_G^*(X)$, and $R = H_G(X)$, the index sets of the two additivity formulae are in 1-1 correspondence and the summands are equal term-by-term under this correspondence. Thus, the result can be understood as a “geometric” interpretation of the algebraic additivity formula.

Proof. We use the following notation: $M = H_G^*(X)$, $R = H_G(X)$, $N = H_{C_G(A,c)}^*(c)$, and $S = H_{C_G(A,c)}(c)$.

Let’s apply the algebraic additivity formula to M as an R -module (Theorem 4.6)

$$\deg(M) = \sum_{\mathfrak{p} \in \mathcal{D}(M)} * \ell_{R_{[\mathfrak{p}]}}(M_{[\mathfrak{p}]}) \cdot \deg(R/\mathfrak{p}).$$

Recall that $\mathcal{D}(M) \doteq \{\mathfrak{q} \in \text{Spec}(R) : * \dim(R/\mathfrak{q}) = * \dim_R(M)\}$. By Quillen’s Theorem 2.7, there is a bijective correspondence between $\mathcal{D}(M)$ and $\mathcal{Q}'_{max}(G, X)$.

Now, the *length factor in each summand of the degree formula may be re-written using the previous theorem (5.4):

$$* \ell_{R_{[\mathfrak{p}]}}(M_{[\mathfrak{p}]}) = \frac{1}{|W|} * \ell_{R_{[\mathfrak{p}]}}(N_{[\mathfrak{p}]})$$

Let $[A, c]$ be any element of $\mathcal{Q}'_{max}(G, X)$, and consider the corresponding primes $\mathfrak{p} \doteq \ker(\text{res}_A^G : H_G^*(X) \rightarrow H_A/\sqrt{0})$, and $\mathfrak{p}^C \doteq \ker(\text{res}_A^G : H_{C_G(A,c)}^*(c) \rightarrow H_A/\sqrt{0})$.

In Lemma 5.1, we compared the sum formula for the degree of N given its R and S -module structures. In particular, we showed that, in either case, the additivity formula for the degree of N has only one summand, which may be computed using either module structure:

$$\begin{aligned}\deg(N) &= * \ell_{S_{[\mathfrak{p}^C]}}(N_{[\mathfrak{p}^C]}) \deg(S/\mathfrak{p}^C) \\ &= * \ell_{R_{[\mathfrak{p}]}}(N_{[\mathfrak{p}]}) \deg(R/\mathfrak{p}).\end{aligned}$$

We therefore have the computation:

$$\begin{aligned}\deg(M) &= \sum_{\mathfrak{p} \in \mathcal{D}(M)} * \ell_{R_{[\mathfrak{p}]}}(M_{[\mathfrak{p}]}) \cdot \deg(R/\mathfrak{p}) \\ &= \sum_{[A,c] \in \mathcal{Q}'_{max}(G,X)} \frac{1}{|W_G(A,c)|} * \ell_{R_{[\mathfrak{p}]}}(N_{[\mathfrak{p}]}) \deg(R/\mathfrak{p}) \\ &= \sum_{[A,c] \in \mathcal{Q}'_{max}(G,X)} \frac{1}{|W_G(A,c)|} * \ell_{S_{[\mathfrak{p}^C]}}(N_{[\mathfrak{p}^C]}) \deg(S/\mathfrak{p}^C) \\ &= \sum_{[A,c] \in \mathcal{Q}'_{max}(G,X)} \frac{1}{|W_G(A,c)|} \deg(N). \quad \square\end{aligned}$$

Finally, we pass on an open problem, posed by the referee. For which other classes of groups, whether actual groups or homotopical groups such as fusion systems, does the main theorem hold? Probably, such a generalization is best stated and studied using the mechanisms developed around Lannes' T -functor, in the category of unstable algebras over the Steenrod algebra.

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