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CYCLIC A_∞-ALGEBRAS AND CYCLIC HOMOLOGY

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Abstract

We provide a new description of the complex computing the Hochschild homology of an H-unitary A_∞-algebra A as a derived tensor product $A \otimes^{\infty}_{A^e} A$ such that: **(1)** there is a canonical morphism from it to the complex computing the cyclic homology of A that was introduced by Kontsevich and Soibelman, **(2)** this morphism induces the map I in the well-known SBI sequence, and **(3)** $H^0((A \otimes^{\infty}_{A^e} A)^{\#})$ is canonically isomorphic to the space of morphisms from A to $A^{\#}$ in the derived category of A_∞-bimodules. As direct consequences we obtain previous results of Cho and Cho–Lee, as well as the fact that Koszul duality establishes a bijection between (resp., almost exact) d-Calabi–Yau structures and (resp., strong) homotopy inner products, extending a result proved by Van den Bergh.

1. Introduction

In their very interesting article [16], the authors provided a new description of Hochschild and cyclic (co)homology, based on noncommutative (formal) geometry. In particular, they showed that, in characteristic zero, the complex computing the cyclic cohomology of a homologically unitary A_∞-algebra A is quasi-isomorphic to a(n even shift of) complex $\Omega^2_{\mathrm{cyc,cl}}(A[1])$ of closed cyclic 2-forms. By combining this result with a formal version of Darboux's theorem, they showed that a closed cyclic 2-form induces an isomorphism class of symplectic structures on the minimal model $H^\bullet(A)$ of A, if $H^\bullet(A)$ is finite dimensional. On the other hand, as noted by Cho in [4], a constant closed cyclic 2-form on a finite dimensional A_∞-algebra A is precisely the same as a strict isomorphism of A_∞-bimodules between A and its dual $A^{\#}$. Moreover, he found an equivalent description for the existence of a symplectic structure on a minimal model (see [4], Thm. 4.1, but also [5], Thm. 3.6). In their pursuit of understanding the results in [16], Cho and Lee found an explicit description of the quasi-isomorphism of A_∞-bimodules between $H^\bullet(A)$ and its dual $H^\bullet(A)^{\#}$ stated in [16], that they called *strong homotopy inner product* (see [5]). Their proof is however somehow *ad hoc* as well as computationally highly involved, and the proof of several steps are omitted.

The goal of this article is to show that the mentioned results by Cho in [4], and Cho and Lee in [5] can be directly deduced from a new description of the complex

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computing the Hochschild homology of A, which is closer to the complex $\Omega^2_{\mathrm{cyc,cl}}(A[1])$ in [**16**]. In order to express our results more clearly, we consider the dual noncommutative Cartan calculus of A: in this case, the complex computing the cyclic homology is a quotient of a(n even shift of) space $\mho^2_{\mathrm{cyc}}(A[1])$, dual to $\Omega^2_{\mathrm{cyc}}(A[1])$. Inspired by [**18**], given an A_∞-algebra A and two A_∞-bimodules M and N, we introduce the tensor product $M \otimes^\infty_{A^e} N$, and prove that if A is H-unitary, then $A \otimes^\infty_{A^e} A$ computes the Hochschild homology $HH_\bullet(A)$ of A (see Proposition 4.4). Moreover, from the explicit expression of the complex $A \otimes^\infty_{A^e} A$ we directly obtain a map from it to $\mho^2_{\mathrm{cyc}}(A[1])$, since the latter is a symmetrized version of the former, and we show that this morphism induces the map I in the SBI sequence (see Proposition 4.5, *cf.* [**5**], Prop. 6.1). Furthermore, the explicit expression of $A \otimes^\infty_{A^e} A[d]$ easily tells us that the space of morphisms of A_∞-bimodules from A to the shift of the graded dual $A^\#[-d]$ in the derived category is in clear correspondence with $H^0(A \otimes^\infty_{A^e} A[d])^\#$ (see Proposition 5.1, *cf.* [**5**], Lemma 6.5). The results from Cho and Lee are just obtained by combining the previous two statements (see Theorem 5.3; *cf.* [**4**], Thm. 4.1, or [**5**], Thm. 3.6, as well as the results in [**5**], Section 7). As another application, we also obtain that a strongly smooth pseudo-compact local augmented dg algebra is (resp., almost exact) d-Calabi–Yau if and only if its Koszul dual has a (resp., strong) homotopy inner product, extending Thm. 11.1 in [**25**], but with a completely different proof (see Theorem 6.6).

The structure of the article is as follows. Section 2 is devoted to provide the basic material we will use. This includes the noncommutative Cartan calculus in Subsection 2.2, as well as the basic definitions concerning A_∞-algebras, their A_∞-bimodules and the associated Hochschild and cyclic homologies in Subsection 2.3. All these results are by no means new, with the possible exception of some previously unnoticed results in Subsection 2.3 (*e.g.* Lemma 2.5). In the first part of Section 3 we recall the proof of the mentioned theorem of [**16**], and we provide then the notion of homotopy inner products and some basic results.

In Section 4, we prove the first two main results of this article. First, if A is an H-unitary A_∞-algebra and M a right H-unitary A_∞-bimodule, then $M \otimes^\infty_{A^e} A$ computes the Hochschild homology $H_\bullet(A, M)$ (see Proposition 4.4). Secondly, we construct a morphism from $A \otimes^\infty_{A^e} A$ to $\mho^2_{\mathrm{cyc}}(A[1])$ inducing the map I in the SBI sequence (see Proposition 4.5). In Section 5, we show that $H^0((M \otimes^\infty_{A^e} N)^\#)$ is in correspondence with the space of morphisms from M to the graded dual $N^\#$ of N in the derived category $\mathcal{D}(A^e)$ of A_∞-bimodules (see Proposition 5.1). We deduce from this the main result in [**4**], namely Thm. 4.1 (see Theorem 5.3), as well as the explicit relations in [**5**], Section 6 (see Propositions 4.5 and 5.1).

To be completely fair, one might argue that the complex $A \otimes^\infty_{A^e} A$ that we introduce in this article and that computes the Hochschild homology of A is not completely new, for it is built from well-known constructions. Moreover, it might be well-known to some experts, but we have not seen it used before in the literature. For example, in case A is a unitary (dg) algebra, $A \otimes^\infty_{A^e} A$ is precisely the complex obtained by applying the functor $A \otimes_{A^e} (-)$ to the resolution of A in the category of A-bimodules given by the tensor product of the bar resolution of A with itself over A.

Finally, after reviewing some basic results concerning pseudo-compact local dg algebras in Section 6, as well as their Hochschild homology (see Proposition 6.3), we use our previous results to show that Koszul duality establishes a bijection between (resp., almost exact) d-Calabi–Yau structures and (resp., strong) homotopy inner products (see Theorem 6.6).

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2. Preliminaries

2.1. Basic notation

In this article, we work over a fixed field k. We write $\mathbb{N}$ for the set of (strictly) positive integers and $\mathbb{N}_0$ for the set of nonnegative integers. Given $\bar{i} \in \mathbb{Z}^p$, also written $(i_1, \dots, i_p)$, we set $|\bar{i}| = i_1 + \dots + i_p$. We recall that, if $V = \oplus_{\bar{n}\in\mathbb{Z}^p} V^{\bar{n}}$ is a multigraded vector space (or just graded, for short), $V[\bar{m}]$ is a graded vector space over k whose $\bar{n}$-th homogeneous component $V[\bar{m}]^{\bar{n}}$ is given by $V^{\bar{m}+\bar{n}}$, for all $\bar{n}, \bar{m} \in \mathbb{Z}^p$, and it is called the *shift* of V. For $\bar{d} \in \mathbb{Z}^p$, $s_{V,\bar{d}} \colon V \to V[\bar{d}]$ is the *suspension morphism*, whose underlying map is the identity of V, and has degree $-\bar{d}$. Moreover, $V[(1,0,\dots,0)]$ and $s_{V,(1,0,\dots,0)}$ will be simply denoted by $V[1]$ and s_V (or s if V is clear from the context), resp., whereas $V[(0,1,0,\dots,0)]$ and $s_{V,(0,1,0,\dots,0)}$ will be denoted by ΣV and t_V (or just t if V is clear), resp. All morphisms between vector spaces will be k-linear (satisfying further requirements if they are so decorated). All unadorned tensor products $\otimes$ would be over k. If V is a graded vector space, $V^{\#}$ will denote its graded dual. Moreover, all the signs appearing in this work are obtained from the Koszul sign rule in the symmetric monoidal category of (differential) graded vector spaces with the (symmetric) braiding

$$\tau_{V,W} \colon V \otimes W \to W \otimes V \tag{2.1}$$

given by $v \otimes w \mapsto (-1)^{\bar{n}\cdot\bar{m}} w \otimes v$, where $v \in V^{\bar{n}}$, $w \in V^{\bar{m}}$ and $\bar{n} \cdot \bar{m} = \sum_{i=1}^{p} n_i m_i$.[1] The first component n_1 of $\bar{n} = (n_1, \dots, n_p) \in \mathbb{Z}^p$ of $v \in V^{\bar{n}}$ will be called *cohomological degree*, and it will also be written $|v|$, whereas the second n_2, if it exists, will be called *weight*. If $f \colon (M, d_M) \to (N, d_N)$ is a *closed* morphism of dg vector spaces, also called a *morphism of complexes*, *i.e.* f has zero degree and $f \circ d_M = d_N \circ f$, the *cone* of f is by definition the dg vector space $\mathrm{Co}(f) = M[1] \oplus N$ with differential ∂ given by $\partial|_N = d_N$ and $\partial|_{M[1]} = -s_M \circ d_M \circ s_M^{-1} + f \circ s_M^{-1}$.

We recall that if W is a graded vector space, then $T^c W = \oplus_{n\in\mathbb{N}_0} W^{\otimes n}$ (resp., $\bar{T}^c W = \oplus_{n\in\mathbb{N}} W^{\otimes n}$) is the cofree cocomplete coaugmented graded coalgebra (resp., cofree cocomplete noncounitary graded coalgebra) cogenerated by W, where the coproduct is given by deconcatenation. We denote by $\pi_W \colon T^c W \to W$ the canonical projection. If A is a nonunitary dg algebra, then, its *augmentation* $A^+ = k \oplus A$ is the unique augmented dg algebra such that the canonical inclusion $A \to A^+$ is a morphism of nonunitary dg algebras, the canonical inclusion $k \to A^+$ is its unit and the canonical projection of $A^+ \to k$ is the augmentation. The correspondence $A \mapsto A^+$ is

[1] The authors of [16] never explicitly stated which precise braiding the Koszul sign rule should be applied to. One can check that the “correct” signs cannot be obtained by using the usual braiding associated to the total degree.

clearly functorial and it induces an equivalence between the categories of nonunitary dg algebras and that of augmented dg algebras, the quasi-inverse being the map that associates the kernel of the augmentation to each augmented dg algebra. Analogously, if C is a noncounitary dg coalgebra, we define similarly a structure of coaugmented dg coalgebra on $C^+ = k \oplus C$, called the *coaugmentation* of C. All the corresponding dual results also hold in this case.

2.2. Basics on noncommutative Cartan calculus

We mainly present the definitions given in [**16**], but in coalgebra terms. All the results there can be obtained by taking the dual of the ones we provide here. The reason for doing so is to facilitate the comparison to Hochschild and cyclic homology from [**10**].

Let $V = \oplus_{n \in \mathbb{Z}} V^n$ be a cohomologically graded vector space. We will regard it as a bigraded vector space, concentrated in weight zero. Define $\mho^\bullet V = T^c(V \oplus \Sigma V)$, where $\Sigma V = V[(0,1)]$. Recall that $t = t_V : V \to \Sigma V$ is the morphism whose underlying map is the identity of V. It has cohomological degree 0 and weight -1. Note that $\mho^\bullet V = \oplus_{m \in \mathbb{N}_0} \mho^m V$, where $\mho^m V$ is the subspace of $\mho^\bullet V$ formed by all homogeneous elements of weight $-m$. Let $\mathbf{q}_{\mathrm{DR}} : \mho^\bullet V \to \mho^\bullet V$ be the unique coderivation satisfying that $\pi_{V \oplus \Sigma V} \circ \mathbf{q}_{\mathrm{DR}}$ is the composition of $\pi_{V \oplus \Sigma V}$ and the map $V \oplus \Sigma V \to V \oplus \Sigma V$ given by $v \oplus t_V(w) \mapsto w$, for all $v, w \in V$. Explicitly,

$$\begin{aligned} &\mathbf{q}_{\mathrm{DR}}\big(\bar{u}_1 t_V(v_1) \cdots \bar{u}_n t_V(v_n) \bar{u}_{n+1}\big) \\ &= \sum_{i=1}^{n} (-1)^{i-1} \bar{u}_1 t_V(v_1) \cdots \bar{u}_{i-1} t_V(v_{i-1}) \bar{u}_i v_i \bar{u}_{i+1} t_V(v_{i+1}) \cdots \bar{u}_n t_V(v_n) \bar{u}_{n+1}, \end{aligned} \tag{2.2}$$

for all $\bar{u}_1, \dots, \bar{u}_{n+1} \in T^c V$, $v_1, \dots, v_n \in V$ and $n \in \mathbb{N}$. It is clear that $\mathbf{q}_{\mathrm{DR}}$ has cohomological degree 0 and weight 1, and that $\mathbf{q}_{\mathrm{DR}} \circ \mathbf{q}_{\mathrm{DR}} = 0$. Set $\mathbf{q}_{\mathrm{DR}^m} = \mathbf{q}_{\mathrm{DR}}|_{\mho^m V}$.

Given a morphism of coaugmented graded coalgebras $F : T^c V \to T^c V$, we define $\hat{F} : \mho^\bullet V \to \mho^\bullet V$ as the unique morphism of coaugmented graded coalgebras such that $p_V \circ \pi_{V \oplus \Sigma V} \circ \hat{F}$ is the composition of the projection $T^c(V \oplus \Sigma V) \to T^c V$ and $\pi_V \circ F$, and $p_{\Sigma V} \circ \pi_{V \oplus \Sigma V} \circ \hat{F}$ is $t_V \circ p_V \circ \pi_{V \oplus \Sigma V} \circ \hat{F} \circ \mathbf{q}_{\mathrm{DR}}$, where the canonical projections are denoted $p_V : V \oplus \Sigma V \to V$ and $p_{\Sigma V} : V \oplus \Sigma V \to \Sigma V$. It is clear that $\hat{F}$ is an automorphism of coaugmented graded coalgebras if F is so, and it satisfies that $\hat{F} \circ \mathbf{q}_{\mathrm{DR}} = \mathbf{q}_{\mathrm{DR}} \circ \hat{F}$. Indeed, this equality is verified if and only if composing it with $\pi_{V \oplus \Sigma V}$ holds, which trivially follows from the definitions.

Given $X \in \mathrm{Coder}(T^c V)$, define the *contraction* $\iota_X : \mho^\bullet V \to \mho^\bullet V$ as the unique coderivation satisfying that $\pi_{V \oplus \Sigma V} \circ \iota_X$ is equal to the composition of the projection $T^c(V \oplus \Sigma V) \to T^c V$, X, π_V, t_V and the canonical inclusion of $\Sigma V \to V \oplus \Sigma V$. It is clear that ι_X has cohomological degree d and weight -1, if X has (cohomological) degree d. It is also easy to verify that

$$[\iota_X, \iota_Y] = \iota_X \circ \iota_Y + (-1)^{|X|.|Y|} \iota_Y \circ \iota_X = 0,$$

for all $X, Y \in \mathrm{Coder}(T^c V)$ homogeneous. Note that the sign in the graded commutator is determined using the Koszul sign rule associated to (2.1). Given a morphism of coaugmented graded coalgebras $F : T^c V \to T^c V$ and $X, X' \in \mathrm{Coder}(T^c V)$ such that $X' \circ F = F \circ X$, then the induced morphism $\hat{F} : \mho^\bullet V \to \mho^\bullet V$ of coaugmented graded coalgebras satisfies that $\hat{F} \circ \iota_X = \iota_{X'} \circ \hat{F}$. Indeed, this equality is verified if and only if composing it with $\pi_{V\oplus\Sigma V}$ holds, which is true if and only if the restriction of the latter equality to TV holds, because the restriction to $\mho^m V$ vanish if $m\in\mathbb{N}$. Since the mentioned restriction to TV is precisely $t_V\circ\pi_V\circ X'\circ F = t_V\circ\pi_V\circ F\circ X$, the claim follows.

Given a coderivation $X\in\mathrm{Coder}(T^cV)$ of degree d, the *Lie derivative* is the map $\mathcal{L}_X\colon\mho^\bullet V\to\mho^\bullet V$ defined via $\mathcal{L}_X=[\mathsf{d}_{\mathrm{DR}},\iota_X]=\mathsf{d}_{\mathrm{DR}}\circ\iota_X+\iota_X\circ\mathsf{d}_{\mathrm{DR}}$. It is easy to see that $\mathcal{L}_X$ is a coderivation of cohomological degree d and weight zero, and that

$$\pi_{V\oplus\Sigma V}\circ\mathcal{L}_X(\bar u)=x(\bar u),\quad \pi_{V\oplus\Sigma V}\circ\mathcal{L}_X\big(\bar u t_V(v)\bar u'\big)=t_V\big(x(\bar u v\bar u')\big),\tag{2.3}$$

and $\pi_{V\oplus\Sigma V}\circ\mathcal{L}_X|_{\mho^m V}=0$, for all $m\in\mathbb{N}\backslash\{1\}$, $\bar u,\bar u'\in T^cV$ and $v\in V$, where we write $x=\pi_V\circ X$. From the two previous paragraphs we see that, given a morphism of coaugmented graded coalgebras $F\colon T^cV\to T^cV$ and $X,X'\in\mathrm{Coder}(T^cV)$ such that $X'\circ F=F\circ X$, then $\hat F\colon\mho^\bullet V\to\mho^\bullet V$ satisfies that $\hat F\circ\mathcal{L}_X=\mathcal{L}_{X'}\circ\hat F$.

The following result is easy to verify.

Fact 2.1. *Let $X,Y\in\mathrm{Coder}(T^cV)$. Then,*

$$[\mathsf{d}_{\mathrm{DR}},\mathcal{L}_X]=0,\qquad[\mathcal{L}_X,\iota_Y]=\iota_{[X,Y]},\qquad\textit{and}\qquad[\mathcal{L}_X,\mathcal{L}_Y]=\mathcal{L}_{[X,Y]}.$$

We recall that, given a coaugmented graded coalgebra C with coproduct Δ_C, one defines the graded vector space $C^\natural=\mathrm{Ker}(\Delta_C-\tau_{C,C}\circ\Delta_C)\subseteq C$. Note that any coderivation $d\in\mathrm{Coder}(C)$ satisfies that $d(C^\natural)\subseteq C^\natural$. We shall usually denote the induced map $d|_{C^\natural}\colon C^\natural\to C^\natural$ simply by d. Set $\mho^\bullet_{\mathrm{cyc}}V=(\mho^\bullet V)^\natural$, which is a dg vector subspace of $\mho^\bullet V$ for the differential d_{DR} by the previous comments. Moreover, the coderivations ι_X and $\mathcal{L}_X$ also induce maps from $\mho^\bullet_{\mathrm{cyc}}V$ to itself. Analogously, note that if $f\colon C\to D$ is a morphism of coaugmented graded coalgebras, then $f(C^\natural)\subseteq D^\natural$. We also set $\mho^m_{\mathrm{cyc}}V=\mho^\bullet_{\mathrm{cyc}}V\cap\mho^m V$, for all $m\in\mathbb{N}_0$.

We recall the following well-known result, called the (formal) *Poincaré lemma.*

Lemma 2.2. *Let V be a cohomologically graded vector space over a field k of zero characteristic. Then, the complex $(\mho^\bullet_{\mathrm{cyc}}V,\mathsf{d}_{\mathrm{DR}})$ is quasi-isomorphic to k (in zero cohomological degree and weight).*

Proof. Consider $X_{\mathrm{eu}}\in\mathrm{Coder}(T^cV)$ defined as follows. For $n\in\mathbb{N}_0$, set $X_{\mathrm{eu}}|_{V^{\otimes n}}$ as the composition of $n\mathrm{id}_{V^{\otimes n}}$ and the canonical inclusion $V^{\otimes n}\to T^cV$. The explicit expression (2.3) of $\mathcal{L}_{X_{\mathrm{eu}}}$ implies that $\pi_{V\oplus\Sigma V}\circ\mathcal{L}_{X_{\mathrm{eu}}}|_{V\oplus\Sigma V}=\mathrm{id}_{V\oplus\Sigma V}$, which in turn tells us that $\mathcal{L}_{X_{\mathrm{eu}}}|_{(V\oplus\Sigma V)^{\otimes n}}=n\mathrm{id}_{(V\oplus\Sigma V)^{\otimes n}}$, for all $n\in\mathbb{N}_0$. This gives in particular that $\mathrm{Ker}(\mathcal{L}_{X_{\mathrm{eu}}})=k$. Moreover, the identity $\mathcal{L}_{X_{eu}}=[\mathsf{d}_{\mathrm{DR}},\iota_{X_{\mathrm{eu}}}]$ and the assumption $\mathrm{char}(k)=0$ tell us that $(\mho^\bullet_{\mathrm{cyc}}V,\mathsf{d}_{\mathrm{DR}})$ is quasi-isomorphic to $(\mathrm{Ker}(\mathcal{L}_{X_{\mathrm{eu}}}),\mathsf{d}_{\mathrm{DR}})$, and the result follows. $\square$

Assume that the field k has characteristic different from 2. Given $\omega\in(\mho^2_{\mathrm{cyc}}V)^\#$, where $(-)^\#$ denotes the graded dual, we write the formal infinite sum $\omega=\sum_{\ell\in\mathbb{N}_0}\omega_\ell$, where

$$\omega_\ell\in\bigg(\Big(\bigoplus_{p+q+r=\ell}V^{\otimes p}\otimes\Sigma V\otimes V^{\otimes q}\otimes\Sigma V\otimes V^{\otimes r}\Big)_{\mathrm{cyc}}\bigg)^\#\subseteq(\mho^2_{\mathrm{cyc}}V)^\#,\tag{2.4}$$

for all $\ell\in\mathbb{N}_0$, and the last inclusion is given by extending ω_ℓ by zero on the direct summands indexed by $\ell'\neq\ell$. The sum is well defined, since, given $\alpha\in\mho^2_{\mathrm{cyc}}V$, for all but a finite number of indices $\ell \in \mathbb{N}_0$, $\omega_\ell(\alpha)$ vanishes. Set $\omega_{>0} = \sum_{\ell\in\mathbb{N}} \omega_\ell$. Note that $\omega_0 \in ((\Sigma V \otimes \Sigma V)^{\mathbb{S}_2})^\# \simeq ((\Sigma V \otimes \Sigma V)^\#)^{\mathbb{S}_2} \subseteq ((\Sigma V)^{\otimes 2})^\#$, where the total map sends ω_0 to $t(v) \otimes t(w) \mapsto \omega_0(t(v) \otimes t(w) - (-1)^{|v|.|w|} t(w) \otimes t(v))/2$, for $v, w \in V$.

A *pre-symplectic structure* of degree d on a graded vector space V is a homogeneous element $\omega \in (\mho^2_{\mathrm{cyc}} V)^\#$ of cohomological degree $2 - d$ such that $\omega \circ \mathfrak{q}_{\mathrm{DR}} = 0$. Denote by $\mathrm{PSp}_{\mathrm{nc}}(V)$ the graded vector subspace of $(\mho^2_{\mathrm{cyc}} V)^\#$ spanned by the homogeneous pre-symplectic structures on V. A pre-symplectic structure ω is *nondegenerate* if the graded symmetric bilinear form $\Sigma V \otimes \Sigma V \to k$ induced by ω_0 is nondegenerate, *i.e.* it induces an isomorphism of graded vector spaces $\Sigma V \to (\Sigma V)^\#$, where we write as before $\omega = \omega_0 + \omega_{>0}$. Note that a graded vector space V provided with a nondegenerate pre-symplectic structure ω is necessarily locally finite dimensional. Notice also that any element $\omega_0 \in ((\Sigma V \otimes \Sigma V)^{\mathbb{S}_2})^\# \subseteq (\mho^2_{\mathrm{cyc}} V)^\#$ satisfies that $\omega_0 \circ \mathfrak{q}_{\mathrm{DR}} = 0$. We remark that if ω is a (resp., nondegenerate) pre-symplectic structure of degree d on V and $F\colon T^c V \to T^c V$ is an automorphism of coaugmented graded algebras, then $\omega' = \omega \circ \hat{F}$ is also a (resp., nondegenerate) pre-symplectic structure of degree d, and if $X\colon T^c V \to T^c V$ is a coderivation of degree N, then $\omega \circ \mathcal{L}_X$ is a pre-symplectic structure of degree $d - N$.

The next result is the well-known (formal) *Darboux theorem.*

Theorem 2.3. *Let V be a graded vector space over a field k of zero characteristic and $\omega \in \mho^2_{\mathrm{cyc}} V$ a nondegenerate pre-symplectic structure of degree N on V. There exists an automorphism $F\colon T^c V \to T^c V$ of coaugmented graded coalgebras such that $\omega' = \omega \circ \hat{F}$ satisfies that $\omega'_{>0}$ vanishes.*

Proof. It suffices to show that if $\omega = \omega_0 + \sum_{\ell \geqslant \ell_0} \omega_\ell$ is a pre-symplectic structure of degree N, for $\ell_0 \geqslant 1$, there exists an automorphism $F\colon T^c V \to T^c V$ of coaugmented graded coalgebras such that $\omega' = \omega \circ \hat{F}$ satisfies that $\omega' = \omega'_0 + \sum_{\ell \geqslant \ell_0 + 1} \omega'_\ell$. To prove this latter statement, note that $\omega \circ \mathfrak{q}_{\mathrm{DR}} = 0$ implies in particular $\omega_{\ell_0} \circ \mathfrak{q}_{\mathrm{DR}}$ vanishes. By Lemma 2.2, which can be applied since $\ell_0 > 0$, there exists

$$\theta_{\ell_0+1} \in \Bigg(\Big(\bigoplus_{p+q=\ell_0+1} V^{\otimes p} \otimes \Sigma V \otimes V^{\otimes q}\Big)_{\mathrm{cyc}}\Bigg)^\#$$

such that $\omega_{\ell_0} = \theta_{\ell_0+1} \circ \mathfrak{q}_{\mathrm{DR}}$. Define $f_{\ell_0+1}\colon V^{\otimes \ell_0+1} \to V$ by means of

$$\theta_{\ell_0+1}\big(v_1 \cdots v_{\ell_0+1} t_V(w)\big) = \omega_0\Big(t_V\big(f_{\ell_0+1}(v_1 \cdots v_{\ell_0+1})\big) t_V(w)\Big),$$

for all homogeneous $v_1, \ldots, v_{\ell_0+1}, w \in V$. Since ω_0 is nondegenerate, then the map f_{ℓ_0+1} is uniquely determined by the previous equality. Let $F\colon T^c V \to T^c V$ be the unique morphism of coaugmented graded coalgebras satisfying that $\pi_V \circ F|_V = \mathrm{id}_V$, $\pi_V \circ F|_{V^{\otimes \ell_0+1}} = -f_{\ell_0+1}$ and $\pi_V \circ F|_{V^{\otimes n}}$ vanishes if $n \in \mathbb{N}\backslash\{1, \ell_0 + 1\}$. The reader can easily verify that $\omega' = \omega \circ \hat{F}$ satisfies the required property. □

2.3. A_∞-algebras and their bimodules

For the basics on dg algebras we refer to [**1**] (see also [**12**]). For A_∞-algebras and their bimodules we refer to [**18**] (see also [**21**]), even though we follow the sign convention in [**15**] (see also [**13**]). We give the basic definitions needed in the sequel as well as some probably well-known results on weak units that we couldn't find in the literature. For the rest of this section we consider cohomologically graded vector spaces, so the braiding (2.1) coincides with the usual one.

2.3.1. Basic definitions

A *nonunitary A_∞-algebra* A is a coderivation B_A of cohomological degree 1 on the noncounitary graded tensor algebra $\bar{T}^c(A[1]) = \oplus_{n\in\mathbb{N}} A[1]^{\otimes n}$ provided with the deconcatenation coproduct, such that $B_A \circ B_A = 0$. The previous noncounitary dg coalgebra is called the *(nonunitary) bar construction* of A and is denoted by $B(A)$. By the equivalence between noncounitary and coaugmented dg coalgebras, a nonunitary A_∞-algebra can also be defined as a coderivation B_A^+ of cohomological degree 1 on the coaugmented graded tensor algebra $T^c(A[1]) = \oplus_{n\in\mathbb{N}_0} A[1]^{\otimes n}$ provided with the deconcatenation coproduct, such that $B_A^+ \circ B_A^+ = 0$ and both $B_A^+|_k$ and the composition of B_A^+ with the counit $T^c(A[1]) \to k$ vanish. If $n \in \mathbb{N}$ we will denote an element $s(a_1) \otimes \cdots \otimes s(a_n) \in A[1]^{\otimes n}$ by $[a_1|\cdots|a_n]$, where $a_1, \dots, a_n \in A$ and $s = s_A\colon A \to A[1]$ is the suspension on A, whereas $[]$ will denote the unit element of $k \subseteq T^c(A[1])$. From now on, we will usually drop the adjective nonunitary if there is no risk of ambiguity.

By [**18**], Lemme 1.1.2.2, there is a linear bijection between the vector space of coderivations of $\bar{T}^c(A[1])$ and the space of linear maps from $\bar{T}^c(A[1])$ to $A[1]$, given by sending a coderivation B to $\pi_{A[1]} \circ B$, where $\pi_{A[1]}\colon \bar{T}^c(A[1]) \to A[1]$ is the canonical projection. Hence, B_A is uniquely determined by $\pi_{A[1]} \circ B_A = \sum_{i\in\mathbb{N}} b_i^A$ for maps of the form $b_i^A\colon A[1]^{\otimes i} \to A[1]$. Then, this collection of maps satisfies the identities

$$\sum_{(r,s,t)\in\mathcal{I}_n} b^A_{r+1+t} \circ (\mathrm{id}_{A[1]}^{\otimes r} \otimes b_s^A \otimes \mathrm{id}_{A[1]}^{\otimes t}) = 0, \tag{SI(n)}$$

for $n \in \mathbb{N}$, where $\mathcal{I}_n = \{(r,s,t) \in \mathbb{N}_0 \times \mathbb{N} \times \mathbb{N}_0 : r + s + t = n\}$. Reciprocally, starting from a collection of maps $b_i^A\colon A[1]^{\otimes i} \to A[1]$ fulfilling the previous properties we obtain an A_∞-algebra structure. Note that $H^\bullet(A, -s_A^{-1} \circ b_1 \circ s_A)$ is a (nonunitary) graded algebra for the product induced by $-s_A^{-1} \circ b_2 \circ s_A^{\otimes 2}$. A *(strictly) unitary A_∞-algebra* A is an A_∞-algebra provided with a (necessarily unique) map $\eta_A\colon k \to A$ of cohomological degree 0 such that $b_i^A \circ (\mathrm{id}_{A[1]}^{\otimes r} \otimes (s_A \circ \eta_A) \otimes \mathrm{id}_{A[1]}^{\otimes(i-r-1)}) = 0$, for all $i \in \mathbb{N}\backslash\{2\}$ and $r \in \{0, \dots, i-1\}$, and η_A is a unit for the (nonassociative) product $-s_A^{-1} \circ b_2 \circ s_A^{\otimes 2}$ on A. A *homologically unitary* A_∞-algebra A is an A_∞-algebra such that the graded algebra $H^\bullet(A, -s_A^{-1} \circ b_1 \circ s_A)$ is unitary. Moreover, an A_∞-algebra A is called *H-unitary* if the underlying complex of $B(A)$ is quasi-isomorphic to zero. Any unitary A_∞-algebra is clearly homologically unitary, and any homologically unitary A_∞-algebra is H-unitary (see [**18**], Cor. 4.1.2.7).

Given two A_∞-algebras $(A, b_\bullet^A)$ and $(A', b_\bullet^{A'})$, a *morphism of A_∞-algebras* from A to A' is a morphism of noncounitary dg coalgebras from $B(A)$ to $B(A')$. By [**18**], Lemme 1.1.2.2, such a morphism is uniquely determined by its composition with the canonical projection $\pi_W\colon \bar{T}^c(W) \to W$. As a consequence, any morphism $\tilde{F}$ of noncounitary dg coalgebras from $B(A)$ to $B(A')$ is uniquely determined by the map $\pi_{A'[1]} \circ \tilde{F} = \sum_{i\in\mathbb{N}} F_i$, where $F_i\colon A[1]^{\otimes i} \to A'[1]$. The fact that $\tilde{F}$ is a morphism of noncounitary dg coalgebras means exactly that $\{F_i\}_{i\in\mathbb{N}}$ satisfies that

$$\sum_{(r,s,t)\in\mathcal{I}_n} F_{r+1+t} \circ (\mathrm{id}_{A[1]}^{\otimes r} \otimes b_s^A \otimes \mathrm{id}_{A[1]}^{\otimes t}) = \sum_{q\in\mathbb{N}} \sum_{\bar{i}\in\mathbb{N}^{q,n}} b_q^{A'} \circ (F_{i_1} \otimes \cdots \otimes F_{i_q}), \tag{MI(n)}$$

for $n \in \mathbb{N}$, where $\mathbb{N}^{q,n}$ is the subset of $\mathbb{N}^q$ of elements $\bar{i} = (i_1, \dots, i_q)$ satisfying that $|\bar{i}| = i_1 + \cdots + i_q = n$. If A and A' are unitary, one further requires the condition that $F_1 \circ s_A \circ \eta_A = s_{A'} \circ \eta_{A'}$ and $F_i \circ (\mathrm{id}_{A[1]}^{\otimes r} \otimes (s_A \circ \eta_A) \otimes \mathrm{id}_{A[1]}^{\otimes(i-r-1)}) = 0$, for all $i \in \mathbb{N}\backslash\{1\}$ and $r \in \{0, \dots, i-1\}$. A morphism $F_\bullet$ is called *strict* if $F_i = 0$ for all $i \in \mathbb{N}\backslash\{1\}$. The notions of *identity* and *composition* of morphisms of A_∞-algebras are clear. If A is an A_∞-algebra, $A^+ = k \oplus A$ is canonically an *augmented* A_∞-algebra, *i.e.* it has a unique structure of unitary A_∞-algebra with unit $k \to A^+$ such that the canonical projection $\epsilon_{A^+} \colon A^+[1] \to k[1]$, called the *augmentation* of A^+, is a strict morphism of unitary A_∞-algebras. As for the case of dg algebras, the category of nonunitary A_∞-algebras with morphisms of A_∞-algebras is equivalent to the category of augmented A_∞-algebras provided with morphisms of unitary A_∞-algebras commuting with the augmentations.

Given an A_∞-algebra A, a *nonunitary A_∞-bimodule over A* is a graded vector space M equipped with a bicoderivation B_M on the graded counitary $B(A)^+$-bicomodule $B(A)^+ \otimes M[1] \otimes B(A)^+$ such that $B_M \circ B_M = 0$. We shall denote the previous bicomodule by $B(A, M, A)$, and call it the *bar construction* of M. As usual, we will drop the adjective nonunitary if it causes no confusion. Since $B(A)^+ \otimes M[1] \otimes B(A)^+$ is a cofree graded counitary bicomodule, a bicoderivation is uniquely determined by its composition with $\epsilon_{B(A)^+} \otimes \mathrm{id}_{M[1]} \otimes \epsilon_{B(A)^+}$ (see [**18**], Lemme 2.1.2.1), which is a sum of mappings of the form $b_{p,q}^M \colon A[1]^{\otimes p} \otimes M[1] \otimes A[1]^{\otimes p} \to M[1]$, for $p, q \in \mathbb{N}_0$. Then, the collection of maps $\{b_{p,q}^M\}_{p,q\in\mathbb{N}_0}$ satisfies the following identities

$$\sum_{(r,s,t)\in\mathcal{I}_{n'+n''+1}} \tilde{b}_{r,t}^M \circ (\mathrm{id}^{\otimes r} \otimes \tilde{b}_s \otimes \mathrm{id}^{\otimes t}) = 0 \qquad (\mathrm{BI}(n', n''))$$

in $\mathcal{H}om(A[1]^{\otimes n'} \otimes M[1] \otimes A[1]^{\otimes n''}, M[1])$ for all $n', n'' \in \mathbb{N}_0$, where $\tilde{b}_s$ is interpreted as the corresponding multiplication map b_s of A if either $r + s \leqslant n'$ or $s + t \leqslant n''$, and it is understood as $b_{n'-r,n''-t}^M$ else. In the first case, $\tilde{b}_{r,t}^M$ is $b_{n'-s+1,n''}^M$ if $r + s \leqslant n'$ or $b_{n',n''-s+1}^M$ if $s + t \leqslant n''$, and it is $b_{r,t}^M$ else. Moreover, $\mathrm{id}^{\otimes r}$ is $\mathrm{id}_{A[1]}^{\otimes r}$ and $\mathrm{id}^{\otimes t}$ is $\mathrm{id}_{A[1]}^{\otimes(n'-r-s)} \otimes \mathrm{id}_{M[1]} \otimes \mathrm{id}_{A[1]}^{\otimes n''}$ if $r + s \leqslant n'$; $\mathrm{id}^{\otimes r}$ is $\mathrm{id}_{A[1]}^{\otimes n'} \otimes \mathrm{id}_{M[1]} \otimes \mathrm{id}_{A[1]}^{\otimes(n''-s-t)}$ and $\mathrm{id}^{\otimes t}$ is $\mathrm{id}_{A[1]}^{\otimes t}$ if $s + t \leqslant n''$; and $\mathrm{id}^{\otimes r}$ is $\mathrm{id}_{A[1]}^{\otimes r}$ and $\mathrm{id}^{\otimes t}$ is $\mathrm{id}_{A[1]}^{\otimes t}$ else. Reciprocally, given any collection of maps $b_{p,q}^M \colon A[1]^{\otimes p} \otimes M[1] \otimes A[1]^{\otimes q} \to M[1]$ fulfilling the previous properties, it defines an A_∞-bimodule structure on M over A. If A is unitary, one further imposes that

$$\begin{aligned} b_{p,q}^M &\circ \big(\mathrm{id}_{A[1]}^{\otimes r} \otimes (s_A \circ \eta_A) \otimes \mathrm{id}_{A[1]}^{\otimes(p-r-1)} \otimes \mathrm{id}_{M[1]} \otimes \mathrm{id}_{A[1]}^{\otimes q}\big) \\ &= b_{p,q}^M \circ \big(\mathrm{id}_{A[1]}^{\otimes p} \otimes \mathrm{id}_{M[1]} \otimes \mathrm{id}_{A[1]}^{\otimes s} \otimes (s_A \circ \eta_A) \otimes \mathrm{id}_{A[1]}^{\otimes(q-s-1)}\big) = 0, \end{aligned} \tag{2.5}$$

for all $(p, q) \in \mathbb{N}_0^2\backslash\{(0, 1), (1, 0)\}$, $r \in \{0, \dots, p-1\}$ and $s \in \{0, \dots, q-1\}$, and also that $b_{1,0}^A \circ ((s_A \circ \eta_A) \otimes s_M) = -s_M = b_{0,1}^M \circ (s_M \otimes (s_A \circ \eta_A))$. Note that a nonunitary (resp., unitary) A_∞-algebra is also a nonunitary (resp., unitary) A_∞-bimodule over itself for the structure maps $b_{p,q} = b_{p+q+1}$, where $p, q \in \mathbb{N}_0$. In this case, the underlying complexes of $B(A, A, A)$ and $B(A)$ are identical. Moreover, given a nonunitary A_∞-bimodule M over a nonunitary A_∞-algebra A, it can be canonically regarded as a unitary A_∞-bimodule over A^+ uniquely extending the A_∞-bimodule structure over A. There are obvious versions of left (resp., right) A_∞-module, for which all the previous definitions as well as the ones below (together with the all results in this subsection) also hold.

Given two nonunitary (resp., unitary) A_∞-bimodules M and N over a nonunitary (resp., unitary) A_∞-algebra A, a *morphism* from M to N is a morphism of counitary dg bicomodules $F\colon B(A,M,A) \to B(A,N,A)$, which is uniquely determined by its composition with $\epsilon_{B(A)^+} \otimes \mathrm{id}_{N[1]} \otimes \epsilon_{B(A)^+}$, which we will simply write as $\sum_{p,q\in\mathbb{N}_0} F_{p,q}\colon A[1]^{\otimes p} \otimes M[1] \otimes A[1]^{\otimes q} \to N[1]$. The collection of these morphisms $F_{p,q}\colon A[1]^{\otimes p} \otimes M[1] \otimes A[1]^{\otimes q} \to N[1]$ satisfies

$$\sum_{(r,s,t)\in\mathcal{I}_{n'+n''+1}} F_{r',t'} \circ (\mathrm{id}^{\otimes r} \otimes \tilde{b}_s \otimes \mathrm{id}^{\otimes t}) = \sum_{(a,k,l,b)\in\mathbb{N}_{0,n',n''}} b^N_{a,b} \circ (\mathrm{id}^{\otimes a}_{A[1]} \otimes F_{k,l} \otimes \mathrm{id}^{\otimes b}_{A[1]}) \qquad (\mathrm{MBI}(n',n''))$$

for all $n',n'' \in \mathbb{N}_0$, where $\mathbb{N}_{0,n',n''}$ is the subset of $\mathbb{N}_0^4$ of elements (a,k,l,b) such that $a+k=n'$ and $l+b=n''$, and where we should understand $\tilde{b}_s$ as b^A_s if either $r+s \leqslant n'$ or $s+t \leqslant n''$, or as $b^M_{n'-r,n''-t}$ else. The indices (r',t') are completely determined from the previous cases. Furthermore, in the unitary case we have that $F_{p,q} \circ (\mathrm{id}^{\otimes r}_{A[1]} \otimes (s_A \circ \eta_A) \otimes \mathrm{id}^{\otimes t}_{A[1]})$ vanishes for $r \neq p$ and $(p,q) \notin \{(0,0)\}$. Reciprocally, given any collection of maps $F_{p,q}\colon A[1]^{\otimes p} \otimes M[1] \otimes A[1]^{\otimes q} \to N[1]$ fulfilling the previous properties, it defines a morphism of A_∞-bimodules from M to N over A. We say that it is *strict* if $F_{p,q}$ vanishes for all $(p,q) \neq (0,0)$, and a *quasi-isomorphism* if $F_{0,0}$ is a quasi-isomorphism from $(M[1], b^M_{0,0})$ to $(N[1], b^N_{0,0})$.

We say that an A_∞-bimodule M over a homologically unitary A_∞-algebra is *homologically unitary* if $H^\bullet(M, -s_M^{-1} \circ b^M_{0,0} \circ s_M)$ is a unitary graded bimodule over $H^\bullet(A, -s_M^{-1} \circ b^A_1 \circ s_A)$. Similarly, an A_∞-bimodule M over a nonunitary A_∞-algebra A is *H-unitary* if the underlying complex of $B(A,M,A)$ is quasi-isomorphic to zero. A right (resp., left) A_∞-module M is called *right (resp., left) H-unitary* if the underlying complex of the corresponding bar construction $B(M,A)$ (resp., $B(A,M)$) is quasi-isomorphic to zero. Since any A_∞-bimodule is in particular a right and left A_∞-module, this definition also applies to it. An A_∞-bimodule M over a homologically unitary A_∞-algebra is H-unitary if and only if it is homologically unitary (see [**18**], Lemme 4.1.3.7).

Fact 2.4. *Let A be an H-unitary A_∞-algebra. Then, the A_∞-bimodule (resp., right A_∞-module, left A_∞-module) A is H-unitary (resp., right H-unitary, left H-unitary).*

Proof. It is clear that the underlying complexes of $B(A,A,A)$ and of $B(A)$ coincide, proving the result in the A_∞-bimodule case. The same comments apply to the bar constructions of the right A_∞-module (resp., left A_∞-module) A. □

We also have the following sufficient criterion.

Lemma 2.5. *A right (resp., left) H-unitary A_∞-bimodule M is H-unitary.*

Proof. We prove the case of a right H-unitary A_∞-bimodule M, since the other is analogous. Let $\{F_\bullet B(A)^+\}_{\bullet\in\mathbb{N}_0}$ be the usual primitive filtration of $B(A)^+$, *i.e.* $F_n B^+(A) = \oplus_{m=0}^n A[1]^{\otimes m}$. This is clearly an exhaustive increasing filtration of the dg coalgebra $B(A)^+$. Consider the filtration $\{F_\bullet B(A,M,A)\}_{\bullet\in\mathbb{N}_0}$ of $B(A,M,A)$ given by

$F_\bullet B(A,M,A) = (F_\bullet B(A)^+) \otimes M \otimes B(A)^+$. Then, $\{F_\bullet B(A,M,A)\}_{\bullet\in\mathbb{N}_0}$ is an increasing exhaustive filtration of dg comodules of $B(A,M,A)$. It is clear that the underlying complex of the associated graded construction is $B(A)^+ \otimes B(M,A)$. Since $B(M,A)$ is quasi-isomorphic to zero, the Künneth formula implies that the associated graded of $B(A,M,A)$ is quasi-isomorphic to zero, so the same holds for $B(A,M,A)$. □

Remark 2.6. The converse of the previous lemma is in general false, and, more generally, there are H-unitary A_∞-bimodules that are neither right nor left H-unitary. Indeed, note first that a right (resp., left) module M over a nonunitary algebra A is right (resp., left) H-unitary if and only if $\mathrm{Tor}_\bullet^{A^+}(M,k) = 0$ (resp., $\mathrm{Tor}_\bullet^{A^+}(k,M) = 0$). In case of a bimodule M, it is H-unitary if and only if $\mathrm{Tor}_\bullet^{(A^+)^e}(M,k) = 0$. Let B be the quotient of the quiver algebra kQ, where Q is the quiver $\overset{e_1}{\bullet} \overset{\alpha}{\leftarrow} \overset{e_2}{\bullet} \overset{\beta}{\leftarrow} \overset{e_2}{\bullet}$, modulo the relation $\alpha\beta$. Define $A = k.e_1 \oplus k.\alpha \oplus k.e_3 \oplus k.\beta$. It is an H-unitary algebra (this is the example $k.(e_{1,1},0) \oplus k.(e_{1,2},0) \oplus k.(0,e_{1,1}) \oplus k.(0,e_{2,1}) \subseteq M_2(k) \times M_2(k)$ given in [**26**], p. 607, by identifying e_1, e_3, α and β with $(e_{1,1},0)$, $(0,e_{1,1})$, $(e_{1,2},0)$ and $(0,e_{2,1})$, resp., where $e_{i,j} \in M_2(k)$ is the matrix with (a,b)-th entry $\delta_{i,a}\delta_{j,b}$), and $A^+ \simeq B$. It is easy to see that the dual B-bimodule A^* has a projective resolution $(P_\bullet)_{\bullet\in\mathbb{N}_0}$ of length 2, with

$$P_0 = B^e.(e_3 \otimes e_2^{op}) \oplus B^e.(e_2 \otimes e_1^{op}), \quad P_1 = B^e.(e_2 \otimes e_2^{op})^{\oplus 2},$$
$$\text{and } P_2 = B^e.(e_2 \otimes e_3^{op}) \oplus B^e.(e_1 \otimes e_2^{op}),$$

where $B^e.(e_i \otimes e_j^{op})$ denotes the submodule of B^e generated by $e_i \otimes e_j^{op} \in B^e$. From the canonical isomorphism $k \otimes_{B^e} M \simeq M/(A.M + M.A)$ for any B-bimodule M and a simple computation, we get that $k \otimes_{B^e} P_j$ vanishes for all j, so A^* is an H-unitary bimodule over A, but it is neither right H-unitary nor left H-unitary, since $A^* \otimes_B k \simeq A^*/(A^*.A) \neq 0$ and $k \otimes_B A^* \simeq A^*/(A.A^*) \neq 0$.

2.3.2. Duals and suspensions

Given an A_∞-bimodule $(M, b^M_{\bullet,\bullet})$ over an A_∞-algebra A and $d \in \mathbb{Z}$, we define the *shifted A_∞-bimodule* $M[d]$ as follows. The underlying graded vector space is the usual shift $M[d]$ and the bicoderivation $B_{M[d]}$ is the unique one satisfying that

$$\begin{tikzcd}
B(A,M,A) \arrow[r, "B_M"] \arrow[d, "\mathrm{id}_{B(A)^+}\otimes s_{M[1],d}\otimes \mathrm{id}_{B(A)^+}"] & B(A,M,A) \arrow[r, "\epsilon_{B(A)^+}\otimes \mathrm{id}_{M[1]}\otimes\epsilon_{B(A)^+}"] \arrow[d, "\mathrm{id}_{B(A)^+}\otimes s_{M[1],d}\otimes \mathrm{id}_{B(A)^+}"] & M[1] \arrow[d, "s_{M[1],d}"] \\
B(A,M[d],A) \arrow[r, "(-1)^d B_{M[d]}"] & B(A,M[d],A) \arrow[r, "\epsilon_{B(A)^+}\otimes \mathrm{id}_{M[d+1]}\otimes\epsilon_{B(A)^+}"] & M[d+1]
\end{tikzcd} \tag{2.6}$$

commutes. Indeed,

$$b^{M[d]}_{p,q} \colon A[1]^{\otimes p} \otimes M[d+1] \otimes A[1]^{\otimes q} \to M[d+1]$$

is uniquely defined by

$$b^{M[d]}_{p,q} \circ (\mathrm{id}^{\otimes p}_{A[1]} \otimes s_{M[1],d} \otimes \mathrm{id}^{\otimes q}_{A[1]}) = (-1)^d s_{M[1],d} \circ b^M_{p,q},$$

for all $p,q \in \mathbb{N}_0$. It is easy to verify that the maps $\{b^{M[d]}_{p,q}\}_{p,q\in\mathbb{N}_0}$ satisfy the unitary condition if $\{b^M_{p,q}\}_{p,q\in\mathbb{N}_0}$ do and they indeed provide a structure of A_∞-bimodule over A. The previous shift construction is functorial. Indeed, if $F_{\bullet,\bullet} \colon M \to N$ is a morphism of A_∞-bimodules over the A_∞-algebra A, then it defines a morphism $F_{\bullet,\bullet}[d]\colon M[d] \to N[d]$ of A_∞-bimodules by

$$F_{p,q}[d] \circ (\mathrm{id}_{A[1]}^{\otimes p} \otimes s_{M[1],d} \otimes \mathrm{id}_{A[1]}^{\otimes q}) = s_{N[1],d} \circ F_{p,q},$$

for all $p, q \in \mathbb{N}_0$. We leave the reader to verify that $F_{\bullet,\bullet}[d]$ is indeed a morphism of A_∞-bimodules over A, and that the shift induces a covariant functor from the category of A_∞-bimodules over A to itself. Note that the $F_{\bullet,\bullet}[d]\colon M[d] \to N[d]$ is a quasi-isomorphism if $F_{\bullet,\bullet}\colon M \to N$ is so.

Since the suspension functor is a self-equivalence, from now on we will equivalently describe an A_∞-bimodule M by its *unshifted bar construction* $B^u(A, M, A)$, defined as $B(A, M[-1], A)$. The composition of its bicoderivation $B_{M[-1]}$, denoted by B_M^u, with $\epsilon_{B(A)^+} \otimes \mathrm{id}_M \otimes \epsilon_{B(A)^+}$ is a sum of ${}^u b_{p,q}\colon A[1]^{\otimes p} \otimes M \otimes A[1]^{\otimes q} \to M$ for $p, q \in \mathbb{N}_0$. Moreover, given a morphism $F\colon B(A, M, A) \to B(A, N, A)$ of A_∞-bimodules, we set ${}^u F\colon B^u(A, M, A) \to B^u(A, N, A)$ as $F[-1]$. Its composition with the tensor product $\epsilon_{B(A)^+} \otimes \mathrm{id}_N \otimes \epsilon_{B(A)^+}$ is also a sum of maps ${}^u F_{p,q}\colon A[1]^{\otimes p} \otimes M \otimes A[1]^{\otimes q} \to N$, for $p, q \in \mathbb{N}_0$.

We recall that, given an A_∞-bimodule M over an A_∞-algebra A, its *(graded) dual A_∞-bimodule* $M^\#$ is defined as follows. The underlying graded space is given by the usual graded dual $M^\#$ of M and the bicoderivation $B_{M^\#}^u$ is the unique one satisfying that the diagram

$$\begin{tikzcd}
M^\# \otimes B^u(A, M, A) \arrow[r, "\mathrm{id}_{M^\#} \otimes B_M^u"] \arrow[dd, "S_{1234}"] & M^\# \otimes B^u(A, M, A) \arrow[r, "\mathrm{id}_{M^\#} \otimes \epsilon_{B(A)^+} \otimes \mathrm{id}_M \otimes \epsilon_{B(A)^+}"] & M^\# \otimes M \arrow[d, "\mathrm{ev}_M"] \\
 & & k \\
B^u(A, M^\#, A) \otimes M \arrow[r, "-B_{M^\#}^u \otimes \mathrm{id}_M"] & B^u(A, M^\#, A) \otimes M \arrow[r, "\epsilon_{B(A)^+} \otimes \mathrm{id}_{M^\#} \otimes \epsilon_{B(A)^+} \otimes \mathrm{id}_M"] & M^\# \otimes M \arrow[u, "\mathrm{ev}_M"']
\end{tikzcd}$$

commutes, where $S_{1234}\colon M^\# \otimes B(A)^+ \otimes M \otimes B(A)^+ \to B(A)^+ \otimes M^\# \otimes B(A)^+ \otimes M$ is the map defined by

$$\lambda \otimes b \otimes m \otimes b' \mapsto (-1)^\epsilon b' \otimes \lambda \otimes b \otimes m \quad \text{and} \quad \epsilon = |b'|(|\lambda| + |b| + |m|),$$

for homogeneous $b, b' \in B(A)^+$, $\lambda \in M^\#$ and $m \in M$, and $\mathrm{ev}_M\colon M^\# \otimes M \to k$ is the evaluation map. More explicitly, ${}^u b_{p,q}^{M^\#}\colon A[1]^{\otimes p} \otimes M^\# \otimes A[1]^{\otimes q} \to M^\#$ is defined by

$$\begin{aligned} {}^u b_{p,q}^{M^\#}&(sa_1, \dots, sa_p, \lambda, sa'_1, \dots, sa'_q)(m) \\ &= -(-1)^{\sigma'} \lambda\big({}^u b_{q,p}^M(sa'_1, \dots, sa'_q, m, sa_1, \dots, sa_p)\big), \end{aligned}$$

where $\sigma' = |\lambda| + \big(\sum_{j=1}^p |sa_j|\big)\big(|m| + |\lambda| + \sum_{i=1}^q |sa'_i|\big)$, for all homogeneous $m \in M$, $\lambda \in M^\#$ and $a_1, \dots, a_p, a'_1, \dots, a'_q \in A$. It is clear that the maps $\{{}^u b_{p,q}^{M^\#}\}_{p,q \in \mathbb{N}_0}$ satisfy the required unitary conditions if the maps $\{{}^u b_{p,q}^M\}_{p,q \in \mathbb{N}_0}$ do so.

Moreover, the previous (graded) dual construction is in fact functorial. To wit, if ${}^u F_{\bullet,\bullet}\colon M \to N$ is a morphism of A_∞-bimodules over the A_∞-algebra A, then it defines a morphism ${}^u F_{\bullet,\bullet}^\#\colon N^\# \to M^\#$ of A_∞-bimodules between the corresponding graded duals, given as the unique morphism of graded bicomodules such that

$$\begin{tikzcd}
N^{\#} \otimes B^u(A,M,A) \arrow[r, "\mathrm{id}_{N^{\#}}\otimes {}^uF"] \arrow[dd, "S_{1234}"] & N^{\#} \otimes B^u(A,N,A) \arrow[r, "\mathrm{id}_{N^{\#}}\otimes\epsilon_{B(A)^+}\otimes\mathrm{id}_N\otimes\epsilon_{B(A)^+}"] & N^{\#}\otimes N \arrow[d, "\mathrm{ev}_N"] \\
 & & k \\
B^u(A,N^{\#},A)\otimes M \arrow[r, "{}^uF^{\#}\otimes\mathrm{id}_M"] & B^u(A,M^{\#},A)\otimes M \arrow[r, "\epsilon_{B(A)^+}\otimes\mathrm{id}_{M^{\#}}\otimes\epsilon_{B(A)^+}\otimes\mathrm{id}_M"] & M^{\#}\otimes M \arrow[u, "\mathrm{ev}_M"']
\end{tikzcd}$$

commutes. More explicitly, the maps $\{{}^uF^{\#}_{p,q}\}_{p,q\in\mathbb{N}_0}$ are defined by

$$\begin{aligned}
{}^uF^{\#}_{p,q}(sa_1,\dots,sa_p,&\lambda,sa'_1,\dots,sa'_q)(m) \\
&= (-1)^{\rho'}\lambda\big({}^uF_{q,p}(sa'_1,\dots,sa'_q,m,sa_1,\dots,sa_p)\big),
\end{aligned}$$

where $\rho'=\big(\sum_{j=1}^p|sa_j|\big)\big(|m|+|\lambda|+\sum_{i=1}^q|sa'_i|\big)$, for all homogeneous $\lambda\in N^{\#}$, $m\in M$ and $a_1,\dots,a_p,a'_1,\dots,a'_q\in A$. It is clear that the collection of maps $\{{}^uF^{\#}_{p,q}\}_{p,q\in\mathbb{N}_0}$ satisfies the required unitary conditions, for the maps $\{{}^uF_{p,q}\}_{p,q\in\mathbb{N}_0}$ do so, and that the graded dual induces a contravariant functor from the category of A_∞-bimodules over A to itself. If M and N are locally finite dimensional, ${}^uF^{\#}_{\bullet,\bullet}\colon N^{\#}\to M^{\#}$ is a quasi-isomorphism if ${}^uF_{\bullet,\bullet}\colon M\to N$ is so.

Fact 2.7. *If M is an H-unitary A_∞-bimodule over an A_∞-algebra and $d\in\mathbb{Z}$, then $M[d]$ also.*

Proof. The statement is a direct consequence of the commutativity of the left square in diagram (2.6). □

Remark 2.8. The previous constructions can be carried out also for right (resp., left) A_∞-modules over an A_∞-algebra. In this case, the graded dual sends right (resp., left) A_∞-modules to left (resp., right) A_∞-modules. All the definitions are directly obtained from the previous ones by dropping the tensor factor $B(A)^+$ from the side of $B^u(A,M,A)$ that is not involved, in order to obtain $B^u(M,A)$ (resp., $B^u(A,M)$). Moreover, the previous fact also holds in that situation, with the analogous proof.

Remark 2.9. Note that the graded dual $M^{\#}$ of an H-unitary A_∞-bimodule is in general not H-unitary, as the following example shows. Let $\ell\geqslant 2$ be an integer and set B_ℓ to be the nonunitary algebra given as the direct colimit $\cup_{n\in\mathbb{N}_0}t^{\ell^{-n}}.k[t^{\ell^{-n}}]$ of the diagram of nonunitary algebras obtained from the obvious inclusions

$$t^{\ell^{-n}}.k\big[t^{\ell^{-n}}\big]\longrightarrow t^{\ell^{-(n+1)}}.k\big[t^{\ell^{-(n+1)}}\big]$$

of nonunitary algebras, for all $n\in\mathbb{N}_0$. Finally, define the nonunitary algebra A_ℓ given by $B_\ell/(B_\ell.t)$. This is Example 4.7, (3), in [**26**], where it is also shown that A_ℓ is an H-unitary algebra. Moreover, it is easy to see that the set

$$\{t^{p/\ell^n} : n,p\in\mathbb{N}_0 \text{ and } 1\leqslant p\leqslant \ell^n\}$$

is a basis of A_ℓ. We claim that the A_ℓ-bimodule given by the dual A^*_ℓ is not H-unitary. Since $k\otimes_{(A^+_\ell)^e}A^*_\ell\simeq A^*_\ell/(A_\ell.A^*_\ell+A^*_\ell.A_\ell)$, it suffices to show that $A^*_\ell\neq A_\ell.A^*_\ell$, for A_ℓ is commutative. This follows directly from the assertion that any $\lambda\in A_\ell.A^*_\ell$ satisfies that $\lambda(t) = 0$. Indeed, it suffices to prove it for λ of the form $t^{p/\ell^n}.\mu$, where $\mu \in A_\ell^*$, t^{p/ℓ^n}, $n \in \mathbb{N}_0$ and $p \in \{1, \dots, \ell^n\}$, since any $\lambda \in A_\ell.A_\ell^*$ is a linear combination of elements of the previous type. Finally, we note that $(t^{p/\ell^n}.\mu)(t) = \mu(t.t^{p/\ell^n}) = \mu(0) = 0$, which proves our last assertion, and, as a consequence, A_ℓ^* is not an H-unitary bimodule over A_ℓ.

Let M be an A_∞-bimodule over an A_∞-algebra A. Define $\iota_M : M \to (M^\#)^\#$ as the map given by $\iota_M(m)(\lambda) = (-1)^{|\lambda|}\lambda(m)$, for all $m \in M$ and $\lambda \in M^\#$. Then, ι_M is a strict morphism of A_∞-bimodules, that is natural in M. Using the previous definitions it is rather long but straightforward to check the following result.

Fact 2.10. *Let M be an A_∞-bimodule over an A_∞-algebra A and let $d \in \mathbb{Z}$. Then, the canonical map $\theta_{M,d} : M^\#[d] \to (M[-d])^\#$ defined by*

$$\theta_{M,d}(s_{M^\#,d}\lambda)(s_{M,-d}m) = (-1)^{d|\lambda|}\lambda(m),$$

for all $\lambda \in M^\#$ and $m \in M$, is a strict isomorphism of A_∞-bimodules, natural in M.

2.3.3. Basics on Hochschild and cyclic homology

2.3.3.1. THE HOCHSCHILD HOMOLOGY OF AN A_∞-ALGEBRA We follow [10], Section 3. Given a dg bicomodule N over a coaugmented dg coalgebra C, with left and right coactions $\rho_\ell : N \to C \otimes N$ and $\rho_r : N \to N \otimes C$, define $N^\natural = \mathrm{Ker}(\rho_\ell - \tau_{N,C} \circ \rho_r)$. It is easy to verify that $N^\natural$ is a complex with the induced differential, and that $(-)^\natural$ defines a functor from the category of dg bicomodules over C to the category of dg vector spaces. Moreover, it is clear that $(-)^\natural$ sends a homotopy h between endomorphisms $F, G : N \to N$ of dg bicomodules (*i.e.* a morphism $h : N \to N$ of the underlying graded bicomodules such that $F - G = d_N \circ h + h \circ d_N$) to a homotopy of the corresponding endomorphisms of dg vector spaces. Note that, if C is a coaugmented dg coalgebra, $C^\natural$ defined in Subsection 2.2 coincides with the one here if C is regarded as a dg bicomodule with left and right coactions given by its coproduct. Moreover, notice that the canonical inclusion

$$j_M : (M, {}^u b^M_{0,0}) \to \big(B^u(A, M, A), B^u_M\big) \tag{2.7}$$

is a morphism of complexes with image included in $B^u(A, M, A)^\natural$.

The next result gives a more usual way of regarding the underlying graded vector space of the complex $B^u(A, M, A)^\natural$. For a proof, see [10], Prop. 3.6.

Lemma 2.11. *The maps*

$$\epsilon_{B(A)^+} \otimes \mathrm{id}_M \otimes \mathrm{id}_{B(A)^+}|_{B^u(A,M,A)^\natural} : B^u(A, M, A)^\natural \to M \otimes B(A)^+$$

and $S_{123} \circ (\mathrm{id}_M \otimes \Delta_{B(A)^+}) : M \otimes B(A)^+ \to B^u(A, M, A)^\natural$ are inverse morphisms of graded vector spaces, where $S_{123} : M \otimes B(A)^+ \otimes B(A)^+ \to B(A)^+ \otimes M \otimes B(A)^+$ is given by $m \otimes b \otimes b' \mapsto (-1)^{|b'|(|m|+|b|)} b' \otimes m \otimes b$, for all homogeneous elements $m \in M$ and $b, b' \in B(A)^+$.

We will denote by $N^\sharp$ the dg vector space defined as the image of $N^\natural$ under the isomorphism of the previous lemma.

Definition 2.12. Let A be an A_∞-algebra and let M be an A_∞-bimodule over A. Define the complex $C_\bullet(A, M) = B^u(A, M, A)^\sharp$. If A and M are H-unitary, then

$C_\bullet(A, M)$ is called the *Hochschild homology complex* and its homology $H_\bullet(A, M)$ is called the *Hochschild homology of A with coefficients in M*. They coincide with the usual definitions for dg algebras and dg bimodules. We usually write $HH_\bullet(A)$ instead of $H_\bullet(A, A)$.

Remark 2.13. We have not defined Hochschild homology for any A_∞-algebra with coefficients in any A_∞-bimodule, and it would be wrong to believe that $C_\bullet(A, M)$ computes it in general, as it is already noticed from studying the Hochschild homology of nonunitary algebras A (with coefficients in the standard bimodule A).

2.3.3.2. THE CYCLIC HOMOLOGY OF AN A_∞-ALGEBRA Let A be an A_∞-algebra and let A^+ be the augmentation of A. Using the inclusion $A \to A^+$, we see that A^+ is an A_∞-bimodule over A. Denote the differential of the complex $C_\bullet(A, A^+)$ by b. As proved in [**10**], Thm. 3.7, $C_\bullet(A, A^+)$ is a dg module over the dg algebra $\Lambda = k[\epsilon]/(\epsilon^2)$ with zero differential, where ϵ has cohomological degree -1. The action of ϵ is given by an endomorphism B of $C_\bullet(A, A^+)$, which is described as follows. Define the map

$$\sigma\colon B(A)^+ \to B^u(A, A^+, A)$$

sending 1_k to 1_{A^+}, and $[a_1|\cdots|a_n]$ to

$$1_{A^+} \otimes [a_1|\cdots|a_n] + \sum_{i=1}^{n-1} [a_1|\cdots|a_i] \otimes 1_{A^+} \otimes [a_{i+1}|\cdots|a_n] + [a_1|\cdots|a_n] \otimes 1_{A^+},$$

as well as the mapping

$$\varphi\colon B^u(A, A^+, A) \to B(A)^+$$

sending $[a_1|\cdots|a_n] \otimes 1_{A^+} \otimes [b_1|\cdots|b_m]$ to zero, and $[a_1|\cdots|a_n] \otimes a \otimes [b_1|\cdots|b_m]$ to $[a_1|\cdots|a_n|a|b_1|\cdots|b_m]$, if $a \in A$. Set $B = \sigma^\natural \circ \varphi^\natural$. Then, $C_\bullet(A, A^+)$ is a dg module over the dg algebra $\Lambda = k[\epsilon]/(\epsilon^2)$ with zero differential, where ϵ has cohomological degree -1 and the right action of ϵ is given by B. Note that the canonical projection

$$\mathcal{p}_A\colon C_\bullet(A, A^+) \to k \tag{2.8}$$

induced by $(\epsilon_{B(A)^+} \otimes \epsilon_{A^+} \otimes \epsilon_{B(A)^+})^\natural$ is a morphism of dg modules over Λ, where k has the trivial dg module structure.

Consider the (graded) pseudo-compact algebra $k[\![u]\!]$, where u has cohomological degree -2, and let W be a graded pseudo-compact left module W over $k[\![u]\!]$. Consider the complex $C_\bullet(A, A^+)[\![u]\!]\hat{\otimes}_{k[\![u]\!]}W$, with the differential $b + uB$. The mapping (2.8) induces a morphism of complexes from $C_\bullet(A, A^+)[\![u]\!]\hat{\otimes}_{k[\![u]\!]}W$ to $k[\![u]\!]\hat{\otimes}_{k[\![u]\!]}W \simeq W$. The cohomology of the kernel $CC_\bullet(A, W)$ of this map is denoted by $HC_\bullet(A, W)$. If $W = k$, then $HC_\bullet(A, W)$ is called the *Hochschild homology* $H_\bullet(A, A)$ of A with coefficients in A. It coincides with the notion considered in Definition 2.12 if A is H-unitary, since the canonical inclusion $C_\bullet(A, A) \to \operatorname{Ker}(\mathcal{p}_A)$ is a quasi-isomorphism. Indeed, its cokernel is isomorphic to $B(A)$, which is acyclic. If $W = k(\!(u)\!)$ (resp., $W = k(\!(u)\!)/uk[\![u]\!]$, $W = k[\![u]\!]$), the complex $CC_\bullet(A, W)$ is denoted $CP_\bullet(A)$ (resp., $CC_\bullet(A)$, $CC_\bullet^-(A)$) and its homology is called the *periodic cyclic homology* $HP_\bullet(A)$ (resp., the *cyclic homology* $HC_\bullet(A)$, the *negative cyclic homology* $HC_\bullet^-(A)$) of A.

The commutative diagram of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & k[\![u]\!] \arrow[r] \arrow[d] & k(\!(u)\!) \arrow[r] \arrow[d] & k(\!(u)\!)/k[\![u]\!] \arrow[r] \arrow[d, "\mathrm{id}_{k(\!(u)\!)/k[\![u]\!]}"] & 0 \\ 0 \arrow[r] & k \arrow[r] & k(\!(u)\!)/uk[\![u]\!] \arrow[r] & k(\!(u)\!)/k[\![u]\!] \arrow[r] & 0 \end{tikzcd} \tag{2.9}$$

of pseudo-compact $k[\![u]\!]$-modules (where $k \simeq k[\![u]\!]/uk[\![u]\!]$ and the two left vertical maps are the canonical projections) induces the commutative diagram

$$\begin{tikzcd} \cdots \arrow[r] & HC_{\bullet-1}(A) \arrow[r, "B'"] \arrow[d, "\mathrm{id}_{HC_{\bullet-1}(A)}"] & HC^-_\bullet(A) \arrow[r, "I'"] \arrow[d, "h"] & HP_\bullet(A) \arrow[r, "S'"] \arrow[d, "p"] & HC_{\bullet-2}(A) \arrow[r] \arrow[d, "\mathrm{id}_{HC_{\bullet-2}(A)}"] & \cdots \\ \cdots \arrow[r] & HC_{\bullet-1}(A) \arrow[r, "B"] & HH_\bullet(A) \arrow[r, "I"] & HC_\bullet(A) \arrow[r, "S"] & HC_{\bullet-2}(A) \arrow[r] & \cdots \end{tikzcd} \tag{2.10}$$

We note that $B^u(A, A, A)$ is canonically a subcomplex of $B^u(A, A^+, A)$, whose cokernel is isomorphic to $B(A)^+ \otimes B(A)^+$. By applying the functor $(-)^\sharp$ we see that $C_\bullet(A, A)$ is a subcomplex of $C_\bullet(A, A^+)$, whose cokernel is $B(A)^+$. Moreover, the complex $C_\bullet(A, A^+)[1]$ is isomorphic to the cone of the morphism of complexes

$$f\colon B(A)^+ \to C_\bullet(A, A)[1]$$

such that $f|_k$ is zero, and, for $n \in \mathbb{N}$, $f|_{A[1]^{\otimes n}}$ is given by $\mathrm{id}_{A[1]^{\otimes n}} - t_n$, where

$$t_n([a_1|\cdots|a_n]) = (-1)^{|sa_n|\sum_{i=1}^{n-1}|sa_i|}[a_n|a_1|\cdots|a_{n-1}],$$

for all homogeneous $a_1, \dots, a_n \in A$. Note that $f|_{A[1]} = 0$. One also defines the morphism of complexes $N\colon C_\bullet(A, A) \to B(A)[1]$, where $N[1]|_{A[1]^{\otimes n}}$ is given by the composite $s_{B(A),2} \circ (\sum_{j=0}^{n-1} t_n)$. It is easy to prove that $f \circ N[-1] = 0 = N[1] \circ f$.

Set $C^\lambda_\bullet(A)$ as the cokernel $\mathrm{Coker}(f[-1])$, and denote its homology by $H^\lambda_\bullet(A)$. Define the map $\bar{\pi}\colon C_\bullet(A, A^+)(\!(u)\!)/uk[\![u]\!] \to C^\lambda_\bullet(A)$ as zero on $B(A)^+.u^{-\ell}$ and on $C_\bullet(A, A).u^{-\ell-1}$ for all $\ell \in \mathbb{N}_0$, and as the canonical projection $C_\bullet(A, A) \to C^\lambda_\bullet(A)$ on $C_\bullet(A, A)$. It clearly induces a map $HC_\bullet(A) \to H^\lambda_\bullet(A)$, which is an isomorphism if the characteristic of k is zero (see [19], Thm. 2.1.5). Moreover, if $\operatorname{char} k = 0$ and A is H-unitary, then $s_{B(A)[1],-2} \circ N$ induces a quasi-isomorphism from the complex $\mathrm{Coker}(f[-1]) = C^\lambda_\bullet(A)$ to $\mathrm{Ker}(f[-1]) \cap B(A)[-1] = B(A)^\natural[-1]$ (see [22], Lemma 1.2 and comments below).

3. Symplectic structures and cyclic A_∞-algebras

3.1. Generalities

From now on we assume that the characteristic of the field is different from 2. Let A be an A_∞-algebra and let B^+_A be the differential of $B(A)^+$. Since $\mathcal{L}_{B^+_A}$ and $\mathbf{q}_{\mathrm{DR}}$ anticommute, $(\mho^m(A[1]), \mathcal{L}_{B^+_A})$ and $(\mho^m_{\mathrm{cyc}}(A[1]), \mathcal{L}_{B^+_A})$ are complexes for all $m \in \mathbb{N}_0$, and $\mathbf{q}_{\mathrm{DR}}$ induces morphisms between the corresponding complexes. They will be provided with these differentials unless otherwise indicated. A *symplectic structure* of degree d on A is by definition a nondegenerate pre-symplectic structure ω of degree d on the graded vector space $V = A[1]$ satisfying that $\omega \circ \mathcal{L}_{B^+_A} = 0$. A symplectic structure ω is *constant* if $\omega_{>0}$ defined after (2.4) vanishes.

The following is well-known (*cf.* [4], Lemma 3.1 and [5], Lemma 4.1). Moreover, condition (ii) in the lemma below is the usual definition of a *d-cyclic (or symplectic) A_∞-algebra*, which was introduced in [16], Def. 10.1.

Lemma 3.1. *Let A be a finite dimensional A_∞-algebra and set $V = A[1]$. Given a homogeneous linear map $\omega\colon \Sigma V \otimes \Sigma V \to k$ of cohomological degree $2-d$, define $\Gamma = \omega \circ (t_V \otimes t_V)\colon V \otimes V \to k$ and $f\colon A \to A^{\#}[-d]$ by*

$$(s_{A^{\#}[-d],d} f(a))(b) = (-1)^{|a|}\Gamma(sa, sb),$$

for all $a, b \in A$. Then, the following are equivalent

(i) ω is a constant symplectic structure of degree d on A;

(ii) Γ is a nondegenerate graded anti-symmetric map (i.e. $\Gamma \circ \tau_{V,V} = -\Gamma$) that satisfies

$$\Gamma(b_n(sa_1, \dots, sa_n), sa_0) = (-1)^{\epsilon}\Gamma(b_n(sa_0, \dots, sa_{n-1}), sa_n), \tag{3.1}$$

for all homogeneous $a_0, \dots, a_n \in A$, where $\epsilon = |sa_0|(\sum_{i=1}^{n} |sa_i|)$;

(iii) f is a strict isomorphism of A_∞-bimodules such that $f = (f^{\#} \circ \theta_{A^{\#},d})[-d] \circ \iota_A$, as morphisms of graded vector spaces.

Proof. First, note that the finite dimensional assumption on A implies that the graded dual $A^{\#}$ is equal to the usual dual A^*. It is clear that the nondegeneracy conditions for ω and Γ are equivalent, as well as the corresponding graded symmetric and graded antisymmetric conditions, respectively. Moreover, a straightforward computation shows that $\omega \circ \mathscr{L}_{B_A^+} = 0$ is equivalent to (3.1). On the other hand, it is clear that (3.1) is tantamount to f being a strict morphism of A_∞-bimodules, whereas the nondegeneracy of Γ is equivalent to f being bijective, and the graded antisymmetry property of Γ is tantamount to $f = (f^{\#} \circ \theta_{A^{\#},d})[-d] \circ \iota_A$. $\square$

The next result is a direct consequence of (2.3) and the definition of $(-)^{\natural}$.

Fact 3.2. *Let A be an A_∞-algebra and let B_A^+ be the differential of $B(A)^+$. Then,*

(i) $(\mho^0 A[1], \mathscr{L}_{B_A^+}) = B(A)^+$, which implies $(\mho^0_{\mathrm{cyc}}(A[1]), \mathscr{L}_{B_A^+}) = (B(A)^+)^{\natural}$;

(ii) the map $\mathrm{id}_{B(A)^+} \otimes (t_{A[1]} \circ s_A) \otimes \mathrm{id}_{B(A)^+}$ from $B^u(A, A, A)$ to $(\mho^1 A[1], -\mathscr{L}_{B_A^+})$ is an isomorphism of dg bicomodules over $B(A)^+$, so it induces an isomorphism of complexes $C_\bullet(A, A) \to (\mho^1_{\mathrm{cyc}}(A[1]), -\mathscr{L}_{B_A^+})$.

The following result is essential in the sequel (*cf.* [16], Subsection 7.2, p. 178).

Lemma 3.3. *There is a homogeneous linear map $\psi\colon \Sigma(T^c V) \to \mho^1 V$ of cohomological degree and weight 0 such that the diagram*

$$\begin{tikzcd}
 & \mho^1 V \arrow[d, "\rho_r"] \\
 & \mho^1 V \otimes T^c V \arrow[d, "\mathfrak{q}_{\mathrm{DR}} \otimes \mathrm{id}_{T^c V}"] \\
\Sigma(T^c V) \arrow[ruu, dashed, bend left, "\psi"] \arrow[r, "\Delta_{T^c V} - \tau_{T^c V, T^c V} \circ \Delta_{T^c V}"] & T^c V \otimes T^c V
\end{tikzcd} \tag{3.2}$$

commutes, where ρ_r denotes the standard right coaction of T^cV on $\mho^1 V$. More explicitly, set $\psi|_{\Sigma k} = 0$ and $\psi|_{\Sigma V} = 0$, and, for $n \geqslant 2$ and $v_1, \dots, v_n \in V$, set $\psi(t(v_1 \cdots v_n))$ as

$$\sum_{i=1}^{n} (-1)^{\epsilon_{1,i}\epsilon_{i+1,n}} v_{i+1} \cdots v_n t(v_1) v_2 \cdots v_i - \sum_{i=0}^{n-1} (-1)^{\epsilon_{1,i}\epsilon_{i+1,n}} v_{i+1} \cdots v_{n-1} t(v_n) v_1 \cdots v_i, \tag{3.3}$$

where $\epsilon_{j,j'} = \sum_{\ell=j}^{j'} |v_\ell|$ for $1 \leqslant j \leqslant j' \leqslant n$. It satisfies that $\mathrm{Ker}(\psi) = \Sigma(T^cV)^\natural$ and $\mathrm{Im}(\psi) = \mho^1_{\mathrm{cyc}} V \cap \mathrm{Ker}(\mathfrak{q}_{\mathrm{DR}})$, so it induces a linear isomorphism

$$\bar{\psi}\colon (\Sigma \bar{T}^c V)/\big(\Sigma(\bar{T}^c V)^\natural\big) \to \mho^1_{\mathrm{cyc}} V \cap \mathrm{Ker}(\mathfrak{q}_{\mathrm{DR}}).$$

Proof. The reader can check that the map ψ given by (3.3) satisfies (3.2). The inclusion $\mathrm{Im}(\psi) \subseteq \mho^1_{\mathrm{cyc}} V \cap \mathrm{Ker}(\mathfrak{q}_{\mathrm{DR}})$ follows from (2.2) and (3.3), $\mathrm{Ker}(\psi) \subseteq \Sigma(T^cV)^\natural$ from (3.2) and $\Sigma(T^cV)^\natural \subseteq \mathrm{Ker}(\psi)$ from (3.3). Set $\varpi \in \mho^2_{\mathrm{cyc}} V$ as the symmetrizer of $t(v_1)v_2 \cdots v_{i-1} t(v_i) v_{i+1} \cdots v_n$ in $\mho^2 V$, for $v_1, \dots, v_n \in V$, *i.e.*

$$\varpi = -\sum_{\ell=2}^{i} (-1)^{\epsilon_{1,\ell-1}\epsilon_{\ell,n}} v_\ell \cdots v_{i-1} t(v_i) v_{i+1} \cdots v_n t(v_1) v_2 \cdots v_{\ell-1} + \sum_{\ell=i+1}^{n+1} (-1)^{\epsilon_{1,\ell-1}\epsilon_{\ell,n}} v_\ell \cdots v_n t(v_1) v_2 \cdots v_{i-1} t(v_i) v_{i+1} \cdots v_{\ell-1}.$$

Then,

$$\mathfrak{q}_{\mathrm{DR}^2}(\varpi) = -\sum_{j=1}^{i} (-1)^{\epsilon_{1,j-1}\epsilon_{j,n}} \psi\big(t(v_j \cdots v_n v_1 v_2 \cdots v_{j-1})\big). \tag{3.4}$$

The previous identity and Lemma 2.2 show that $\mho^1_{\mathrm{cyc}} V \cap \mathrm{Ker}(\mathfrak{q}_{\mathrm{DR}}) \subseteq \mathrm{Im}(\psi)$. □

Fact 3.4. *Let A be an A_∞-algebra, B_A^+ be the differential of $B(A)^+$ and $V = A[1]$. Then, the map ψ defined in Lemma 3.3 is a morphism of complexes, where $\Sigma(T^cV)$ is endowed with the differential $t_{T^cV} \circ \mathcal{L}_{B_A^+}|_{T^cV} \circ t_{T^cV}^{-1}$ and $\mho^1 V$ with $\mathcal{L}_{B_A^+}|_{\mho^1 V}$.*

Proof. This is a lengthy but straightforward computation that follows from (2.3) and (3.3). □

Let $(A, b_\bullet)$ be an H-unitary A_∞-algebra. Recall the map $j_A\colon A \to B^u(A,A,A)^\natural$ given by (2.7) for the standard A_∞-bimodule A. By composing it with the isomorphism in Lemma 2.11 and the canonical map $C_\bullet(A,A) \to \mathrm{Ker}(\mathfrak{p}_A)$ mentioned after (2.8), it gives a morphism of complexes $(A, {}^u b_{0,0}) \to \mathrm{Ker}(\mathfrak{p}_A)$ and *a fortiori* $(A, {}^u b_{0,0}) \to CC_\bullet(A)$, by using (2.9). One declares that a homogeneous cocycle $\lambda \in CC_\bullet(A)^\#$ is *homologically nondegenerate* if the composition $\bar{\lambda}$ of $A \to CC_\bullet(A)$ with λ induces a nondegenerate bilinear form $\bar{\lambda} \circ \bar{m}_2\colon H^\bullet(A)^{\otimes 2} \to k$, where $\bar{m}_2$ is the product on $H^\bullet(A)$ induced by $-s_A^{-1} \circ b_2 \circ s_A^{\otimes 2}$. Note that a homogeneous cocycle $\lambda' \in CC_\bullet(A)^\#$ that is cohomologous to a homologically nondegenerate cocycle $\lambda \in CC_\bullet(A)^\#$ is also homologically nondegenerate.

The next result is due to Kontsevich and Soibelman (see [16], Thm. 10.7).

Theorem 3.5. *Let A be an H-unitary A_∞-algebra over a field k of characteristic zero. There is a quasi-isomorphism $\bar{\theta}$ from $(\mho^2_{\mathrm{cyc}}(A[1])/\operatorname{Im}(\mathbf{q}_{\mathrm{DR}^3}))$ to $B(A)^\natural[1]$, where the differential of the former is induced by $\leftthreetimes_{B_A^+}$ and we omitted the weight. Hence, there is a quasi-isomorphism from $(B(A)^\natural)^\#[-1]$ to $(\mathrm{PSp}_{\mathrm{nc}}(A[1]), \leftthreetimes^\#_{B_A^+})$, so $CC_\bullet(A)[2]$ and $\mho^2_{\mathrm{cyc}}(A[1])/\operatorname{Im}(\mathbf{q}_{\mathrm{DR}^3})$ are quasi-isomorphic, as well as their graded duals. Moreover, a pre-symplectic form ω induces a nondegenerate bilinear form on $H^\bullet(\Sigma(A[1]), \leftthreetimes_{B_A^+})$ if and only if the corresponding functional is homologically nondegenerate.*

Proof. Fix $V = A[1]$ and write $\mathbf{q}_{\mathrm{DR}}$ for the restriction of this differential to $\mho^\bullet_{\mathrm{cyc}}V$. By Lemma 2.2, the complex $(\mho^2_{\mathrm{cyc}}V/\operatorname{Im}(\mathbf{q}_{\mathrm{DR}^3}))$ is precisely $(\mho^2_{\mathrm{cyc}}V/\operatorname{Ker}(\mathbf{q}_{\mathrm{DR}^2}))$, and $\mathbf{q}_{\mathrm{DR}^2}$ induces an isomorphism from the latter to $\Sigma\operatorname{Im}(\mathbf{q}_{\mathrm{DR}^2}) = \Sigma\operatorname{Ker}(\mathbf{q}_{\mathrm{DR}^1})$, where we have used Lemma 2.2. By Fact 3.4, ψ gives an isomorphism from the complex $\Sigma(B(A)/B(A)^\natural)$ to $(\operatorname{Ker}(\mathbf{q}_{\mathrm{DR}^1}), \leftthreetimes_{B_A^+})$. Since A has is H-unitary, $B(A)$ is quasi-isomorphic to 0. Then, by the snake lemma applied to the short exact sequence of complexes

$$0 \to B(A)^\natural \to B(A) \to B(A)/B(A)^\natural \to 0$$

there is an quasi-isomorphism of complexes $B(A)/B(A)^\natural \to B(A)^\natural[1]$. Then, the map $\theta = \Sigma(B_A \circ \bar{\psi}^{-1}) \circ \mathbf{q}_{\mathrm{DR}^2}$ from $(\mho^2_{\mathrm{cyc}}V/\operatorname{Im}(\mathbf{q}_{\mathrm{DR}^3}))$ to $\Sigma B(A)^\natural[1]$ induces the claimed quasi-isomorphism $\bar{\theta}$, if we drop the immaterial shift on the weight. Finally, $CC_\bullet(A)$ is canonically identified with the complex $B(A)^\natural[-1]$, as recalled in the last paragraph of Subsection 2.3.3, which provides the claimed quasi-isomorphism between $(\mho^2_{\mathrm{cyc}}(A[1])/\operatorname{Im}(\mathbf{q}_{\mathrm{DR}^3}))$ and $CC_\bullet(A)[2]$. By taking the graded dual one obtains a quasi-isomorphism from $(B(A)^\natural)^\#[-1]$ to $(\mathrm{PSp}_{\mathrm{nc}}(A[1]), \leftthreetimes^\#_{B_A^+})$, and thus the latter complex is quasi-isomorphic to $CC_\bullet(A)^\#[-2]$.

To prove the last part, we first note that the composition of $A \to CC_\bullet(A)$, the projection $CC_\bullet(A) \to C^\lambda_\bullet(A)$ and $N\colon C^\lambda_\bullet(A) \to B(A)^\natural[-1]$ is simply the usual inclusion $A \to B(A)^\natural[-1]$. It suffices to show that if $\lambda \in (B(A)^\natural[-1])^\#$ is homogeneous, then the map $\hat{\lambda}$ given as the composition of $A \to B(A)^\natural[-1]$ and λ induces a nondegenerate bilinear form $\hat{\lambda} \circ \bar{m}_2\colon H^\bullet(A)^{\otimes 2} \to k$ if and only if the constant part ω_0 of $\omega = \lambda \circ s_{\Sigma B(A)^\natural[1],(-2,-1,0,\dots,0)} \circ \theta$ is nondegenerate. Using (3.4), we see that

$$\omega_0\big(t(v)t(w) - (-1)^{|v|.|w|}t(w)t(v)\big) = -\hat{\lambda}'\big(b_1(v)w\big) - (-1)^{|v|}\hat{\lambda}'\big(vb_1(w)\big) - \hat{\lambda}'\big(b_2(v,w)\big).$$

for $v = s(a)$ and $w = s(b)$, with $a, b \in A$ homogeneous elements, where $\hat{\lambda}' = \hat{\lambda} \circ s_A^{-1}$. Hence, the composition of $(s_{\Sigma H^\bullet(A)} \circ t_{H^\bullet(A)})^{\otimes 2}$ with $H^\bullet(\Sigma V, \leftthreetimes_{B_A^+})^{\otimes 2} \to k$ induced by ω_0 is equal to $\hat{\lambda} \circ \bar{m}_2$. The theorem follows. □

One obtains the following direct consequence, which is implicit in [**16**].

Corollary 3.6. *Let A be an H-unitary A_∞-algebra over a field of characteristic zero and having finite dimensional cohomology, provided with a symplectic structure ω of degree d. Then, there exist an A_∞-algebra A' provided with a constant symplectic structure of degree d, i.e. a d-cyclic structure, and a quasi-isomorphism $F\colon A \to A'$ of A_∞-algebras.*

Proof. Let $\tilde{A} = H^\bullet(A)$ provided with a quasi-isomorphic A_∞-algebra structure to that of A. Note that $\tilde{A}$ is a *a fortiori* H-unitary. We will denote the differential of the bar construction of its augmentation $\tilde{A}^+$ by $B^+_{\tilde{A}}$. Then, the complexes $C^\bullet(A, A)$ and $C^\bullet(\tilde{A}, \tilde{A})$ computing Hochschild homology are quasi-isomorphic. By the previous theorem, $\tilde{A}$ is provided with a symplectic structure $\tilde{\omega}$ of degree d. By the formal Darboux theorem, there is an automorphism F of the coaugmented graded coalgebra $T^c(\tilde{A}[1])$ sending $\tilde{\omega}$ to a constant nondegenerate pre-symplectic structure of the form $\hat{\omega} = \tilde{\omega} \circ \hat{F}^{-1}$ of degree d on $\tilde{A}$, where $\hat{F}$ is the automorphism of $\mho^\bullet(\tilde{A}[1])$ induced by F as explained in the third paragraph of Subsection 2.2. By transport of structures, there is a differential on $T^c(\tilde{A}[1])$ such that F is an isomorphism of coaugmented dg coalgebras. Let us denote the new A_∞-algebra structure on $\tilde{A}$ by $\hat{A}$. Since $\hat{F}$ commutes with the Lie derivatives induced by the differentials $B^+_{\tilde{A}}$ and $B^+_{\hat{A}}$, we have that

$$\hat{\omega} \circ \mathcal{L}_{B^+_{\hat{A}}} \circ \hat{F} = \hat{\omega} \circ \hat{F} \circ \mathcal{L}_{B^+_{\tilde{A}}} = \tilde{\omega} \circ \mathcal{L}_{B^+_{\tilde{A}}} = 0.$$

As a consequence, $\hat{\omega}$ is a symplectic structure of degree d on $\hat{A}$. □

3.2. Homotopy inner products

Lemma 3.1 motivates the following definition.

Definition 3.7. Let $d \in \mathbb{Z}$. A *homotopy inner product of degree d* on an A_∞-algebra $(A, b_\bullet)$ is a quasi-isomorphism ${}^uF_{\bullet,\bullet} \colon A \to A^\#[-d]$ of A_∞-bimodules. It is called *symmetric* if $({}^uF^\#_{\bullet,\bullet} \circ \theta_{A^\#,d})[-d] \circ \iota_A = {}^uF_{\bullet,\bullet}$.

Fact 3.8. *Let A and B be two locally finite dimensional A_∞-algebras. Assume that A is provided with a (resp., symmetric) homotopy inner product ${}^uF_{\bullet,\bullet} \colon A \to A^\#[-d]$ of degree d, and let $G_\bullet \colon B \to A$ be a quasi-isomorphism of A_∞-algebras. Then, B is endowed with a (resp., symmetric) homotopy inner product of degree d.*

Proof. Note that $G_\bullet$ induces a quasi-isomorphism ${}^u\tilde{G}_{\bullet,\bullet} \colon B \to A$ of A_∞-bimodules over B via ${}^u\tilde{G}_{p,q} = G_{p+q+1} \circ (\mathrm{id}^{\otimes p}_{B[1]} \otimes s_B \otimes \mathrm{id}^{\otimes q}_{B[1]})$. It is easy to see that the map ${}^uH_{\bullet,\bullet} = {}^u\tilde{G}^\#_{\bullet,\bullet}[-d] \circ {}^uF_{\bullet,\bullet} \circ {}^u\tilde{G}_{\bullet,\bullet}$ is a quasi-isomorphism of A_∞-bimodules over B, so B has a homotopy inner product of degree d. If the homotopy inner product $F_{\bullet,\bullet}$ is symmetric, *i.e.* $({}^uF^\#_{\bullet,\bullet} \circ \theta_{A^\#,d})[-d] \circ \iota_A = {}^uF_{\bullet,\bullet}$, then we see that

$$\begin{aligned}
\big({}^uH^\#_{\bullet,\bullet} \circ \theta_{B^\#,d}\big)[-d] \circ \iota_B &= \big({}^u\tilde{G}^\#_{\bullet,\bullet} \circ {}^uF^\#_{\bullet,\bullet} \circ ({}^u\tilde{G}^\#_{\bullet,\bullet}[-d])^\# \circ \theta_{B^\#,d}\big)[-d] \circ \iota_B \\
&= \big({}^u\tilde{G}^\#_{\bullet,\bullet} \circ {}^uF^\#_{\bullet,\bullet} \circ \theta_{A^\#,d} \circ {}^u\tilde{G}^{\#\#}_{\bullet,\bullet}[d]\big)[-d] \circ \iota_B \\
&= {}^u\tilde{G}^\#_{\bullet,\bullet}[-d] \circ \big({}^uF^\#_{\bullet,\bullet} \circ \theta_{A^\#,d}\big)[-d] \circ \iota_A \circ {}^u\tilde{G}_{\bullet,\bullet} \\
&= {}^u\tilde{G}^\#_{\bullet,\bullet}[-d] \circ {}^uF_{\bullet,\bullet} \circ {}^u\tilde{G}_{\bullet,\bullet} = {}^uH_{\bullet,\bullet},
\end{aligned}$$

so the homotopy inner product ${}^uH_{\bullet,\bullet}$ is also symmetric. □

Lemma 3.9. *Let A be a locally finite dimensional homologically unitary A_∞-algebra provided with a homotopy inner product ${}^uF_{\bullet,\bullet} \colon A \to A^\#[-d]$ of degree d. Then, it also has a symmetric homotopy inner product.*

Proof. Let A be an A_∞-algebra with a homotopy inner product ${}^uF_{\bullet,\bullet} \colon A \to A^\#[-d]$ of degree d. Set $B = H^\bullet(A)$ the cohomology of A, provided with the minimal A_∞-algebra structure induced by the theorem of Kadeishvili, and let $G \colon B \to A$ be the quasi-isomorphism of A_∞-algebras. By Fact 3.8, there exists an homotopy inner product ${}^uH_{\bullet,\bullet}\colon B \to B^\#[-d]$ of degree d. Since B is minimal, ${}^uH_{0,0}$ is an isomorphism of graded vector spaces. Moreover, by only considering the underlying unitary graded algebra structure of B given by $-s_B \circ b_2^B \circ s_B^{\otimes 2}$, ${}^uH_{0,0}$ is in fact an isomorphism of graded bimodules, where $B^\#[-d]$ has the underlying graded bimodule structure induced by its A_∞-bimodule structure. Setting $h = s_{B^\#[-d],d} \circ {}^uH_{0,0}$ and $h' = s_{B^\#[-d],d} \circ ({}^uH_{0,0}^\# \circ \theta_{B^\#,d})[-d] \circ \iota_B$, the previous definitions tell us that $h'(a)(b) = (-1)^{|a||b|}h(b)(a)$, for all homogeneous $a, b \in B$. The fact that ${}^uH_{0,0}$ is a morphism of graded bimodules yields $h(1_B)a = h(a) = (-1)^{d|a|}ah(1_B)$ for all homogeneous $a \in B$, which in turn implies that

$$\begin{aligned} h(a)(b) &= (-1)^{d|a|}\big(ah(1_B)\big)(b) = (-1)^{|a||b|}h(1_B)(ba) \\ &= (-1)^{|a||b|}\big(h(1_B)b\big)(a) = (-1)^{|a||b|}h(b)(a), \end{aligned}$$

and we have as a consequence that $({}^uH_{0,0}^\# \circ \theta_{B^\#,d})[-d] \circ \iota_B = {}^uH_{0,0}$. We set finally ${}^uK_{\bullet,\bullet} = (({}^uH_{\bullet,\bullet}^\# \circ \theta_{B^\#,d})[-d] \circ \iota_B + {}^uH_{\bullet,\bullet})/2$, which is well defined since $\mathrm{char}(k) \neq 2$. It satisfies that ${}^uK_{0,0} = {}^uH_{0,0}$, so ${}^uK_{\bullet,\bullet}$ is a symmetric homotopy inner product on B. By applying Fact 3.8 once more, we see that A is provided with a symmetric homotopy inner product. □

Remark 3.10. The proof of Lemma 3.9 also shows that the symmetry condition in the definition of d-cyclic A_∞-algebra A is superfluous if A is minimal and homologically unitary.

4. Another version of the Hochschild homology

We will consider in this section another version of the Hochschild and the cyclic homology of A_∞-algebras, that is more convenient when one wants to compare to the spaces of morphisms of A_∞-bimodules, which we will study in the next section.

Let M and N be two A_∞-bimodules over the A_∞-algebra A. Consider the cotensor product $B^u(A, M, A) \,\square^{B(A)^+}\, B^u(A, N, A)$, with the induced differential. It is clearly a dg bicomodule over $B(A)^+$. The mapping

$$i_{M,N}\colon B^u(A, M, A) \,\square^{B(A)^+}\, B^u(A, N, A) \to B(A)^+ \otimes M \otimes B(A)^+ \otimes N \otimes B(A)^+ \tag{4.1}$$

given as the restriction of $b \otimes m \otimes b' \otimes b'' \otimes n \otimes b''' \mapsto b \otimes m \otimes \epsilon_{B(A)^+}(b')b'' \otimes n \otimes b'''$, for all $b, b', b'', b''' \in B(A)^+$, $m \in M$ and $n \in N$, is an isomorphism of graded bicomodules over $T^c(A[1])$. Indeed, the inverse is given explicitly by the tensor product $\mathrm{id}_{B(A)^+} \otimes \mathrm{id}_M \otimes \Delta_{B(A)^+} \otimes \mathrm{id}_N \otimes \mathrm{id}_{B(A)^+}$.

The following result is immediate.

Fact 4.1. *Let A be an A_∞-algebra and $m \in \mathbb{N}$. Set $V = A[1]$.*

(i) The map (4.1) gives an isomorphism of graded bicomodules between the m-th cotensor power $(\mho^1 V)^{\square^{B(A)^+} m}$ and $\mho^m V$, and the differential of the former induced by $\otimes_{B_A^+}$ on $\mho^1 V$ is identified with $\otimes_{B_A^+}$ on $\mho^m V$ under the previous isomorphism.

(ii) *Let* $\sigma_m\colon (\mho^1 V)^{\square^{B(A)^+} m} \to (\mho^1 V)^{\square^{B(A)^+} m}$ *be the cyclic permutation induced by*

$$\varpi_1 \otimes \cdots \otimes \varpi_m \mapsto -(-1)^{m+|\varpi_1|(\sum_{j=2}^{m}|\varpi_j|)}\varpi_2 \otimes \cdots \otimes \varpi_m \otimes \varpi_1,$$

for $\varpi_1, \dots, \varpi_m \in \mho^1 V$ *homogeneous, and* $N_m = \sum_{\ell=0}^{m-1} \sigma_m^\ell$. *Then, using the identification in the previous item,* σ_m *restricts to a map* $(\mho^m V)^\natural \to (\mho^m V)^\natural$ *that commutes with* $\eth_{B_A^+}$ *such that the image of the restriction of* N_m *is* $\mho^m_{\rm cyc} V$.[2]

(iii) *By Fact 3.2 and (4.1),* $(\mathrm{id}_{B(A)^+} \otimes (t_{A[1]} \circ s_A))^{\otimes m} \otimes \mathrm{id}_{B(A)^+}$ *induces an isomorphism of dg bicomodules from* $B^u(A, A, A)^{\square^{B(A)^+} m}$ *to* $(\mho^m(A[1]), (-1)^m \eth_{B_A^+})$ *over* $B(A)^+$.

The previous cotensor product allows to introduce the following notion, the first part of which was defined in [18], Subsubsection 4.1.1.

Definition 4.2. Let M and N be two A_∞-bimodules over an A_∞-algebra A. The *tensor product* $M \otimes_A^\infty N$ is the graded vector space $M \otimes B(A)^+ \otimes N$ with the A_∞-bimodule structure over A whose (unshifted) bar construction is precisely given by $B(A)^+ \otimes M \otimes B(A)^+ \otimes N \otimes B(A)^+$, provided with the dg bicomodule structure over $B(A)^+$ obtained by transport of structures using (4.1). Moreover, we define the *tensor product* $M \otimes_{A^e}^\infty N$ of M and N as the dg vector space $B^u(A, M \otimes_A^\infty N, A)^\natural$.

Consider also the unique map

$$\rho\colon B(A)^+ \otimes M \otimes B(A)^+ \otimes A^+ \otimes B(A)^+ \to B(A)^+ \otimes M \otimes B(A)^+ \tag{4.2}$$

of graded bicomodules over $T^c(A[1])$ satisfying that $\bar\rho = (\epsilon_{B(A)^+} \otimes \mathrm{id}_M \otimes \epsilon_{B(A)^+}) \circ \rho$ is such that $\bar\rho|_{A[1]^{\otimes p} \otimes M \otimes A[1]^{\otimes p'} \otimes A \otimes A[1]^{\otimes p''}}$ is

$$^u b_{p,p'+p''+1} \circ \big(\mathrm{id}_{A[1]}^{\otimes p} \otimes \mathrm{id}_M \otimes \mathrm{id}_{A[1]}^{\otimes p'} \otimes s_A \otimes \mathrm{id}_{A[1]}^{\otimes p''}\big),$$

$\bar\rho|_{A[1]^{\otimes p} \otimes M \otimes A[1]^{\otimes p'} \otimes k.1_{A^+} \otimes A[1]^{\otimes p''}}$ is zero if $p + p' + p'' > 0$, and $\bar\rho(m \otimes 1_{A^+}) = m$. It is lengthy but straightforward to verify that ρ is a morphism of dg bicomodules over $B(A)^+$, *i.e.* a morphism of A_∞-bimodules over A from $M \otimes_A^\infty A^+$ of M. Indeed, the restriction of (MBI(n', n'')) for ρ to $A[1]^{\otimes n'} \otimes M \otimes A[1]^{\otimes i} \otimes A \otimes A[1]^{\otimes n''}$ is equivalent to (BI(n', n'')) for M with $n'' + i + 1$ instead of n'', whereas the restriction of (MBI(n', n'')) for ρ to $A[1]^{\otimes n'} \otimes M \otimes A[1]^{\otimes i} \otimes k.1_{A^+} \otimes A[1]^{\otimes n''}$ is precisely the tautology $^u b_{n',n''+i} = {}^u b_{n',n''+i}$ for the structure maps of M.

Lemma 4.3. *Let* M *be an* A_∞*-bimodule over an* A_∞*-algebra* A*. The map (4.2) is a weak equivalence of dg bicomodules over* $B(A)^+$ *for the model category structure defined in [18], Thm. 2.2.2.2. Equivalently,* ρ *induces a quasi-isomorphism of* A_∞*-bimodules over* A *from* $M \otimes_A^\infty A^+$ *to* M.

Proof. The equivalence is a consequence of [18], Prop. 2.4.1.5. To prove the latter, it suffices to show that the cone $(\mathrm{Co}(\bar\rho_{0,0}), \partial)$ is acyclic, where $\bar\rho_{0,0} = \bar\rho|_{M \otimes T^c A[1] \otimes A}$. In order to do so, consider the linear endomorphism r of $\mathrm{Co}(\bar\rho_{0,0})$ given by sending $m \in M$ to $s(m \otimes 1_{A^+})$, $s(m \otimes [a_1|\cdots|a_n] \otimes a)$ to $(-1)^\epsilon s(m \otimes [a_1|\cdots|a_n|a] \otimes 1_{A^+})$ for $n \in \mathbb{N}_0$, and $s(m \otimes [a_1|\cdots|a_n] \otimes 1_{A^+})$ to zero if $n \in \mathbb{N}$, for all $m \in M$ and $a_1, \dots, a_n, a \in A$, with $\epsilon = |m| + \sum_{j=1}^n |sa_j|$. It is clear that $r \circ \partial + \partial \circ r = \mathrm{id}_{\mathrm{Co}(\bar\rho_{0,0})}$. □

[2] Note that $\mho^m_{\rm cyc} V = \mho^m V \cap (\mho^\bullet V)^\natural$ is a strict graded vector subspace of $(\mho^m V)^\natural$, if $m \geqslant 2$.

Proposition 4.4. *Let A be an A_∞-algebra and M an A_∞-bimodule over A. Then, there is a quasi-isomorphism from $M \otimes^\infty_{A^e} A^+$ to $C_\bullet(A, M)$. If M is further assumed to be right H-unitary, the inclusion of $M \otimes^\infty_{A^e} A$ inside of $M \otimes^\infty_{A^e} A^+$ is a quasi-isomorphism of complexes.*

Proof. Since a morphism of dg bicomodules $F\colon B^u(A, M, A) \to B^u(A, N, A)$ over $B(A)^+$ is a weak equivalence if and only if it is a homotopy equivalence (see [**18**], Prop. 2.4.1.1), the functor $(-)^\natural$ sends weak equivalences to quasi-isomorphisms. Thus, applying $(-)^\natural$ to (4.2), we get that $B^u(A, M, A)^\natural$ and $B^u(A, M \otimes^\infty_A A^+, A)^\natural$ are quasi-isomorphic, and the first statement follows.

To prove the last part, note first that the cokernel of the inclusion of complexes $M \otimes^\infty_{A^e} A \to M \otimes^\infty_{A^e} A^+$ is exactly $M \otimes^\infty_{A^e} k$, where k is the trivial A_∞-bimodule over A. It suffices to show that the latter tensor product is quasi-isomorphic to zero. Note that the underlying complex of $M \otimes^\infty_A k$ is exactly the bar construction of the underlying right A_∞-module of M over A. As the latter is quasi-isomorphic to zero, the zero morphism $M \otimes^\infty_A k \to 0$ is a quasi-isomorphism of A_∞-bimodules. In consequence, the associated bar constructions are weakly equivalent or, equivalently, homotopy equivalent, which in turn implies that $M \otimes^\infty_{A^e} k$ is quasi-isomorphic to zero, since the latter is isomorphic to $B^u(A, M \otimes^\infty_A k, A)^\natural$. □

We define the morphism of dg vector spaces (of cohomological degree -2 and weight 2) given by the following composition

$$\mathrm{sym}\colon A \otimes^\infty_{A^e} A \xrightarrow{\simeq} \big(\mho^2(A[1])\big)^\natural \xrightarrow{\mathrm{id}+\sigma_2} \big(\mho^2_{\mathrm{cyc}}(A[1]), \mathfrak{D}_{B^+_A}\big), \tag{4.3}$$

where the left isomorphism is induced by the map in Lemma 2.11 and s_A, and σ_2 was defined in Fact 4.1. More explicitly, sym sends $a_0[a_1|\cdots|a_n]a_{n+1}[a_{n+2}|\cdots|a_m]$ (for homogeneous $a_0, \ldots, a_m \in A$ and $n, m \in \mathbb{N}_0$ such that $m \geqslant n+1$) to $-(-1)^{\epsilon_{0,n}}$ times

$$\sum_{\ell=1}^{m-n} (-1)^{\varepsilon_\ell} s(a_{n+\ell+1}) \cdots s(a_m) t\big(s(a_0)\big) s(a_1) \cdots s(a_n) t\big(s(a_{n+1})\big) s(a_{n+2}) \cdots s(a_{n+\ell})$$
$$- \sum_{\ell=1}^{n+1} (-1)^{\varepsilon'_\ell} s(a_\ell) \cdots s(a_n) t\big(s(a_{n+1})\big) s(a_{n+2}) \cdots s(a_m) t\big(s(a_0)\big) s(a_1) \cdots s(a_{\ell-1}),$$

where $\varepsilon_\ell = \epsilon_{n+\ell+1,m}\epsilon_{0,n+\ell}$, $\varepsilon'_\ell = \epsilon_{\ell,m}\epsilon_{0,\ell-1}$, and $\epsilon_{j,j'}$ was defined in Lemma 3.3. It is clear that sym is surjective.

We provide the following result relating our previous description of Hochschild homology with the one for cyclic homology in Theorem 3.5. Together with Corollary 3.6, it gives a more natural proof of [**5**], Prop. 6.1.

Proposition 4.5. *Let A be an A_∞-algebra. Then we define the extended mapping $\overline{\mathrm{sym}}\colon A \otimes^\infty_{A^e} A \to (\mho^2_{\mathrm{cyc}}(A[1])/\mathrm{Im}(\mathfrak{q}_{\mathrm{DR}^3})$ as the composition of* sym *given in* (4.3) *and the canonical projection, as well as $\overline{\mathrm{ssym}} = s_{\mho^2_{\mathrm{cyc}}(A[1])/\mathrm{Im}(\mathfrak{q}_{\mathrm{DR}^3}),-2} \circ \overline{\mathrm{sym}}$. If A is H-unitary and* $\mathrm{char}(k) = 0$*, the latter map induces $HH_\bullet(A) \to HC_\bullet(A)$, which is precisely the morphism I from* (2.10) *using the quasi-isomorphism between $C_\bullet(A, A)$ and $A \otimes^\infty_{A^e} A$, and the quasi-isomorphism between*

$$CC_\bullet(A) \quad \text{and} \quad (\mho^2_{\mathrm{cyc}}(A[1])/\mathrm{Im}(\mathfrak{q}_{\mathrm{DR}^3}), \mathfrak{D}_{B^+_A})[-2]$$

explained in Theorem 3.5.

Proof. From Fact 4.1 and (iii), we see that there is a map of dg $B(A)^+$-bicomodules of the form $F\colon B^u(A, A \otimes^\infty_A A, A) \to \mho^2(A[1])$. Hence, the composition of the isomorphism of Lemma 2.11 and $F^\natural$ is a map of complexes from $A \otimes^\infty_{A^e} A$ to $\mho^2(A[1])^\natural$, whose composition with $\mho^2(A[1])^\natural \to \mho^2_{\mathrm{cyc}}(A[1])$ given by Fact 4.1 and (ii), is sym. This proves the first statement. Assume now that A is H-unitary and the field k has zero characteristic. In that case, Fact 2.4 tells us that the underlying right A_∞-module structure of A is H-unitary. We recall that, when using the identification between $CC_\bullet(A)$ and $B(A)^\natural[-1]$ given as the composition of the canonical projection $\bar\pi\colon CC_\bullet(A, A^+) \to C^\lambda_\bullet(A)$ and the morphism $s_{B(A)[1],-2} \circ N$ recalled in the end of Subsection 2.3.3, the map I is induced by the restriction of the previous composition to $C_\bullet(A, A).u^0$. Then, the later statement is a direct consequence of the following commutative diagram

$$\begin{tikzcd}
A \otimes^\infty_{A^e} A \arrow[rrr, "\mathrm{sym}"] \arrow[d, "\mathrm{inc}"] & & & \mho^2_{\mathrm{cyc}}(A[1]) \arrow[d, "\theta"] \\
A \otimes^\infty_{A^e} A^+ \arrow[r, "\rho^\natural"] & C_\bullet(A, A) \arrow[r, "\bar\pi|_{C_\bullet(A,A)}"] & C^\lambda_\bullet(A) \arrow[r, "N"] & B(A)^\natural[1],
\end{tikzcd}$$

where the map $\theta = B_A \circ \bar\psi^{-1} \circ \mathbf{q}_{\mathrm{DR}^2}$ was defined in the proof of Theorem 3.5. □

5. Homology and morphisms of A_∞-bimodules

For later use, we recall that if A is a unitary A_∞-algebra over k, the *derived category* $\mathcal{D}_\infty(A^e)$ is defined as the triangulated category given as the localization of the dg category $\mathrm{Mod}_\infty(A^e)$ of (unitary) A_∞-bimodules by quasi-isomorphisms. This is the standard definition in case A is augmented (see [**18**], Déf. 2.5.2.1), whereas the usual definition in the general nonunitary case is the kernel of the functor

$$k \otimes^\infty_{A^+} (-) \otimes^\infty_{A^+} k\colon \mathcal{D}_\infty\big((A^+)^e\big) \to \mathcal{D}_\infty(k^e) \tag{5.1}$$

(see [**18**], Déf. 4.1.2.1 and 4.2.0.1). As noted in [**18**], Rk. 4.1.3.5, an object M of $\mathcal{D}_\infty((A^+)^e)$ is in the kernel of the previous functor if and only if the underlying complex of its bar construction $B(A, M, A)$ is quasi-isomorphic to zero. In any case, as proved in [**18**], Thm. 4.2.0.4, (see also [**18**], Thm. 4.1.3.1) the latter is equivalent to the definition we provided if A is strictly unitary. Moreover, $\mathcal{D}_\infty(A^e)$ is also triangulated equivalent to the quotient of $\mathrm{Mod}_\infty(A^e)$ by homotopies of (unitary) A_∞-bimodules if A is unitary (see [**18**], Cor. 2.4.2.2, Thm. 4.1.3.1 and 4.2.0.4). We remark that if A is a unitary dg algebra, the faithful functor $\mathrm{Mod}_{dg}(A^e) \to \mathrm{Mod}_\infty(A^e)$ given by inclusion induces an equivalence of triangulated categories $\mathcal{D}_{dg}(A^e) \to \mathcal{D}_\infty(A^e)$, where $\mathcal{D}_{dg}(A^e)$ is the triangulated category given as the localization of the dg category $\mathrm{Mod}_{dg}(A^e)$ of (unitary) dg A-bimodules by quasi-isomorphisms (see [**18**], Lemmes 2.4.2.3 and 4.1.3.8). The previous definitions can be generalized to the nonunitary case, for which the previous results also hold (see [**18**], Déf. 4.1.3.9 and Cor. 4.1.3.1). Moreover, any quasi-isomorphism of A_∞-algebras $f_\bullet\colon A \to B$ induces an equivalence of triangulated categories $\mathcal{D}_\infty(B^e) \to \mathcal{D}_\infty(A^e)$ (see [**18**], Thm. 4.1.2.4).

The reason for studying the complexes considered in the previous subsection is justified by the following result. Combined with Proposition 4.5, it gives a more natural proof of [**5**], Lemma 6.5.

Proposition 5.1. *Let A be an A_∞-algebra and let M and N be two A_∞-bimodules over A. Then, the space of cocycles $Z^0((M \otimes^\infty_{A^e} N)^\#)$ is canonically isomorphic to the space of morphisms $\mathrm{Hom}_{\mathrm{Mod}_\infty((A^+)^e)}(N, M^\#) = \mathrm{Hom}_{\mathrm{Mod}_\infty(A^e)}(N, M^\#)$. Moreover, if $M^\#$ and N are H-unitary, the previous map induces an isomorphism between $H^0((M \otimes^\infty_{A^e} N)^\#)$ and the space of morphisms $\mathrm{Hom}_{\mathcal{D}_\infty(A^e)}(N, M^\#)$.*

Proof. Using the identification (4.1), we can transfer the differential on $(M \otimes^\infty_{A^e} N)^\#$ to endow $(M \otimes B(A)^+ \otimes N \otimes B(A)^+)^\#$ with a structure of dg vector space, which is identified to a structure of dg vector space on $\mathcal{H}om(B(A)^+ \otimes N \otimes B(A)^+, M^\#)$. We leave to the reader the rather long but elementary verification that the kernel of the differential of the latter is precisely the image of

$$\begin{aligned}\mathrm{Hom}_{\mathrm{coMod}_{dg}(B(A)^+)}&\big(B^u(A, N, A), B^u(A, M^\#, A)\big)\\ &= \mathrm{Hom}_{\mathrm{coMod}_{dg}(B^+(A^+)}\big(B^u(A, N, A), B^u(A, M^\#, A)\big)\end{aligned}$$

under the map $F \mapsto (\epsilon_{B(A)^+} \otimes \mathrm{id}_{M^\#} \otimes \epsilon_{B(A)^+}) \circ F$. Moreover, this identification also sends the set of coboundaries $B^0((M \otimes^\infty_{A^e} N)^\#)$ to the equivalence relation generated by homotopies , so it induces an isomorphism between $H^0((M \otimes^\infty_{A^e} N)^\#)$ and the space of morphisms $\mathrm{Hom}_{\mathcal{D}_\infty((A^+)^e)}(N, M^\#)$. Since $M^\#$ and N are H-unitary, they are in the kernel of the functor (5.1) defining the derived category $\mathcal{D}_\infty(A^e)$, so $\mathrm{Hom}_{\mathcal{D}_\infty((A^+)^e)}(N, M^\#)$ coincides with $\mathrm{Hom}_{\mathcal{D}_\infty(A^e)}(N, M^\#)$. □

Note that the map $\ell_d \colon (A \otimes^\infty_{A^e} A)[d] \to A[d] \otimes^\infty_{A^e} A$ sending

$$s_{A \otimes^\infty_{A^e} A, d}(a \otimes \bar{a} \otimes b \otimes \bar{b}) \quad \text{to} \quad s_{A,d}(a) \otimes \bar{a} \otimes b \otimes \bar{b}$$

is an isomorphism of complexes, for $a, b \in A$ and $\bar{a}, \bar{b} \in B(A)^+$. Propositions 4.5 and 5.1 naturally lead to the following notion.

Definition 5.2. Let $(A, b_\bullet)$ be an A_∞-algebra, ${}^uF_{\bullet,\bullet} \colon A \to A^\#[-d]$ a quasi-isomorphism of A_∞-bimodules and $\tilde{f} \in Z^0((A \otimes^\infty_{A^e} A)^\#[-d])$ be the associated cycle. More precisely, $\tilde{f}$ is the image under $\theta^{-1}_{A \otimes^\infty_{A^e} A, -d} \circ \ell_d^\#$ of the element in $Z^0((A[d] \otimes^\infty_{A^e} A)^\#)$ corresponding to $\theta_{A,-d} \circ {}^uF_{\bullet,\bullet}$ under Proposition 5.1. ${}^uF_{\bullet,\bullet}$ is called a *strong homotopy inner product of degree d* if there is a pre-symplectic structure ω of degree d on A such that $\tilde{f} = \theta^{-1}_{A \otimes^\infty_{A^e} A, -d}(\omega \circ s_{(\mho^2_{\mathrm{cyc}}(A[1]))[d], -d} \circ (\mathrm{sym}[d]))$. Note that $\omega \circ \mathfrak{N}_{B^+_A} = 0$, since sym is surjective. By writing down the correspondences, we see that ${}^uF_{\bullet,\bullet}$ is a strong homotopy inner product if and only if it is a symmetric homotopy inner product satisfying

$$\begin{aligned}\big(s_{A^\#[-d],d} &\circ {}^uF_{p+q+1,r}(\bar{a}_1 s a_2 \bar{a}_2, a_3, \bar{a}_3)\big)(a_1)\\ &= (-1)^{|\bar{a}_2| + |sa_2|}\big(s_{A^\#[-d],d} \circ {}^uF_{p,q+r+1}(\bar{a}_1, a_2, \bar{a}_2 s a_3 \bar{a}_3)\big)(a_1)\\ &+ (-1)^{|a_2| + d(|\bar{a}_1| + |sa_2|)}\big(s_{A^\#[-d],d} \circ {}^uF_{q,r+p+1}(\bar{a}_2, a_3, \bar{a}_3 s a_1 \bar{a}_1)\big)(a_2),\end{aligned}$$

for all homogeneous $a_1, a_2, a_3 \in A$, and $\bar{a}_1 \in A[1]^{\otimes p}$, $\bar{a}_2 \in A[1]^{\otimes q}$ and $\bar{a}_3 \in A[1]^{\otimes r}$. This is precisely the notion introduced by Cho and Lee in [**6**] (see Def. 2.2).

The main result in [**4**], namely Thm. 4.1 (see also [**5**], Thm. 3.6), is now a direct consequence of our Propositions 5.1 and 4.4, together with Corollary 3.6.

Theorem 5.3. *Let A be an H-unitary A_∞-algebra having finite dimensional cohomology over a field of characteristic zero. Assume that $A^{\#}$ is an H-unitary A_∞-bimodule (e.g. if A is homologically unitary). Let ${}^uF_{\bullet,\bullet}\colon A \to A^{\#}[-d]$ be a morphism of A_∞-bimodules. Then, ${}^uF_{\bullet,\bullet}$ is a strong homotopy inner product if and only if there is a quasi-isomorphism of A_∞-algebras $G\colon B \to A$ such that B is finite dimensional and ${}^u\tilde{G}^{\#}_{\bullet,\bullet}[-d] \circ {}^uF_{\bullet,\bullet} \circ {}^u\tilde{G}_{\bullet,\bullet}$ is a d-cyclic structure on B, where ${}^u\tilde{G}_{\bullet,\bullet}$ is the associated morphism of A_∞-bimodules over B recalled in the proof of Fact 3.8.*

Proof. Note first that, if $\tilde{f} \in Z^0((A \otimes^{\infty}_{A^e} A)^{\#}[-d])$ is the associated cycle to ${}^uF_{\bullet,\bullet}$ and $[\tilde{f}]$ is its cohomology class using Proposition 5.1, then Proposition 4.5 tells us that $[\tilde{f}]$ is in the image of the map $HC_\bullet(A,A)^{\#}[-d] \to H_\bullet(A,A)^{\#}[-d]$ induced by $I\colon H_\bullet(A,A) \to HC_\bullet(A,A)$. In other words, ${}^uF_{\bullet,\bullet}$ is induced by a symplectic structure ω of degree d on A. By Corollary 3.6, there is a quasi-isomorphism of A_∞-algebras $G\colon B \to A$ such that B is finite dimensional and $\omega \circ \hat{G}$ is constant. Applying again Propositions 4.5 and 5.1, we obtain that ${}^u\tilde{G}^{\#}_{\bullet,\bullet}[-d] \circ {}^uF_{\bullet,\bullet} \circ {}^u\tilde{G}_{\bullet,\bullet}$ is a d-cyclic structure on B. The converse is immediate, since any d-cyclic structure is *a fortiori* a strong homotopy inner product, and they are invariant under quasi-isomorphisms of A_∞-algebras for the stated transformation. □

A more concrete application of the previous result is the following homological description of the cyclic structure of a (homologically) unitary A_∞-algebra. It gives a direct proof of the equivalence between Definition 5.2 and the *Calabi–Yau* condition on a compact A_∞-algebras introduced [**16**], that it is called *compact Calabi–Yau* in [**7**] and *right Calabi–Yau* in [**2**].

Corollary 5.4. *Let A be a (homologically) unitary A_∞-algebra over a field of characteristic zero. Let B be any finite dimensional minimal homologically unitary A_∞-algebra quasi-isomorphic to A. Then, B has a d-cyclic structure if and only if there is a quasi-isomorphism $A \to A^{\#}[-d]$ of A_∞-bimodules whose cohomology class is in the image of the map $HC_\bullet(A,A)^{\#}[-d] \to H_\bullet(A,A)^{\#}[-d]$.*

6. Application to Calabi–Yau algebras

6.1. Basics on Calabi–Yau algebras

In this section we recall the basic material on pseudo-compact Calabi–Yau algebras. For further details, see [**25**], Sections 2–8 and 12–15.

Recall that the category of pseudo-compact dg vector spaces is formed by all dg vector spaces provided with a decreasing filtration of dg vector spaces each of whose terms is of finite codimension and such that the induced topology is complete (see [**9**], IV, 3–4). It is a symmetric monoidal category for the completed tensor product, and the coproducts in the category of pseudo-compact dg vector spaces coincide with the product of the underlying dg vector spaces. The notions of pseudo-compact (resp., augmented) unitary dg algebras and their pseudo-compact (unitary) dg (bi)modules are clear, as well as that of pseudo-compact (resp., coaugmented) counitary dg coalgebras are their pseudo-compact (counitary) dg (bi)comodules. A pseudo-compact augmented dg algebra A is called *local* if the kernel of the augmentation of A is the unique maximal dg ideal A. The category of local pseudo-compact augmented dg algebras is in fact (contravariantly) equivalent to the category of cocomplete coaugmented dg coalgebras, and the category of pseudo-compact (unitary) dg (bi)modules over a local pseudo-compact (unitary) dg algebra A is (contravariantly) equivalent to the category of (counitary) dg (bi)comodules over the corresponding cocomplete coaugmented dg coalgebra $(A')^{\mathrm{op}}$, where X' denotes the continuous dual of X. Indeed, the contravariant functors $X \mapsto X'$, and $Y \mapsto \mathbb{D}(Y)$ given by taking the (ungraded) dual space $\mathrm{Hom}(Y, k)$ provided with the topology obtained by the duals of finite dimensional subspaces, are quasi-inverse to each other. The categories of local pseudo-compact augmented dg algebras and that of pseudo-compact (unitary) dg (bi)modules over a local pseudo-compact augmented dg algebra are endowed with model structures by means of the previous contravariant equivalence (see [**18**], Thm. 1.3.1.2, and [**20**], Section 8.2). This allows to define the *bar construction* $B^+(A)$ of a local pseudo-compact augmented dg algebra A as the pseudo-compact dg coalgebra $\mathbb{D}(\Omega^+(A'))$, as well as the universal twisting cochain $\tau_A \colon B^+(A) \to A$, which is the dual of the couniversal twisting cochain $\tau^{A'} \colon A' \to \Omega^+(A')$ (*cf.* [**14**], Section 2.1).

If A is a local pseudo-compact augmented dg algebra, the *derived category* is $\mathcal{D}_{pcdg}(A)$ is the localization of the category of pseudo-compact (unitary) dg modules over A by weak equivalences. Recall that the minimal thick triangulated subcategory of $\mathcal{D}_{pcdg}(A)$ containing A is denoted by $\mathrm{Perf}_{pcdg}(A)$, and their objects are generically called *perfect*. As usual, A^e denotes the completed tensor product of A and A^{op}, and we identify the category of pseudo-compact dg bimodules over A with the category of pseudo-compact dg modules over A^e. We say that A is *homologically smooth* if A belongs to $\mathrm{Perf}_{pcdg}(A^e)$. We recall that in this article a (triangulated) subcategory $\mathcal{S}$ of a (triangulated) category $\mathcal{T}$ is be definition always full and *strict*, *i.e.* $\mathcal{S}$ contains all objects in $\mathcal{T}$ isomorphic to an object in $\mathcal{S}$. We also recall that a triangulated subcategory $\mathcal{S}$ of a triangulated category is called *thick* if it is closed under direct summands. We will say that a local pseudo-compact augmented dg algebra A is *strongly smooth* if it is homologically smooth and k belongs to $\mathrm{Perf}_{pcdg}(A)$.

For latter use, we recall that A^e is a pseudo-compact A^e-bimodule, with the *outer* and *inner* actions given by

$$a(c \otimes d)b = ac \otimes db, \text{ and } a(c \otimes d)b = (-1)^{|a||b|+|a||c|+|b||d|} cb \otimes ad,$$

respectively. It is clear that the pseudo-compact dg algebras A^e and $(A^e)^{op}$ are isomorphic via $a \otimes b \mapsto (-1)^{|a||b|} b \otimes a$, interchanging the outer and inner actions.

The following definition is due to Ginzburg in [**11**] (see also [**16**, **23**, **24**]).

Definition 6.1. Let $d \in \mathbb{Z}$. A local pseudo-compact augmented dg algebra A is called *d-Calabi–Yau* if it is homologically smooth and there is an isomorphism

$$\mathbb{R}\mathcal{H}om_{A^e}(A, A^e) \to A[-d], \tag{6.1}$$

in $\mathcal{D}_{pcdg}(A^e)$, where A^e is provided with the outer action, and the inner action induces a structure of A^e-module on $\mathbb{R}\mathcal{H}om_{A^e}(A, A^e)$.

Remark 6.2. In the first definition by Ginzburg, the previous isomorphism f was also assumed to satisfy the duality condition $f = \mathbb{R}\mathcal{H}om_{A^e}(f, A^e)[-d]$. However, this is always verified (see [**25**], Prop. 14.1).

We first recall that the *Hochschild homology* $HH_\bullet(A)$ of a local pseudo-compact dg algebra A can be defined as the homology of the complex $A \otimes^{\mathbb{L}}_{A^e} A$ in $\mathcal{D}_{pcdg}(k)$.

This is equivalent to the usual definition involving the bar resolution of A, since the latter is a semifree resolution of A in the category of pseudo-compact dg A-bimodules. Moreover, if A is augmented, then the usual Hochschild complex computing $A \otimes^{\mathbb{L}}_{A^e} A$ is $A \otimes_{\tau_A} B^+(A)$, where $\tau_A \colon B^+(A) \to A$ is the universal twisting cochain. By its very definition, the latter complex is obtained by applying the functor $\mathbb{D}$ to the complex $\Omega^+(A') \otimes_{\tau_{A'}} A'$, computing the Cartier–Doi homology $HH_\bullet(A')$ (see [3,8]). This fact together with a well-known result on resolutions of cofibrant dg algebras implies the following (*cf.* [25], Cor. 15.2).

Proposition 6.3. *Let A be a local pseudo-compact augmented dg algebra. Then,*

$$HH_\bullet(A) \simeq \mathbb{D}\Big(HH_\bullet\big(B^+(A)'\big)\Big).$$

Proof. As the complex $A \otimes_{\tau_A} B^+(A)$ computing the Hochschild homology $HH_\bullet(A)$ is obtained by applying the functor $\mathbb{D}$ to the complex $\Omega^+(A') \otimes_{\tau_{A'}} A'$ computing the Cartier–Doi homology $HH_\bullet(A')$, we see that $HH_\bullet(A) \simeq \mathbb{D}(HH_\bullet(A'))$. It suffices to show that $HH_\bullet(A')$ is isomorphic to the Hochschild homology $HH_\bullet(B^+(A)')$ of the dg algebra $B^+(A)'$, *i.e.* $\Omega^+(A') \otimes_{\tau_{A'}} A'$ and $B^+(A)' \otimes_{\tau_{B^+(A)'}} B^+(B^+(A)')$ are quasi-isomorphic complexes. The result now follows from the fact that, by definition, $B^+(A)' = \Omega^+(A')$, and that [14], Fact 2.5, tells us that the complexes $\Omega^+(A') \otimes_{\tau_{A'}} A'$ and $\Omega^+(A') \otimes_{\tau_{\Omega^+(A')}} B^+(\Omega^+(A'))$ are quasi-isomorphic. □

We recall that, given three local pseudo-compact augmented dg algebras B, C and D, as well as a pseudo-compact dg B^{op}-module M, a pseudo-compact dg $C \otimes B^{op}$-module N, and a pseudo-compact dg $C \otimes D^{op}$-module P, such that M is in the minimal thick triangulated subcategory of $\mathcal{D}_{pcdg}(C^{op})$ containing the right pseudo-compact dg B^{op}-module B^{op}, then the morphism

$$M \otimes^{\mathbb{L}}_B \mathbb{R}\mathcal{H}om_C(N, P) \to \mathbb{R}\mathcal{H}om_C(\mathbb{R}\mathcal{H}om_{B^{op}}(M, N), P)$$

induced by the map sending $m \otimes f$ to the mapping $\phi \mapsto f(\phi(m))$ is an isomorphism in $\mathcal{D}_{pcdg}(D^{op})$. As a consequence, if A is d-Calabi–Yau, we see that

$$\mathbb{R}\mathcal{H}om_{A^e}\big(\mathbb{R}\mathcal{H}om_{A^e}(A, A^e), A[-d]\big) \simeq A \otimes^{\mathbb{L}}_{A^e} \mathbb{R}\mathcal{H}om_{A^e}(A^e, A[-d]) \simeq A \otimes^{\mathbb{L}}_{A^e} A[-d] \tag{6.2}$$

in $\mathcal{D}_{pcdg}(k)$. As a consequence, the isomorphism (6.1) is given by an element $\xi \in A \otimes^{\mathbb{L}}_{A^e} A[-d]$ of homological degree zero, and it is unique up to quasi-isomorphism, *i.e.* it is uniquely determined by a Hochschild homology class $HH_d(A)$, which we are going to denote also by ξ.

The next definitions make sense for both usual dg algebras or for local pseudo-compact augmented ones, taking into account that for the latter we use completed tensor products and the model structure we explained before. Denote $A \otimes^{\mathbb{L}}_{A^e} A$ simply by $C_\bullet(A, A)$. As recalled previously, this complex is endowed with a dg module structure over the dg algebra $\Lambda = k[\epsilon]/(\epsilon^2)$, where ϵ has degree -1 and $d(\epsilon) = 0$. For the following definitions, we recall that k is provided with the trivial action of Λ and the zero differential. We recall that the cyclic homology of a pseudo-compact dg algebra A is given as the homology of the complex $CC_\bullet(A, A) = C_\bullet(A, A) \otimes^{\mathbb{L}}_\Lambda k$. Analogously, we recall that the negative cyclic homology of A is given as the homology of $CC^-_\bullet(A, A) = \mathbb{R}\mathcal{H}om_\Lambda(k, C_\bullet(A, A))$. Even though this is not the usual phrasing of the notions of cyclic and negative cyclic homology, respectively, it can be easily seen that they are equivalent to the usual notions we recalled previously by picking the obvious semifree resolution of the dg Λ-module k. Recall that the map I in (2.10) coincides with

$$C_\bullet(A,A) = C_\bullet(A,A) \otimes k \to C_\bullet(A,A) \otimes^{\mathbb{L}}_{\Lambda} k = CC_\bullet(A,A).$$

The following definition appears in [**7**] as *(strong) smooth Calabi–Yau* and in [**2**] as *left Calabi–Yau*, and it is apparently due to Kontsevich and Vlassopoulos (see [**17**]).

Definition 6.4. Let A be a local pseudo-compact augmented dg algebra and let $d \in \mathbb{Z}$. It is said to be an *almost exact* d-Calabi–Yau algebra, if it is d-Calabi–Yau with isomorphism (6.1) given by a Hochschild homology class $\xi \in HH_d(A)$ that is in the image of the canonical map $h_d \colon HC^-_d(A) \to HH_d(A)$.

Remark 6.5. The isomorphism (6.1) is not part of the structure of a (resp., almost exact) d-Calabi–Yau algebra: we only require its existence. In [**25**], Def. 7.3, the author calls a d-Calabi–Yau algebra A with isomorphism (6.1) given by a homology class $\xi \in HH_d(A)$ *exact* if ξ belongs to the image of the canonical map

$$h_d \circ B_{d-1} \colon HC_{d-1}(A) \to HH_d(A),$$

where this latter map is denoted simply by B (as usual in the literature). It is clear that an exact d-Calabi–Yau algebra is almost exact.

6.2. Calabi–Yau dg algebras and homotopy inner products

We now provide a direct application of our main result, Theorem 5.3, which extends [**25**], Thm. 11.1, but with a completely different proof. The proof is based on some discussions with Greg Stevenson, but the last part is closer to the one sketched in [**7**], Thm. 25.

Theorem 6.6. *Let A be a local pseudo-compact augmented dg algebra over a field of characteristic zero and let A' be the cocomplete coaugmented dg coalgebra given by the continuous dual. Assume A is strongly smooth. Let $E = \Omega^+(A')$ be the cobar construction of A', so $E \simeq \mathbb{R}\mathcal{H}om_A(k,k)$. Then, A is (resp., almost exact) d-Calabi–Yau if and only if E has a (resp., strong) homotopy inner product of degree d.*

Proof. We establish first some basic results. The strong smoothness assumption on A implies that k is in $\mathrm{Perf}_{pcdg}(A^e)$, and that E has finite dimensional cohomology. Indeed, the first statement follows from considering $P \otimes P^{\mathrm{op}} \simeq k \otimes k \simeq k$ in $\mathcal{D}_{pcdg}(A^e)$, where $P \simeq k$ is a finite free resolution in $\mathcal{D}_{pcdg}(A)$, whereas the second follows from $E \simeq \mathbb{R}\mathcal{H}om_A(k,k)$ in $\mathcal{D}_{dg}(k)$, which is a compact object for it is the dual of $k \otimes^{\mathbb{L}}_A k \simeq P \otimes^{\mathbb{L}}_A P^{\mathrm{op}}$, which is compact in $\mathcal{D}_{dg}(k)$. On the other hand consider the functor

$$T = (-) \otimes^{\mathbb{L}}_{A^e} k \colon \mathcal{D}_{pcdg}(A^e) \to \mathcal{D}_{dg}(E^e),$$

where k is regarded as a dg A^e-E^e-bimodule, that is pseudo-compact as left dg A^e-module, as well as

$$T' = (-) \otimes^{\mathbb{L}}_{E^e} k \colon \mathcal{D}_{dg}(E^e) \to \mathcal{D}_{pcdg}(A^e),$$

where k is a dg E^e-A^e-bimodule, that is pseudo-compact as right dg A^e-module. It is clear that T is the composition of the equivalence $\mathcal{D}_{pcdg}(A^e) \to \mathcal{D}_{dg}((A')^e)$ together with $(-)\otimes_{\tau_{A'}} E^e$, where $\mathcal{D}_{dg}((A')^e)$ denotes the coderived category of the cocomplete coaugmented dg coalgebra A', whereas T' is the composition of $(-)\otimes_{\tau_{A'}} (A')^e$ together with the equivalence $\mathcal{D}_{dg}((A')^e) \to \mathcal{D}_{pcdg}(A^e)$. As explained in [**20**], Section 8.4, these functors come from a Quillen adjunction at the level of the categories of pseudo-compact dg modules and dg modules, respectively, that induce a Quillen equivalence between the corresponding derived categories.

The adjunction between the derived tensor product and the derived homomorphism functor tells us that

$$T(X)^{\#} = (k \otimes^{\mathbb{L}}_{A^e} X)^{\#} \simeq \mathbb{R}\mathcal{H}om_{A^e}(X,k), \text{ for all } X \in \mathcal{D}_{pcdg}(A^e),$$

where we have used that

$$\mathbb{R}\mathcal{H}om_k(-,k) = \mathcal{H}om_k(-,k) = (-)^{\#}.$$

As a consequence, if $T(X)$ or $T(X)^{\#}$ has finite dimensional cohomology, then the other has as well and $(T(X)^{\#})^{\#} \simeq T(X)$ in $\mathcal{D}_{dg}(E^e)$. As explained before, since

$$T(A[-d]) = A[-d] \otimes^{\mathbb{L}}_{A^e} k \simeq k \otimes^{\mathbb{L}}_{A} k[-d] \text{ and } k \in \mathrm{Perf}_{pcdg}(A),$$

$T(A[-d])$ is a compact object of $\mathcal{D}_{dg}(E^e)$. As a consequence,

$$T(A[-d]) \simeq k \otimes^{\mathbb{L}}_{A} k[-d] \simeq E^{\#}[-d]$$

in $\mathcal{D}_{dg}(E^e)$. Note also that

$$\begin{aligned} T\big(\mathbb{R}\mathcal{H}om_{A^e}(A,A^e)\big) &= \mathbb{R}\mathcal{H}om_{A^e}(A,A^e) \otimes^{\mathbb{L}}_{A^e} k \simeq \mathbb{R}\mathcal{H}om_{A^e}(A, A^e \otimes^{\mathbb{L}}_{A^e} k) \\ &\simeq \mathbb{R}\mathcal{H}om_{A^e}(A,k) \simeq \mathbb{R}\mathcal{H}om_{A}(k,k) = E \end{aligned}$$

in $\mathcal{D}_{dg}(E^e)$, where we used in the first isomorphism that k belongs to $\mathrm{Perf}_{pcdg}(A^e)$.

By Proposition 6.3, there is an isomorphism $A \otimes^{\mathbb{L}}_{A^e} A \simeq \mathbb{D}(E \otimes^{\mathbb{L}}_{E^e} E)$ in $\mathcal{D}_{pcdg}(k)$, which combined with (6.2) gives an isomorphism

$$\mathbb{R}\mathcal{H}om_{A^e}\big(\mathbb{R}\mathcal{H}om_{A^e}(A,A^e), A[-d]\big) \simeq \mathbb{D}(E \otimes^{\mathbb{L}}_{E^e} E[-d]) \tag{6.3}$$

in $\mathcal{D}_{pcdg}(k)$. Notice that the map (6.3) induces an isomorphism between the space of morphisms $f\colon \mathbb{R}\mathcal{H}om_{A^e}(A,A^e) \to A[-d]$ in $\mathcal{D}_{pcdg}(A^e)$ and that of linear forms $\beta \in (E \otimes^{\mathbb{L}}_{E^e} E[-d])^{\#}$ homogeneous of degree zero, *i.e.* morphisms $g\colon E[d] \to E^{\#}$ in $\mathcal{D}_{dg}(E^e)$. We claim that $f\colon \mathbb{R}\mathcal{H}om_{A^e}(A,A^e) \to A[-d]$ is an isomorphism in $\mathcal{D}_{pcdg}(A^e)$ if and only if its image under the map (6.3) induces an isomorphism $g\colon E \to E^{\#}[-d]$ in $\mathcal{D}_{dg}(E^e)$. By the comments in the previous paragraph, $T(f)$ defines a morphism $E \to E^{\#}[-d]$ in $\mathcal{D}_{dg}(E^e)$. This is precisely the image of f under (6.3). Since T is an equivalence, it is clear that $T(f)$ is an isomorphism if and only if f is so, which proves the claim. Hence, A is d-Calabi–Yau if and only if E has a homotopy inner product of degree d.

Let us now prove that A is exact d-Calabi–Yau if and only if E has a strong homotopy inner product of degree d. Assume that A is d-Calabi–Yau (or, equivalently, that E has a homotopy inner product of degree d). Let $\xi \in HH_d(A)$ be the homology class associated to the isomorphism $f\colon \mathbb{R}\mathcal{H}om_{A^e}(A,A^e) \to A[-d]$ in $\mathcal{D}_{pcdg}(A^e)$, and $\lambda \in HH_d(E)'$ the linear functional on the Hochschild homology associated to the corresponding isomorphism $g\colon E \to E^{\#}[-d]$ in $\mathcal{D}_{dg}(E^e)$ under (6.3). We will show that $\xi = h_d(\chi)$, for some $\chi \in HC_d^-(A)$ if and only if $\lambda\colon C_\bullet(E,E) \to k$ factors through

$$I\colon C_\bullet(E,E) \to C_\bullet(E,E) \otimes^{\mathbb{L}}_\Lambda k = CC_\bullet(E),$$

i.e. λ is in the image of $I^\#$. This follows from the commutative diagram

$$\begin{tikzcd}
CC_\bullet^-(A) \arrow[rr, "h"] \arrow[d, "\wr"'] & & C_\bullet(A,A) \arrow[dd, "\wr"'] \\
\mathbb{R}\mathcal{H}om_\Lambda\big(k, C_\bullet(A,A)\big) \arrow[d, "\wr"'] & & \\
\mathbb{R}\mathcal{H}om_\Lambda\Big(k, \mathbb{R}\mathcal{H}om\big(C_\bullet(E,E),k\big)\Big) \arrow[d, "\wr"'] & & \mathbb{R}\mathcal{H}om\big(C_\bullet(E,E),k\big) \\
\mathbb{R}\mathcal{H}om\big(C_\bullet(E,E)\otimes^{\mathbb{L}}_\Lambda k, k\big) \arrow[rr, "\simeq"] & & \mathbb{R}\mathcal{H}om\big(CC_\bullet(E),k\big) \arrow[u, "I^\#"]
\end{tikzcd}$$

in $\mathcal{D}_{dg}(k)$ (given by forgetting the topologies on the respective pseudo-compact vector spaces), where the upper right vertical map and the middle left vertical map are given by Proposition 6.3, the bottom left vertical map is just adjunction between the derived tensor product and the derived homomorphism space, and the upper left vertical map and the bottom horizontal map are identifications described in the previous subsection. The claimed equivalence follows from Corollary 5.4. □

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