

DERIVED UNIVERSAL MASSEY PRODUCTS

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Abstract

We define an obstruction to the formality of a differential graded algebra over a graded operad defined over a commutative ground ring. This obstruction lives in the derived operadic cohomology of the algebra. Moreover, it determines all operadic Massey products induced on the homology algebra, hence the name of derived universal Massey product.

Introduction

Massey products are a well-known secondary operation on the homology $H_*(A)$ of a differential graded associative \mathbb{k} -algebra A which is defined for any three homogeneous elements $x, y, z \in H_*(A)$ such that $x \cdot y = 0 = y \cdot z$. It yields an element of the quotient

$$\langle x, y, z \rangle \in \frac{H_{|x|+|y|+|z|+1}(A)}{x \cdot H_{|y|+|z|+1}(A) + H_{|x|+|y|+1}(A) \cdot z},$$

often regarded as a subset (coset) of the numerator, and the denominator is called the indeterminacy.

When \mathbb{k} is a field, the universal Massey product of A is the Hochschild cohomology class

$$\gamma_A = \{m_3\} \in HH^{3,-1}(H_*(A), H_*(A))$$

of the first non-trivial piece of a minimal model for A , which is a minimal A -infinity algebra

$$(H_*(A), m_3, m_4, \dots, m_n, \dots)$$

built upon the homology algebra $H_*(A)$ which is ∞ -quasi-isomorphic to A . Here m_3 is an always-defined fully-determined operation

$$m_3: H_*(A) \otimes H_*(A) \otimes H_*(A) \longrightarrow H_*(A)$$

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of degree 1 which determines all Massey products since

$$m_3(x, y, z) \in \langle x, y, z \rangle$$

whenever the Massey product is defined. Despite the choice, the universal Massey product is a quasi-isomorphism invariant of A . Moreover, it is an obstruction to formality.

This class goes back to Kadeishvili's [13]. It was studied in depth by Benson, Krause, and Schwede in [4]. They apply it to finite group cohomology. More precisely, given a finite group G and a graded module X over the Tate cohomology algebra $\widehat{H}^*(G)$, they show that the universal Massey product of the endomorphism algebra of a complete resolution of \mathbb{k} as a trivial G -module defines an obstruction for X to be realizable as a direct summand of an $\widehat{H}^*(G)$ -module of the form $\widehat{H}^*(G, M)$, where M is a G -module. A similar application is obtained in [10] replacing Tate cohomology with the homology of a differential graded algebra. It has also been considered by Kaledin in [14]. He used it to show that, given a sheaf of DG-algebras on a scheme in characteristic zero, the set of points with formal fiber is closed under specialization, see also [17]. We also considered this class in [23] to characterize the existence and uniqueness of enhancements for finite triangulated categories (finite here means equivalent to the category of projective modules over an Artin algebra).

The name *universal Massey product* is inspired by the similar notion of universal Toda bracket due to Baues and his collaborators [2, 3], see also [28].

Dimitrova [6] extended the definition of the universal Massey product to differential graded algebras A over a graded operad \mathcal{O} defined over a ground field \mathbb{k} . Her definition simplifies if \mathcal{O} is a Koszul operad in characteristic zero or a non-symmetric Koszul operad in positive characteristic. In that case, the universal Massey product is an invariant living in operadic cohomology in the sense of [16],

$$\gamma_A^{\mathcal{O}} = \{\varphi\} \in H_{\mathcal{O}}^{2,-1}(H_*(A), H_*(A)),$$

it is an obstruction to formality, the representing cocycle φ is defined from a minimal \mathcal{O}_{∞} -model of A , and any representing cocycle determines all operadic Massey products [24]. This generalizes the associative case (operadic cohomology over the associative operad essentially coincides with Hochschild's up to a degree shift). The Lie version of ordinary Massey products was introduced by Allday [1] and Retah [27].

In this paper we extend the theory of universal Massey products to algebras A over a graded operad \mathcal{O} over a commutative ground ring \mathbb{k} . We work with either of the following assumptions:

- \mathcal{O} is a Koszul operad and $\mathbb{Q} \subset \mathbb{k}$.
- \mathcal{O} is a non-symmetric Koszul operad (no restrictions on \mathbb{k}).

Under these standing assumptions, we consider derived operadic cohomology. Here the word “derived” has the usual homotopical meaning. Indeed, the category of differential graded \mathcal{O} -algebras carries a model structure with quasi-isomorphisms as weak equivalences and surjections as fibrations. Hence derived operadic cohomology can be constructed by means of cofibrant replacements. We define an invariant derived operadic cohomology class in the expected bidegree

$$\gamma_A^{\mathcal{O}} = \{\varphi\} \in dH_{\mathcal{O}}^{2,-1}(H_*(A), H_*(A)).$$

This class coincides with Dimitrova’s for \mathbb{k} a field (there is no need to derive in this case) and, in general, it computes the operadic Massey products defined in [24], so it deserves to be called *derived universal Massey product*. As in previous cases, it is also an obstruction to formality. This is illustrated with explicit examples in §5 and §6 for \mathcal{O} the associative, commutative, and Lie operads, and also the initial operad, which is not an uninteresting example within this theory.

A representing cocycle φ is defined from a derived minimal model in the sense of [19]. These minimal models extend the classical theory over fields to commutative rings. They were first defined by Sagave [29] in the associative case. He also defined a derived universal Massey product living in a cohomology theory which is unfortunately not invariant (it is *dependent* on the choice of resolutions used to define it, see Remark 2.8). Nevertheless, this cohomology is not that far away from derived Hochschild cohomology (also known as Shukla cohomology), and our φ in the associative case is a slight variation (actually a simplification) of Sagave’s cocycle (see Remark 4.2). Therefore, this paper can be regarded as a generalization of Sagave’s derived universal Massey product to algebras over Koszul operads after some fixing.

The results in this paper have many potential applications. We could tackle the realizability problems for Tate cohomology considered in [4] over commutative rings instead of just fields. Moreover, we could try to promote the aforementioned result of [14, 17] to sheaves of operadic algebras on schemes defined over rings. Furthermore, and this is the application we would be most interested in, we could address the study of finite triangulated categories over rings, extending [23]. This would shed some light on the existence of the exotic triangulated categories introduced in [26].

We start this paper with a few elementary facts on derived operadic cohomology (§1). We then split this cohomology for graded algebras (§2) using certain bicomplex-shaped resolutions arising from the Cartan–Eilenberg homotopy theory of algebras in bicomplexes [19, §2]. Derived universal Massey products will later be defined in one of these pieces (§4). In order to achieve this goal, we recall in §3 what we need from the theory of derived minimal models.

In §5 we consider the case where \mathcal{O} is the initial operad $\mathbf{1}$. Algebras over the initial operad are just graded modules (or complexes if they are equipped with a differential). This case is interesting because, by invariance, it sometimes allows to show that the universal Massey product of a differential graded \mathcal{O} -algebra A (with \mathcal{O} any other admissible operad) is non-trivial by just looking at its underlying complex. More precisely, it suffices that some

$$H_{n+1}(A) \hookrightarrow \frac{A_{n+1}}{d(A_{n+2})} \xrightarrow{d} \ker[A_n \xrightarrow{d} A_{n-1}] \twoheadrightarrow H_n(A)$$

is a non-trivial \mathbb{k} -module extension. However, this can only happen if \mathbb{k} has global dimension ≥ 2 .

Finally, in §6, we show examples of \mathcal{O} -algebras (for \mathcal{O} the associative, commutative, and Lie operads) with \mathbb{k} hereditary (i.e. of global dimension ≤ 1) which have non-trivial derived universal Massey product. For this, we establish a connection between derived universal Massey products and certain operadic Massey products, in the sense of [24], associated to torsion elements.

Throughout this paper, \mathbb{k} is a commutative ground ring and $\mathcal{O} = \mathcal{P}(E, R)$ is a graded \mathbb{k} -projective quadratic Koszul operad with generating \mathbb{S} -module E , which is

trivial in arity 0, i.e. $E(0) = 0$, and relations \mathbb{S} -module $R \subset E \circ_{(1)} E$. We refer the reader to [8] for operadic Koszul duality over commutative rings. Nonetheless, we use terminology and notation from [16] since we feel it is more commonly employed. We stress that \mathcal{O} is *graded*, i.e. it has trivial differential, hence differential graded \mathcal{O} -algebras A have graded homology \mathcal{O} -algebras $H_*(A)$. The operad \mathcal{O} is naturally weighted and we denote by $\mathcal{O}^{(n)}$ its weight n part. The weighting is a non-negative grading characterized by the fact that E has weight 1. We have $\mathcal{O}^{(0)} = \mathbb{1}$, the \mathbb{S} -module given by \mathbb{k} concentrated in arity 1 and degree 0, and $1 \in \mathbb{k} = \mathbb{1}(1)_0 \subset \mathcal{O}(1)_0$ is the operadic unit. The \mathbb{S} -module $\mathbb{1}$ is the monoidal unit for the composition product whose (co)monoids are (co)operads, hence it is the initial operad (and the final cooperad). We also have $\mathcal{O}^{(1)} = E$ and $\mathcal{O}^{(2)} = (E \circ_{(1)} E)/R$. We assume that either $\mathbb{Q} \subset \mathbb{k}$ or \mathcal{O} is non-symmetric (or rather the symmetrization of a non-symmetric operad). In this way we can apply all results from [19].

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1. Derived operadic cohomology

In this section we recall operadic cohomology from [16] and consider its derived functor. By [12, 11, 21, 22], under our standing assumptions, the category of differential graded \mathcal{O} -algebras carries a model structure transferred from the category Ch of chain complexes. Fibrations are surjections and weak equivalences are quasi-isomorphisms, respectively.

The *suspension* sX of a chain complex X is defined by shifting one degree up and changing the sign of the differential. Equivalently, s is defined as a functor by the existence of a natural degree 1 isomorphism $s: X \rightarrow sX$.

The *Koszul dual cooperad* $\mathcal{O}^i = \mathcal{C}(sE, s^2R)$ is cogenerated by sE with corelations $s^2R \subset s^2(E \circ_{(1)} E) \cong (sE) \circ_{(1)} (sE)$. It is naturally weighted with $(\mathcal{O}^i)^{(0)} = \mathbb{1}$, $(\mathcal{O}^i)^{(1)} = sE$, and $(\mathcal{O}^i)^{(2)} = s^2R$. We denote by

$$\kappa: \mathcal{O}^i \longrightarrow \mathcal{O}$$

the composite

$$\mathcal{O}^i \twoheadrightarrow sE \xrightarrow{s^{-1}} E \hookrightarrow \mathcal{O},$$

where the first arrow is the projection onto the weight 1 part of \mathcal{O}^i , the second one is the desuspension, and the third one is the inclusion of the weight 1 part of \mathcal{O} .

Given $\mu \in \mathcal{O}^i(r)$ we use the following Sweedler-like formula for its infinitesimal decomposition, like in [16, §10.1.2],

$$\Delta_{(1)}: \mathcal{O}^i \longrightarrow \mathcal{O}^i \circ_{(1)} \mathcal{O}^i, \quad \Delta_{(1)}(\mu) = \sum_{(\mu)} (\mu^{(1)} \circ_l \mu^{(2)}) \cdot \sigma. \tag{1.1}$$

Here $\sigma \in \mathbb{S}_r$ is a permutation, which could be different in each summand, the same as the cooperad elements $\mu^{(1)}$ and $\mu^{(2)}$ and the index l . The summation index (μ) is just a symbol to make it distinguishable from other summations with specific indexing sets or non-specified indexing sets, like e.g. (2.5).

Definition 1.1 ([16, §12.4]). The *operadic cochain complex* $C_{\mathcal{O}}^*(A, M)$ of a differential graded \mathcal{O} -algebra A with coefficients in an A -module M is given by

$$C_{\mathcal{O}}^n(A, M) = \prod_{p \in \mathbb{Z}} \text{Hom}(\mathcal{O}^i(A)_p, M_{p-n}).$$

Here $\mathcal{O}^i(A) = \bigoplus_{r \geq 1} \mathcal{O}^i(r) \otimes_{\mathbb{S}_r} A^{\otimes r}$ and $\mathcal{O}^i(A)$ is its degree p part. The degree $+1$ differential is defined as follows for $\psi \in C_{\mathcal{O}}^n(A, M)$,

$$\begin{aligned} d(\psi)(\mu; x_1, \dots, x_r) &= d(\psi(\mu; x_1, \dots, x_r)) \\ &+ \sum_{(\mu)} (-1)^{\eta_n} \kappa(\mu^{(1)})(x_{\sigma^{-1}(1)}, \dots, \psi(\mu^{(2)}; x_{\sigma^{-1}(l)}, \dots), \dots) \\ &- \sum_{(\mu)} (-1)^{n+\eta_1} \psi(\mu^{(1)}; x_{\sigma^{-1}(1)}, \dots, \kappa(\mu^{(2)})(x_{\sigma^{-1}(l)}, \dots), \dots) \\ &- \sum_{i=1}^r (-1)^{n+\beta} \psi(\mu; x_1, \dots, d(x_i), \dots, x_r). \end{aligned} \quad (1.2)$$

Here

$$\begin{aligned} \eta_n &= \alpha_{\sigma} + |\mu^{(1)}|n + (|\mu^{(2)}| + n) \sum_{m=1}^{l-1} |x_{\sigma^{-1}(m)}|, & \beta &= |\mu| + \sum_{j=1}^{i-1} |x_j|, \\ \alpha_{\sigma} &= \sum_{\substack{s < t \\ \sigma(s) > \sigma(t)}} |x_s||x_t|. \end{aligned} \quad (1.3)$$

The cohomology of $C_{\mathcal{O}}^*(A, M)$ is called *operadic cohomology* and denoted by

$$H_{\mathcal{O}}^*(A, M).$$

The *derived operadic cohomology* of A with coefficients in M is defined as

$$dH_{\mathcal{O}}^*(A, M) = H_{\mathcal{O}}^*(B, M),$$

where B is a cofibrant resolution of A , given by a trivial fibration of \mathcal{O} -algebras $B \rightarrow A$ with cofibrant source which turns M into a B -module. This trivial fibration induces a “non-derived to derived” comparison map

$$H_{\mathcal{O}}^*(A, M) \longrightarrow dH_{\mathcal{O}}^*(A, M).$$

We prove below that the derived operadic cohomology is well defined (Corollary 1.5).

The bivariant functoriality of (derived) operadic cohomology is clear and the comparison map is natural.

Remark 1.2. The operadic cochain complex admits a decreasing filtration by the

weight in \mathcal{O}^i , i.e. $\psi \in F_n C_{\mathcal{O}}(A, M)$ if $\psi(\mu; x_1, \dots, x_r) = 0$ for μ of weight $< n$,

$$\dots \subset F_{n+1} C_{\mathcal{O}}(A, M) \subset F_n C_{\mathcal{O}}(A, M) \subset \dots$$

Since κ vanishes except in weight 1, $\mu^{(2)}$ and $\mu^{(1)}$ in the second and third lines of (1.2) have strictly less weight than μ . This shows that, if $\psi \in F_n$, the second and third lines of (1.2), are cochains in F_{n+1} , and the first and last lines are cochains in F_n . Therefore the previous filtration is indeed a filtration and, moreover,

$$E_0(C_{\mathcal{O}}(A, M)) = \text{Hom}_{\text{Ch}}(\mathcal{O}^i(A), M),$$

the inner hom in Ch, which does not take into account either the algebra structure of A or the module structure of M . The filtration is exhaustive since $F_0 = C_{\mathcal{O}}(A, M)$. It is not bounded above, but it is weakly convergent in the sense of [20, Definition 3.1] since $\bigcap_n F_n C_{d\mathcal{O}}(A, M) = 0$. In addition, it is complete in the sense of [20, Definition 3.8] by the definition of $C_{\mathcal{O}}(A, M)$ as a product, see [20, Lemma 3.10]. Hence, we can apply the following useful lemma to it.

Lemma 1.3 ([20, Theorem 3.9]). *If $f: X \rightarrow Y$ is a morphism of complete and exhaustive filtered complexes which induces an isomorphism on some page E^n of the corresponding spectral sequences, then f is a quasi-isomorphism.*

Lemma 1.4. *Given a quasi-isomorphism $f: B \rightarrow A$ between differential graded \mathcal{O} -algebras with underlying cofibrant complexes and an A -module M , the induced morphism in cohomology $f^*: H_{\mathcal{O}}^*(A, M) \rightarrow H_{\mathcal{O}}^*(B, M)$ is an isomorphism.*

Proof. The induced map on complexes $f^*: C_{\mathcal{O}}^*(A, M) \rightarrow C_{\mathcal{O}}^*(B, M)$ is obviously compatible with the filtration in Remark 1.2. The induced morphism between E_0 terms $\text{Hom}_{\text{Ch}}(\mathcal{O}^i(A), M) \rightarrow \text{Hom}_{\text{Ch}}(\mathcal{O}^i(B), M)$ is $\text{Hom}_{\text{Ch}}(\mathcal{O}^i(f), M)$. By Lemma 1.3, it suffices to prove that this morphism is a quasi-isomorphism. Since all objects in Ch are fibrant, it is enough to show that $\mathcal{O}^i(f): \mathcal{O}^i(B) \rightarrow \mathcal{O}^i(A)$ is a quasi-isomorphism between cofibrant complexes. This will follow from [9, Proposition 11.5.3] since \mathcal{O} is cofibrant as an \mathbb{S} -module in the sense of [9, §11.4]. This is indeed a consequence of our standing assumptions since \mathcal{O} is \mathbb{k} -projective and moreover, each $\mathcal{O}(r)$ is \mathbb{S}_r -projective because \mathcal{O} is the symmetrization of a non-symmetric operad or the ground ring contains \mathbb{Q} and Maschke’s theorem applies. Any arity-wise \mathbb{S}_r -projective \mathbb{S} -module with trivial differential is cofibrant. \square

The same argument shows homotopy invariance in the second variable M , but we will not use it.

Corollary 1.5. *The derived cohomology $dH_{\mathcal{O}}^n(A, M)$ of an \mathcal{O} -algebra A is independent of choices. Moreover, an \mathcal{O} -algebra quasi-isomorphism $B \rightarrow A$ induces an isomorphism in derived cohomology $f^*: dH_{\mathcal{O}}^*(A, M) \cong dH_{\mathcal{O}}^*(B, M)$ for any A -module M . Furthermore, if an \mathcal{O} -algebra B has underlying cofibrant complex then the comparison map is an isomorphism $H_{\mathcal{O}}^*(B, M) \cong dH_{\mathcal{O}}^*(B, M)$. In particular, combining both, for any A , a quasi-isomorphism $B \rightarrow A$ whose source has underlying cofibrant complex induces an isomorphism $dH_{\mathcal{O}}^*(A, M) \cong H_{\mathcal{O}}^*(B, M)$.*

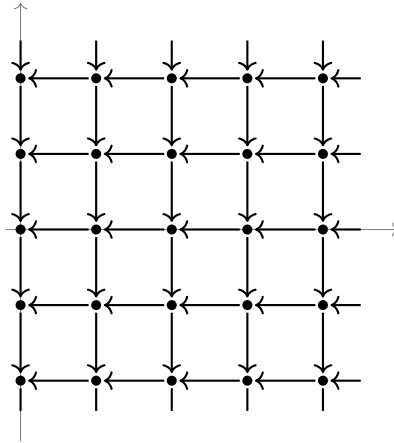
2. Cohomology of graded algebras

In this section we concentrate on the derived operadic cohomology of *graded* \mathcal{O} -algebras and modules, i.e. with trivial differential. We show how to compute it from bicomplex-shaped resolutions and then we split it by introducing an extra grading.

We consider *bicomplexes* X concentrated in the right half plane, i.e. $X_{p,q} = 0$ for $p < 0$, with anti-commuting *horizontal* and *vertical differentials* d_1 and d_0 , respectively,

$$d_1: X_{p,q} \longrightarrow X_{p-1,q}, \quad d_0: X_{p,q} \longrightarrow X_{p,q-1}, \quad d_1^2 = 0 = d_0^2, \quad d_0d_1 + d_1d_0 = 0.$$

They look like



We will not distinguish between a bicomplex and its total complex, i.e. we regard a bicomplex X as a complex equipped with a splitting,

$$X_n = \bigoplus_{n=p+q} X_{p,q}, \tag{2.1}$$

concentrated in horizontal degrees $p \geq 0$, whose differential d decomposes as $d = d_0 + d_1$ with d_0 and d_1 like above. Here p is called *horizontal degree*, q is the *vertical degree*, the pair (p, q) is the *bidegree*, and $n = p + q$ is the *total degree*. *Morphisms* of bicomplexes are chain maps preserving bidegrees. An E^2 -*equivalence* is a morphism which induces an isomorphism between the E^2 pages of the spectral sequences associated to the increasing filtration by the horizontal degree (the first spectral sequence in the sense of [20, Theorem 2.15]) of the source and target bicomplexes. The category bCh of bicomplexes is closed symmetric monoidal with the usual tensor product of complexes and the obvious assignation of horizontal and vertical degrees, compare [25, Definition 2.7]. The symmetry constraint uses the Koszul sign rule with respect to the total degree. We can regard complexes as bicomplexes concentrated in horizontal degree 0. This is compatible with the monoidal structures. In this way, we can consider \mathcal{O} -algebras in bicomplexes, that we call *bicomplex \mathcal{O} -algebras*, see [19, §3]. A bicomplex \mathcal{O} -algebra A is just a bicomplex equipped with an \mathcal{O} -algebra structure whose structure maps decompose as

$$\mathcal{O}(r)_s \otimes A_{p_1, q_1} \otimes \cdots \otimes A_{p_r, q_r} \longrightarrow A_{p_1 + \cdots + p_r, s + q_1 + \cdots + q_r}.$$

Definition 2.1. A bicomplex \mathcal{O} -algebra is *minimal* if it has trivial vertical differential, $d_0 = 0$. A *horizontal resolution* of a graded \mathcal{O} -algebra A consists of a \mathbb{k} -projective minimal bicomplex \mathcal{O} -algebra A_H equipped with an E^2 -equivalence $\rho: A_H \rightarrow A$.

Remark 2.2. In this case, being an E^2 -equivalence is the same as saying that ρ exhibits A as the horizontal homology of A_H . Equivalently, each horizontal complex $(A_H)_{*,q}$ is a resolution of A_q via ρ .

The underlying complex of a minimal bicomplex \mathcal{O} -algebra is a direct sum of bounded below \mathbb{k} -projective complexes. Such complexes are cofibrant. Therefore, the following result is a direct consequence of Corollary 1.5.

Corollary 2.3. *If A is a graded \mathcal{O} -algebra, M is a graded A -module, and in addition $\rho: A_H \rightarrow A$ is a horizontal resolution, then this map induces an isomorphism*

$$dH_{\mathcal{O}}^n(A, M) \cong H_{\mathcal{O}}^n(A_H, M).$$

We now establish the existence of horizontal resolutions, even a preferred one.

Proposition 2.4. *Any graded \mathcal{O} -algebra A has a horizontal resolution $\rho: A'_H \rightarrow A$ such that, if $\rho: A_H \rightarrow A$ is another horizontal resolution, then there exists at least one E^2 -equivalence $g: A'_H \rightarrow A_H$ such that $\rho g = \rho'$. Moreover, if $h: A \rightarrow B$ is a graded \mathcal{O} -algebra morphism and $\rho: B_H \rightarrow B$ is a horizontal resolution then there exists a bicomplex \mathcal{O} -algebra morphism $g: A'_H \rightarrow B_H$ such that $\rho g = h\rho'$.*

Proof. The category of bicomplex \mathcal{O} -algebras can be endowed with the so-called Cartan–Eilenberg semi-model structure [19, §2]. A cofibrant replacement $\tilde{A} \rightarrow A$ in this semi-model category is an E^2 -equivalence. The induced morphism between E^1 terms is another E^2 -equivalence $\rho': A'_H = H_*^v(\tilde{A}) \rightarrow A$. The source is the vertical homology of \tilde{A} , which is a minimal bicomplex \mathcal{O} -algebra by construction. Moreover, it is \mathbb{k} -projective by [19, Proposition 2.4] and [25, Theorem 4.1]. Hence ρ' is a horizontal resolution.

If ρ is another horizontal resolution then ρ is a trivial fibration in the Cartan–Eilenberg semi-model structure by [25, Theorem 4.1] and Remark 2.2. Therefore, there is a lift

$$\begin{array}{ccc} & & A_H \\ & \nearrow \tilde{g} & \downarrow \rho \\ \tilde{A} & \longrightarrow & A \end{array}$$

and we can take g to be the map induced by \tilde{g} in vertical homology. The final part of the statement is a slight generalization of this. □

Using horizontal resolutions, we can split the derived operadic cohomology of a graded \mathcal{O} -algebra with coefficients in a graded module. In the non-derived case this splitting is straightforward.

Proposition 2.5. *Given a graded \mathcal{O} -algebra A , a graded A -module M , and a horizontal resolution $A_H \rightarrow A$, the operadic cochain complex $C_{\mathcal{O}}^*(A_H, M)$ splits as follows*

$$C_{\mathcal{O}}^*(A_H, M) = \prod_{t \in \mathbb{Z}} s^{-t} C_{\mathcal{O}}^{*,t}(A_H, M).$$

Here $C_{\mathcal{O}}^{*,t}(A_H, M)$ is the cochain complex defined as

$$C_{\mathcal{O}}^{w,t}(A_H, M) = \bigoplus_{u=0}^w \prod_{p \in \mathbb{Z}} \text{Hom}((\mathcal{O}^i)^{(u)}(A_H)_{w-u, p-w+u}, M_{p-w-t}).$$

Here $(\mathcal{O}^i)^{(u)}(A_H) = \bigoplus_{r \geq 1} (\mathcal{O}^i)^{(u)}(r) \otimes_{\mathbb{S}_r} (A_H)^{\otimes r}$ is bigraded because A_H is and we regard \mathcal{O}^i as concentrated in horizontal degree 0. The differential of $\psi = (\psi_0, \dots, \psi_w) \in C_{\mathcal{O}}^{w,t}(A_H, M)$ is given by

$$\begin{aligned} d(\psi)_u(\mu; x_1, \dots, x_r) = & \sum_{(\mu)} (-1)^{t+\eta w+t} \kappa(\mu^{(1)})(x_{\sigma^{-1}(1)}, \dots, \psi_{u-1}(\mu^{(2)}; x_{\sigma^{-1}(l)}, \dots), \dots) \\ & - \sum_{(\mu)} (-1)^{w+\eta_1} \psi_{u-1}(\mu^{(1)}; x_{\sigma^{-1}(1)}, \dots, \kappa(\mu^{(2)})(x_{\sigma^{-1}(l)}, \dots), \dots) \\ & - \sum_{i=1}^r (-1)^{w+\beta} \psi_u(\mu; x_1, \dots, d_1(x_i), \dots, x_r) \end{aligned} \quad (2.2)$$

for $0 \leq u \leq w+1$, using the convention $\psi_{-1} = \psi_{w+1} = 0$. The constants β and η are in (1.3). Moreover, the induced cohomological splitting

$$dH_{\mathcal{O}}^n(A, M) = \prod_{w+t=n} dH_{\mathcal{O}}^{w,t}(A, M)$$

is independent of the choice of horizontal resolution and is natural with respect to graded \mathcal{O} -algebra morphisms and module morphisms.

Proof. From the very definition of operadic cochain complex we have that

$$C_{\mathcal{O}}^n(A_H, M) = \prod_{u \geq 0} \prod_{s \geq 0} \prod_{p \in \mathbb{Z}} \text{Hom}((\mathcal{O}^i)^{(u)}(A_H)_{s, p-s}, M_{p-n}).$$

Here we use the weight grading of \mathcal{O}^i ($u \geq 0$) and the bigrading of $(\mathcal{O}^i)^{(u)}(A_H)$ in the statement, which is concentrated in non-negative horizontal degrees ($s \geq 0$). If we make the change of variables $s = w - u$ then $u \leq w$ since $s \geq 0$, hence

$$C_{\mathcal{O}}^n(A_H, M) = \prod_{w \geq 0} \prod_{u=0}^w \prod_{p \in \mathbb{Z}} \text{Hom}((\mathcal{O}^i)^{(u)}(A_H)_{w-u, p-w+u}, M_{p-n}).$$

Here, the middle product is actually direct a sum since it has finitely many factors. Calling $t = n - w$, we have that

$$C_{\mathcal{O}}^n(A_H, M) = \prod_{w+t=n} C_{\mathcal{O}}^{w,t}(A_H, M).$$

Note that $C_{\mathcal{O}}^{w,t}(A_H, M) = 0$ for $w < 0$ because the direct sum is empty in this case.

We now check that the operadic complex differential (co)restricts to

$$d: C_{\mathcal{O}}^{w,t}(A_H, M) \longrightarrow C_{\mathcal{O}}^{w+1,t}(A_H, M).$$

This is a consequence of the following observations about formula (1.2):

- The first term vanishes because M has trivial differential.
- In the second term, the weight of $\mu^{(2)}$ is one less than the weight of μ because κ vanishes in weight $\neq 1$.
- The same happens in the third term with $\mu^{(1)}$.
- In the last term, d is the differential of A_H , which is just $d = d_1$ by minimality. Hence, d reduces the horizontal degree of x_s by one.

This proves the chain complex splitting of $C_{\mathcal{O}}^*(A_H, M)$ in the statement and the formula (2.2) for the differential of $C_{\mathcal{O}}^{*,t}(A_H, M)$.

The splitting is well defined in derived cohomology because different splittings associated to different horizontal resolutions compare well to the preferred horizontal resolution. Here we use the E^2 -equivalence in Proposition 2.4. Moreover, given a graded \mathcal{O} -algebra morphism $f: B \rightarrow A$ the induced map in derived operadic cohomology $f^*: dH_{\mathcal{O}}^n(A, M) \rightarrow dH_{\mathcal{O}}^n(B, M)$ can be computed by using a compatible map between horizontal resolutions $g: B_H \rightarrow A_H$ as in Proposition 2.4. Since this map is bigraded, f^* preserves the splittings. \square

We will use several times the following instances of (2.2).

Lemma 2.6. *Let $\psi \in C_{\mathcal{O}}^{w,t}(A_H, M)$. Recall that $(\mathcal{O}^i)^{(u)}$ is $\mathbf{1}$, sE , and s^2R for $u = 0, 1, 2$, respectively. First,*

$$d(\psi)_u(1; x) = (-1)^{w+1} \psi_u(1; d_1(x)). \tag{2.3}$$

Given $\mu \in E(r)$,

$$\begin{aligned} d(\psi)_u(s\mu; x_1, \dots, x_r) &= \sum_{i=1}^r (-1)^{t+(\beta+1)(w+t)} \mu(x_1, \dots, \psi_{u-1}(1; x_i), \dots, x_r) \\ &\quad - (-1)^w \psi_{u-1}(1; \mu(x_1, \dots, x_r)) + \sum_{i=1}^r (-1)^{w+\beta} \psi_u(s\mu; x_1, \dots, d_1(x_i), \dots, x_r), \end{aligned} \tag{2.4}$$

where β is as in (1.3) Given an arbitrary element in $(\mathcal{O}^i)^{(2)}$, expressed with a Sweedler-like notation as in (1.1) (we do not use any summation index for this finite sum)

$$\Gamma = \sum (\mu^{(1)} \circ_l \mu^{(2)}) \cdot \sigma \in R(r) \subset (E \circ_{(1)} E)(r), \tag{2.5}$$

$$\begin{aligned} d(\psi)_u(s^2\Gamma; x_1, \dots, x_r) &= \sum (-1)^{\theta_{w,t}} \mu^{(1)}(x_{\sigma^{-1}(1)}, \dots, \psi_{u-1}(s\mu^{(2)}; x_{\sigma^{-1}(l)}, \dots), \dots) \\ &\quad + \sum (-1)^{w+\delta} \psi_{u-1}(s\mu^{(1)}; x_{\sigma^{-1}(1)}, \dots, \mu^{(2)}(x_{\sigma^{-1}(l)}, \dots), \dots) \\ &\quad - \sum_{i=1}^r (-1)^{w+\omega} \psi_u(s^2\Gamma; x_1, \dots, d_1(x_i), \dots, x_r). \end{aligned} \tag{2.6}$$

Here,

$$\begin{aligned}\omega &= |\Gamma| + \sum_{j=1}^{i-1} |x_j|, & \delta &= \alpha_\sigma + |\mu^{(2)}| \sum_{i=1}^{l-1} |x_{\sigma^{-1}(i)}|, \\ \theta_{w,t} &= \alpha_\sigma + w + (|\mu^{(1)}|)(w+t+1) + (|\mu^{(2)}| + 1 + w+t) \sum_{i=1}^{l-1} |x_{\sigma^{-1}(i)}|,\end{aligned}\tag{2.7}$$

with α_σ as in (1.3).

Proof. Each of the three formulas in the statement follows by substituting in (2.2) the corresponding Sweedler-like formula for the infinitesimal decomposition in \mathcal{O}^i :

$$\begin{aligned}\Delta_{(1)}(1) &= 1 \circ_1 1, \\ \Delta_{(1)}(s\mu) &= 1 \circ_1 (s\mu) + \sum_{i=1}^r (s\mu) \circ_i 1, \\ \Delta_{(1)}(s^2\Gamma) &= 1 \circ_1 (s^2\Gamma) + \sum_{i=1}^r (s^2\Gamma) \circ_i 1 + \sum (-1)^{|\mu^{(1)}|} ((s\mu^{(1)}) \circ_l (s\mu^{(2)})) \cdot \sigma.\end{aligned}\tag{2.8}$$

□

Example 2.7. The bigraded derived operadic cohomology of algebras and modules with trivial differential can be computed as follows for the classical operads, compare [16, 12.4.1].

1. For differential graded associative algebras, operadic cohomology in positive degrees coincides with Hochschild's up to a shift. Moreover,

$$C_{\mathcal{A}}^{w,t}(A_H, M) = \bigoplus_{u=0}^w \prod_{\substack{p \in \mathbb{Z} \\ i_0 + \dots + i_u + u = w \\ j_0 + \dots + j_u = p - w}} \text{Hom}((A_H)_{i_0, j_0} \otimes \dots \otimes (A_H)_{i_u, j_u}, M_{p-w-t})$$

and the differential of $\psi = (\psi_0, \dots, \psi_w) \in C_{\mathcal{A}}^{w,t}(A_H, M)$ is given by the following formula for $0 \leq u \leq w+1$,

$$\begin{aligned}d(\psi)_u(x_0, \dots, x_u) &= (-1)^{w+(w+t+u-1)|x_0|} x_0 \psi_{u-1}(x_1, \dots, x_u) \\ &\quad - \sum_{i=0}^{u-1} (-1)^{w+i} \psi_{u-1}(x_0, \dots, x_i x_{i+1}, \dots, x_u) \\ &\quad - (-1)^{w+u} \psi_{u-1}(x_0, \dots, x_{u-1}) x_u \\ &\quad - \sum_{i=0}^u (-1)^{w+u+\sum_{j=0}^{s-1} |x_j|} \psi_u(x_0, \dots, d_1(x_i), \dots, x_u).\end{aligned}$$

Here we understand that $\psi_{-1} = 0 = \psi_{w+1}$.

2. In the differential graded commutative case, we have Harrison's instead of Hochschild cohomology and

$$C_{\mathcal{C}}^{w,t}(A_H, M) \subset C_{\mathcal{A}}^{w,t}(A_H, M)$$

is the subcomplex of cochains $\psi = (\psi_0, \dots, \psi_w)$ such that each ψ_u vanishes on

shuffles in the sense of [19, Example 5.18 (2)].

3. In the case of differential graded Lie algebras, we obtain the Chevalley–Eilenberg cohomology. More precisely,

$$C_C^{w,t}(A_H, M) \subset \bigoplus_{u=0}^w \prod_{\substack{p \in \mathbb{Z} \\ i_0 + \dots + i_u + u = w \\ j_0 + \dots + j_u = p - w}} \text{Hom}((A_H)_{i_0, j_0} \otimes \dots \otimes (A_H)_{i_u, j_u}, M_{p-w-t})$$

consists of the elements $\psi = (\psi_0, \dots, \psi_w)$ such that each ψ_u is skew-symmetric in the sense of [19, Example 5.18 (3)] and, given $0 \leq u \leq w + 1$,

$$\begin{aligned} d(\psi)_u(x_0, \dots, x_u) &= - \sum_{i=1}^u (-1)^{w+i+|x_i|} \prod_{k=i+1}^u |x_k| [\psi_{u-1}(\dots, \widehat{x}_i, \dots), x_i] \\ &\quad - \sum_{0 \leq i < j \leq u} (-1)^{z(i,j)} \psi_{u-1}([x_i, x_j], \dots, \widehat{x}_i, \dots, \widehat{x}_j, \dots) \\ &\quad - \sum_{i=0}^u (-1)^{w+u+\sum_{j=0}^{s-1} |x_j|} \psi_u(x_0, \dots, d_1(x_i), \dots, x_u) \end{aligned}$$

with the convention $\psi_{-1} = 0 = \psi_{w+1}$, where

$$z(i, j) = w + i + j - 1 + (|x_i| + |x_j|) \sum_{k=0}^{i-1} |x_k| + |x_j| \sum_{k=i+1}^{j-1} |x_k|.$$

Remark 2.8. Sagave’s definition of derived cohomology for $\mathcal{O} = \mathcal{A}$ the associative operad does not coincide with ours. Roughly speaking, the reason is that he uses horizontal resolutions as coefficients, rather than graded modules.

More precisely, Sagave uses a Hochschild-like cohomology $HH^{a,t}(E, F)$ for bicomplex \mathcal{A} -algebras E with coefficients in E -bimodules F also in bicomplexes. Both E and F are required to have trivial vertical differential. He claims that his definition is invariant in both variables under E^2 -equivalences $E \rightarrow F$ between \mathbb{k} -projective minimal bicomplex \mathcal{A} -algebras. Unfortunately this is not true, as we now show with an example.

Sagave’s $HH^{0,0}(E, E)$ coincides with the central elements of E in bidegree $(0, 0)$. The (horizontal) differential of E does not play any role in horizontal degree 0. Hence, we can take the unital bicomplex \mathcal{A} -algebras $E = \mathbb{Z}$ concentrated in bidegree $(0, 0)$ and $F = \mathbb{Z}[x, y]/(y^2)$ with x of bidegree $(0, 0)$, y of bidegree $(1, 0)$, and $d_1(y) = x$. Clearly, the unital map $E \rightarrow F$ is an E^2 -equivalence. However $HH^{0,0}(E, E) = \mathbb{Z}$ but $HH^{0,0}(F, F) = \mathbb{Z}[x]$. Sagave’s cohomology is not invariant in higher horizontal degrees either, however computing explicit examples becomes more complicated and it is not really worth to include them here.

3. Derived homotopy algebras

In this section we briefly recall from [19] the theory of derived homotopy algebras. They are filtered homotopy algebras which allow the construction of minimal models over rings, even for algebras with torsion homology. This theory was initiated by

Sagave [29] in the associative case, which attracted some attention [15, 5], and then continued in [18, 19] over other operads.

Recall that a *homotopy \mathcal{O} -algebra* is an algebra over the operad $\mathcal{O}_\infty = \Omega\mathcal{O}^i$, the cobar construction of the Koszul dual cooperad \mathcal{O}^i of \mathcal{O} .

Remark 3.1. The structure of a homotopy \mathcal{O} -algebra can be described in terms of the Koszul dual cooperad \mathcal{O}^i . An \mathcal{O}_∞ -algebra is a complex A equipped with structure morphisms

$$\begin{aligned} \mathcal{O}^i(r)_{n_0} \otimes A_{n_1} \otimes \cdots \otimes A_{n_r} &\longrightarrow A_{n_0-1+n_1+\cdots+n_r}, \\ \mu \otimes x_1 \otimes \cdots \otimes x_r &\mapsto \mu(x_1, \dots, x_r), \end{aligned}$$

satisfying $1(x) = 0$, where $1 \in \mathbb{k} = \mathbf{1}(1)_0 = (\mathcal{O}^i)^{(0)}(1)_0 \subset \mathcal{O}^i(1)_0$, and certain formulas related to the infinitesimal decomposition of \mathcal{O}^i , see [16, §10.1.2] or [19, Remark 5.2].

Derived homotopy \mathcal{O} -algebras are split filtered homotopy \mathcal{O} -algebras. Let us make this structure precise.

Definition 3.2. A *derived \mathcal{O}_∞ -algebra* or *derived homotopy \mathcal{O} -algebra* A is an \mathcal{O}_∞ -algebra such that:

1. For each $n \in \mathbb{Z}$, the degree n module splits as

$$A_n = \bigoplus_{p+q=n} A_{p,q}$$

with $A_{p,q} = 0$ if $p < 0$, like bicomplexes, see §2.

2. If we denote

$$F_m A_n = \bigoplus_{\substack{p+q=n \\ p \leq m}} A_{p,q},$$

the differential of A satisfies $d(F_m A_n) \subset F_m A_{n-1}$, so A is a *filtered complex*.

3. The \mathcal{O}_∞ -algebra structure is compatible with the filtration, i.e. the structure maps (co)restrict to

$$\mathcal{O}^i(r)_{n_0} \otimes F_{m_1} A_{n_1} \otimes \cdots \otimes F_{m_r} A_{n_r} \longrightarrow F_{m_1+\cdots+m_r} A_{n_0-1+n_1+\cdots+n_r}.$$

Remark 3.3. This is not the original definition in [19, Definition 5.1], but an equivalent one from [19, Corollary 5.21]. The differential and the structure maps of A are therefore determined by pieces, $i \geq 0$,

$$\begin{aligned} d_i : A_{p,q} &\longrightarrow A_{p-i,q-1+i}; \\ \mathcal{O}^i(r)_s \otimes A_{p_1,q_1} \otimes \cdots \otimes A_{p_r,q_r} &\longrightarrow A_{p_1+\cdots+p_r-i,s-1+q_1+\cdots+q_r+i}, \\ \mu \otimes x_1 \otimes \cdots \otimes x_r &\mapsto \mu_i(x_1, \dots, x_r). \end{aligned}$$

The derived homotopy \mathcal{O} -algebra equations can be rewritten in terms of these pieces, see [19, Proposition 5.10]. One of them is

$$1_i(x) = 0, \quad i \geq 0, \tag{3.1}$$

where 1 is as in Remark 3.1. See Lemma 3.14 below for some of these equations.

Remark 3.4. For the three classical operads (associative, commutative, and Lie), derived homotopy algebras look similar, since they have operations of the same bidegrees, but they satisfy quite different laws, see [19, Example 5.18] for full details.

A derived homotopy associative algebra A has operations d_i and $m_{i,r}: A^{\otimes r} \rightarrow A$ of bidegree $(-i, r - 2 + i)$ for $r \geq 2$ and $i \geq 0$. A derived homotopy commutative algebra too, and the $m_{i,r}$ vanish on shuffles. In the case of derived homotopy Lie algebras, these operations are skew-symmetric and denoted by $\ell_{i,r}: A^{\otimes r} \rightarrow A$.

If $\mathcal{O} = \mathbb{1}$ is the initial operad, an \mathcal{O} -algebra is just a complex and a derived homotopy \mathcal{O} -algebra is known as a *twisted complex*. Its only structure maps are the d_i , $i \geq 0$.

Remark 3.5. Usual morphisms between \mathcal{O}_∞ -algebras are not enough for many purposes, hence this category is often enlarged by means of the so-called ∞ -morphisms. An ∞ -morphism between \mathcal{O}_∞ -algebras $f: A \rightsquigarrow B$ is given by structure morphisms

$$\begin{aligned} \mathcal{O}^i(r)_{n_0} \otimes A_{n_1} \otimes \cdots \otimes A_{n_r} &\longrightarrow B_{n_0+n_1+\cdots+n_r}, \\ \mu \otimes x_1 \otimes \cdots \otimes x_r &\mapsto f(\mu)(x_1, \dots, x_r), \end{aligned}$$

satisfying some equations connected with the decomposition and the infinitesimal decomposition of \mathcal{O}^i , see [16, §10.2.3] and [19, Remark 5.23].

The *underlying morphism* of an ∞ -morphism $f: A \rightsquigarrow B$ refers to the chain map $f(1): A \rightarrow B$, where $1 \in \mathcal{O}^i$ is the element explained in Remark 3.1.

Definition 3.6. Given two derived \mathcal{O}_∞ -algebras A and B , a *derived ∞ -morphism* $f: A \rightsquigarrow B$ is an ∞ -morphism whose structure maps (co)restrict to

$$\mathcal{O}^i(r)_{n_0} \otimes F_{m_1} A_{n_1} \otimes \cdots \otimes F_{m_r} A_{n_r} \longrightarrow F_{m_1+\cdots+m_r} A_{n_0+n_1+\cdots+n_r}.$$

Remark 3.7. The original definition is in [19, Definition 5.24] and this equivalent description corresponds to [19, Corollary 5.27]. The structure maps of $f: A \rightsquigarrow B$ are determined by pieces, $i \geq 0$,

$$\begin{aligned} \mathcal{O}^i(r)_s \otimes A_{p_1, q_1} \otimes \cdots \otimes A_{p_r, q_r} &\longrightarrow B_{p_1+\cdots+p_r-i, s+q_1+\cdots+q_r+i}, \\ \mu \otimes x_1 \otimes \cdots \otimes x_r &\mapsto f(\mu)_i(x_1, \dots, x_r). \end{aligned}$$

The derived ∞ -morphism equations can be rewritten in terms of these pieces, see [19, Proposition 5.25]. Some of these equations will be recalled in Lemma 3.15 below.

Remark 3.8. A bicomplex \mathcal{O} -algebra is exactly the same thing as a derived homotopy \mathcal{O} -algebra such that $d_i = 0$ for $i \geq 2$ and

$$\mu_i(x_1, \dots, x_r) = 0$$

if $i \geq 1$ or if $\mu \in \mathcal{O}^i$ has weight $\neq 1$. Given $\mu \in E(r)$, the formula

$$(s\mu)_0(x_1, \dots, x_r) = \mu(x_1, \dots, x_r) \tag{3.2}$$

relates the derived homotopy \mathcal{O} -algebra structure, on the left, and the bicomplex \mathcal{O} -algebra structure, on the right.

A differential graded \mathcal{O} -algebra is the same as a bicomplex \mathcal{O} -algebra concentrated in horizontal degree 0, so $d_1 = 0$ and $d = d_0$.

A bicomplex \mathcal{O} -algebra morphism is the same as a derived ∞ -morphism f such that $f(\mu)_i = 0$ for $i \geq 1$ or μ of weight ≥ 1 , i.e. the only possibly non-trivial part is $f(1)_0$.

Definition 3.9. A derived ∞ -morphism $f: A \rightsquigarrow B$ between derived homotopy \mathcal{O} -algebras is an E^2 -equivalence if the filtration-preserving chain map $f(1)_0: A \rightarrow B$ induces an isomorphism between the E^2 pages of the spectral sequences associated to the filtrations.

A derived homotopy \mathcal{O} -algebra is *minimal* if $d_0 = 0$. If A is a differential graded \mathcal{O} -algebra, a (*derived*) *minimal model* is an E^2 -equivalence $A_M \rightsquigarrow A$ whose source A_M is a minimal and \mathbb{k} -projective derived homotopy \mathcal{O} -algebra.

Under our standing assumptions, minimal models exist, and there is actually a preferred one which allows for comparison.

Theorem 3.10 ([19, Theorem 7.25 and Proposition 7.26]). *Any differential graded \mathcal{O} -algebra A has a minimal model $A'_M \rightsquigarrow A$ such that, if $A_M \rightsquigarrow A$ is another minimal model, then there exists an E^2 -equivalence $A_M \rightsquigarrow A'_M$ which induces the identity in $H_*(A)$ on the E^2 terms of the corresponding spectral sequences. Moreover, when $f: B \rightarrow A$ is an \mathcal{O} -algebra morphism and $B_M \rightsquigarrow B$ is a minimal model then there exists a derived ∞ -morphism $B_M \rightsquigarrow A'_M$ which induces $H_*(f)$ on E^2 terms.*

In [19, Examples 6.6, 6.8, and 6.11] we explicitly compute minimal models for some of the algebras in Examples 6.7, 6.8, and 6.9 below.

Remark 3.11. In a minimal model, d_1 is a bidegree $(-1, 0)$ differential on A_M by minimality. We call it *horizontal differential*. Its homology is the E^2 term of the spectral sequence of A_M , so it is isomorphic to $H_*(A)$ concentrated in horizontal degree 0 since f is an E^2 -equivalence (in particular the complexes defined by d_1 are exact in positive horizontal degrees). This isomorphism is induced by $f(1)_0: A_M \rightarrow A$, which is trivial in positive horizontal degrees for degree reasons and maps horizontal degree 0 elements in A_M to cycles in A by minimality.

Remark 3.12. By [19, Remark 6.3], any minimal derived homotopy \mathcal{O} -algebra has an underlying minimal bicomplex \mathcal{O} -algebra structure given by restricting to $d_0 = 0$, d_1 , and the operations $(s\mu)_0$, $\mu \in E$, i.e. (3.2), see Remark 3.8. Minimality is essential here. Moreover, if $f: A \rightsquigarrow B$ is a derived ∞ -morphism then $f(1)_0: A \rightarrow B$ is a morphism between the underlying bicomplex \mathcal{O} -algebras.

Remark 3.13. Given a differential graded \mathcal{O} -algebra A , any minimal model for it $f: A_M \rightsquigarrow A$ produces a horizontal resolution $\rho: A_H \rightarrow H_*(A)$ of the homology graded \mathcal{O} -algebra of A . Here, A_H is the bicomplex \mathcal{O} -algebra underlying A_M in the sense of Remark 3.12 and ρ is the composite

$$A_H \rightarrow Z_*(A) \twoheadrightarrow H_*(A),$$

where $Z_*(A) \subset A$ are the cycles, the second arrow is the natural projection onto homology, and the first arrow is the corestriction of $f(1)_0: A_M \rightarrow A$, see Remark 3.11.

In particular, if $f: A_M \rightsquigarrow A$ and $f': A'_M \rightsquigarrow A$ are both minimal models while $g: A_M \rightsquigarrow A'_M$ is an E^2 -equivalence inducing the identity in the E^2 terms of the corresponding spectral sequences (it is $H_*(A)$ concentrated in horizontal degree 0 in both cases), then the induced horizontal resolutions satisfy

$$\rho'g(1)_0 = \rho.$$

In the following two lemmas, for a *minimal* derived homotopy \mathcal{O} -algebra, we use the operations in Remarks 3.3 and 3.7, the underlying bicomplex \mathcal{O} -algebra structure in Remark 3.12, and the notation (3.2) for $\mu \in E$. Recall that $(\mathcal{O}^i)^{(u)}$ is $\mathbf{1}, sE, s^2R$ for $u = 0, 1, 2$, respectively.

Lemma 3.14. *The following equations hold within any minimal derived homotopy \mathcal{O} -algebra: $d_0 = 0, d_1^2 = 0,$*

$$d_1d_2 + d_2d_1 = 0. \tag{3.3}$$

Given $\mu \in E(r),$

$$\begin{aligned} 0 &= d_1((s\mu)_1(x_1, \dots, x_r)) + d_2(\mu(x_1, \dots, x_r)) \\ &\quad - \sum_{i=1}^r (-1)^\beta (\mu(x_1, \dots, d_2(x_i), \dots, x_r) + (s\mu)_1(x_1, \dots, d_1(x_i), \dots, x_r)), \end{aligned} \tag{3.4}$$

where β is as in (1.3). Given $\Gamma \in R(r)$ like in (2.5),

$$\begin{aligned} 0 &= d_1((s^2\Gamma)_0(x_1, \dots, x_r)) + \sum_{i=1}^r (-1)^\omega (s^2\Gamma)_0(x_1, \dots, d_1(x_i), \dots, x_r) \\ &\quad - \sum_{(\mu)} (-1)^\delta \left(\mu^{(1)}(x_{\sigma^{-1}(1)}, \dots, (s\mu^{(2)})_1(x_{\sigma^{-1}(l)}, \dots), \dots) \right. \\ &\quad \left. + (s\mu^{(1)})_1(x_{\sigma^{-1}(1)}, \dots, \mu^{(2)}(x_{\sigma^{-1}(l)}, \dots), \dots) \right), \end{aligned} \tag{3.5}$$

where ω and δ been defined in (2.7) With the Sweedler notation in (1.1), we also have

$$0 = \sum_{(\mu)} (-1)^{\eta_1} \mu_0^{(1)}(x_{\sigma^{-1}(1)}, \dots, \mu_0^{(2)}(x_{\sigma^{-1}(l)}, \dots), \dots), \tag{3.6}$$

where η_1 is as in (1.3).

Proof. The proof is straightforward. We just need to take the general formulas in [19, Proposition 5.10] and substitute some specific values, concretely $d_0 = 0$ (the minimality condition), (3.1), and (2.8). \square

Lemma 3.15. *Let $f: A \rightsquigarrow B$ be an ∞ -morphism from a minimal derived homotopy \mathcal{O} -algebra A to a differential graded \mathcal{O} -algebra B . Then*

$$df(1)_0 = 0, \tag{3.7}$$

$$df(1)_1(x) - f(1)_0d_1(x) = 0, \tag{3.8}$$

$$df(1)_2(x) - f(1)_0d_2(x) - f(1)_1d_1(x) = 0. \tag{3.9}$$

Given $\mu \in E(r),$

$$\begin{aligned} df(s\mu)_1(x_1, \dots, x_r) &+ \sum_{i=1}^r (-1)^\beta f(s\mu)_0(x_1, \dots, d_1(x_i), \dots, x_r) \\ &= f(1)_0(s\mu)_1(x_1, \dots, x_r) + f(1)_1\mu(x_1, \dots, x_r) \\ &\quad - \sum_{i=1}^r \mu(f(1)_0(x_1), \dots, f(1)_1(x_i), \dots, f(1)_0(x_r)), \end{aligned} \tag{3.10}$$

where β has been defined in (1.3).

Suppose now that $g: A \rightsquigarrow B$ is an ∞ -morphism between minimal derived homotopy \mathcal{O} -algebras A, B . We have

$$d_1g(1)_1(x) + d_2g(1)_0(x) - g(1)_0d_2(x) - g(1)_1d_1(x) = 0. \tag{3.11}$$

Moreover, for $\mu \in E(r)$,

$$\begin{aligned} d_1g(s\mu)_0(x_1, \dots, x_r) + \sum_{i=1}^r (-1)^\beta g(s\mu)_0(x_1, \dots, d_1(x_i), \dots, x_r) = \\ g(1)_0((s\mu)_1(x_1, \dots, x_r)) + g(1)_1(\mu(x_1, \dots, x_r)) \\ - \sum_{i=1}^r \mu(g(1)_0(x_1), \dots, g(1)_1(x_i), \dots, g(1)_0(x_r)) - (s\mu)_1(g(1)_0(x_1), \dots, g(1)_0(x_r)). \end{aligned} \tag{3.12}$$

For $\Gamma \in R(r)$ like in (2.5),

$$\begin{aligned} 0 = g(1)_0((s^2\Gamma)_0(x_1, \dots, x_r)) - (s^2\Gamma)_0(g(1)_0(x_1), \dots, g(1)_0(x_r)) \\ - \sum (-1)^\delta g(s\mu^{(1)})_0(x_{\sigma^{-1}(1)}, \dots, \mu^{(2)}(x_{\sigma^{-1}(l)}, \dots), \dots) \\ - \sum (-1)^\varepsilon \mu^{(1)}(g(1)_0(x_{\sigma^{-1}(1)}), \dots, g(s\mu^{(2)})_0(x_{\sigma^{-1}(l)}, \dots), \dots), \end{aligned} \tag{3.13}$$

where

$$\varepsilon = \alpha_\sigma + |\mu^{(1)}| + (|\mu^{(2)}| + 1) \sum_{i=1}^{l-1} |x_{\sigma^{-1}(i)}|, \tag{3.14}$$

and α_σ and δ have been defined in (1.3) and (2.7), respectively.

Proof. All this follows from [19, Proposition 5.25], (3.1), and the fact that the infinitesimal decomposition $\Delta_{(1)}: \mathcal{O}^i \rightarrow \mathcal{O}^i \circ_{(1)} \mathcal{O}^i$ together with the decomposition $\Delta: \mathcal{O}^i \rightarrow \mathcal{O}^i \circ \mathcal{O}^i$ of the Koszul dual cooperad \mathcal{O}^i satisfy (2.8) and

$$\begin{aligned} \Delta(1) &= (1; 1), \\ \Delta(s\mu) &= (1; s\mu) + (s\mu; 1, \dots, 1), \\ \Delta(s^2\Gamma) &= (1; s^2\Gamma) + (s^2\Gamma; 1, \dots, 1) + \sum (-1)^{|\mu^{(1)}|} (s\mu^{(1)}; 1, \overset{l-1}{\cdot}, 1, s\mu^{(2)}, 1, \dots, 1) \cdot \sigma. \end{aligned}$$

□

4. Derived universal Massey products

We are finally ready to define the derived cohomology class this paper takes its name from.

Definition 4.1. Given an \mathcal{O} -algebra A in Ch , its *derived universal Massey product*

$$\gamma_A^\mathcal{O} = \{\varphi\} \in dH_{\mathcal{O}}^{2,-1}(H_*(A), H_*(A))$$

is the derived operadic cohomology class represented by the cochain φ defined in the following way. Choose a minimal model $f: A_M \rightsquigarrow A$, consider the induced horizontal

resolution $\rho: A_H \rightarrow H_*(A)$ in Remark 3.13, and define the representing cocycle $\varphi = (\varphi_0, \varphi_1, \varphi_2) \in C_{\mathcal{O}}^{2,-1}(A_H, H_*(A))$ by the following formulas,

$$\begin{aligned} \varphi_0(1; x) &= \rho(d_2(x)), \\ \varphi_1(s\mu; x_1, \dots, x_r) &= \rho((s\mu)_1(x_1, \dots, x_r)), \quad \mu \in E(r), \\ \varphi_2(s^2\Gamma; x_1, \dots, x_r) &= \rho((s^2\Gamma)_0(x_1, \dots, x_r)), \quad \Gamma \in R(r). \end{aligned}$$

See Proposition 2.5 for notation and recall that $(\mathcal{O}^i)^{(u)}$ is $\mathbb{1}$, sE , and s^2R for $u = 0, 1, 2$, respectively.

Remark 4.2. With the notation in Example 2.7 and Remark 3.4, the derived universal Massey product of a differential graded associative algebra A with minimal model $f: A_M \rightsquigarrow A$ is represented by the bidegree $(2, -1)$ cochain $(\rho d_2, \rho m_{1,2}, \rho m_{0,3})$. The same formula holds in the commutative case, and also in the Lie case replacing m with ℓ . The case $\mathcal{O} = \mathbb{1}$ will be considered in §5 below.

Notice that the associative derived universal Massey product is indeed defined by the same pieces of a minimal model as in [29, Proposition 5.4], but composed with ρ and living in a different cohomology theory, see Remark 2.8.

The following result shows that the derived universal Massey product is a well-defined natural invariant of differential graded \mathcal{O} -algebras.

Theorem 4.3. *In the setting of Definition 4.1, the following statements hold:*

1. *The cochain φ in is indeed a cocycle.*
2. *Its cohomology class is independent of the choice of minimal model.*
3. *Given an \mathcal{O} -algebra morphism $f: A \rightarrow B$, the morphisms induced by $H_*(f)$ in derived operadic cohomology*

$$dH_{\mathcal{O}}^{2,-1}(H_*(A), H_*(A)) \longrightarrow dH_{\mathcal{O}}^{2,-1}(H_*(A), H_*(B)) \longleftarrow dH_{\mathcal{O}}^{2,-1}(H_*(B), H_*(B))$$

take $\gamma_A^{\mathcal{O}}$ and $\gamma_B^{\mathcal{O}}$ to the same class in the middle.

4. *Given a weight-preserving morphism of operads satisfying our standing assumptions $f: \mathcal{P} \rightarrow \mathcal{O}$, the morphism induced by f on derived operadic cohomology*

$$H_{\mathcal{O}}^{2,-1}(H_*(A), H_*(A)) \longrightarrow H_{\mathcal{P}}^{2,-1}(H_*(A), H_*(A))$$

takes $\gamma_A^{\mathcal{O}}$ to $\gamma_A^{\mathcal{P}}$.

Proof. We start with (1). We will see that $d(\varphi) = 0$ as a consequence of the minimal derived homotopy \mathcal{O} -algebra equations for A_M . By (2.3),

$$d(\varphi)_0(1; x) = -\varphi_0(1; d_1(x)) = -\rho(d_2 d_1(x)) = \rho d_1 d_2(x) = 0.$$

Here we also use (3.3) and the fact that $\rho d_1 = 0$ (see Remark 2.2).

Given $\mu \in E(r)$, using (2.4) and the definition of φ

$$\begin{aligned} d(\varphi)_1(s\mu; x_1, \dots, x_r) &= \sum_{i=1}^r (-1)^{\beta} \underbrace{\mu(x_1, \dots, \rho(d_2(x_i)), \dots, x_r)}_{\rho(\mu(x_1, \dots, d_2(x_i), \dots, x_r))} \\ &\quad - \rho d_2(\mu(x_1, \dots, x_r)) + \sum_{i=1}^r (-1)^{\beta} \rho((s\mu)_1(x_1, \dots, d_1(x_i), \dots, x_r)). \end{aligned}$$

Here, in the underbrace we use that $H_*(A)$ is regarded as an A_H -module via the

bicomplex \mathcal{O} -algebra morphism $\rho: A_H \rightarrow H_*(A)$. This vanishes by (3.4) as a consequence of $\rho d_1 = 0$.

For $\Gamma \in R(r)$ as in (2.5), using now (2.6)

$$\begin{aligned} d(\varphi)_2(s^2\Gamma; x_1, \dots, x_r) = & \sum (-1)^{\theta_{2,-1}} \underbrace{\mu^{(1)}(x_{\sigma^{-1}(1)}, \dots, \rho((s\mu^{(2)})_1(x_{\sigma^{-1}(l)}, \dots)), \dots)}_{\rho(\mu^{(1)}(x_{\sigma^{-1}(1)}, \dots, (s\mu^{(2)})_1(x_{\sigma^{-1}(l)}, \dots)), \dots)} \\ & + \sum (-1)^\delta \rho((s\mu^{(1)})_1(x_{\sigma^{-1}(1)}, \dots, \mu^{(2)}(x_{\sigma^{-1}(l)}, \dots)), \dots) \\ & - \sum_{i=1}^r (-1)^\omega \rho((s^2\Gamma)_0(x_1, \dots, d_1(x_i), \dots, x_r)). \end{aligned}$$

This vanishes by (3.5) since $\rho d_1 = 0$ and $\theta_{2,-1} = \delta$.

Finally, for $\mu \in (\mathcal{O}^i)^{(3)}$, using (2.2) and the notation (1.1),

$$\begin{aligned} d(\varphi)_3(\mu; x_1, \dots, x_r) = & \sum_{(\mu)} (-1)^{\eta_1-1} \kappa(\mu^{(1)})(x_{\sigma^{-1}(1)}, \dots, \varphi_2(\mu^{(2)}; x_{\sigma^{-1}(l)}, \dots), \dots) \\ & - \sum_{(\mu)} (-1)^{\eta_1} \varphi_2(\mu^{(1)}; x_{\sigma^{-1}(1)}, \dots, \kappa(\mu^{(2)})(x_{\sigma^{-1}(l)}, \dots), \dots) \\ = & - \sum_{(\mu)'} (-1)^{\eta_1} \underbrace{\mu_0^{(1)}(x_{\sigma^{-1}(1)}, \dots, \rho(\mu_0^{(2)}(x_{\sigma^{-1}(l)}, \dots)), \dots)}_{\rho(\mu_0^{(1)}(x_{\sigma^{-1}(1)}, \dots, \mu_0^{(2)}(x_{\sigma^{-1}(l)}, \dots)), \dots)} \\ & - \sum_{(\mu)''} (-1)^{\eta_1} \rho(\mu_0^{(1)}(x_{\sigma^{-1}(1)}, \dots, \mu_0^{(2)}(x_{\sigma^{-1}(l)}, \dots)), \dots) \\ = & - \sum_{(\mu)} (-1)^{\eta_1} \rho \mu_0^{(1)}(x_{\sigma^{-1}(1)}, \dots, \mu_0^{(2)}(x_{\sigma^{-1}(l)}, \dots), \dots). \end{aligned}$$

Here, $(\mu)'$ (resp. $(\mu)''$) runs over the summands of (1.1) with $\mu^{(1)}$ (resp. $\mu^{(2)}$) of weight 1. In the last step, we use that the weights of $\mu^{(1)}$ and $\mu^{(2)}$ add up to 3, hence $\sum_{(\mu)'} + \sum_{(\mu)''}$ comprises all summands in $\sum_{(\mu)}$ with $\mu^{(i)}$ of weight $\neq 0$. These last summands can be omitted by (3.1). The final result vanishes by (3.6).

Let us tackle (2). It suffices to show that, in the situation of Theorem 3.10, the cocycles φ and φ' defined by $f: A_M \rightsquigarrow A$ and $f': A'_M \rightsquigarrow A$ give rise to the same cohomology class. In order to compare them, we use the E^2 -equivalence $g: A_M \rightsquigarrow A'_M$ in that theorem. The underlying morphism of bicomplex \mathcal{O} -algebras $g(1)_0: A_H \rightarrow A'_H$ induces a map of complexes,

$$g(1)_0^*: C_{\mathcal{O}}^{*,t}(A'_H, H_*(A)) \longrightarrow C_{\mathcal{O}}^{*,t}(A_H, H_*(A))$$

which takes φ' to

$$\begin{aligned} g(1)_0^*(\varphi')_0(1; x) &= \rho'(d_2(g(1)_0(x))), \\ g(1)_0^*(\varphi')_1(s\mu; x_1, \dots, x_r) &= \rho'((s\mu)_1(g(1)_0(x_1), \dots, g(1)_0(x_r))), \quad \mu \in E(r), \\ g(1)_0^*(\varphi')_2(s^2\Gamma; x_1, \dots, x_r) &= \rho'((s^2\Gamma)_0(g(1)_0(x_1), \dots, g(1)_0(x_r))), \quad \Gamma \in R(r). \end{aligned}$$

We define the cochain $\psi \in C_{\mathcal{O}}^{1,-1}(A_H, H_*(A))$ as

$$\begin{aligned}\psi_0(1; x) &= \rho'(g(1)_1(x)), \\ \psi_1(s\mu; x_1, \dots, x_r) &= \rho'(g(s\mu)_0(x_1, \dots, x_r)), \quad \mu \in E.\end{aligned}$$

Let us check that $d(\psi) = g(1)_0^*(\varphi') - \varphi$.

By (2.3) and (3.11),

$$\begin{aligned}d(\psi)_0(1; x) &= \rho'(g(1)_1 d_1(x)) \\ &= \rho'(d_1 g(1)_1(x) + d_2 g(1)_0(x) - g(1)_0 d_2(x)) \\ &= \rho' d_2 g(1)_0(x) - \rho d_2(x) \\ &= g(1)_0^*(\varphi')_0(1; x) - \varphi_0(1; x).\end{aligned}$$

Here we use that $\rho' d_1 = 0$ like above. We also use that $\rho' g(1)_0 = \rho$, see Remark 3.13.

Given $\mu \in E(r)$, by (2.4) and (3.12),

$$\begin{aligned}d(\psi)_1(s\mu; x_1, \dots, x_r) &= - \sum_{i=1}^r \underbrace{\mu(x_1, \dots, \rho'(g(1)_1(x_i)), \dots, x_r)}_{\rho'(\mu(g(1)_0(x_1), \dots, g(1)_1(x_i), \dots, g(1)_0(x_r)))} \\ &\quad + \rho'(g(1)_1(\mu(x_1, \dots, x_r))) - \sum_{i=1}^r (-1)^\beta \rho'(g(s\mu)_0(x_1, \dots, d_1(x_i), \dots, x_r)) \\ &= \rho' d_1 g(s\mu)_0(x_1, \dots, x_r) - \rho'(g(1)_0(s\mu)_1(x_1, \dots, x_r) \\ &\quad + (s\mu)_1(g(1)_0(x_1), \dots, g(1)_0(x_r))) \\ &= -\rho((s\mu)_1(x_1, \dots, x_r)) + \rho'((s\mu)_1(g(1)_0(x_1), \dots, g(1)_0(x_r))) \\ &= -\varphi_1(s\mu; x_1, \dots, x_r) + g(1)_0^*(\varphi')_1(s\mu; x_1, \dots, x_r).\end{aligned}$$

Here we use again the equations $\rho' d_1 = 0$ and $\rho' g(1)_0 = \rho$, and the fact that ρ' is a bicomplex \mathcal{O} -algebra morphism.

Given $\Gamma \in R(r)$, by (2.6) and (3.13),

$$\begin{aligned}d(\psi)_2(s^2\Gamma; x_1, \dots, x_r) &= \\ &\quad \sum (-1)^{\theta_{1,-1}} \underbrace{\mu^{(1)}(x_{\sigma^{-1}(1)}, \dots, \rho'(g(s\mu^{(2)})_0(x_{\sigma^{-1}(l)}, \dots)), \dots)}_{\rho'(\mu^{(1)}(g(1)_0(x_{\sigma^{-1}(1)}, \dots, g(s\mu^{(2)})_0(x_{\sigma^{-1}(l)}, \dots)), \dots))} \\ &\quad + \sum (-1)^{1+\delta} \rho'(g(s\mu)_0(x_{\sigma^{-1}(1)}, \dots, (s\mu^{(2)})_0(x_{\sigma^{-1}(l)}, \dots)), \dots)) \\ &= -\rho'(g(1)_0((s^2\Gamma)_0(x_1, \dots, x_r)) + (s^2\Gamma)_0(g(1)_0(x_1), \dots, g(1)_0(x_r))) \\ &= -\rho((s^2\Gamma)_0(x_1, \dots, x_r)) + \rho'(s^2\Gamma)_0(g(1)_0(x_1), \dots, g(1)_0(x_r)) \\ &= -\varphi_2(s^2\Gamma; x_1, \dots, x_r) + g(1)_0^*(\varphi')_2(s^2\Gamma; x_1, \dots, x_r).\end{aligned}$$

Here we use once again $\rho' d_1 = 0$, $\rho' g(1)_0 = \rho$, and that ρ' is a bicomplex \mathcal{O} -algebra morphism. We also use that $\theta_{1,-1} = \epsilon + 1$, see (2.7) and (3.14).

Part (3) is a slight generalization of (2) which also uses Theorem 3.10, concretely the last part. Hence we skip it.

Lastly, in (4) any minimal model for A as an \mathcal{O} -algebra can be restricted to a minimal model as a \mathcal{P} -algebra and the statement holds already at the level of the cochains defined by a given minimal model. Such a restriction is possible because the

operad map f preserves weights, so it gives rise to a cooperad map between their Koszul duals $\mathcal{P}^i \rightarrow \mathcal{O}^i$. □

The derived universal Massey product is an obstruction to formality.

Corollary 4.4. *If A is a formal differential graded \mathcal{O} -algebra then $\gamma_A^{\mathcal{O}} = 0$.*

Proof. By Theorem 4.3, we can directly assume that $A = H_*(A)$, i.e. A has trivial differential. Then a horizontal resolution is also a minimal model for A , compare Remark 3.8. The cocycle φ in Definition 4.1 is trivial if A_M is just a bicomplex \mathcal{O} -algebra, hence we are done. □

Remark 4.5. If \mathbb{k} is a field, deriving operadic cohomology is unnecessary, see Corollary 1.5. Moreover, we can construct a minimal model for any differential graded \mathcal{O} -algebra A on its homology $H_*(A)$, and the derived universal Massey product coincides with Dimitrova’s class since representing cocycles agree on the nose, compare [24, §4]. The same happens for \mathbb{k} a commutative ring if $H_*(A)$ is \mathbb{k} -projective, compare [24, Remark 4.2].

5. Chain complexes

The case of $\mathcal{O} = \mathbf{1}$ the initial operad does not lack of interest. Its algebras are chain complexes, its derived homotopy algebras are twisted complexes, and its derived ∞ -morphisms are twisted maps. We do not need any hypotheses on the ground ring because this operad is non-symmetric.

We here compute the derived cohomology of a graded module and the derived universal Massey product of a chain complex. Any operad \mathcal{O} satisfying our assumptions admits an operad map $\mathbf{1} \rightarrow \mathcal{O}$, the unit. By naturality, it is sometimes possible to check that the derived universal Massey product of an \mathcal{O} -algebra is non-trivial by just looking at its underlying complex, see Theorem 4.3. We will see below, in Example 5.2, that this is the case for the differential graded associative algebra in [29, Example 5.7].

A graded $\mathbf{1}$ -algebra A is just a graded module and a horizontal resolution A_H is a sequence of projective resolutions of the modules A_n , $n \in \mathbb{Z}$. Moreover, a graded A -module M is just another graded module, which can be completely unrelated to A for there is not structure intertwining both. The factors of the operadic cochain complex in Proposition 2.5 are

$$C_{\mathbf{1}}^{w,t}(A_H, M) = \prod_{q \in \mathbb{Z}} \text{Hom}((A_H)_{w,q}, M_{q-t})$$

with differential $\partial: C_{\mathbf{1}}^{w,t}(E, M) \rightarrow C_{\mathbf{1}}^{w+1,t}(E, M)$ is $d(\psi)(x) = -(-1)^w f d_1(x)$. Therefore, the derived operadic cohomology of A with coefficients in M is

$$dH_{\mathbf{1}}^{w,t}(A, M) = \prod_{q \in \mathbb{Z}} \text{Ext}_{\mathbb{k}}^w(A_q, M_{q-t}).$$

In particular, if A is now a chain complex (regarded as a $\mathbf{1}$ -algebra), its derived

universal Massey product lies in

$$dH_{\mathbb{1}}^{2,-1}(H_*(A), H_*(A)) = \prod_{q \in \mathbb{Z}} \text{Ext}_{\mathbb{k}}^2(H_q(A), H_{q+1}(A)).$$

We have the following intrinsic characterization of the derived universal Massey product of a chain complex.

Proposition 5.1. *Given a chain complex A ,*

$$\cdots \rightarrow A_{q+1} \xrightarrow{d} A_q \rightarrow \cdots,$$

its derived universal Massey product $\gamma_A^{\mathbb{1}}$ is represented by the opposite of the following sequence of extensions, $q \in \mathbb{Z}$,

$$H_{q+1}(A) \hookrightarrow \frac{A_{q+1}}{d(A_{q+2})} \xrightarrow{d} \ker[A_q \xrightarrow{d} A_{q-1}] \twoheadrightarrow H_q(A).$$

Proof. Let $f: A_M \rightsquigarrow A$ be a minimal model, which consists of maps, $i \geq 0$,

$$f_i = f(1)_i: (A_M)_{i,n-i} \rightarrow A_n$$

satisfying certain equations. The derived universal Massey product is the sequence of extension classes represented by the following composites, $q \in \mathbb{Z}$,

$$(A_M)_{2,q} \xrightarrow{d_2} (A_M)_{0,q+1} \xrightarrow{f_0} Z_{q+1}(A) \twoheadrightarrow H_{q+1}(A),$$

see Definition 4.1 and Remark 3.13.

Let us consider the following piece of the twisted map $f: A_M \rightsquigarrow A$,

$$\begin{array}{ccccccc} A_{q+2} & & & & & & \\ \downarrow d & \swarrow f_2 & & & & & \\ A_{q+1} & \xleftarrow{f_0} & (A_M)_{0,q+1} & \xleftarrow{d_1} & (A_M)_{1,q+1} & & \\ \downarrow d & \swarrow f_2 & \swarrow f_1 & \swarrow d_2 & & & \\ A_q & \xleftarrow{f_0} & (A_M)_{0,q} & \xleftarrow{d_1} & (A_M)_{1,q} & \xleftarrow{d_1} & (A_M)_{2,q} \\ \downarrow d & \swarrow f_1 & \swarrow d_2 & & & & \\ A_{q-1} & \xleftarrow{f_0} & (A_M)_{0,q-1} & \xleftarrow{d_1} & (A_M)_{1,q-1} & \xleftarrow{d_1} & (A_M)_{2,q-1}. \end{array}$$

The following diagram commutes by (3.7), (3.8), and (3.9),

$$\begin{array}{ccccccc} (A_M)_{2,q} & \xrightarrow{d_1} & (A_M)_{1,2} & \xrightarrow{d_1} & (A_M)_{0,q} & \twoheadrightarrow & H_q(A) \\ \downarrow d_2 & & \downarrow f_1 & & \downarrow f_0 & & \parallel \\ (A_M)_{0,q+1} & & & & & & \\ \downarrow -f_0 & & & & & & \\ Z_{q+1}(A) & & & & & & \\ \downarrow & & & & & & \\ H_{q+1}(A) & \hookrightarrow & \frac{A_{q+1}}{d(A_{q+2})} & \xrightarrow{d} & \ker[d: A_q \rightarrow A_{q-1}] & \twoheadrightarrow & H_q(A). \end{array}$$

Here, on top, we find part of the horizontal resolution A_H of $H_q(A)$ arising from the minimal model A_M as in Remark 3.13. The bottom line is the extension in the statement. This proves the proposition. \square

Example 5.2. The example considered by Sagave in [29, Example 5.7] over $\mathbb{k} = \mathbb{Z}/(p^2)$ is the unital differential graded associative algebra $A = \mathbb{k}[x]/(x^2)$ with $|x| = 1$ and $d(x) = p$. The underlying chain complex is

$$\cdots \rightarrow 0 \rightarrow \mathbb{Z}/(p^2) \xrightarrow{p} \mathbb{Z}/(p^2) \rightarrow 0 \rightarrow \cdots$$

concentrated in degrees 1 and 2. By Proposition 5.1, γ_A^1 is represented by the extension,

$$\mathbb{Z}/(p) \hookrightarrow \mathbb{Z}/(p^2) \xrightarrow{p} \mathbb{Z}/(p^2) \twoheadrightarrow \mathbb{Z}/(p).$$

It is well known that this extension generates

$$dH_{\mathbb{1}}^{2,-1}(H_*(A), H_*(A)) = \text{Ext}_{\mathbb{Z}/(p^2)}^2(\mathbb{Z}/(p), \mathbb{Z}/(p)) \cong \mathbb{Z}/(p).$$

Hence $\gamma_A^1 \neq 0$ and $0 \neq \gamma_A^A \in dH_A^{2,-1}(H_*(A), H_*(A))$ by the naturality of derived universal Massey products with respect to restriction of scalars along the unit $\mathbb{1} \rightarrow \mathcal{A}$.

If we replace \mathbb{Z} and $p \in \mathbb{Z}$ with $\mathbb{Q}[t]$ and $t \in \mathbb{Q}[t]$ then A is a differential graded commutative algebra with $0 \neq \gamma_A^{\mathcal{O}} \in dH_{\mathcal{O}}^{2,-1}(H_*(A), H_*(A))$ for $\mathcal{O} = \mathbb{1}, \mathcal{A}, \mathcal{C}$.

6. Torsion Massey products

In this section we give an easy criterion to show that some \mathcal{O} -algebras over rings like $\mathbb{k} = \mathbb{Z}, \mathbb{Q}[t]$ have non-trivial derived universal Massey product. The theory in §5 does not apply here because these rings have global dimension ≤ 1 .

We will consider the following secondary operation in the cohomology of a differential graded \mathcal{O} -algebra.

Definition 6.1. Let $\mu \in E(r)$ be an arity r generator of our operad \mathcal{O} and $t_i \in \mathbb{k}$, $1 \leq i \leq r$, scalars such that $\sum_{i=1}^r t_i = 0$. Given a differential graded \mathcal{O} -algebra A and $x_i \in H_*(A)$ with $t_i x_i = 0$, $1 \leq i \leq r$, choose representatives $y_i \in A_{|x_i|}$ of the x_i and elements $z_i \in A_{|x_i|+1}$ with

$$d(z_i) = t_i y_i,$$

for all $1 \leq i \leq r$. The *torsion Massey product*

$$\langle x_1, \dots, x_r \rangle_{t_1, \dots, t_r}^{\mu} \in \frac{H_{|\mu|+|x_1|+\dots+|x_r|+1}(A)}{\sum_{i=1}^r \mu(x_1, \dots, x_{i-1}, H_{|x_i|+1}(A), x_{i+1}, \dots, x_r)} \tag{6.1}$$

is the element represented by the homology class of

$$\sum_{i=1}^r (-1)^{\beta} \mu(y_1, \dots, y_{i-1}, z_i, y_{i+1}, \dots, y_r) \in A_{|\mu|+|x_1|+\dots+|x_r|+1}, \tag{6.2}$$

where β is as in (1.3). This formula defines a cycle because, if we take d , we obtain $\sum_{i=1}^r t_i$ times $\mu(y_1, \dots, y_r)$. The denominator in (6.1) is known as the *indeterminacy* of the torsion Massey product.

Remark 6.2. The torsion Massey product is the operadic Massey product, in the sense of [24], associated to the following relation in \mathcal{O} ,

$$\sum_{i=1}^r \mu \circ_i (t_i \cdot 1) = \sum_{i=1}^r t_i (\mu \circ_i 1) = \left(\sum_{i=1}^r t_i \right) \mu = 0.$$

Strictly speaking, this is not a relation in \mathcal{O} in the sense we are using this word in this paper, i.e. it is not given by an element of R . However, this kind of relation suffices to define an operadic Massey product, see [24, Remark 4.2]. In particular, the different choices of y_i, z_i produce all possible elements of the coset

$$\langle x_1, \dots, x_r \rangle_{t_1, \dots, t_r}^\mu \subset H_{|\mu|+|x_1|+\dots+|x_r|+1}(A).$$

Moreover, torsion Massey products are invariant under differential graded \mathcal{O} -algebra morphisms $f: A \rightarrow B$, in the sense that the induced map $f_*: H_*(A) \rightarrow H_*(B)$ satisfies

$$f(\langle x_1, \dots, x_r \rangle_{t_1, \dots, t_r}^\mu) \subset \langle f(x_1), \dots, f(x_r) \rangle_{t_1, \dots, t_r}^\mu \subset H_{|\mu|+|x_1|+\dots+|x_r|+1}(B)$$

whenever the left hand side is defined. As a consequence, if A is formal all torsion Massey products contain 0.

The condition on the t_i implies that $t_r = -\sum_{i=1}^{r-1} t_i$, hence we can omit t_r from notation,

$$\langle x_1, \dots, x_r \rangle_{t_1, \dots, t_{r-1}}^\mu = \langle x_1, \dots, x_r \rangle_{t_1, \dots, t_{r-1}, t_r}^\mu.$$

In the examples below we will consider cases where the generator μ has arity $r = 2$. In this case $\langle x_1, x_2 \rangle_t^\mu$ is defined whenever $t \cdot x_1 = 0 = t \cdot x_2$ for $t \in \mathbb{k}$.

Torsion Massey products can only be non-trivial for $r \geq 2$. If $t_i = 0$ for all i then the corresponding torsion Massey products vanish because then the homology class of (6.2) lies in the indeterminacy.

Now we show how to compute torsion Massey products from a minimal model. Actually, from the pieces of a minimal model defining the derived universal Massey product.

Theorem 6.3. *Suppose that we have a differential graded \mathcal{O} -algebra A , a minimal model $f: A_M \rightsquigarrow A$, a generator $\mu \in E(r)$, and elements $x_i \in H_*(A)$ and $t_i \in \mathbb{k}$ such that $t_i x_i = 0$, $1 \leq i \leq r$, and $\sum_{i=1}^r t_i = 0$. Choose elements $u_i \in (A_M)_{0, |x_i|}$ such that $f(1)_0(u_i) \in A_{|x_i|}$ represents x_i , $v_i \in (A_M)_{1, |x_i|}$ satisfying $d_1(v_i) = t_i u_i$, $1 \leq i \leq r$, and*

$$w \in (A_M)_{2, |\mu|+\sum_{i=1}^r |x_i|}$$

such that

$$d_1(w) = \sum_{i=1}^r (-1)^\beta (s\mu)_0(u_1, \dots, u_{i-1}, v_i, u_{i+1}, \dots, u_r),$$

where β is as in (1.3). Then, for ρ as in Remark 3.13,

$$\sum_{i=1}^r (-1)^\beta \rho(s\mu)_1(u_1, \dots, v_i, \dots, u_r) - \rho d_2(w) \in H_{|\mu|+\sum_{i=1}^r |x_i|+1}(A)$$

belongs to the torsion Massey product $\langle x_1 \dots, x_r \rangle_{t_1, \dots, t_r}^\mu$.

Proof. First, notice that the elements u_i , v_i , and w must exist (although they are not unique) by Remark 3.11. Since $f(1)_0$ is a chain map, we can choose

$$y_i = f(1)_0(u_i)$$

for the computation of the torsion Massey product following Definition 6.1.

By (3.8),

$$df(1)_1(v_i) = t_i f(1)_0(u_i),$$

so we can choose

$$z_i = f(1)_1(v_i).$$

By (3.10) for $(x_1, \dots, x_r) = (u_1, \dots, u_{i-1}, v_i, u_{i+1}, \dots, u_r)$, since $d_1(u_j) = 0$ and $f(1)_1(u_j) = 0$ for degree reasons,

$$\begin{aligned} df(s\mu)_1(u_1, \dots, v_i, \dots, u_r) &+ (-1)^\beta t_i f(s\mu)_0(u_1, \dots, u_r) \\ &= f(1)_0(s\mu)_1(u_1, \dots, v_i, \dots, u_r) + f(1)_1(s\mu)_0(u_1, \dots, v_i, \dots, u_r) \\ &\quad - \mu(y_1, \dots, z_i, \dots, y_r). \end{aligned}$$

With the previous choices, using this last formula, the cycle in (6.2) is

$$\begin{aligned} \sum_{i=1}^r (-1)^\beta (f(1)_0(s\mu)_1(u_1, \dots, v_i, \dots, u_r) &+ f(1)_1(s\mu)_0(u_1, \dots, v_i, \dots, u_r)) \\ &- \sum_{i=1}^r t_i f(s\mu)_0(u_1, \dots, u_r) - \sum_{i=1}^r (-1)^\beta df(s\mu)_1(u_1, \dots, v_i, \dots, u_r). \end{aligned}$$

The first summation in the second line vanishes because $\sum_{i=1}^r t_i = 0$. Moreover, the last summation in the second line is a boundary, so if we discard it we obtain another cycle in the same homology class. Hence, the torsion Massey product contains the homology class of

$$\sum_{i=1}^r (-1)^\beta f(1)_0(s\mu)_1(u_1, \dots, v_i, \dots, u_r) + f(1)_1 d_1(w).$$

By (3.9), this cycle coincides with

$$\sum_{i=1}^r (-1)^\beta f(1)_0(s\mu)_1(u_1, \dots, v_i, \dots, u_r) + df(1)_2(w) - f(1)_0 d_2(w).$$

We can discard the middle term because it is a boundary. The homology class of this cycle is the one in the statement, see Remark 3.13 \square

Remark 6.4. In [24, §2], given a differential graded \mathcal{O} -algebra A , a relation $\Gamma \in R(r)$ in \mathcal{O} , and $x_1, \dots, x_r \in H_*(A)$ satisfying certain vanishing conditions we defined a coset

$$\langle x_1, \dots, x_r \rangle_\Gamma \subset H_{|\Gamma| + \sum_{i=1}^r |x_i| + 1}(A)$$

that we called *Massey product*, since it generalizes Massey’s and Allday–Retah’s in the associative and Lie cases, respectively.

In [24, §3], we established a connection between these Massey products and non-derived minimal models whenever they exist (under strong projectivity and cofibrancy assumptions). Derived minimal models always exist (as recalled in Theorem 3.10) and essentially the same proof shows that, given elements $u_i \in (A_M)_{0,|x_i|}$ such that $f(1)_0(u_i) \in A_{|x_i|}$ represents x_i ,

$$f(1)_0((s^2\Gamma)_0(u_1, \dots, u_r)) \in A_{|\Gamma|+\sum_{i=1}^r |x_i|+1}$$

is a cycle whose homology class belongs to $\langle x_1, \dots, x_r \rangle_\Gamma$. Hence, derived minimal models also compute these *non-torsion Massey products*.

Remark 6.5. Theorem 6.3 and Remark 6.4 assert that torsion Massey products and operadic Massey products associated to elements of R can be computed from the cocycle representing the derived universal Massey product in Definition 4.1. We can actually use any representative. More precisely, given a differential graded \mathcal{O} -algebra A , a horizontal resolution $\rho: A_H \rightarrow H_*(A)$ and a representative $\psi \in C_{\mathcal{O}}^{2,-1}(A_H, H_*(A))$ of $\gamma_A^{\mathcal{O}}$, we can replace A_M , $\rho(s\mu)_1$, ρd_2 , and $\rho(s^2\Gamma)_0$ in Theorem 6.3 and Remark 6.4 with A_H , $\psi_1(s\mu; \dots)$, $\psi_0(1; \dots)$, and $\psi_2(s^2\Gamma; \dots)$, respectively. This is tedious but straightforward to check, compare [24, Proposition 4.1]. We here include a full proof of the following weaker statement.

Proposition 6.6. *If A is a differential graded \mathcal{O} -algebra with trivial derived universal Massey product, then all torsion Massey products vanish in $H_*(A)$.*

Proof. Let $\psi \in C_{\mathcal{O}}^{1,-1}(A_H, A)$ be a cochain satisfying $d(\psi) = \varphi$. With the notation in Theorem 6.3, using (2.4), (2.3), and that $d_1(u_i) = 0$ and $\psi_0(1; u_i) = 0$ for degree reasons,

$$\begin{aligned} & \sum_{i=1}^r (-1)^\beta d(\psi)_1(s\mu; u_1, \dots, v_i, \dots, u_r) - d(\psi)_0(1; w) = \\ & - \sum_{i=1}^r (-1)^\beta \mu(u_1, \dots, \psi_0(1; v_i), \dots, u_r) + \sum_{i=1}^r (-1)^\beta \psi_0(1; \mu(u_1, \dots, v_i, \dots, u_r)) \\ & - \sum_{i=1}^r \underbrace{\psi_1(s\mu; u_1, \dots, d_1(v_i), \dots, u_r)}_{t_i \psi_1(s\mu; u_1, \dots, u_r)} - \psi_0(1; d_1(w)) = \\ & - \sum_{i=1}^r (-1)^\beta \mu(x_1, \dots, \psi_0(1; v_i), \dots, x_r). \end{aligned}$$

Here we use that $\sum_{i=1}^r t_i = 0$, and the fact that A is regarded as an A_H -module via the bicomplex \mathcal{O} -algebra map $f: A_H \rightarrow A$, which takes u_i to x_i . Clearly, this element belongs to the indeterminacy. \square

In order to conclude this paper, we now exhibit several examples of differential graded \mathcal{O} -algebras over rings of global dimension 1, for \mathcal{O} the classic operads, with some non-vanishing torsion Massey product, and hence non-trivial derived universal Massey product. In particular, these differential graded \mathcal{O} -algebras are not formal. In these examples, we cannot detect the non-triviality of derived universal Massey

products by using operadic Massey products associated to elements of R . These are either not defined except in trivial situations or they vanish for degree reasons.

Example 6.7. Let $\mathcal{O} = \mathcal{A}$ be the associative operad with the usual presentation, with generating \mathbb{S} -module E concentrated in arity 2 and degree 0, generated by μ , the operation representing the associative product. Hence $E(2)$ is a free \mathbb{S}_2 -module of rank 1. Moreover, $R \subset E \circ_{(1)} E$ is generated by the associativity relation $\mu \circ_1 \mu - \mu \circ_2 \mu$.

It would be natural to consider torsion Massey products associated to $\mu \in E$, and so we do in Example 6.8 below, but we here consider the commutator

$$\ell = \mu - \mu \cdot (1\ 2) \in E.$$

Assume A is a differential graded associative algebra which in addition is unital. Suppose that there exists $t \in \mathbb{k}$ such that the unit $1 \in H_*(A)$ satisfies $t \cdot 1 = 0$. Hence $t \cdot H_*(A)$, so

$$\langle 1, x \rangle_t^\ell \in \frac{H_{|x|+1}}{[H_1(A), x]}$$

is defined for all $x \in H_*(A)$. Here

$$[y, z] = yz - (-1)^{|y||z|}zy$$

denotes the commutator bracket, which is the operation corresponding to μ . For the computation of the indeterminacy we have used that $1 \in H_*(A)$ is in the graded center. If $H_*(A)$ is commutative the indeterminacy vanishes.

For $p \in \mathbb{Z} = \mathbb{k}$ a prime, Dugger and Shipley consider in [7] the differential graded unital associative algebra

$$A = \frac{\mathbb{Z}\langle e, x^{\pm 1} \rangle}{(e^2, ex + xe - x^2)}, \quad |e| = |x| = 1,$$

with differential

$$d(e) = p, \quad d(x) = 0.$$

Its homology is

$$H_*(A) = \mathbb{Z}/(p)\langle x^{\pm 1} \rangle,$$

which is commutative (always in the graded sense) if and only if $p = 2$. We will not distinguish between cycles in A and their homology classes so as not to overload notation.

We are going to compute $\langle 1, x \rangle_p^\ell$. For $p \neq 2$,

$$[x, x] = 2x^2 \in H_2(A)$$

is a generator so $H_2(A) = [H_1(A), x]$ and the torsion Massey product trivially vanishes. However, for $p = 2$ the indeterminacy vanishes by commutativity and the second defining relation of A shows that

$$\langle 1, x \rangle_2^\ell = x^2 \in H_2(A).$$

The algebra A is actually not formal for any prime p . This follows from [7]. An alternative proof based on derived minimal models is given in [19, Example 8.8].

Example 6.8. Let $\mathbb{k} = \mathbb{Q}[t]$ and let $\mathcal{O} = \mathcal{C}$ be the commutative operad with the usual presentation, which is like the presentation of \mathcal{A} in Example 6.7 but requiring $\mu = \mu \cdot (1\ 2)$, so $E(2)$ is a free \mathbb{k} -module of rank 1 with trivial action of \mathbb{S}_2 .

We consider the graded commutative (non-unital) algebra A in [19, Example 6.8] generated by

$$x, y, x_t, y_t$$

in degrees $|x| = |y| = 2$ and $|x_t| = |y_t| = 3$, subject to the relations

$$x^2, y^2, xy, xx_t, yy_t, x_t y.$$

We endow it with the differential defined by

$$d(x) = d(y) = 0, \quad d(x_t) = tx, \quad d(y_t) = ty.$$

Its homology $H_*(A)$ has basis $\{x, y, xy_t\}$ over \mathbb{Q} , not over \mathbb{k} . Again, we do not distinguish between cycles in A and their classes. Actually, $t \cdot H_*(A) = 0$, so torsion Massey products associated to μ and t are always defined. The product in $H_*(A)$ is trivial, hence the indeterminacy always vanishes. Nevertheless,

$$\langle x, y \rangle_t^\mu = -xy_t \in H_5(A).$$

The relations in the previous differential graded algebra were designed so as to have a fully-computed small homology algebra. This was necessary in [19]. However, here we can work without relations. More precisely, if B is the free graded commutative algebra with the same generators (and no relations) and we define the differential in the same way, then

$$\langle x, y \rangle_t^\mu = x_t y - xy_t \in H_5(B) \cong \mathbb{Q}$$

is a generator and the indeterminacy vanishes because $H_3(B) = 0$.

Using the canonical operad morphism $\mathcal{A} \rightarrow \mathcal{C}$ we see that these examples work in exactly the same way when regarded as associative algebras.

Example 6.9. Consider again $\mathbb{k} = \mathbb{Q}[t]$ and let $\mathcal{O} = \mathcal{L}$ be the Lie operad with the standard presentation, with E concentrated in arity 2 and degree 0, generated by ℓ , the operation representing the Lie bracket, which must satisfy $\ell \cdot (1\ 2) = -\ell$. Hence $E(2)$ is a free \mathbb{k} -module of rank 1 equipped with the sign action of \mathbb{S}_2 . The \mathbb{S} -module of relations $R \subset E \circ_{(1)} E$ is generated by the Jacobi relation $(\ell \circ_1 \ell) \cdot [() + (1\ 2\ 3) + (3\ 2\ 1)]$.

The algebra A we now consider, introduced in [19, Example 6.11], behaves similarly to that in Example 6.8, despite they are algebras over different operads. It is a graded Lie algebra generated by

$$x, y, x_t, y_t$$

in the same degrees as above subject to the relations saying that all triple brackets vanish, and also all binary brackets except for $[x, y_t], [x_t, y_t]$ and their symmetric. Then $H_*(A)$ has \mathbb{Q} -linear basis $\{x, y, [x, y_t]\}$ and $t \cdot H_*(A) = 0$, so torsion Massey products associated to μ and t are always defined. The graded Lie algebra $H_*(A)$ is abelian, hence indeterminacies vanish, and

$$\langle x, y \rangle_t^\mu = -[x, y_t] \in H_5(A).$$

We can also work without relations in this case, since the necessity of full homology computations in [19] is not present in this paper. More precisely, if B is the free graded

Lie algebra with the same generators as the previous A and we define the differential in the same way, then

$$\langle x, y \rangle_t^\mu = [x_t, y] - [x, y_t] \in H_5(B) \cong \mathbb{Q}$$

is a generator and the indeterminacy vanishes because $H_3(B) = 0$.

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