MULTICATEGORIES MODEL ALL CONNECTIVE SPECTRA

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Abstract

There is a free construction from multicategories to permutative categories, left adjoint to the endomorphism multicategory construction. The main result shows that these functors induce an equivalence of homotopy theories. This result extends a similar result of Thomason, that permutative categories model all connective spectra.

Contents

1. Introduction

Thomason proved in [**Tho95**] that symmetric monoidal categories model all connective spectra, in the sense that there is an equivalence of homotopy categories between that of SMCat and of $\mathsf{Sp}_{\geqslant 0}$ with respect to stable equivalences. See [**Man10**] and [**GJO17b**] for further refinements of this result to an equivalence of homotopy *theories*

$$\left(\mathsf{PermCat}, \mathcal{S}\right) \sim \left(\mathsf{Sp}_{\geqslant 0}, \mathcal{S}\right) \tag{1.1}$$

in the sense of Definition 2.7 below. In (1.1),

- $\mathsf{PermCat}$ denotes the category of small permutative categories and strict monoidal functors,

- $\mathsf{Sp}_{\geqslant 0}$ denotes the category of connective symmetric spectra, and

- $\mathcal{S}$ denotes the class of stable equivalences.

The equivalence is induced by Segal's K-theory construction [**Seg74**]

$$K \colon \mathsf{PermCat} \longrightarrow \mathsf{Sp}_{\geqslant 0}.$$

We give further explanations of the relevant background material in Sections 3, 4, and 7.

In this article we show that there is an equivalence of homotopy theories

$$\big(\mathsf{Multicat}, \mathcal{S}\big) \sim \big(\mathsf{PermCat}, \mathcal{S}\big). \tag{1.2}$$

The equivalence is induced by a free functor

$$F \colon \mathsf{Multicat} \longrightarrow \mathsf{PermCat}$$

that we describe in Sections 5 and 6 below. We prove the homotopy equivalence (1.2) in Theorem 7.3. Combining (1.2) with (1.1) above gives the following main result.

Theorem 1.1. *There is an equivalence of homotopy theories*

$$\big(\mathsf{Multicat}, \mathcal{S}\big) \sim \big(\mathsf{Sp}_{\geqslant 0}, \mathcal{S}\big)$$

induced by

$$\mathsf{Multicat} \xrightarrow{\ F\ } \mathsf{PermCat} \xrightarrow{\ K\ } \mathsf{Sp}_{\geqslant 0}.$$

Motivation

Multicategories, as a context for stable homotopy, have several advantages related to the symmetric monoidal closed structure known as the Boardman–Vogt tensor product [**BV73**]. This monoidal product does not restrict to $\mathsf{PermCat}$, but it does provide $\mathsf{PermCat}$ with the structure of a multicategory. This perspective appears in work of Elmendorf–Mandell [**EM06, EM09**], which goes on to develop a multifunctorial K-theory extending that of Segal. A detailed explanation of the relevant theory is given in [**JY∞**, Part 2]. In particular, see [**JY∞**, 10.3.32 and 10.8].

The category of small Waldhausen categories, useful in the study of algebraic K-theory, also admits the structure of a closed multicategory that is not symmetric monoidal. See Zakharevich [**Zak18**] for a proof and further explanation. Work of Bohmann–Osorno [**BO20**] develops a multifunctorial K-theory for Waldhausen categories similar to that of Elmendorf–Mandell.

The conclusion of Theorem 1.1 shows that, although richer in algebraic structure, multicategories model the same homotopy theory as that of permutative categories, namely, connective stable homotopy theory. Thus, while the category of small multicategories provides a context for multiplicative homotopy theory, it does not introduce exotic or spurious homotopy types.

Outline

Sections 2 through 4 fix notation and terminology by giving relevant background for complete Segal spaces, permutative categories, and multicategories. In Section 5 we define the free construction F, together with unit and counit natural transformations. In Section 6 we show that these define an adjunction of 2-categories. Section 7 contains the definitions of stable equivalences and, in Theorem 7.3, the proof that F is an equivalence of homotopy theories.

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2.   Equivalences of homotopy theories

In this section we review the theory of complete Segal spaces due to Rezk [**Rez01**]. An equivalence of homotopy theories (Definition 2.7) is an equivalence of fibrant replacements in the complete Segal space model structure. For further context and development we refer the reader to [**DK80, Hir03, Toë05, BK12**].

Complete Segal spaces

Definition 2.1. A bisimplicial set X is a *complete Segal space* if

- it is fibrant in the Reedy model structure on bisimplicial sets,

- for each $n \geqslant 2$ the Segal map

$$X(n) \longrightarrow X(1) \times_{X(0)} \cdots \times_{X(0)} X(1)$$

  is a weak equivalence of simplicial sets, and

- the morphism

$$X(0) \cong \mathsf{Map}(\Delta[0], X) \longrightarrow \mathsf{Map}(E, X) \tag{2.1}$$

  is a weak equivalence of simplicial sets, where E is the discrete nerve of the category consisting of two isomorphic objects and (2.1) is induced by the unique morphism $E \longrightarrow \Delta[0]$.

Remark 2.2. The definition of complete Segal space given above is equivalent to that given in [**Rez01**, Section 6] by [**Rez01**, 6.4].

Theorem 2.3 ([**Rez01**, 7.2]). *There is a simplicial closed model structure on the category of bisimplicial sets, called the* complete Segal space model structure, *that is given as a left Bousfield localization of the Reedy model structure and in which the fibrant objects are precisely the complete Segal spaces.*

Relative categories

Definition 2.4. A *relative category* is a pair $(\mathsf{C}, \mathcal{W})$ consisting of a category C and a subcategory $\mathcal{W}$ containing all of the objects of C. A *relative functor*

$$F \colon (\mathsf{C}, \mathcal{W}) \longrightarrow (\mathsf{C}', \mathcal{W}')$$

is a functor from C to C' that sends morphisms of $\mathcal{W}$ to those of $\mathcal{W}'$.

Definition 2.5. Suppose $(\mathsf{C}, \mathcal{W})$ is a relative category and A is another category. We let

$$(\mathsf{C}, \mathcal{W})^{\mathsf{A}}$$

denote the subcategory of C^{A} whose objects are functors $\mathsf{A} \longrightarrow \mathsf{C}$ and whose morphisms are those natural transformations with components in $\mathcal{W}$.

Definition 2.6. Suppose $(\mathsf{C}, \mathcal{W})$ is a relative category. The *classification diagram* of $(\mathsf{C}, \mathcal{W})$ is the bisimplicial set

$$\mathsf{N}^{\mathsf{cl}}(\mathsf{C}, \mathcal{W}) = \mathsf{Ner}\left((\mathsf{C}, \mathcal{W})^{\Delta[\bullet]}\right)$$

given by

$$n \longmapsto \mathsf{Ner}\left((\mathsf{C}, \mathcal{W})^{\Delta[n]}\right)$$

where $\Delta[n]$ denotes the category consisting of n composable arrows.

Definition 2.7. Suppose $(\mathsf{C}, \mathcal{W})$ is a relative category. We say that a bisimplicial set $R\,\mathsf{N}^{\mathsf{cl}}(\mathsf{C}, \mathcal{W})$ is a *homotopy theory of* $(\mathsf{C}, \mathcal{W})$ if it is a fibrant replacement of $\mathsf{N}^{\mathsf{cl}}(\mathsf{C}, \mathcal{W})$ in the complete Segal space model structure. We say that a relative functor

$$F \colon (\mathsf{C}, \mathcal{W}) \longrightarrow (\mathsf{C}', \mathcal{W}')$$

is an *equivalence of homotopy theories* if the induced morphism $R\,\mathsf{N}^{\mathsf{cl}}\,F$ between homotopy theories is a weak equivalence in the complete Segal space model structure.

Remark 2.8. For readers familiar with the notions of hammock localization as well as DK-equivalence [**DK80**], Barwick and Kan have shown in [**BK12**, 1.8] that a relative functor

$$F \colon (\mathsf{C}, \mathcal{W}) \longrightarrow (\mathsf{C}', \mathcal{W}')$$

is an equivalence of homotopy theories if and only if it induces a DK-equivalence between hammock localizations. In that case, F induces equivalences between mapping simplicial sets and between categories of components. In particular, if F is an equivalence of homotopy theories then the induced functor between categorical localizations

$$\mathsf{C}[\mathcal{W}^{-1}] \longrightarrow \mathsf{C}'[(\mathcal{W}')^{-1}]$$

is an equivalence.

Proposition 2.9 ([**GJO17b**, 2.8]). *Suppose*

$$F \colon (\mathsf{C}, \mathcal{W}) \longrightarrow (\mathsf{C}', \mathcal{W}')$$

is a relative functor and suppose that F induces a weak equivalence of simplicial sets

$$\mathsf{Ner}\left((\mathsf{C}, \mathcal{W})^{\Delta[n]}\right) \longrightarrow \mathsf{Ner}\left((\mathsf{C}', \mathcal{W}')^{\Delta[n]}\right) \tag{2.2}$$

for each n. Then F is an equivalence of homotopy theories.

Proof. The assumption that (2.2) is a weak equivalence for each n means that

$$\mathsf{N}^{\mathsf{cl}}\,F \colon \mathsf{N}^{\mathsf{cl}}(\mathsf{C}, \mathcal{W}) \longrightarrow \mathsf{N}^{\mathsf{cl}}(\mathsf{C}', \mathcal{W}')$$

is a weak equivalence between classification diagrams in the Reedy model structure [**Rez01**, Section 2.4]. Thus $\mathsf{N}^{\mathsf{cl}}\,F$ is a weak equivalence in the complete Segal space model structure because it is a localization of the Reedy model structure. As a consequence, $\mathsf{N}^{\mathsf{cl}}\,F$ induces a weak equivalence between the homotopy theories given by fibrant replacements. $\square$

The following application of Proposition 2.9 is a special case of [**GJO17b**, 2.9] that is suitable for our purposes.

Proposition 2.10. *Suppose given relative functors*

$$F \colon (\mathsf{C}, \mathcal{W}) \rightleftarrows (\mathsf{C}', \mathcal{W}') \colon E$$

such that

- *F is left adjoint to E and*
- *each component of the unit, respectively counit, for $F \dashv E$ is a morphism in $\mathcal{W}$, respectively $\mathcal{W}'$.*

Then F and E are equivalences of homotopy theories.

Proof. The adjunction $F \dashv E$ induces an adjunction

$$F^{\Delta[n]} \colon \mathsf{C}^{\Delta[n]} \rightleftarrows \mathsf{C}'^{\Delta[n]} \colon E^{\Delta[n]}$$

for each n. Since the components of the unit and counit are morphisms in $\mathcal{W}$ and $\mathcal{W}'$, respectively, they continue to induce an adjunction when F and E are restricted to the subcategories $(\mathsf{C}, \mathcal{W})^{\Delta[n]}$ and $(\mathsf{C}', \mathcal{W}')^{\Delta[n]}$. A natural transformation between functors induces a simplicial homotopy on nerves, and hence the result follows from Proposition 2.9. $\qquad\square$

3. Permutative categories

In this section we define permutative categories, strict monoidal functors, and monoidal natural transformations. See [**JS93, ML98, JY21, Yau∞I, Yau∞II**] for further discussion in the more general context of plain/braided/symmetric monoidal structure.

Definition 3.1. A *permutative category* $(\mathsf{C}, \oplus, e, \xi)$ consists of

- a category C,
- a functor $\oplus \colon \mathsf{C} \times \mathsf{C} \longrightarrow \mathsf{C}$, called the *monoidal sum*,
- an object $e \in \mathsf{C}$, called the *monoidal unit*, and
- a natural isomorphism ξ called the *symmetry isomorphism* with components

$$\xi_{X,Y} \colon X \oplus Y \longrightarrow Y \oplus X$$

for objects X and Y of C.

The monoidal sum is required to be associative and unital, with e as its unit. The symmetry isomorphism ξ is required to make the following symmetry and hexagon diagrams commute for objects $X, Y, Z \in \mathsf{C}$.

$$(3.1)$$

A permutative category is also called a *strict symmetric monoidal category*. The strictness refers to the conditions that the monoidal sum be strictly associative and unital.

Definition 3.2. Suppose C and D are permutative categories. A *symmetric monoidal functor*

$$(P, P^2, P^0) \colon \mathsf{C} \longrightarrow \mathsf{D}$$

consists of a functor $P \colon \mathsf{C} \longrightarrow \mathsf{D}$ together with natural transformations

$$PX \oplus PY \xrightarrow{\ P^2\ } P(X \oplus Y) \quad\text{and}\quad e \xrightarrow{\ P^0\ } Pe$$

for objects $X, Y \in \mathsf{C}$, called the *monoidal constraint* and *unit constraint*, respectively. These data satisfy the following associativity, unity, and symmetry axioms.

Associativity The following diagram is commutative for all objects $X, Y, Z \in \mathsf{C}$.

$$
\begin{array}{c}
(PX \oplus PY) \oplus PZ = PX \oplus (PY \oplus PZ) \\[2mm]
\end{array}
\tag{3.2}
$$

with maps $P^2 \oplus 1_{PZ}$, $1_{PX} \oplus P^2$, $P(X \oplus Y) \oplus PZ$, $PX \oplus P(Y \oplus Z)$, P^2, P^2, $P((X \oplus Y) \oplus Z) = P(X \oplus (Y \oplus Z))$

Unity The following two diagrams are commutative for all objects $X \in \mathsf{C}$.

$$
\begin{array}{cc}
e \oplus PX = PX & PX \oplus e = PX \\
\end{array}
\quad\text{and}\quad
\tag{3.3}
$$

$P^0 \oplus 1_{PX}$, $Pe \oplus PX \xrightarrow{\ P^2\ } P(e \oplus X)$ and $1_{PX} \oplus P^0$, $PX \oplus Pe \xrightarrow{\ P^2\ } P(X \oplus e)$

Symmetry The following diagram is commutative for all objects $X, Y \in \mathsf{C}$.

$$
\begin{array}{ccc}
PX \oplus PY & \xrightarrow{\ \xi_{PX,PY}\ } & PY \oplus PX \\
{\scriptstyle P^2}\downarrow & & \downarrow{\scriptstyle P^2} \\
P(X \oplus Y) & \xrightarrow{\ P\xi_{X,Y}\ } & P(Y \oplus X)
\end{array}
\tag{3.4}
$$

This finishes the definition of a symmetric monoidal functor. A *strict symmetric monoidal functor* is a symmetric monoidal functor P such that the monoidal constraint P^2 and the unit constraint P^0 are both identity natural transformations.

Definition 3.3. Suppose $P, Q \colon \mathsf{C} \longrightarrow \mathsf{D}$ are strict symmetric monoidal functors between permutative categories. A *monoidal natural transformation*

$$\alpha \colon P \longrightarrow Q$$

is a natural transformation between the underlying functors such that the following monoidal and unit conditions hold:

$$\alpha_{X \oplus Y} = \alpha_X \oplus \alpha_Y \quad\text{and}\quad \alpha_e = 1_e \tag{3.5}$$

for all objects X and Y in C.

Definition 3.4. We let $\mathsf{PermCat}$ denote the 2-category of small permutative categories, strict symmetric monoidal functors, and monoidal natural transformations.

Identities and compositions are given by those of the underlying functors and natural transformations.

Remark 3.5. There are several weaker variants for symmetric monoidal structure both on categories and on functors thereof. Thomason [**Tho95**, 1.9.2] shows that each of the standard variants have equivalent stable homotopy theories. See [**GJO17a**] for a general treatment via 2-dimensional monad theory.

Here we restrict to permutative categories and strict symmetric monoidal functors because they have the most direct comparison with multicategories. For more on this point, see Remark 6.5 below.

4. Multicategories

In this section we review the definitions of multicategories, multifunctors, and multinatural transformations. Several of the details here will be needed for our explanation of the free functor in Sections 5 and 6 below. For further details and context we refer the reader to [**JY∞**, Chapter 5] and [**Yau16**].

Definition 4.1. Suppose C is a class.

- Denote by

$$\mathsf{Prof}(C) = \coprod_{n \geqslant 0} C^{\times n}$$

  the class of finite ordered sequences of elements in C. An element in $\mathsf{Prof}(C)$ is called a *C-profile*.

- A typical C-profile of length $n = \mathrm{len}\langle c \rangle$ is denoted by $\langle c \rangle = (c_1, \ldots, c_n) \in C^n$ or by $\langle c_i \rangle_i$ to indicate the indexing variable. The empty C-profile is denoted by $\langle \rangle$.

- We let $\oplus$ denote the concatenation of profiles, and note that $\oplus$ is an associative binary operation with unit given by the empty tuple $\langle \rangle$.

- An element in $\mathsf{Prof}(C) \times C$ is denoted as $\big(\langle c \rangle \,;\, c' \big)$ with $c' \in C$ and $\langle c \rangle \in \mathsf{Prof}(C)$.

Definition 4.2. A *multicategory* $(\mathsf{M}, \gamma, 1)$ consists of the following data.

- M is equipped with a class $\mathsf{Ob}\,\mathsf{M}$ of *objects*. We write $\mathsf{Prof}(\mathsf{M})$ for $\mathsf{Prof}(\mathsf{Ob}\,\mathsf{M})$.

- For $c' \in \mathsf{Ob}\,\mathsf{M}$ and $\langle c \rangle = (c_1, \ldots, c_n) \in \mathsf{Prof}(\mathsf{M})$, M is equipped with a set of *n-ary operations*

$$\mathsf{M}\big(\langle c \rangle \,;\, c' \big) = \mathsf{M}\big(c_1, \ldots, c_n \,;\, c' \big)$$

  with *input profile* $\langle c \rangle$ and *output* c'.

- For $\big(\langle c \rangle \,;\, c' \big) \in \mathsf{Prof}(\mathsf{M}) \times \mathsf{Ob}\,\mathsf{M}$ as above and a permutation $\sigma \in \Sigma_n$, M is equipped with an isomorphism of sets

$$\mathsf{M}\big(\langle c \rangle \,;\, c' \big) \xrightarrow[\cong]{\sigma} \mathsf{M}\big(\langle c \rangle \sigma \,;\, c' \big),$$

  called the *right action* or the *symmetric group action*, in which

$$\langle c \rangle \sigma = (c_{\sigma(1)}, \ldots, c_{\sigma(n)})$$

  is the right permutation of $\langle c \rangle$ by σ.

- For $c \in \mathsf{Ob}\,\mathsf{M}$, M is equipped with an element

$$1_c \in \mathsf{M}(c\,;\,c),$$

  called the *c-colored unit*.
- For $c'' \in \mathsf{Ob}\,\mathsf{M}$, $\langle c' \rangle = (c'_1, \ldots, c'_n) \in \mathsf{Prof}(\mathsf{M})$, and for each $j \in \{1, \ldots, n\}$, $\langle c_j \rangle = (c_{j,1}, \ldots, c_{j,k_j}) \in \mathsf{Prof}(\mathsf{M})$ let $\langle c \rangle = \oplus_j \langle c_j \rangle \in \mathsf{Prof}(\mathsf{M})$ be the concatenation of the $\langle c_j \rangle$. Then M is equipped with a map

$$\mathsf{M}(\langle c' \rangle\,;\,c'') \times \prod_{j=1}^{n} \mathsf{M}(\langle c_j \rangle\,;\,c'_j) \xrightarrow{\;\gamma\;} \mathsf{M}(\langle c \rangle\,;\,c'') \qquad (4.1)$$

  called the *composition* or *multicategorical composition*.

These data are required to satisfy the following axioms.

Symmetric Group Action For $(\langle c \rangle\,;\,c') \in \mathsf{Prof}(\mathsf{M}) \times \mathsf{Ob}\,\mathsf{M}$ with $n = \mathrm{len}\langle c \rangle$ and $\sigma, \tau \in \Sigma_n$, the following diagram commutes.

$$
\begin{array}{ccc}
\mathsf{M}(\langle c \rangle\,;\,c') & \xrightarrow{\;\sigma\;} & \mathsf{M}(\langle c \rangle \sigma\,;\,c') \\
& \searrow{\scriptstyle \sigma\tau} & \downarrow{\scriptstyle \tau} \\
& & \mathsf{M}(\langle c \rangle \sigma\tau\,;\,c')
\end{array}
\qquad (4.2)
$$

Moreover, the identity permutation in Σ_n acts as the identity map on $\mathsf{M}(\langle c \rangle\,;\,c')$.

Associativity Suppose given

- $c''' \in \mathsf{Ob}\,\mathsf{M}$,
- $\langle c'' \rangle = (c''_1, \ldots, c''_n) \in \mathsf{Prof}(\mathsf{M})$,
- $\langle c'_j \rangle = (c'_{j,1}, \ldots, c'_{j,k_j}) \in \mathsf{Prof}(\mathsf{M})$ for each $j \in \{1, \ldots, n\}$, and
- $\langle c_{j,i} \rangle = (c_{j,i,1}, \ldots, c_{j,i,\ell_{j,i}}) \in \mathsf{Prof}(\mathsf{M})$ for each $j \in \{1, \ldots, n\}$ as well as each $i \in \{1, \ldots, k_j\}$,

such that $k_j = \mathrm{len}\langle c'_j \rangle > 0$ for at least one j. For each j, let $\langle c_j \rangle = \oplus_{i=1}^{k_j} \langle c_{j,i} \rangle$ denote the concatenation of the $\langle c_{j,i} \rangle$. Let $\langle c \rangle = \oplus_{j=1}^{n} \langle c_j \rangle$ denote the concatenation of the $\langle c_j \rangle$ and let $\langle c' \rangle = \oplus_{j=1}^{n} \langle c'_j \rangle$ denote the concatenation of the $\langle c'_j \rangle$. Then the *associativity diagram* below commutes.

$$
\begin{array}{c}
\mathsf{M}(\langle c' \rangle\,;\,c''') \times \prod_{j=1}^{n}\left[\prod_{i=1}^{k_j} \mathsf{M}(\langle c_{j,i} \rangle\,;\,c'_{j,i})\right] \\[2mm]
\nearrow{\scriptstyle (\gamma,1)} \qquad \searrow{\scriptstyle \gamma} \\
\mathsf{M}(\langle c'' \rangle\,;\,c''') \times \left[\prod_{j=1}^{n} \mathsf{M}(\langle c'_j \rangle\,;\,c''_j)\right] \times \prod_{j=1}^{n}\left[\prod_{i=1}^{k_j} \mathsf{M}(\langle c_{j,i} \rangle\,;\,c'_{j,i})\right] \qquad \mathsf{M}(\langle c \rangle\,;\,c''') \\[2mm]
\downarrow{\scriptstyle \mathrm{permute}\,\cong} \\
\mathsf{M}(\langle c'' \rangle\,;\,c''') \times \prod_{j=1}^{n}\left[\mathsf{M}(\langle c'_j \rangle\,;\,c''_j) \times \prod_{i=1}^{k_j} \mathsf{M}(\langle c_{j,i} \rangle\,;\,c'_{j,i})\right] \\[2mm]
\searrow{\scriptstyle (1, \prod_j \gamma)} \qquad \nearrow{\scriptstyle \gamma} \\
\mathsf{M}(\langle c'' \rangle\,;\,c''') \times \prod_{j=1}^{n} \mathsf{M}(\langle c_j \rangle\,;\,c''_j)
\end{array}
\qquad (4.3)
$$

Unity Suppose $c' \in \mathsf{Ob}\,\mathsf{M}$.

1. If $\langle c \rangle = (c_1, \ldots, c_n) \in \mathsf{Prof}(\mathsf{M})$ has length $n \geqslant 1$, then the following *right unity diagram* is commutative. Here $\mathbf{1}$ is the one-point set and $\mathbf{1}^n$ is its n-fold Cartesian product.

$$
\begin{array}{ccc}
\mathsf{M}(\langle c \rangle\,;\,c') \times \mathbf{1}^n & \xrightarrow{\ \ \rho\ \ } & \mathsf{M}(\langle c \rangle\,;\,c') \\[2pt]
{\scriptstyle 1 \times (\times_j 1_{c_j})}\Big\downarrow & & \Big\downarrow{\scriptstyle 1} \\[2pt]
\mathsf{M}(\langle c \rangle\,;\,c') \times \prod\limits_{j=1}^{n} \mathsf{M}(c_j\,;\,c_j) & \xrightarrow{\ \ \gamma\ \ } & \mathsf{M}(\langle c \rangle\,;\,c')
\end{array}
\tag{4.4}
$$

2. For any $\langle c \rangle \in \mathsf{Prof}(\mathsf{M})$, the *left unity diagram* below is commutative.

$$
\begin{array}{ccc}
\mathbf{1} \times \mathsf{M}(\langle c \rangle\,;\,c') & \xrightarrow{\ \ \lambda\ \ } & \mathsf{M}(\langle c \rangle\,;\,c') \\[2pt]
{\scriptstyle 1_{c'} \times 1}\Big\downarrow & & \Big\downarrow{\scriptstyle 1} \\[2pt]
\mathsf{M}(c'\,;\,c') \times \mathsf{M}(\langle c \rangle\,;\,c') & \xrightarrow{\ \ \gamma\ \ } & \mathsf{M}(\langle c \rangle\,;\,c')
\end{array}
\tag{4.5}
$$

Equivariance Suppose that in the definition of γ (4.1), $\mathrm{len}\langle c_j \rangle = k_j \geqslant 0$.

1. For each $\sigma \in \Sigma_n$, the following *top equivariance diagram* is commutative.

$$
\begin{array}{ccc}
\mathsf{M}(\langle c' \rangle\,;\,c'') \times \prod\limits_{j=1}^{n} \mathsf{M}(\langle c_j \rangle\,;\,c'_j) & \xrightarrow{\ (\sigma,\sigma^{-1})\ } & \mathsf{M}(\langle c' \rangle\sigma\,;\,c'') \times \prod\limits_{j=1}^{n} \mathsf{M}(\langle c_{\sigma(j)} \rangle\,;\,c'_{\sigma(j)}) \\[2pt]
{\scriptstyle \gamma}\Big\downarrow & & \Big\downarrow{\scriptstyle \gamma} \\[2pt]
\mathsf{M}(\langle c_1 \rangle, \ldots, \langle c_n \rangle\,;\,c'') & \xrightarrow{\ \sigma\langle k_{\sigma(1)}, \ldots, k_{\sigma(n)} \rangle\ } & \mathsf{M}(\langle c_{\sigma(1)} \rangle, \ldots, \langle c_{\sigma(n)} \rangle\,;\,c'')
\end{array}
\tag{4.6}
$$

Here $\sigma\langle k_{\sigma(1)}, \ldots, k_{\sigma(n)} \rangle \in \Sigma_{k_1 + \cdots + k_n}$ is right action of the block permutation that permutes the n consecutive blocks of lengths $k_{\sigma(1)}$, ..., $k_{\sigma(n)}$ as σ permutes $\{1, \ldots, n\}$, leaving the relative order within each block unchanged.

2. Given permutations $\tau_j \in \Sigma_{k_j}$ for $1 \leqslant j \leqslant n$, the following *bottom equivariance diagram* is commutative.

$$
\begin{array}{ccc}
\mathsf{M}(\langle c' \rangle\,;\,c'') \times \prod\limits_{j=1}^{n} \mathsf{M}(\langle c_j \rangle\,;\,c'_j) & \xrightarrow{\ (1,\times_j \tau_j)\ } & \mathsf{M}(\langle c' \rangle\,;\,c'') \times \prod\limits_{j=1}^{n} \mathsf{M}(\langle c_j \rangle\tau_j\,;\,c'_j) \\[2pt]
{\scriptstyle \gamma}\Big\downarrow & & \Big\downarrow{\scriptstyle \gamma} \\[2pt]
\mathsf{M}(\langle c_1 \rangle, \ldots, \langle c_n \rangle\,;\,c'') & \xrightarrow{\ \tau_1 \times \cdots \times \tau_n\ } & \mathsf{M}(\langle c_1 \rangle\tau_1, \ldots, \langle c_n \rangle\tau_n\,;\,c'')
\end{array}
\tag{4.7}
$$

Here the block sum $\tau_1 \times \cdots \times \tau_n \in \Sigma_{k_1 + \cdots + k_n}$ is the image of $(\tau_1, \ldots, \tau_n)$ under the canonical inclusion

$$
\Sigma_{k_1} \times \cdots \times \Sigma_{k_n} \longrightarrow \Sigma_{k_1 + \cdots + k_n}.
$$

This finishes the definition of a multicategory.

Moreover, we define the following.

- A multicategory is *small* if its class of objects is a set.
- An *operad* is a multicategory with one object. If M is an operad, then its set of n-ary operations is denoted by M_n.

- The *initial operad* I consists of a single object $*$ and its unit operation.
- The *terminal multicategory* T consists of a single object $*$ and a single n-ary operation ι_n for each $n \geqslant 0$.

Example 4.3 (Endomorphism operad). Suppose M is a multicategory and c is an object of M. Then $\mathsf{End}(c)$ is the operad consisting of the single object c and n-ary operation object

$$\mathsf{End}(c)_n = \mathsf{M}(\langle c \rangle \,;\, c),$$

where $\langle c \rangle$ denotes the constant n-tuple at c. The symmetric group action, unit, and composition of $\mathsf{End}(c)$ are given by those of M.

Example 4.4 (Endomorphism multicategory). Suppose $(\mathsf{C}, \oplus, e, \xi)$ is a small permutative category. Then the *endomorphism multicategory* $\mathsf{End}(\mathsf{C})$ is the small multicategory with object set $\mathsf{Ob}\,\mathsf{C}$ and with

$$\mathsf{End}(\mathsf{C})(\langle X \rangle \,;\, Y) = \mathsf{C}(X_1 \oplus \cdots \oplus X_n, Y)$$

for $Y \in \mathsf{Ob}\,\mathsf{C}$ and $\langle X \rangle = (X_1, \cdots, X_n) \in (\mathsf{Ob}\,\mathsf{C})^{\times n}$. An empty $\oplus$ means the unit object e.

Example 4.5 (Underlying category). Each multicategory $(\mathsf{M}, \gamma, 1)$ has an underlying category with

- the same objects,
- identities given by the colored units, and
- composition given by

$$\mathsf{M}(b \,;\, c) \times \mathsf{M}(a \,;\, b) \xrightarrow{\ \gamma\ } \mathsf{M}(a \,;\, c)$$

for objects $a, b, c \in \mathsf{M}$.

The 2-category of small multicategories

Definition 4.6. A *multifunctor* $F \colon \mathsf{M} \longrightarrow \mathsf{N}$ between multicategories M and N consists of

- an object assignment $F \colon \mathsf{Ob}\,\mathsf{M} \longrightarrow \mathsf{Ob}\,\mathsf{N}$ and
- for each $(\langle c \rangle \,;\, c') \in \mathsf{Prof}(\mathsf{M}) \times \mathsf{Ob}\,\mathsf{M}$ with $\langle c \rangle = (c_1, \ldots, c_n)$, a component morphism

$$F \colon \mathsf{M}(\langle c \rangle \,;\, c') \longrightarrow \mathsf{N}(F\langle c \rangle \,;\, Fc'),$$

where $F\langle c \rangle = (Fc_1, \ldots, Fc_n)$.

These data are required to preserve the symmetric group action, the colored units, and the composition in the following sense.

Symmetric Group Action For each $(\langle c \rangle \,;\, c')$ as above and each choice of permutation $\sigma \in \Sigma_n$, the following diagram is commutative.

$$
\begin{array}{ccc}
\mathsf{M}(\langle c \rangle \,;\, c') & \xrightarrow{\ F\ } & \mathsf{N}(F\langle c \rangle \,;\, Fc') \\
{\scriptstyle \sigma}\big\downarrow{\scriptstyle \cong} & & {\scriptstyle \sigma}\big\downarrow{\scriptstyle \cong} \\
\mathsf{M}(\langle c \rangle\sigma \,;\, c') & \xrightarrow{\ F\ } & \mathsf{N}(F\langle c \rangle\sigma \,;\, c')
\end{array}
\qquad (4.8)
$$

Units For each $c \in \mathsf{Ob}\,\mathsf{M}$, the following diagram is commutative.

$$\begin{array}{ccc} & & \mathsf{M}(c\,;\,c) \\ & {}^{1_c}\nearrow & \\ \mathbf{1} & & \downarrow F \\ & {}_{1_{Fc}}\searrow & \\ & & \mathsf{N}(Fc\,;\,Fc) \end{array} \qquad (4.9)$$

Composition For c'', $\langle c'\rangle$, and $\langle c\rangle = \oplus_j \langle c_j\rangle$ as in the definition of γ (4.1), the following diagram is commutative.

$$\begin{array}{ccc} \mathsf{M}(\langle c'\rangle\,;\,c'') \times \prod\limits_{j=1}^{n} \mathsf{M}(\langle c_j\rangle\,;\,c_j') & \xrightarrow{(F,\prod_j F)} & \mathsf{N}(F\langle c'\rangle\,;\,Fc'') \times \prod\limits_{j=1}^{n} \mathsf{N}(F\langle c_j\rangle\,;\,Fc_j') \\ \gamma \downarrow & & \downarrow \gamma \\ \mathsf{M}(\langle c\rangle\,;\,c'') & \xrightarrow{\quad F \quad} & \mathsf{N}(F\langle c\rangle\,;\,Fc'') \end{array} \qquad (4.10)$$

This finishes the definition of a multifunctor.

Moreover, we define the following.

- A multifunctor $\mathsf{M} \longrightarrow \mathsf{N}$ is also called an M-*algebra in* N.
- For another multifunctor $G\colon \mathsf{N} \longrightarrow \mathsf{P}$ between multicategories, where P has object class $\mathsf{Ob}\,\mathsf{P}$, the *composition* $GF\colon \mathsf{M} \longrightarrow \mathsf{P}$ is the multifunctor defined by composing the assignments on objects

$$\mathsf{Ob}\,\mathsf{M} \xrightarrow{\ F\ } \mathsf{Ob}\,\mathsf{N} \xrightarrow{\ G\ } \mathsf{Ob}\,\mathsf{P}$$

and the morphisms on n-ary operations

$$\mathsf{M}(\langle c\rangle\,;\,c') \xrightarrow{\ F\ } \mathsf{N}(F\langle c\rangle\,;\,Fc') \xrightarrow{\ G\ } \mathsf{P}(GF\langle c\rangle\,;\,GFc').$$

- The *identity multifunctor* $1_{\mathsf{M}}\colon \mathsf{M} \longrightarrow \mathsf{M}$ is defined by the identity assignment on objects and the identity morphism on n-ary operations.
- An *operad morphism* is a multifunctor between two multicategories with one object.

Definition 4.7. Suppose $F, G\colon \mathsf{M} \longrightarrow \mathsf{N}$ are multifunctors as in Definition 4.6. A *multinatural transformation* $\theta\colon F \longrightarrow G$ consists of component morphisms

$$\theta_c \in \mathsf{N}(Fc\,;\,Gc) \quad \text{for} \quad c \in \mathsf{Ob}\,\mathsf{M}$$

such that the following *naturality diagram* commutes for each $(\langle c\rangle\,;\,c') \in \mathsf{Prof}(\mathsf{M}) \times \mathsf{Ob}\,\mathsf{M}$ with $\langle c\rangle = (c_1, \ldots, c_n)$.

$$\begin{array}{ccc} \mathbf{1} \times \mathsf{M}(\langle c\rangle\,;\,c') & \xrightarrow{\ \theta_{c'} \times F\ } & \mathsf{N}(Fc'\,;\,Gc') \times \mathsf{N}(F\langle c\rangle\,;\,Fc') \\ {}^{\lambda^{-1}}\nearrow & & \searrow {}^{\gamma} \\ \mathsf{M}(\langle c\rangle\,;\,c') & & \mathsf{N}(F\langle c\rangle\,;\,Gc') \\ {}_{\rho^{-1}}\searrow & & \nearrow {}_{\gamma} \\ \mathsf{M}(\langle c\rangle\,;\,c') \times \prod\limits_{j=1}^{n}\mathbf{1} & \xrightarrow[\quad]{\ G \times \prod\limits_{j=1}^{n}\theta_{c_j}\ } & \mathsf{N}(G\langle c\rangle\,;\,Gc') \times \prod\limits_{j=1}^{n} \mathsf{N}(Fc_j\,;\,Gc_j) \end{array} \qquad (4.11)$$

This finishes the definition of a multinatural transformation.

Moreover, we define the following.

- The *identity multinatural transformation* $1_F \colon F \longrightarrow F$ has components

$$(1_F)_c = 1_{Fc} \quad \text{for} \quad c \in \mathsf{Ob}\,\mathsf{M}.$$

Definition 4.8. Suppose $\theta \colon F \longrightarrow G$ is a multinatural transformation between multifunctors as in Definition 4.7.

1. Suppose $\beta \colon G \longrightarrow H$ is a multinatural transformation for a multifunctor $H \colon \mathsf{M} \longrightarrow \mathsf{N}$. The *vertical composition*

$$\beta\theta \colon F \longrightarrow H \tag{4.12}$$

is the multinatural transformation with components at $c \in \mathsf{Ob}\,\mathsf{M}$ given by the following composites.

$$
\begin{array}{ccc}
\mathbf{1} & \xrightarrow{\quad (\beta\theta)_c \quad} & \mathsf{N}\big(Fc\,;\,Hc\big) \\
{\scriptstyle \lambda^{-1}} \big\downarrow & & \big\uparrow {\scriptstyle \gamma} \\
\mathbf{1} \times \mathbf{1} & \xrightarrow{\quad \beta_c \times \theta_c \quad} & \mathsf{N}\big(Gc\,;\,Hc\big) \times \mathsf{N}\big(Fc\,;\,Gc\big)
\end{array}
$$

2. Suppose $\theta' \colon F' \longrightarrow G'$ is a multinatural transformation for multifunctors $F', G' \colon \mathsf{N} \longrightarrow \mathsf{P}$. The *horizontal composition*

$$\theta' * \theta \colon F'F \longrightarrow G'G \tag{4.13}$$

is the multinatural transformation with components at $c \in \mathsf{Ob}\,\mathsf{M}$ given by the following composites.

$$
\begin{array}{ccc}
\mathbf{1} & \xrightarrow{\quad (\theta' * \theta)_c \quad} & \mathsf{P}\big(F'Fc\,;\,G'Gc\big) \\
{\scriptstyle \lambda^{-1}} \big\downarrow & & \big\uparrow {\scriptstyle \gamma} \\
& & \mathsf{P}\big(F'Gc\,;\,G'Gc\big) \times \mathsf{P}\big(F'Fc\,;\,F'Gc\big) \\
& & \big\uparrow {\scriptstyle 1 \times F'} \\
\mathbf{1} \times \mathbf{1} & \xrightarrow{\quad \theta'_{Gc} \times \theta_c \quad} & \mathsf{P}\big(F'Gc\,;\,G'Gc\big) \times \mathsf{N}\big(Fc\,;\,Gc\big)
\end{array}
$$

This finishes the definition.

Theorem 4.9 ([**JY21**, 2.4.26])**.** *There is a 2-category* Multicat *consisting of the following data.*

- *Its objects are small multicategories.*
- *For small multicategories* M *and* N, *the hom category*

$$\mathsf{Multicat}(\mathsf{M}, \mathsf{N})$$

has

  - *multifunctors* $\mathsf{M} \longrightarrow \mathsf{N}$ *as 1-cells,*
  - *multinatural transformations as 2-cells,*
  - *vertical composition as composition, and*

— *identity multinatural transformations as identity 2-cells.*

- *The identity 1-cell 1_{M} is the identity multifunctor 1_{M}.*

- *Horizontal composition of 1-cells is the composition of multifunctors.*

- *Horizontal composition of 2-cells is that of multinatural transformations.*

Recall from Example 4.4 the endomorphism multicategory $\mathsf{End}(\mathsf{C})$ of a permutative category C. A strict symmetric monoidal functor $F\colon \mathsf{C} \longrightarrow \mathsf{D}$ between permutative categories induces a multifunctor

$$\mathsf{End}(F)\colon \mathsf{End}(\mathsf{C}) \longrightarrow \mathsf{End}(\mathsf{D})$$

by applying F to the operations of $\mathsf{End}(\mathsf{C})$, which are morphisms in C. Similarly, a monoidal natural transformation $\theta\colon F \longrightarrow G$ between strict symmetric monoidal functors induces a multinatural transformation

$$\mathsf{End}(\theta)\colon \mathsf{End}(F) \longrightarrow \mathsf{End}(G)$$

whose components are given by those of θ. See [**JY**∞, Section 5.3] for further explanation of these constructions.

Proposition 4.10 ([**JY**∞, 5.3.6]). *The construction $\mathsf{End}(\mathsf{C})$ of Example 4.4 defines a 2-functor*

$$\mathsf{End}\colon \mathsf{PermCat} \longrightarrow \mathsf{Multicat}$$

from the 2-category of small permutative categories to the 2-category of small multicategories.

Remark 4.11. In [**JY**∞] $\mathsf{PermCat}$ denotes the larger 2-category of small permutative categories and general (not-necessarily-strict) symmetric monoidal functors. The proof of [**JY**∞, 5.3.6] shows that End is 2-functorial with respect to these more general 1-cells. The statement given here is restricted to the subcategory consisting of strict symmetric monoidal functors. In Remark 6.5 we note one reason for this restriction in the present work.

5. Free permutative category on a multicategory

In this section we describe a 2-functor

$$F\colon \mathsf{Multicat} \longrightarrow \mathsf{PermCat}$$

that is left 2-adjoint to the endomorphism 2-functor, End. Our description follows that of [**EM09**, Theorem 4.2], but we include full details here because this free construction is crucial to our main result, Theorem 1.1. The definition makes use of sequences $\langle x \rangle$, indexing functions f, and permutations $\sigma_{g,f}^{k}$. We make the following preliminary definitions.

Definition 5.1. For each natural number $r \geqslant 0$, let $\bar{r}$ denote the finite set with r

elements

$$\overline{r} = \{1, \ldots, r\} \quad \text{for} \quad r \geqslant 1, \quad \text{and} \quad \overline{0} = \varnothing.$$

Suppose $\langle x \rangle$ is a sequence of length r, with each $x_i \in \mathsf{M}$. Suppose

$$f \colon \overline{r} \longrightarrow \overline{s} \quad \text{and} \quad g \colon \overline{s} \longrightarrow \overline{t}$$

are functions of finite sets, for $r, s, t \geqslant 0$. Then we define the following.

- For $j \in \overline{s}$, let

$$\langle x \rangle_{f^{-1}(j)} = \langle x_i \rangle_{i \in f^{-1}(j)} \tag{5.1}$$

  denote the sequence formed by those x_i with $i \in f^{-1}(j)$, ordered as in $\langle x \rangle$. Similarly, for $k \in \overline{t}$, let

$$\langle \phi \rangle_{g^{-1}(k)} = \langle \phi_j \rangle_{j \in g^{-1}(k)},$$

  where $\langle \phi \rangle$ is a length-s sequence of operations in M.

- For $k \in \overline{t}$, let $\sigma_{g,f}^k \in \Sigma_t$ be the unique permutation such that

$$\left[\bigoplus_{j \in g^{-1}(k)} \langle x \rangle_{f^{-1}(j)} \right] \cdot \sigma_{g,f}^k = \langle x \rangle_{(gf)^{-1}(k)}, \tag{5.2}$$

  where the sequence on the left hand side is the concatenation of sequences in the order specified by $g^{-1}(k)$. We will use the action of these permutations on both objects and operations.

Definition 5.2. Suppose M is a multicategory. Define a permutative category $F\mathsf{M}$, called the *free permutative category on* M, as follows.

Objects The objects of $F\mathsf{M}$ are given by the $(\mathsf{Ob}\,\mathsf{M})$-profiles: finite ordered sequences $\langle x \rangle = (x_1, \ldots, x_r)$ of objects of M, with $r \geqslant 0$.

Morphisms Given sequences $\langle x \rangle$ and $\langle y \rangle$ with lengths r and s, respectively, the morphisms from $\langle x \rangle$ to $\langle y \rangle$ in $F\mathsf{M}$ are given by pairs $(f, \langle \phi \rangle)$ consisting of

- a function

$$f \colon \overline{r} \longrightarrow \overline{s}$$

  called the *index map* and
- an ordered sequence of operations

$$\langle \phi \rangle \quad \text{with} \quad \phi_j \in \mathsf{M}\big(\langle x_i \rangle_{i \in f^{-1}(j)} \,;\, y_j\big)$$

  for $j \in \overline{s}$.

The identity morphism on $\langle x \rangle$ is given by $1_{\overline{r}}$ and the tuple of unit operations 1_{x_i}.

Composition The composition of a pair of morphisms

$$\langle x \rangle \xrightarrow{\;(f, \langle \phi \rangle)\;} \langle y \rangle \xrightarrow{\;(g, \langle \psi \rangle)\;} \langle z \rangle$$

is the pair

$$\big(gf, \langle \theta_k \cdot \sigma_{g,f}^k \rangle_{k \in \overline{t}}\big), \tag{5.3}$$

where, for each $k \in \bar{t}$,

$$\theta_k = \gamma \left(\psi_k \, ; \, \langle \phi \rangle_{g^{-1}(k)} \right) \in \mathsf{M} \left(\bigoplus_{j \in g^{-1}(k)} \langle x \rangle_{f^{-1}(j)} \, ; \, z_k \right). \qquad (5.4)$$

Note that the input profile for θ_k is the concatenation of $\langle x \rangle_{f^{-1}(j)}$ for $j \in g^{-1}(k)$. By definition (5.2), the right action of $\sigma_{g,f}^k$ permutes this input profile to $\langle x \rangle_{(gf)^{-1}(k)}$. We check that composition of morphisms is unital and associative in Proposition 5.3 below.

Monoidal Sum The monoidal sum

$$\oplus \colon F\mathsf{M} \times F\mathsf{M} \longrightarrow F\mathsf{M}$$

is given on objects by concatenation of sequences. The monoidal sum of morphisms

$$(f, \langle \phi \rangle) \colon \langle x \rangle \longrightarrow \langle y \rangle \quad \text{and} \quad (f', \langle \phi' \rangle) \colon \langle x' \rangle \longrightarrow \langle y' \rangle,$$

is the pair

$$(f \oplus f', \langle \phi \rangle \oplus \langle \phi' \rangle)$$

where $f \oplus f'$ denotes the composite

$$\overline{r + r'} \cong \bar{r} \coprod \overline{r'} \xrightarrow{f \coprod f'} \bar{s} \coprod \overline{s'} \cong \overline{s + s'}$$

given by the disjoint union of f with f' and the canonical order-preserving isomorphisms. Functoriality of the monoidal sum follows because disjoint union of indexing functions preserves preimages and the operations in a composite (5.3) are determined elementwise for the indexing set of the codomain.

Monoidal Unit The monoidal unit is the empty sequence $\langle \rangle$. The unit and associativity isomorphisms for $\oplus$ are identities.

Symmetry The symmetry isomorphism for sequences $\langle x \rangle$ of length r and $\langle x' \rangle$ of length r' is

$$\xi_{\langle x \rangle, \langle x' \rangle} = \left(\tau_{r,r'}, \langle 1 \rangle \right)$$

where

$$\tau_{r,r'} \colon \overline{r + r'} \cong \bar{r} \coprod \overline{r'} \longrightarrow \overline{r'} \coprod \bar{r} \cong \overline{r' + r}$$

is induced by the block-transposition of $\bar{r}$ with $\overline{r'}$, keeping the relative order within each block fixed.

Concatenation of sequences is strictly associative and unital. The symmetry and hexagon axioms (3.1) follow from the corresponding equalities of block permutations.

This completes the definition of $F\mathsf{M}$ and the data for its permutative structure.

Proposition 5.3. *In the context of Definition 5.2, $F\mathsf{M}$ is a permutative category.*

Proof. Observe that $\sigma_{g,f}^k$ is the identity permutation if either f or g is an identity. This implies that composition is strictly unital. To verify associativity of composition,

suppose given the following objects and morphisms in $F\mathsf{M}$:

$$\langle x \rangle \xrightarrow{(f, \langle \phi \rangle)} \langle y \rangle \xrightarrow{(g, \langle \psi \rangle)} \langle z \rangle \xrightarrow{(h, \langle \mu \rangle)} \langle w \rangle$$

with index maps

$$\overline{r} \xrightarrow{f} \overline{s} \xrightarrow{g} \overline{t} \xrightarrow{h} \overline{u}.$$

The two composites to be compared are

$$\big((h, \langle \mu \rangle)(g, \langle \psi \rangle)\big)(f, \langle \phi \rangle) \quad \text{and} \quad (h, \langle \mu \rangle)\big((g, \langle \psi \rangle)(f, \langle \phi \rangle)\big). \tag{5.5}$$

The index map of both composites is hgf. For each $\ell \in \overline{u}$, the ℓth operations of (5.5) are

$$\gamma\left(\theta_\ell \cdot \sigma_{h,g}^\ell \,;\, \langle \phi \rangle_{(hg)^{-1}(\ell)}\right) \cdot \sigma_{hg,f}^\ell \quad \text{and} \quad \gamma\left(\mu_\ell \,;\, \langle \omega_k \cdot \sigma_{g,f}^k \rangle_{k \in h^{-1}(\ell)}\right) \cdot \sigma_{h,gf}^\ell,$$

respectively, where

$$\theta_\ell = \gamma\left(\mu_\ell \,;\, \langle \psi \rangle_{h^{-1}(\ell)}\right) \quad \text{and} \quad \omega_k = \gamma\left(\psi_k \,;\, \langle \phi \rangle_{g^{-1}(k)}\right).$$

To see that the two composites are equal, we first use the top equivariance diagram (4.6) with $\sigma = \sigma_{h,g}^\ell$. For each $j \in \overline{s}$ the action of σ gives

$$\langle \phi_{\sigma^{-1}(j)} \rangle_{j \in (hg)^{-1}(\ell)} = \bigoplus_{k \in h^{-1}(\ell)} \langle \phi \rangle_{g^{-1}(k)}$$

by (5.2) for h and g. Then by top equivariance, with $r_j = |f^{-1}(j)|$, we have

$$\gamma\left(\theta_\ell \cdot \sigma_{h,g}^\ell \,;\, \langle \phi \rangle_{(gh)^{-1}(\ell)}\right) \cdot \sigma_{hg,f}^\ell = \gamma\left(\theta_\ell \,;\, \bigoplus_{k \in h^{-1}(\ell)} \langle \phi \rangle_{g^{-1}(k)}\right) \cdot \sigma\langle r_{\sigma(1)}, \ldots, r_{\sigma(s)} \rangle \cdot \sigma_{hg,f}^\ell.$$

Next we use the bottom equivariance diagram (4.7) and have

$$\gamma\left(\mu_\ell \,;\, \langle \omega_k \cdot \sigma_{g,f}^k \rangle_{k \in h^{-1}(\ell)}\right) \cdot \sigma_{h,gf}^\ell = \gamma\left(\mu_\ell \,;\, \langle \omega \rangle_{h^{-1}(\ell)}\right) \cdot \left(\bigoplus_{k \in h^{-1}(\ell)} \sigma_{g,f}^k\right) \cdot \sigma_{h,gf}^\ell.$$

To complete the verification that the two composites in (5.5) are equal, observe the following for each $\ell \in \overline{u}$. First, by associativity (4.3) in M we have

$$\gamma\left(\theta_\ell \,;\, \bigoplus_{k \in h^{-1}(\ell)} \langle \phi \rangle_{g^{-1}(k)}\right) = \gamma\left(\gamma\left(\mu_\ell \,;\, \langle \psi \rangle_{h^{-1}(\ell)}\right) \,;\, \bigoplus_{k \in h^{-1}(\ell)} \langle \phi \rangle_{g^{-1}(k)}\right)$$

$$= \gamma\left(\mu_\ell \,;\, \left\langle \gamma\left(\psi_k \,;\, \langle \phi \rangle_{g^{-1}(k)}\right)\right\rangle_{k \in h^{-1}(\ell)}\right)$$

$$= \gamma\left(\mu_\ell \,;\, \langle \omega \rangle_{h^{-1}(\ell)}\right)$$

in

$$\mathsf{M}\left(\bigoplus_{k \in h^{-1}(\ell)}\left(\bigoplus_{j \in g^{-1}(k)} \langle x \rangle_{f^{-1}(j)}\right) \,;\, w_\ell\right).$$

Second, by uniqueness of the permutation in $\Sigma_{|(hgf^{-1})(\ell)|}$ whose right action sends

$$\bigoplus_{k \in h^{-1}(\ell)} \left(\bigoplus_{j \in g^{-1}(k)} \langle x \rangle_{f^{-1}(j)} \right),$$

to $\langle x \rangle_{(hgf)^{-1}(\ell)}$, we have

$$\sigma \langle r_{\sigma(1)}, \ldots, r_{\sigma(s)} \rangle \cdot \sigma^{\ell}_{hg,f} = \left(\bigoplus_{k \in h^{-1}(\ell)} \sigma^{k}_{g,f} \right) \cdot \sigma^{\ell}_{h,gf}.$$

Therefore the two sides of (5.5) are equal for each triple of composable morphisms. This completes the proof that $F\mathsf{M}$ is a category. The symmetric monoidal axioms for $F\mathsf{M}$ are verified as part of Definition 5.2. $\qquad\square$

Definition 5.4. Suppose $H \colon \mathsf{M} \longrightarrow \mathsf{N}$ is a multifunctor. Define a strict symmetric monoidal functor

$$FH \colon F\mathsf{M} \longrightarrow F\mathsf{N}$$

via the following assignment on objects and morphisms. For a sequence $\langle x \rangle$ of length r, define

$$(FH)\langle x \rangle = \langle Hx_i \rangle_{i \in \overline{r}}.$$

For a morphism $(f, \langle \phi \rangle)$, define

$$(FH)(f, \langle \phi \rangle) = (f, \langle H\phi_j \rangle_j). \tag{5.6}$$

The multifunctoriality of H shows that this assignment is functorial on morphisms. Since the monoidal sum is defined by concatenation in $F\mathsf{M}$ and $F\mathsf{N}$, the functor FH is strict monoidal. Compatibility with the symmetry of $F\mathsf{M}$ and $F\mathsf{N}$ follows because FH preserves the index map of each morphism and H preserves unit operations.

Definition 5.5. Suppose $\kappa \colon H \longrightarrow K \colon \mathsf{M} \longrightarrow \mathsf{N}$ is a multinatural transformation. Define a monoidal natural transformation

$$F\kappa \colon FH \longrightarrow FK$$

via components

$$(F\kappa)_{\langle x \rangle} = (1, \langle \kappa_{x_i} \rangle_i) \colon \langle Hx \rangle \longrightarrow \langle Kx \rangle \tag{5.7}$$

for each sequence $\langle x \rangle$ in $F\mathsf{M}$. Naturality of $F\kappa$ follows from multinaturality of κ (4.11) because each $\sigma^{j}_{f,1}$ and $\sigma^{j}_{1,f}$ is an identity permutation and we have

$$(1, \langle \kappa_{y_j} \rangle_j)(f, \langle H\phi_j \rangle_j) = \left(f, \left\langle \gamma\big(\kappa_{y_j} \, ; \, H\phi_j \big) \right\rangle_j \right)$$

$$= \left(f, \left\langle \gamma\big(K\phi_j \, ; \, \langle \kappa_{x_i} \rangle_{i \in f^{-1}(j)} \big) \right\rangle_j \right)$$

$$= (f, \langle K\phi_j \rangle_j)(1, \langle \kappa_{x_i} \rangle_i)$$

for each morphism $(f, \langle \phi \rangle) \colon \langle x \rangle \longrightarrow \langle y \rangle$ in $F\mathsf{M}$.

The monoidal naturality axioms (3.5) for $F\kappa$ follow because the monoidal sum in $F\mathsf{N}$ is given by concatenation of object and operation sequences. The component $(F\kappa)_{\langle \rangle}$ is the identity morphism $(1_{\varnothing}, \langle \rangle) \colon \langle \rangle \longrightarrow \langle \rangle$.

Proposition 5.6. *The free permutative category construction given in Definitions 5.2, 5.4, and 5.5 provides a 2-functor*

$$F \colon \mathsf{Multicat} \longrightarrow \mathsf{PermCat}.$$

Proof. Verification that F produces permutative categories and strict symmetric monoidal functors is given in the definitions and Proposition 5.3. We see that F is functorial because, for each multifunctor H, the symmetric monoidal functor FH is given by applying H componentwise to sequences of objects and operations. Therefore F preserves identity morphisms and composition. The 2-functoriality of F follows similarly because the identities, horizontal, and vertical composites of both multinatural transformations and monoidal natural transformations are determined by their components. □

Example 5.7. The free permutative category on the empty multicategory is the terminal permutative category. Its single object is the empty sequence, and its single morphism is given by the identity index map and the empty sequence of operations.

Example 5.8. Recall from Definition 4.2 that the terminal multicategory T consists of a single object and a unique n-ary operation for each n. The free permutative category $F\mathsf{T}$ is isomorphic to the natural number category N whose objects are given by natural numbers and morphisms are given by morphisms of finite sets

$$\mathsf{N}(r, s) = \mathsf{Set}(\overline{r}, \overline{s}).$$

The natural number $r \in \mathsf{N}$ corresponds to the length-r sequence whose terms are the unique object of T. Each morphism $f \colon \overline{r} \longrightarrow \overline{s}$ corresponds to the morphism

$$(f, \langle \phi \rangle) \in F\mathsf{T}$$

where ϕ_j is the unique operation in T of arity $|f^{-1}(j)|$.

Example 5.9. Recall from Definition 4.2 that the initial operad I consists of a single object and its unit operation. Similarly to Example 5.8, the free permutative category $F\mathsf{I}$ is isomorphic to the permutation category Σ with objects given by natural numbers and morphisms given by permutations

$$\Sigma(r, s) = \begin{cases} \Sigma_r, & \text{if} \quad r = s, \\ \varnothing, & \text{if} \quad r \neq s \end{cases}$$

for each pair of natural numbers r and s.

6. Free permutative category as a left 2-adjoint

Recall from Proposition 4.10 that the endomorphism multicategory $\mathsf{End}(\mathsf{C})$ associated to a permutative category C defines a 2-functor

$$\mathsf{End} \colon \mathsf{PermCat} \longrightarrow \mathsf{Multicat}.$$

Throughout this section we let $E = \mathsf{End}$. The main result of this section is Theorem 6.6, which shows that the free construction F of Section 5 is left 2-adjoint to $E = \mathsf{End}$.

Definition 6.1 (Unit). Suppose M is a multicategory. Define a component

$$\eta = \eta_{\mathsf{M}} \colon \mathsf{M} \longrightarrow EF\mathsf{M}$$

as follows. For an object $w \in \mathsf{M}$ and an operation $\phi \in \mathsf{M}(\langle x \rangle \,;\, y)$, let (w) and (ϕ) denote the corresponding length-1 sequences. For each $r \geqslant 0$, let $\iota_r \colon \bar{r} \longrightarrow \bar{1}$ be the unique function. Then $\eta = \eta_{\mathsf{M}}$ is the following assignment:

$$\eta w = (w) \quad \text{for} \quad w \in \mathsf{M}, \quad \text{and}$$
$$\eta \phi = (\iota_r, (\phi)) \colon \langle x \rangle \longrightarrow (y)$$

where $\phi \in \mathsf{M}(\langle x \rangle \,;\, y)$ and $|\langle x \rangle| = r$. Note that $\langle x \rangle$ is an r-fold concatenation of length-1 sequences (x_i) and the morphism $(\iota_r, (\phi))$ in $F\mathsf{M}$ is an r-ary operation in $EF\mathsf{M}$. Lemma 6.2 shows that each η_{M} is multinatural and that the components are 2-natural with respect to multifunctors and multinatural transformations.

Lemma 6.2. *The components η_{M} of Definition 6.1 define a 2-natural transformation*

$$\eta \colon 1_{\mathsf{Multicat}} \longrightarrow EF.$$

Proof. To check multifunctoriality of each component $\eta = \eta_{\mathsf{M}}$, first note that η preserves unit operations because ι_1 is the identity on $\bar{1}$. Now suppose given

$$\psi \in \mathsf{M}(\langle x' \rangle \,;\, x'') \quad \text{where} \quad |\langle x' \rangle| = s, \quad \text{and}$$

$$\phi_j \in \mathsf{M}(\langle x_j \rangle \,;\, x_j') \quad \text{where} \quad |\langle x_j \rangle| = r_j \quad \text{for} \quad j \in \bar{s}.$$

Let $r = \sum_j r_j$. Then the composite in $EF\mathsf{M}$ of

$$\eta \psi = (\iota_s; (\psi)) \quad \text{and} \quad \langle \eta \phi_j \rangle_{j=1}^s = \langle (\iota_{r_j}, (\phi_j)) \rangle_j$$

is given by composing the morphisms

$$(\iota_s, (\psi)) \quad \text{and} \quad \bigoplus_{j \in \bar{s}} (\iota_{r_j}, \phi_j) = \big(\oplus_{j \in \bar{s}} \, \iota_{r_j}, \langle \phi \rangle \big) \tag{6.1}$$

in $F\mathsf{M}$. For $g = \iota_s$ and $f = \oplus_j \iota_{r_j}$, the permutation $\sigma^1_{g,f}$ of (5.2) is the identity on $\bar{r}$. Therefore the composite of the morphisms (6.1) is

$$\big(\iota_s \circ (\oplus_j \iota_{r_j}), \gamma(\psi \,;\, \langle \phi \rangle) \big) = \big(\iota_r, \gamma(\psi \,;\, \langle \phi \rangle) \big) = \eta \gamma(\psi \,;\, \langle \phi \rangle).$$

Naturality of η with respect to multifunctors H follows because FH is given by applying H termwise to sequences of objects and operations. The index map of each morphism $(f, \langle \phi \rangle)$ is preserved by FH as in (5.6). Similarly, 2-naturality of η with respect to multinatural transformations κ follows because the components of $F\kappa$ have the identity index map and take sequences of the corresponding components of κ as in (5.7). This completes the proof that

$$\eta \colon 1_{\mathsf{Multicat}} \longrightarrow EF$$

is a 2-natural transformation. $\qquad\square$

Definition 6.3 (Counit). Suppose C is a permutative category. Define a component

$$\varepsilon = \varepsilon_{\mathsf{C}} \colon FE\mathsf{C} \longrightarrow \mathsf{C}$$

as follows. For each index map

$$f \colon \overline{r} \longrightarrow \overline{s}$$

and each length-r sequence of objects $\langle x \rangle$ with $x_i \in \mathsf{C}$, let

$$\bigoplus_{i \in \overline{r}} x_i \xrightarrow{\ \xi_f\ } \bigoplus_{j \in \overline{s}} \bigoplus_{i \in f^{-1}(j)} x_i \qquad (6.2)$$

denote the unique morphism in C given by components of ξ permuting the terms of the sum. Uniqueness of this morphism in C follows from the Symmetric Coherence Theorem [**ML98**, XI.1, Th. 1]. Then ε is the following assignment:

$$\varepsilon\langle x \rangle = \bigoplus_i x_i \quad \text{for} \quad \langle x \rangle \in FE\mathsf{C} \quad \text{and} \qquad (6.3)$$

$$\varepsilon(f, \langle \phi \rangle) = \Big(\bigoplus_j \phi_j \Big) \circ \xi_f \qquad (6.4)$$

where $(f, \langle \phi \rangle) \colon \langle x \rangle \longrightarrow \langle y \rangle$ in $FE\mathsf{C}$. Lemma 6.4 shows that each ε_{C} is a strict symmetric monoidal functor and that the components are 2-natural with respect to strict symmetric monoidal functors and monoidal natural transformations.

Lemma 6.4. *The components ε_{C} of Definition 6.3 define a 2-natural transformation*

$$\varepsilon \colon FE \longrightarrow 1_{\mathsf{PermCat}}.$$

Proof. To check that each component $\varepsilon = \varepsilon_{\mathsf{C}}$ is a strict symmetric monoidal functor, first note that $\varepsilon(1, \langle 1_{x_i} \rangle_i)$ is the identity on $\oplus_i x_i$. Now suppose given a composable pair of morphisms in $FE\mathsf{C}$

$$\langle x \rangle \xrightarrow{\ (f, \langle \phi \rangle)\ } \langle y \rangle \xrightarrow{\ (g, \langle \psi \rangle)\ } \langle z \rangle$$

where $|\langle x \rangle| = r$, $|\langle y \rangle| = s$, and $|\langle z \rangle| = t$. The composite $(\varepsilon g)(\varepsilon f)$ appears along the sides and top of the following diagram, while $\varepsilon(gf)$ appears along the bottom with $\theta_k = \gamma\big(\psi_k \,;\, \langle \phi \rangle_{g^{-1}(k)}\big)$ as in (5.4).

In the above diagram, each of the three unlabeled morphisms is the unique morphism in C given by components of ξ permuting terms. The two triangles at left and bottom-left commute by the Symmetric Coherence Theorem. The triangle at bottom-right

commutes by definition of $\sigma_{g,f}^k$ in (5.2). The triangle at top commutes by naturality of ξ and the triangle at right commutes by definition of θ_k. Therefore each component ε_{C} is functorial.

The monoidal sum in $FE\mathsf{C}$ is given by concatenation of sequences and disjoint union of index maps. So ε_{C} is a strict monoidal functor because

- the monoidal sum in C is strictly associative and functorial,
- the morphisms ξ_f are uniquely determined by the index maps f, and
- the empty monoidal sum in C is the monoidal unit.

Therefore we have

$$\varepsilon\langle x\rangle \oplus \varepsilon\langle x'\rangle = \left(\bigoplus_i x_i\right) \oplus \left(\bigoplus_{i'} x_i'\right) = \varepsilon(\langle x\rangle \oplus \langle x'\rangle)$$

and

$$\varepsilon(f, \langle\phi\rangle) \oplus \varepsilon(f', \langle\phi'\rangle) = \left((\oplus_j \phi) \circ \xi_f\right) \oplus \left((\oplus_{j'} \phi') \circ \xi_{f'}\right) = \varepsilon(f \oplus f', \phi \oplus \phi')$$

for objects $\langle x\rangle$, $\langle x'\rangle$ and morphisms $(f, \langle\phi\rangle)$, $(f', \langle\phi'\rangle)$ in $FE\mathsf{C}$. Moreover, ε_{C} is a symmetric monoidal functor because the symmetry in $FE\mathsf{C}$ is given by block permutation of sequences and hence ε sends the symmetry of $FE\mathsf{C}$ to that of C.

Naturality of ε with respect to strict symmetric monoidal functors

$$P \colon \mathsf{C} \longrightarrow \mathsf{D}$$

is commutativity of the following square for each such P.

$$\begin{array}{ccc} FE\mathsf{C} & \xrightarrow{\varepsilon_{\mathsf{C}}} & \mathsf{C} \\ {\scriptstyle FEP}\Big\downarrow & & \Big\downarrow{\scriptstyle P} \\ FE\mathsf{D} & \xrightarrow{\varepsilon_{\mathsf{D}}} & \mathsf{D} \end{array} \qquad (6.5)$$

The above square commutes on objects $\langle x\rangle$ in $FE\mathsf{C}$ because P is strict monoidal and hence

$$P\left(\oplus_i x_i\right) = \oplus_i P x_i.$$

Commutativity on morphisms $(f, \langle\phi\rangle)$ depends on the following.

- By Definition 5.4, FEP does not change the index map f.
- Since P is a strict symmetric monoidal functor, we have

$$P(\xi_f^{\mathsf{C}}) = \xi_f^{\mathsf{D}}$$

  where ξ_f^{C} and ξ_f^{D} are the unique morphisms of (6.2) induced by the symmetry of C and D, respectively.
- Since P is functorial, it preserves the composition of morphisms in the definition of ε (6.4).

For 2-naturality of ε, suppose

$$\alpha \colon P \longrightarrow Q \colon \mathsf{C} \longrightarrow \mathsf{D}$$

is a monoidal natural transformation between strict symmetric monoidal functors.

The compatibility with monoidal constraints in (3.5) implies that the components of α preserve monoidal sums. Therefore, using (5.7), we have

$$\varepsilon(1, \langle \alpha_{x_i} \rangle) = \bigoplus_i \alpha_{x_i} = \alpha_{\oplus_i x_i} = \alpha_{\varepsilon \langle x \rangle}$$

for each $\langle x \rangle \in FE\mathsf{C}$. This completes the proof that

$$\varepsilon \colon FE \longrightarrow 1_{\mathsf{PermCat}}$$

is a 2-natural transformation. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 6.5. Note that the 2-naturality of ε depends on the assumption that P is a *strict* monoidal functor. Without that assumption, the naturality square (6.5) generally does not commute on objects. This requirement is one of our motivations for restricting the morphisms of $\mathsf{PermCat}$ to be strict symmetric monoidal functors.

Theorem 6.6. *There is a 2-adjunction*

$$F \colon \mathsf{Multicat} \rightleftarrows \mathsf{PermCat} \colon \mathsf{End}$$

with $F \dashv \mathsf{End}$.

Proof. The unit and counit

$$\eta \colon 1_{\mathsf{Multicat}} \longrightarrow EF \quad \text{and} \quad \varepsilon \colon FE \longrightarrow 1_{\mathsf{PermCat}}$$

are shown to be 2-natural transformations in Lemmas 6.2 and 6.4. For each small multicategory M, the composite

$$F\mathsf{M} \xrightarrow{\ F\eta_{\mathsf{M}}\ } FEF\mathsf{M} \xrightarrow{\ \varepsilon_{F\mathsf{M}}\ } F\mathsf{M} \qquad\qquad (6.6)$$

is the identity on objects because each sequence $\langle x \rangle$ is equal to the concatenation of length-1 sequences of its entries. For a morphism $(f, \langle \phi \rangle)$ in $F\mathsf{M}$,

$$(\varepsilon_{F\mathsf{M}})(F\eta_{\mathsf{M}})(f, \langle \phi \rangle) = \varepsilon_{F\mathsf{M}} \left(f, \langle (\iota_{|f^{-1}(j)|}, \phi_j) \rangle_{j \in \overline{s}} \right)$$

is the composite of the two morphisms in the following diagram.

The index map for this composite is

$$\overline{r} \xrightarrow{\ \xi_f\ } \overline{r} \xrightarrow{\ \oplus_j \iota_{|f^{-1}(j)|}\ } \overline{s},$$

which is equal to f. The second component of the composite is $\langle \phi \rangle$ because, for each $k \in \overline{s}$, the permutation $\sigma^k_{\oplus_j \iota_{|f^{-1}(j)|}, \xi_f}$ of (5.2) is the identity.

A similar checking of objects and morphisms shows that the composite

$$E\mathsf{C} \xrightarrow{\ \eta_{E\mathsf{C}}\ } EFE\mathsf{C} \xrightarrow{\ E\varepsilon_{\mathsf{C}}\ } E\mathsf{C}$$

is the identity for each small permutative category C because each ξ_{ι_r} is an identity morphism. This completes the proof that $F \dashv E$ is a 2-adjunction. $\qquad\square$

We close this section with one further construction: a componentwise right-adjoint for ε. This will be used in the proof of Theorem 7.3 to show that ε is a componentwise stable equivalence.

Proposition 6.7. *For each permutative category* C *there is an adjunction of categories*

$$\varepsilon_{\mathsf{C}} \colon FE\mathsf{C} \rightleftarrows \mathsf{C} \colon \rho_{\mathsf{C}}$$

where ρ_{C} *is induced by inclusion of length-1 tuples and* $\varepsilon_{\mathsf{C}} \dashv \rho_{C}$.

Proof. Throughout this proof we write $\varepsilon = \varepsilon_{\mathsf{C}}$ and $\rho = \rho_{\mathsf{C}}$. The functor ρ is defined by the assignments

$$\rho x = (x) \quad \text{for} \quad x \in \mathsf{C} \quad \text{and}$$
$$\rho \phi = (1_{\overline{1}}, (\phi))$$

for morphisms ϕ in C. In this definition both (x) and (ϕ) denote length-1 tuples. The composite

$$\mathsf{C} \xrightarrow{\rho} FE\mathsf{C} \xrightarrow{\varepsilon} \mathsf{C}$$

is the identity. The composite

$$FE\mathsf{C} \xrightarrow{\varepsilon} \mathsf{C} \xrightarrow{\rho} FE\mathsf{C}$$

is given by the assignments

$$\langle x \rangle \longmapsto \left(\oplus_i x_i \right)$$
$$(f, \langle \phi \rangle) \longmapsto \left(1_{\overline{1}}, \left(\oplus_j \phi_j \right) \circ \xi_f \right)$$

where $\langle x \rangle$ and $(f, \langle \phi \rangle)$ are objects and morphisms in $FE\mathsf{C}$. A unit for the adjunction (ε, ρ) is provided by the following. Define a natural transformation $\alpha \colon 1_{FE\mathsf{C}} \longrightarrow \rho \varepsilon$ with components

$$\alpha_{\langle x \rangle} = \left(\iota_r, 1_{\oplus_i x_i} \right) \colon \langle x \rangle \longrightarrow \left(\oplus_i x_i \right) \tag{6.7}$$

for a length-r sequence $\langle x \rangle$ in $FE\mathsf{C}$. In (6.7), $\iota_r \colon \overline{r} \longrightarrow \overline{1}$ denotes the unique morphism. For a morphism

$$(f, \langle \phi \rangle) \colon \langle x \rangle \longrightarrow \langle y \rangle \in FE\mathsf{C},$$

where $\langle x \rangle$ has length r and $\langle y \rangle$ has length s, the naturality diagram for α is the following.

$$\begin{array}{ccc}
\langle x \rangle & \xrightarrow{\quad (f, \langle \phi \rangle) \quad} & \langle y \rangle \\
{\scriptstyle \left(\iota_r, 1_{\oplus_i x_i} \right)} \downarrow & & \downarrow {\scriptstyle \left(\iota_s, 1_{\oplus_j y_j} \right)} \\
\left(\oplus_i x_i \right) & \xrightarrow{\left(1_{\overline{1}}, \left(\oplus_j \phi_j \right) \circ \xi_f \right)} & \left(\oplus_j y_j \right)
\end{array} \tag{6.8}$$

In the above diagram, the first component in each composite is ι_r. The second component in each composite is $(\oplus_j \phi_j) \circ \xi_f$ because the right action of $\sigma^1_{\iota_s, f}$ on $\oplus_j \phi_j$ is equal to the composition with ξ_f.

The triangle identities for unit α and counit $1_{1_{\mathsf{C}}}$ hold as follows. First, for an object x of C, the component $\alpha_{(x)}$ is the identity on (x). Second, for an object $\langle x \rangle$

of $FE\mathsf{C}$, the morphism $\varepsilon(\alpha_{\langle x \rangle})$ is the identity on $\oplus_i x_i$. This finishes the proof that $(\varepsilon, \rho) = (\varepsilon_\mathsf{C}, \rho_\mathsf{C})$ is an adjunction of categories. $\square$

Remark 6.8. Note that the components ρ_C in the proof of Proposition 6.7 are not strictly monoidal, because the monoidal sum in $FE\mathsf{C}$ is given by concatenation. For objects x and x' in C, there is a monoidal constraint morphism

$$(\iota_2, 1_{x \oplus x'}) \colon (x, x') \longrightarrow (x \oplus x').$$

There is also a unit constraint morphism

$$(\iota_0, 1_e) \colon \langle \rangle \longrightarrow (e).$$

One can verify that these satisfy associativity, unit, and symmetry axioms to make ρ_C a symmetric monoidal functor.

7. Multicategories model all connective spectra

For permutative categories and for multicategories, we define stable equivalences via the stable equivalences on K-theory spectra. We let SymSp denote the Hovey–Shipley–Smith category of symmetric spectra [**HSS00**]. We let

$$K \colon \mathsf{PermCat} \longrightarrow \mathsf{SymSp}$$

denote Segal's K-theory functor [**Seg74**] that constructs a connective symmetric spectrum from each small permutative category. See [**JY∞**, Chapters 7 and 8] for a review and further references. For the work below we will need the following two facts about stable equivalences.

Remark 7.1 (Stable equivalences).

1. There is a Quillen model structure on SymSp whose weak equivalences are the stable equivalences [**HSS00**, 3.4.4 and 5.3.8]. In particular, the class of stable equivalences includes isomorphisms, is closed under composition, and has the 2-out-of-3 property.
2. If $P \colon \mathsf{C} \longrightarrow \mathsf{D}$ is a strict symmetric monoidal functor whose underlying functor is a left or right adjoint, then KP is a stable equivalence [**JY∞**, 7.2.5 and 7.8.8].

Definition 7.2. A *stable equivalence* between permutative categories is a strict symmetric monoidal functor P such that KP is a stable equivalence of K-theory spectra. A *stable equivalence* between multicategories is a multifunctor H such that FH is a stable equivalence of permutative categories. Equivalently, the stable equivalences of permutative categories are reflected by K and the stable equivalences of multicategories are reflected by F.

Theorem 7.3. *The free functor F and the endomorphism functor End induce equivalences of homotopy theories*

$$F \colon (\mathsf{Multicat}, \mathcal{S}) \, \underset{\longleftarrow}{\overset{\sim}{}} \, (\mathsf{PermCat}, \mathcal{S}) \colon \mathsf{End}$$

where $\mathcal{S}$ denotes the class of stable equivalences in each category.

Proof. By definition of stable equivalences, F is a relative functor. To see that End is a relative functor, we first note that the components of ε are stable equivalences by Proposition 6.7 and Remark 7.1 (2). Naturality of ε and the 2-out-of-3 property for stable equivalences then imply that

$$F\mathsf{End}(P)\colon F\mathsf{End}(\mathsf{C}) \longrightarrow F\mathsf{End}(\mathsf{D})$$

is a stable equivalence whenever P is a stable equivalence. This, in turn, implies that $\mathsf{End}(P)$ is a stable equivalence. Hence End is a relative functor.

The triangle identities for η and ε, together with the 2-out-of-3 property, imply that the components of η are also stable equivalences. Then the result follows from Proposition 2.10. $\qquad\square$

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