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GRAPHS ASSOCIATED TO FOLD MAPS FROM CLOSED SURFACES TO THE PROJECTIVE PLANE

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Abstract

We describe in this paper how to attach a weighted graph to any fold map (i.e., a stable map without cusps) with planar apparent contour from a closed surface to the projective plane. Also we study necessary and sufficient conditions for a weighted graph with non negatively weighted vertices to be the graph of a fold map from a closed surface to the projective plane.

1. Introduction

Stable maps between surfaces can only have fold curves with isolated cusp points on them (Whitney [**13**]). The global description of a stable map between closed surfaces requires the determination of both, the topological type of its regular set in the domain and the isotopy type of the image of its singular set (branch set or apparent contour) in the range surface. For this purpose, we introduced in [**3**] a graph with weights its vertices that codifies the topological type of the regular set of stable maps from oriented surfaces to the plane. Subsequently in [**4, 5, 6**] we studied the properties of such graphs and their behaviour through surgeries and isotopes maps from a closed orientable surface to the plane and to the sphere. We proved that any bipartite weighted graph can be the graph of a stable map from a closed surface to the plane (and hence to the sphere). It is worth mentioning that a bipartite graph is a graph whose vertices can be divided into two disjoint and independent sets U and V such that every edge connects a vertex in U to one in V. For any natural number d, we also characterized those graphs that can be associated with fold maps from closed surfaces of degree d to the sphere. In a more recent paper [**7**], we considered stable maps between non-orientable surfaces and attached them conveniently defined graphs which extend the idea of graph between orientable surfaces. We proved that any weighted graph with non-negatively weighted vertices is the graph of some stable map from a closed surface to the projective plane.

In the present paper we focalize our attention in the particular case of planar fold maps (see Definition 2.2) including the non-orientable case. We shall thus consider stable maps from closed (non necessarily oriented) surfaces to the projective plane

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and will extend the class of graphs previously defined in a way that enables us to distinguish among orientable and non-orientable regions in the complement of the singular set. We analyze necessary and sufficient conditions for such a graph to be attached to a fold map from a closed surface to the projective plane. Our main result states that any one of these graphs can be attached to a conveniently defined stable map from a (non necessarily oriented) surface to the projective plane.

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2. Graphs of stable maps from closed surfaces to the projective plane

Given two closed surfaces M and N with respective genus g_M and g_N and $f,\ g\colon M \to N$ be smooth maps between them. It is said that f is $\mathcal{A}$-*equivalent* (or *equivalent*) to g if there are orientation-preserving diffeomorphisms, $k\colon M \to M$ and $l\colon N \to N$, such that $g \circ k = l \circ f$. A smooth map $f\colon M \to N$ is said to be *stable* if all maps sufficiently close to f, in the Whitney C^∞-topology (see [**1**]), are equivalent to f. The singular set Σf, of a stable map $f\colon M \to N$, is a finite collection of closed regular simple curves on M made of folds points with possible isolated cusp points and the image of Σf, known as the *apparent contour* or *branch set* of f, is a collection of closed curves in N with normal crossings and isolated singularities corresponding to the cusp points of f. Topological information of the map f may be conveniently encoded in a weighted graph from which the pair $(M, \Sigma f)$ may be reconstructed (up to diffeomorphism) (see [**4, 5**]). The edges and vertices of this graph correspond (respectively) to the singular curves and the connected components of the non-singular set. An edge is incident to a vertex if and only if the singular curve corresponding to the edge lies in the frontier of the regular region corresponding to the vertex. In other words, given a stable map $f\colon M \to N$, its graph $\mathcal{G}(f)$ is the *dual graph* of Σf in M. The weight g_v of a vertex v is defined to be the genus of the corresponding region i.e the genus of the closed surface obtained by glueing a disk along each boundary curve of the region. Clearly, the graph of a stable map is an $\mathcal{A}$-invariant.

Our aim in the present paper is to complete the previous work, extending the definition of graph to the class of planar fold maps from closed (non necessarily oriented) surfaces to the projective plane $\mathbb{P}$ and generalizing the already known results on fold maps from closed orientable surfaces to the plane and the sphere ([**5, 6**]) to fold maps from non necessarily orientable surfaces to the $\mathbb{P}$.

We first observe that when the domain surface M is non-orientable, the graph does no need to be bipartite. Moreover, we have two possibilities for a closed curve γ in M (Figure 1):

a) The curve γ has a small neighbourhood homeomorphic to a cylinder.

b) The curve γ has a small neighbourhood homeomorphic to a Möebius band.

In order to clarify the figures, we will use the cross-cap surface as a model of the projective plane and the connected sum of two cross-cap surfaces as the model of the Klein bottle (see Figure 1).

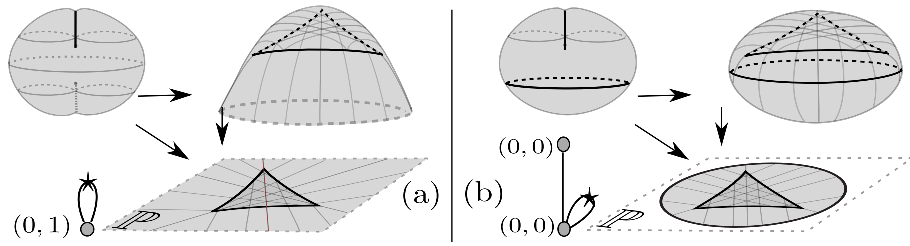

Figure 1: Examples of graphs of stable map with $\star$-loops.

According to this, for a stable map $f\colon M \to \mathbb{P}$ of a closed surface, we shall attach a $\star$ to each edge of the graph of f that corresponds to a singular curve of f having a neighbourhood homeomorphic to a Möebius band. Notice that such an edge always determines a loop, i.e., its both ends correspond to a unique vertex of the graph. Figure 1 illustrates graphs and the apparent contours with three cusps of two stable maps from Klein bottle and projective plane to the $\mathbb{P}$. Here we see that the three cusps appears in the curves having a small neighbourhood homeomorphic to a Möebius band. On the other hand, we observe that in the non-orientable case, the regular set of a stable map may have orientable and non-orientable connected components. We shall reflect this fact in the corresponding graph by attaching a weight $(t, 0)$ to an orientable connected component with genus t and a weight $(0, p)$ to a non-orientable connected component with genus p as illustrated in Figure 2. We denote

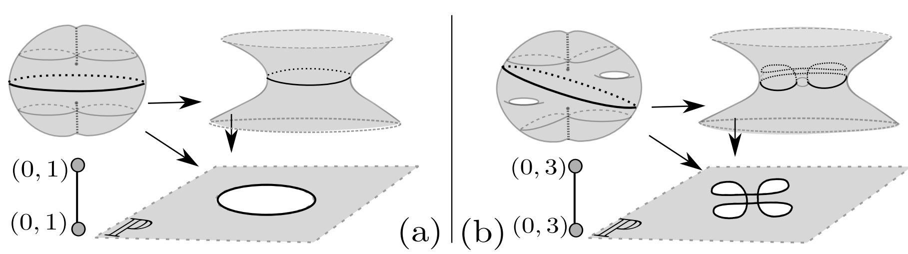

Figure 2: Example of fold maps and their graphs.

by $\mathcal{G}^S_{(T,P)}(V, E)$ a graph having V vertices, E edges, S the number of $\star$-loops, T and P the total sum of the weights of type $(t, 0)$ and $(0, p)$ respectively.

Figure 2 displays two fold maps from closed surfaces to the $\mathbb{P}$, where $P > 0$:

(a) Map from the Klein bottle to $\mathbb{P}$, with an unique singular curve and no double points, whose graph is of type $\mathcal{G}^0_{(0,2)}(2, 1)$;

(b) Map from a surface given by a connected sum of the bitorus with a Klein bottle (i.e., the sum of 6 projective planes) to $\mathbb{P}$, with graph of type $\mathcal{G}^0_{(0,6)}(2, 1)$. This map has a unique singular curve and four double points.

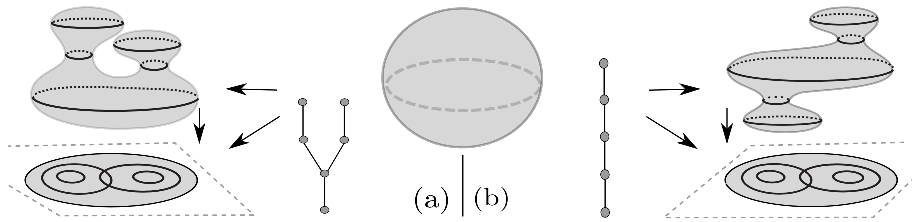

Figure 3: Examples of fold maps from sphere with different graphs.

In Figure 3 we observe that although the two fold maps have equivalent apparent contours, their corresponding graphs are different and thus they are not equivalent as stable maps.

We recall the following result:

Theorem 2.1 ([7]). *Any graph $\mathcal{G}^S_{(T,P)}(V,E)$ is the graph of a stable map from a closed surface M to $\mathbb{P}$. The surface M is orientable if and only if $\mathcal{G}$ is bipartite and $P = S = 0$. Moreover, the genus of M is given by $1 - V + E + T$ in case that M is orientable and by $2(1 - V + E + T) + P - S$ otherwise.*

Given a fold map, $f\colon M \longrightarrow \mathbb{P}$, we denote by D_P a simply connected region (homeomorphic to the disc) in the projective plane $\mathbb{P}$, with boundary $\gamma = D_P \cap F_P$ and by $F_P = \mathbb{P} \setminus D_P$ a Möebius strip, which is the closure of the complement of D_P in $\mathbb{P}$. Observe that $\gamma = D_P \cap F_P$. We can then distinguish among this two possible cases of apparent contour of a stable maps.

Definition 2.2. We say that a stable map $f\colon M \longrightarrow \mathbb{P}$ *has a planar apparent contour* if there is a stable map $h\colon M \longrightarrow \mathbb{P}$ with $h(\Sigma h) \subset D_P$, where the singular sets Σf and Σh are diffeomorphic on M and sets $f(\Sigma f)$ and $h(\Sigma h)$ are diffeomorphic in $\mathbb{P}$. Otherwise, we say that f has a *non-planar apparent contour*. If $h(M) \subset D_P$ then we say that f is a *planar stable map*.

Our purpose in this paper is to determine the possible graph attached to fold maps from closed surfaces (oriented or not) to $\mathbb{P}$, with planar apparent contour, generalizing the already known results on fold maps from closed orientable surfaces to the plane and to the sphere ([**4, 5**]).

We first observe that Figure 1-(b) displays planar stable map with three cusps from projective plane to $\mathbb{P}$, which can be seen as map to the plane (see [**12**]). A possible way to modify the number of cusps of stable maps from projective plane to the $\mathbb{P}$ consists in taking a path in the space of maps from projective plane to the projective plane and make transitions through convenient codimension one strata (see [**2, 10**]). But we observe that all such transitions alter by dois the number of cusps, so we deduce that *there are no planar fold maps from the projective plane*. Moreover, we can assert that any singular curve having a small neighbourhood homeomorphic to a Möebius strip has at least a cusp point, therefore for any fold map we have $S = 0$. For the sake of simplicity, in what follows we shall denote the graph $\mathcal{G}^0_{(T,P)}(V,E)$ as $\mathcal{G}_{(T,P)}(V,E)$.

Given a bipartite graph $\mathcal{G}_{(T,P)}(V,E)$ with weights (t_i, p_i) in its vertices, we can assign labels $\pm$ to them and denote by V^+ (resp. V^-) the total number of vertices labelled with $+$ (resp. with $-$), by T^+ (resp. T^-) the sum of all the t_i weights corresponding to vertices with positive label (resp. with negative label) and by P^+ (resp. P^-) the sum of all the p_i weights corresponding to vertices with positive label (resp. with negative label).

Definition 2.3. A bipartite graph $\mathcal{G}_{(T,P)}(V,E)$ is said to be balanced if it satisfies

$$\mathcal{D} = (V^+ - V^-) - (T^+ - T^-) - (P^+ - P^-) = 0.$$

For fold maps from closed orientable surface to the plane, there is the following result:

Theorem 2.4 ([6])**.** *A bipartite graph $\mathcal{G}_{(T,0)}(V,E)$ is associated with a fold map from a closed orientable surface M to the plane if and only if is balanced, that is, precisely when $(V^+ - V^-) = (T^+ - T^-)$.*

Corollary 2.5. *A bipartite graph $\mathcal{G}_{(T,0)}(V,E)$ is associated with a planar fold map from a closed orientable surface M to the projective plane if and only if is balanced.*

Our aim in this paper is to analyze the case of fold maps with target in the projective plane having a non necessarily bipartite graph and possibly $P > 0$.

3. Codimension one transition and surgeries

We describe in this section how to obtain new stable map by applying convenient codimension one transitions (see [8, 9]) or surgeries (see [5]) to other stable maps.

3.1. Codimension one transition and graphs

In this subsection we will consider transitions that alter only the graph associated to a stable map $f\colon M \to \mathbb{P}$. The *lips* transition, denoted by $\mathbf{L}$, creates new singular curves introducing at the same time couples of cusps. They are the same transitions that change the regular and singular sets of f.

The lips transition ($\mathbf{L}$) increases by one the number of regular regions in M (i.e., vertices in V) and the number of singular curves (i.e., edges in E).

The beaks transition ($\mathbf{B}$) for stable maps from closed surfaces to projective plane can be classified in five different cases (see Figure 4):

- $\mathbf{B}_v^+$: beaks transition increases by one the number of regular regions, i.e., it adds one vertex and one edge on the graph;
- $\mathbf{B}_v^-$: beaks transition decreases by one the number of regular regions, therefore it removes 1 vertex and one edge on the graph;
- $\mathbf{B}_w^+$: beaks transition increases by one the weight, maintains the number of regular regions (vertices) but decreases by one the number of edges;
- $\mathbf{B}_w^-$: beaks transition decreases by one the weight, maintains the number of regular regions (vertices) but increases by one the number of edges;
- $\mathbf{B}_e$: beaks transition does not affect the number of connected components either of the singular set or the regular set, but changes the weight of the two vertices connected by the edge that corresponds to the singular curve at which the transition is performed.

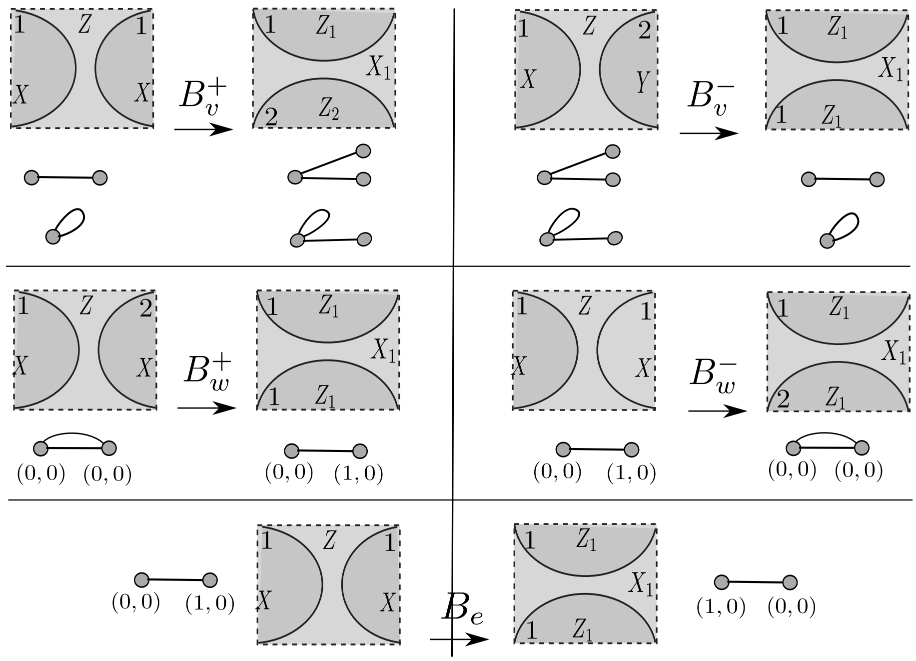

Figure 4: Decomposition of beaks transition.

The five types of beaks transition are illustrated (locally) in Figure 4, where in the picture X, X_1, Y, Z, Z_1 and Z_2 denote (locally) the regular regions where the transitions hold and the numbers 1 and 2 represent the number of singular curves.

On the other hand, in Figure 5 we display four stable maps, together with their corresponding graphs, illustrating the action of beaks transitions that change the weight and vertices of the graph:

(a) Map from the Klein bottle to $\mathbb{P}$ with graph of type $\mathcal{G}_{(0,0)}(2,2)$, having two singular curves.

(b) Map from the Klein bottle to $\mathbb{P}$ with graph of type $\mathcal{G}_{(0,0)}(1,1)$. We observe that in this case, the unique singular curve does not disconnect the regular region. This map has two double points and two cusps and may be obtained through a beaks transition B_v^+ on map (a).

(c) Map from the torus to $\mathbb{P}$ with graph $\mathcal{G}_{(0,0)}(2,2)$. This graph has two double points and two singular curves decomposing the torus into two cylinders.

(d) Map from the torus to $\mathbb{P}$ with a unique singular curve and graph $\mathcal{G}_{(1,0)}(2,1)$. The singular curve separates a disc and a torus with a hole. This map has two double points and two cusps and may be obtained through a beaks transition B_w^+ on map (c).

On the other hand, in Figure 6 we display three stable maps, illustrating an application of beaks transitions B_e exchanging the weights in the vertices of the graph. In the sequel, a lips transitions L on the identity map introduces a new singular curve

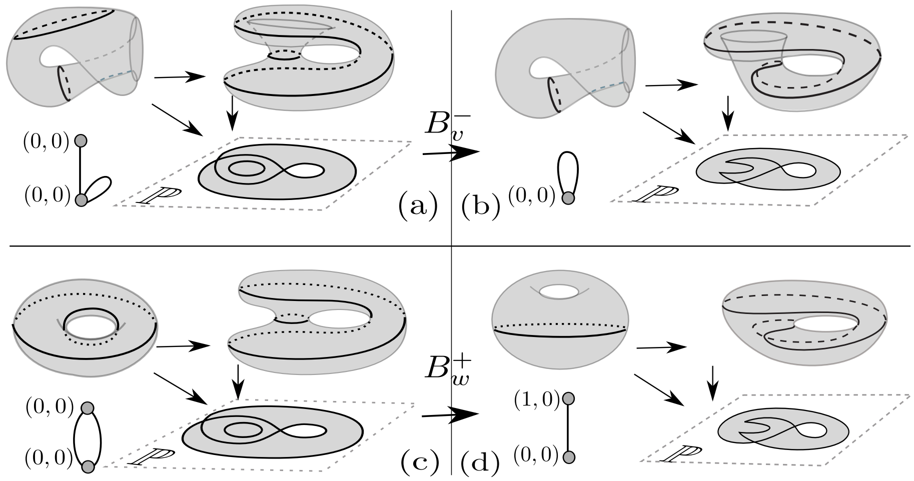

Figure 5: Example of beaks transitions B_v^- and B_w^+.

with two cups. This is followed by a $-B_e$ transition eliminating these two cusps and transforming a region homeomorphic to the disc into a Möebius strip, where the apparent contour is not planar.

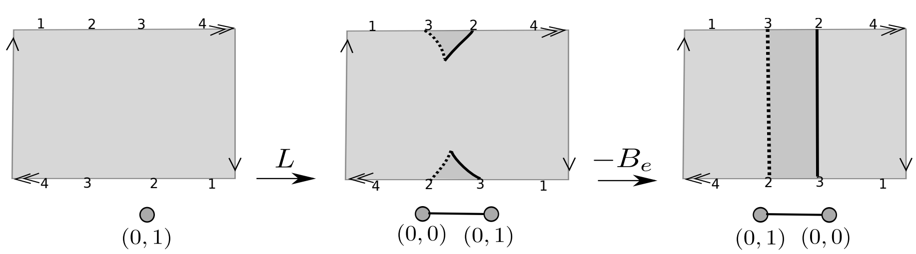

Figure 6: Example of beaks transitions B_e.

We point out that the beaks and lips transitions over the plane alter the number of singular components (see [**9**]) and thus the number of edges of a graph (see Figure 5). Moreover, they also can alter the number of vertices, as well as their corresponding weights.

Observe that in the case of maps with target in the projective plane, the beaks transitions may alter the genus of the affected regions without changing the numbers of singular and regular components (see Figure 6). In such case, just the weights of the two affected vertices are altered, without changing the total sum of the weights.

3.2. Surgeries of stable maps and graphs

We observe that the maps f and h that can be obtained one from the other through convenient codimension 1 transitions lie in the same homotopy class. In order to create new maps belonging to different homotopy classes we can use convenient surgeries as described below.

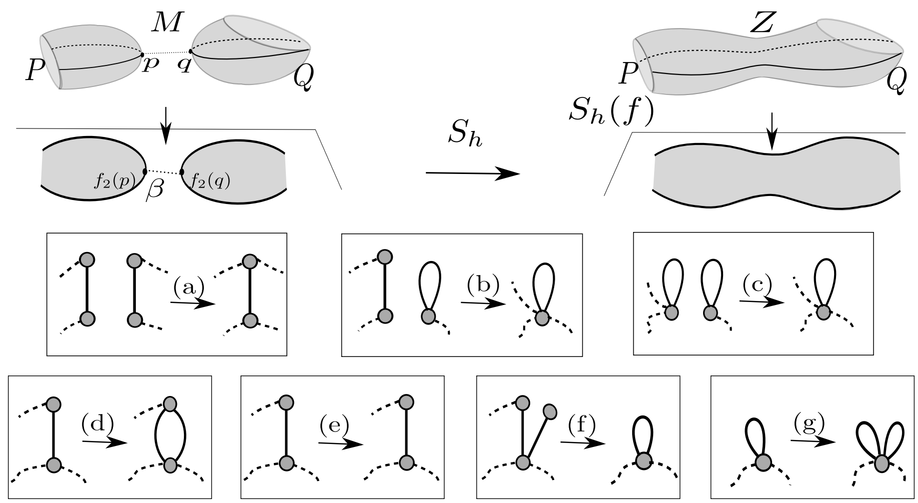

Figure 7: Local horizontal surgery.

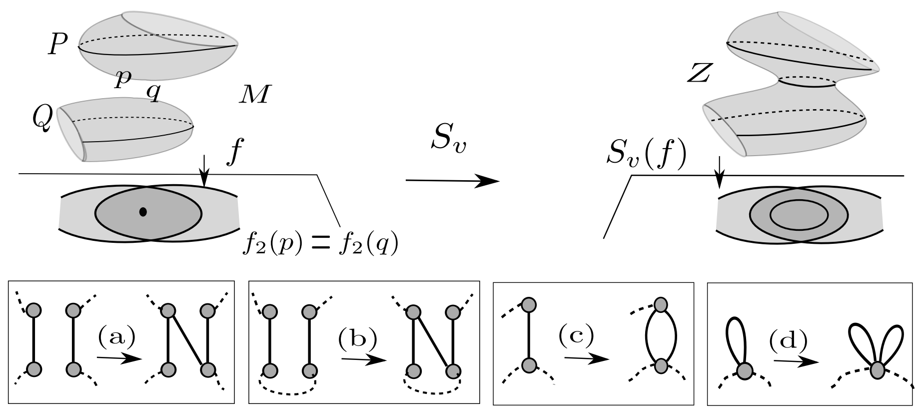

Figure 8: Local vertical surgery.

i) *Horizontal surgery of stable maps (S_h)*: Given a stable map l between two surfaces M and N, a *bridge* is an embedded arc β in N which meets the set of singular values (or apparent contour) in its two end points (and nowhere else), as in Figure 7. The stable map l_β is constructed as follows. The bridge meets $l(M)$ in its end points that we denote by $l(p)$ and $l(q)$, where p and q are singular points of l. Choose small disks in M centred at p and q and replace their interiors by a tube (i.e., an annulus), connecting these two small disks. As illustrated in Figure 7, the map l may be extended over the tube to give the required stable map l_β. In particular, if M is the disjoint union of surfaces P and Q and f and g denote the restrictions of l to P and to Q, respectively, with $p \in P$ and $q \in Q$ then we call the stable map l_β the *horizontal sum* of f and g, and denote it by $f +_h g$. In other words $l = f \cup g$ and $(f \cup g)_\beta = f +_h g$.

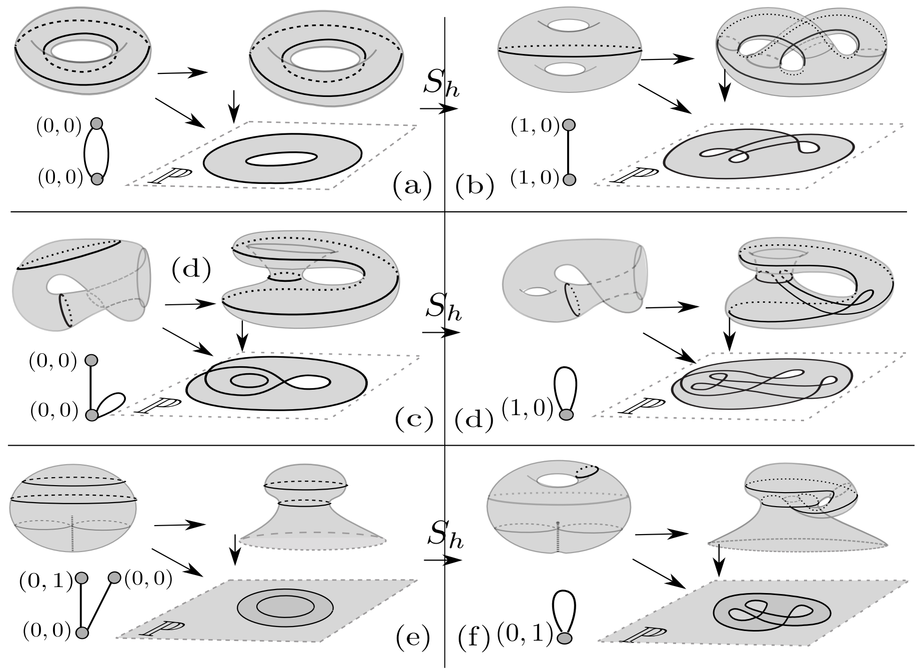

Figure 9: Examples of horizontal surgeries.

ii) *Vertical surgery of stable maps (S_v)*: This is done by identifying two small non-singular disks in the domain, both positive or both negative (as in the Figure 8) whose images in N coincide. These disks are replaced by a tube which is mapped into the plane, with a singular curve running around the middle of the tube. Clearly, this surgery adds a disjoint embedded curve to the branch set. The stable map l_β is said to be the *vertical sum* of f and g and we denote it by $f +_v g$.

In order to illustrate the action of horizontal surgeries on the graphs we display in Figure 9 six fold maps with their corresponding graphs.

(a) This figure corresponds to a fold map from the torus to $\mathbb{P}$ whose graph is of type $\mathcal{G}_{(0,0)}(2, 2)$. It has two singular curves and no double points. This graph has two double points and two singular curves decomposing the torus into two cylinders.

(b) Here we have a stable map from the bitorus to $\mathbb{P}$ with a unique singular curve and four double points, whose graph is of type $\mathcal{G}_{(2,0)}(2, 1)$. This map can be obtained through a horizontal surgery that connects the two singular curves in Figure 9-(a).

(c) This corresponds to a map from the Klein bottle to $\mathbb{P}$ two singular curve, whose graph is of type $\mathcal{G}_{(0,0)}(2, 2)$.

(d) This is a fold map with source the connected sum of four projective planes having six double points. It has a graph of type $\mathcal{G}_{(1,0)}(1, 1)$ and can be obtained through a convenient horizontal surgery on the two singular curves of the map

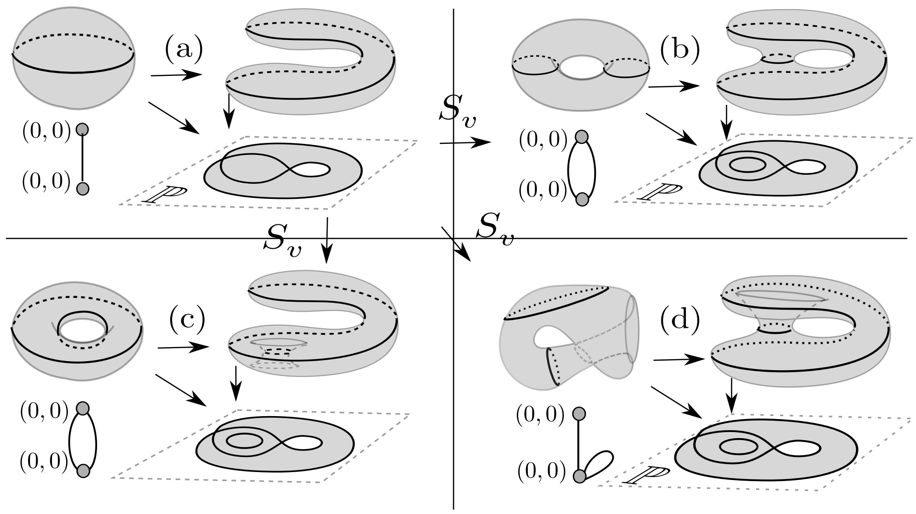

Figure 10: Examples of vertical surgeries.

given in Figure 9-(c).

(e) This figure represents a map from the projective plane to itself with two singular curves which graph is of type $\mathcal{G}_{(0,1)}(3, 2)$.

(f) This is a fold map which its source is the connected sum of three projective planes having four double points and its graph is of type $\mathcal{G}_{(0,1)}(1, 1)$. It can be obtained as the result of a convenient horizontal surgery between the singular curves in Figure 9-(e).

Figure 10 illustrates how to obtain new fold maps by using vertical surgeries: We start with a map from the sphere to $\mathbb{P}$ with graph of type $\mathcal{G}_{(0,0)}(2, 1)$ as displayed in Figure 10-(a). This map has a unique singular curve and two double points. We can apply some convenient vertical surgeries to Figure 10-(a) in order to obtain Figures 10-(b), 10-(c) and 10-(d). Note that in Figures 10-(b) and 10-(c) the surface becomes a torus and the graphs are of type $\mathcal{G}_{(0,0)}(2, 2)$, whereas in Figure 10-(d) the surface is a Klein bottle, but its graph is of type $\mathcal{G}_{(0,0)}(2, 2)$ as well.

We observe that when M is a connected orientable surface, the map $f + g$ obtained through one of the above surgeries is defined on a connected surface Z whose genus is $g(Z) = g(M) + 1$ if Z is orientable and $g(Z) = 2g(M) + 2$ otherwise. If M is a non-orientable surface, then $g(Z) = g(M) + 2$. In Figure 9 we display two stable maps, where Figures 9-(e) and 9-(f) are fold maps, which has a non-planar apparent.

We notice that a vertical surgery adds a new connected component to the singular set (see Figure 10), whereas in the case expanded graph of a horizontal surgery, the number of connected components of the singular set increases by one if p and q are in the same connected component of the singular set or decrease by one if p and q are in different connected components of the singular set (see Figure 9).

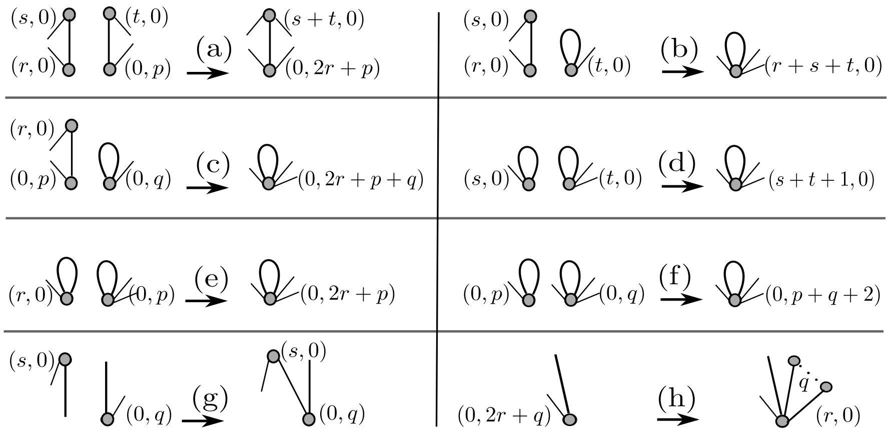

Figure 11: Examples illustrating horizontal surgeries of graphs.

3.3. Surgeries of graphs

The surgeries of stable maps defined above lead in a natural way to surgeries between the corresponding graphs.

1. The connected sum of two oriented regions with respective genus g_1 and g_2 makes a connected region with genus $g_1 + g_2$.
2. The connected sum of an oriented region of genus t with a non-oriented region of genus p is a non-oriented region of genus $2t + p$.

We shall introduce now some natural *surgeries* between graphs. Such surgeries can be performed either by identifying one of the edges (and its respective vertices) of the graph $\mathcal{G}_1$ with an edge (and its respective vertices) of the graph $\mathcal{G}_2$, or through the introduction of a new edge that connects a vertex of the graph $\mathcal{G}_1$ with a vertex of the graph $\mathcal{G}_2$.

Definition 3.1. A *horizontal surgery of two graphs* is carried out by identifying an edge of one of them with an edge of the other one. This gives rise to a new graph and can be done in one of the following ways:

1. The identification of two edges, both of which end at two different vertices (i.e., none of them is a loop) produces an edge that also ends at two different vertices. The weights of the involved vertices are added according to the following rules (see Figure 11-(a)):
   i) $(s, 0) + (t, 0) = (s + t, 0)$,
   ii) $(0, p) + (0, q) = (0, p + q)$,
   iii) $(t, 0) + (0, p) = (0, 2t + p)$.
2. The identification of an edge ending at two different vertices with a loop produces a loop. The weight of the corresponding vertex also follows the above rules (see Figures 11-(b) and 11-(c)), where the weights of the vertices can behave as follows:
   i) $(s, 0) + (t, 0) = (s + t + 1, 0)$,
   ii) $(0, p) + (0, q) = (0, p + q + 2)$,

iii) $(r, 0) + (0, p) = (0, 2(r + 1) + p)$.

3. The identification of two loops produces a loop (see Figures 11-(d), (e) and (f)), where the weights of the vertices can behave as follows:
   i) $(s, 0) + (r, 0) + (t, 0) = (s + r + t, 0)$,
   ii) $(r, 0) + (0, p) + (0, q) = (0, 2r + p + q)$,
   ii) $(0, r) + (0, p) + (0, q) = (0, r + p + q)$.

Definition 3.2. A *vertical surgery of graphs* consists in joining two vertices of different graphs through an extra edge (see Figure 11-(g)).

4. Constructing fold maps with prefixed graphs

Our aim now is to determine all the graphs that can be attached to fold maps from closed surfaces to $\mathbb{P}$. For this purpose we introduce some convenient surgeries.

"Strip-disc" surgery (SD): Let $h\colon M \longrightarrow \mathbb{P}$ be a fold map with planar apparent contour as given in Definition 2.2, whose associated graph is $\mathcal{G}_{(T,P)}(V, E)$. We choose a point y lying in the interior of F_P in the image of h. Suppose that its inverse image $h^{-1}(y)$ has Q points in M, then $M_1 = h^{-1}(F_P)$ is the union of Q Möebius strips embedded in M that do not intersect Σh. We denote by $\gamma_1, \dots, \gamma_Q$, the boundary components of M_1 which are the inverse images of γ through h.

We can construct a planar map (as in Definition 2.2) from h as follows:

By removing the interior of Q Möebius strips in M we can get a new surface M_1 with Q boundary components and genus $g(M) - Q$ in case that M_1 is non-oriented, or $(g(M) - Q)/2$ otherwise. Now, by glueing a disc along each boundary component we obtain a closed surface M_0 with genus $g(M_1) = g(M_0)$.

Figure 11-(h) illustrates the action on the graph of a surgery of type SD ("Disc-strip"). We observe that this leads to a change from $(0, 2r + q)$ to $(r, 0)$ in the weight of one of the vertices v. It also adds q extreme vertices (leaves) of weight $(0, 0)$, all of them connected to v.

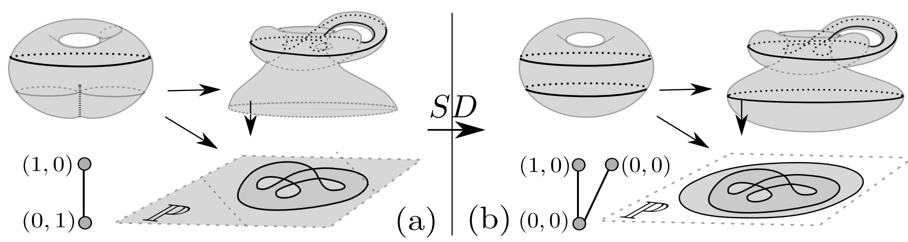

Figure 12: Exchanging a disc with a Möebius strip.

By extending the restriction h_1 of the map h to M_1 over the discs B_i, we obtain a new (not necessarily stable) map $\phi\colon M_0 \longrightarrow \mathbb{P}$, with $\gamma_i \in \Sigma\phi$ and $\phi(\gamma_i) = \gamma$ ($i = 1, \cdots, Q$). Now a convenient perturbation of ϕ over some neighbourhoods of the curves γ_i, leads to a planar fold map $\psi_1\colon M_0 \longrightarrow \mathbb{P}$, where the singular curves ψ and h_1 restricted to the interior of M_1 coincide (up to diffeomorphism) and the set $\{\psi(\gamma_1), \cdots, \psi(\gamma_Q)\} \subset D_P$ is made of simple and disjoint closed curves. A fold map

$\psi\colon M_0 \longrightarrow \mathbb{R}^2$ can now be defined as the composition $\psi = l \circ \psi_1$, where $l\colon D_P \longrightarrow \mathbb{R}^2$ is an embedding.

Remark 4.1. The result of the above "strip-disc" surgery SD on the graph associated to the map consists in substituting the weight $(0, p_i)$ by $(r_i, 0)$, where $p_i = 2r_i + q_i \geqslant 0$ and $Q = \sum_{i=1}^{V} q_i$, connecting q_i extremal vertices (i.e., vertices with degree 1) with a unique edge and weights $(0, 0)$ to each vertex v_i (see Figures 11 and 12).

Definition 4.2. A maximal tree of the graph $\mathcal{G}$ is a connected subgraph (type tree) that contains all the vertices of $\mathcal{G}$.

Definition 4.3. A non-bipartite graph $\mathcal{G}$ with weights $\{(t_j, p_j)_{j=1}^m\}$ in its vertices is said to be *balanced* if it has some balanced maximal tree.

Definition 4.4. We say that two pairs $(t_1, p_1), (t_2, p_2) \in \mathbb{N} \times \mathbb{N}$ are *equivalent* if they satisfy one of the following conditions

1) $t_1 = t_2 = 0$ and $p_1 = p_2$, or
2) $t_1, t_2 \neq 0$ and $2p_1 + t_1 = 2p_2 + t_2$.

We consider now graphs with weights (t_i, p_i) in its vertices.

Definition 4.5. We say that two graphs are *weighted-equivalent* if they are isomorphic as graphs and the corresponding vertices have equivalent weights in the above sense (see Definition 4.4).

Definition 4.6. A graph $\mathcal{G}_{(T,P)}(V, E)$ is said to be a *quasibalanced graph* if it is weighted-equivalent to some balanced graph.

Remark 4.7. Clearly, any balanced graph is quasibalanced too. Also observe that given a quasibalanced graph $\mathcal{G}_{(T,P)}(V, E)$. Any balanced graph weighted-equivalent to it has the form $\mathcal{G}_{(T-k,P+2k)}(V, E)$, for some $k \in I\!N$.

4.1. Realization of quasibalanced trees

Definition 4.8. A graph with a unique edge is said to be *irreducible*. A graph of type $\mathcal{G}_{(T,P)}(2, 1)$ is said to be an *irreducible tree*.

Observe that the horizontal sum or vertical sum of two balanced graphs leads to an balanced graph. The horizontal sum of two irreducible trees leads to an irreducible tree and satisfy the relation

$$S_h(\mathcal{G}_{(P_1,T_1)}(2, 1), \mathcal{G}_{(T_2,P_2)}(2, 1)) = \mathcal{G}_{(T_1+T_2,\, P_1+P_2)}(2, 1).$$

We show in Figure 13 four examples of fold maps to $\mathbb{P}$, associated to irreducible trees, that can be obtained as connected horizontal sums of stable maps. More precisely:

(a) has graph of type $\mathcal{G}_{(2t,0)}(2, 1)$, showing a fold map from the $2t$-tori with a unique singular curve, with $4t$ double points, separating two copies of the t-tori with a hole. This map can be obtained through $t - 1$ horizontal surgeries between t fold maps of the type of Figure 9-(b).

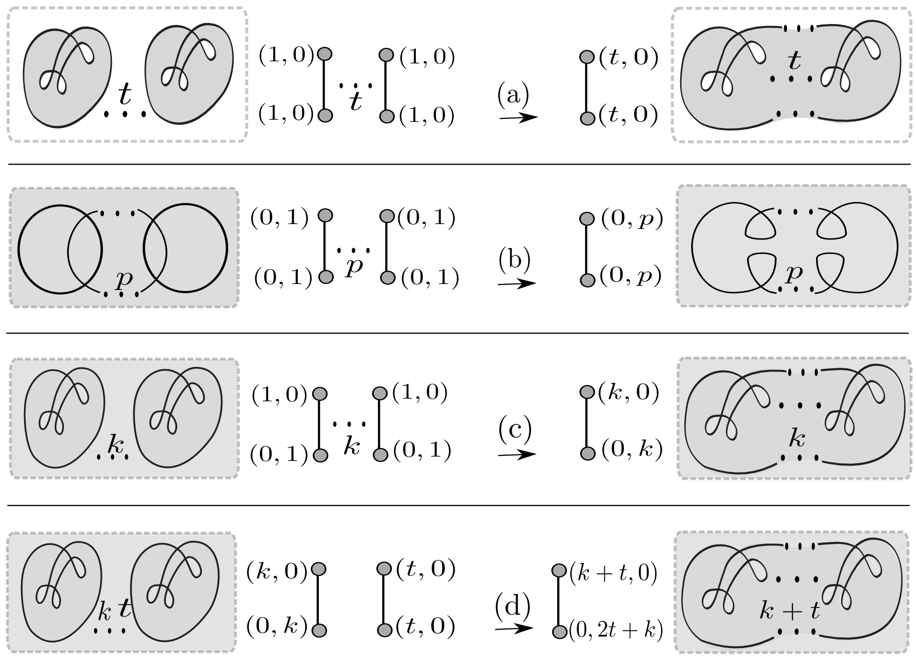

Figure 13: Examples of fold maps with irreducible tree.

(b) has graph of type $\mathcal{G}_{(0,2p)}(2,1)$, showing a fold map, from the $2p$-projective plane to $\mathbb{P}$ with a unique singular curve with $2p-2$ double points, separating two copies of the p-projective plane with a hole. This map can be obtained through $p-1$ horizontal surgeries between p fold maps of the type of Figure 2-(a).

(c) has graph of type $\mathcal{G}_{(0,3k)}(2,1)$, showing a fold map from the $3k$-projective plane to $\mathbb{P}$ with a unique singular curve and $4k$ double points, separating k-tori with a hole and k-projective plane with a hole. This map can be obtained through $k-1$ horizontal surgeries between k fold maps of the type of Figure 12-(a).

(d) has graph of type $\mathcal{G}_{(k+t,k+2t)}(2,1)$, showing a fold map from the $3k+4t$-projective plane to $\mathbb{P}$ with a unique singular curve and $4(k+t)$ double points, separating $k+t$-tori with a hole and $k+2t$-projective plane with a hole. This map can be obtained through one horizontal surgeries between the fold maps (a) and (b) in Figure 12.

Given the graph $\mathcal{G}_{(T,P)}(V,E)$, with $P = 2R^{\pm} + Q^{\pm}$, then $\mathcal{G}^{e}_{(T+R,0)}(V+Q, E+Q)$ will denote the graph obtained by exchanging the weight $(0, 2r^{\pm} + q^{\pm})$ at vertex $v^{\pm}$ by the weight $(r^{\pm}, 0)$ and connecting $q^{\pm}$ vertices of degree 1 in the vertex $v^{\pm}$.

Definition 4.9. The graphs $\mathcal{G}^e_{(T+R,0)}(V+Q,E+Q)$ obtained (as above) from the graph $\mathcal{G}_{(T,P)}(V,E)$, by exchanging weights for vertices, will be called "expanded graphs".

Lemma 4.10. *An irreducible tree of type $\mathcal{G}_{(T,P)}(2,1)$ is the graph of a fold map $f\colon M \longrightarrow \mathbb{P}$ with planar apparent contour if and only if it is a quasibalanced graph.*

The surface M is orientable if and only if $P=0$. Moreover, the genus of M is given by T in case that M is orientable and by $2T+P$ otherwise.

Proof. Suppose that $\mathcal{G}_{(T,P)}(2,1)$ is the tree of a fold map $f\colon M \longrightarrow \mathbb{P}$ with planar apparent contour. Let h be a fold map homotopic to f such that $h(\Sigma h) \subset D_P$ (disk contained in $\mathbb{P}$). We denote by M^+ and M^- the two regular region of h and by $Q^\pm$ the number of preimages of $y \in F_P$ (complementary of D_P in $\mathbb{P}$) in $M^\pm$ corresponding to the vertex $v^\pm$. We denote by $R^\pm$ the genus of orientable region $M^\pm$, complementary of $Q^\pm$ Möebius strips in $M^\pm$. Then the weight $(0,P^\pm)$ in $v^\pm$ satisfies $P^\pm = 2R^\pm + Q^\pm$.

Then we can apply a "Strip-disc" surgeries SD to replace each one of the $Q^\pm$ Möebius strips by a disc, whose boundary is a singular curve for the new fold map $h\colon M_0 \longrightarrow \mathbb{P}$, where M_0 is an orientated surface, with "expanded graph" of type $\mathcal{G}^e_{(T+R,0)}(2+Q,1+Q)$, having $V_0^- = Q^+ + 1$ signed negative vertices and $V_0^+ = Q^- + 1$ signed positive vertices and corresponding weights $T_0^\pm = T^\pm + R^\pm$. It follows from Theorem 2.4 that this graph must satisfy

$$(V_0^+ - V_0^-) - (T_0^+ - T_0^-) = 0. \tag{1}$$

Substitution of $V_0^\pm$ and $T_0^\pm$ in the equality (1) leads to $(T^+ - T^-) + (R^+ - R^-) + (Q^+ - Q^-) = 0$. And from this we can conclude that the graph is quasibalanced. Moreover, it will be balanced provided $R^+ = R^- = 0$.

If $\mathcal{G}_{(T,P)}(2,1)$ is a quasibalanced graph then has five different cases for the weight pairs at its two vertices:

a) $(t,0)$, $(t,0)$: the graph can be realized as in Figure 13-(a).

b) $(0,p)$, $(0,p)$: the graph can be realized as in Figure 13-(b).

c) $(k,0)$, $(0,k)$: the graph can be realized as in Figure 13-(c).

d) $(0,p)$, $(0,2q+r)$, with $p=q+r$: the graph can be realized as connected sum of fold maps associated to two graphs: one with weights $(0,q)$, $(q,0)$ and another with weights $(0,r)$, $(0,r)$.

e) $(t,0)$, $(0,2q+r)$, with $t=q+r$: the graph can be realized as connected sum of fold maps associated to two graphs: one with weights $(q,0)$, $(q,0)$ and another with weights $(r,0)$, $(0,r)$. □

Figure 13 illustrates different ways to realize an irreducible quasibalanced graph of fold maps $f\colon M \longrightarrow \mathbb{P}$ with planar apparent contour.

We shall now extend this result to trees with arbitrary number of vertices.

Lemma 4.11. *Any tree $\mathcal{G}_{(T,P)}(V,V-1)$ of a fold map with planar apparent contour from a closed surface M to $\mathbb{P}$ is a quasibalanced.*

Proof. The proof is analogous to Lemma 4.10. Suppose that $\mathcal{G}_{(T,P)}(V,V-1)$ is the tree, with sign $+$ and $-$ on vertices, associated to fold map $f\colon M \longrightarrow \mathbb{P}$, with planar apparent contour. Let h be a fold map homotopic to f such that $h(\Sigma h) \subset D_P$. We denote by $M^\pm$ the union of the regular regions of h corresponding to the $V^\pm$ vertices of the graph with sign $\pm$. For $y \in F_P$, we denote by $Q^\pm$ the number of preimages of y in $M^\pm$ and by $R^\pm$ the genus of orientable region $M^\pm$, complementary of $Q^\pm$ Möebius strips in $M^\pm$. Thus the weight $P^\pm$ satisfies $P^\pm = 2R^\pm + Q^\pm$.

Then we can apply a "Strip-disc" surgery SD to replace each one of the $Q^\pm$ Möebius strips by a disc, in order to obtain a fold map $f_0\colon M_0 \longrightarrow \mathbb{P}$, associated to the "expanded graph" $\mathcal{G}^e_{(T+R,0)}(V+Q, V-1+Q)$, where M_0 is orientable. Its graph will have $V_0^+ = Q^- + V^+$ positive vertices and $V_0^- = Q^+ + V^-$ negative vertices with corresponding weights $T_0^\pm = T^\pm + R^\pm$. From Theorem 2.4, we know that this graph satisfies

$$(V_0^+ - V_0^-) - (T_0^+ - T_0^-) = 0. \tag{2}$$

By substituying the values $V_0^\pm$ and $T_0^\pm$ in equality (2) and conveniently simplifying, we obtain $(V^+ - V^-) - (T^+ - T^-) - (R^+ - R^-) - (Q^+ - Q^-) = 0$. Finally, from this we get that the graph is quasibalanced. Particularly when $R^+ = R^- = 0$ the graph will be balanced. □

Let's see the converse assertion.

Lemma 4.12. *Any quasibalanced tree $\mathcal{G}_{(T,P)}(V, V-1)$ is the graph of a fold map, with planar apparent contour from a closed surface M to $\mathbb{P}$. The surface M is orientable, with genus T, if and only if, $P = 0$.*

Proof. The case $P = 0$ follows by using Theorem 2.4. Therefore we can suppose $P > 0$. Suppose that $\mathcal{G}_{(T,P)}(V, V-1)$ is a quasibalanced tree with signs $+$ and $-$, such that each edge connects two different vertices, then there is a balanced tree $\mathcal{G}'_{(T+R,Q)}(V, V-1)$ satisfying

$$(V^+ - V^-) - (T^+ - T^-) - (R^+ - R^-) - (Q^+ - Q^+) = 0, \tag{3}$$

where the weights $(0, p_i^\pm)$ in the vertices $v_i^\pm$ satisfy $p_i^\pm = 2r_i^\pm + q_i^\pm$, with $r_i^\pm, q_i^\pm \geqslant 0$ are integer numbers such that $R^\pm = \sum_{i=1}^{V^\pm} r_i$, $Q^\pm = \sum_{i=1}^{V^\pm} q_i$ and $P^\pm = 2R^\pm + Q^\pm$.

We can now construct an auxiliar tree of type $\mathcal{G}'_{(T_0,0)}(V_0, V_0-1)$ as follows: We change the weight $(0, p_i^\pm)$ of the vertex $v_i^\pm$ by $(r_i^\pm, 0)$ and connect $q_i^\mp$ extremal vertices with weights $(0,0)$ to the vertex $v_i^\pm$ (see Figure 11-(h)). Then $T_0^\pm = T^\pm + R^\pm$, $V_0^\pm = V^\pm + Q^\mp$ and from (3) they satisfy $(V_0^+ - V_0^-) - (T_0^+ - T_0^-) = 0$. That is, $\mathcal{G}'_{(T_0,0)}(V_0, V_0-1)$ is a balanced tree (with $P_0 = 0$). Then, from Theorem 2.4 we have that there exists a fold map $h\colon M_0 \longrightarrow \mathbb{P}$, where M_0 is an oriented surface of genus $g(M) = T_0^+ + T_0^-$, whose graph is $\mathcal{G}'_{(T_0,0)}(V_0, V_0-1)$. Notice that the regular region of the map h corresponding to each one of the new extremal vertices $Q^\pm$ in $\mathcal{G}'_{(T_0,0)}(V_0, V_0-1)$, is homeomorphic to a disc whose boundary is a singular curve that corresponds to an extremal edge.

We construct now a new fold map $k\colon M \longrightarrow \mathbb{P}$ with graph $\mathcal{G}_{(T,P)}(V, E)$ as follows: Removing discs D_j, $j = 1, \cdots, q_j^\pm$, with $Q^\pm = q_j^\pm$, whose boundaries are the singular curves corresponding to the extremal edges with weight $(0,0)$, leads to a surface M_0 with $Q^+ + Q^-$ connected components α_i, $i = 1, \cdots, Q$, on its boundary. Now by glueing a Möebius strip along each one of these boundary components, we obtain a closed non-oriented surface M, with genus (by Theorem 2.1)

$$g(M) = 2(1 - V + (V - 1) + T + R) + Q = 2T + P.$$

We can now extend the fold map k on each one of the glued Möebius strips, in a way that k and h coincide on M_0 and k embeds each Möebius strip F_j in $\mathbb{P}$, as illustrated in Figure 12 (i.e., it takes the generator of $H_1(F_j)$ into the generator of the first homology group $\mathbb{H}_1(\mathbb{P})$). We have thus obtained a fold map $f\colon M \longrightarrow \mathbb{P}$, whose graph coincides with $\mathcal{G}_{(T,P)}(V, E)$. $\square$

The following result is a consequence of Lemma 4.11 and 4.12:

Theorem 4.13. *Any tree $\mathcal{G}_{(T,P)}(V, V-1)$ is the graph of a fold map with planar apparent contour from a closed surface M to $\mathbb{P}$ if and only if is a quasibalanced graph.*

4.2. Realization of quasibalanced graph with cycles

We shall see first how to realize any graph with at most two vertices.

The following examples, illustrated in Figure 14, correspond to fold maps with planar apparent contour:

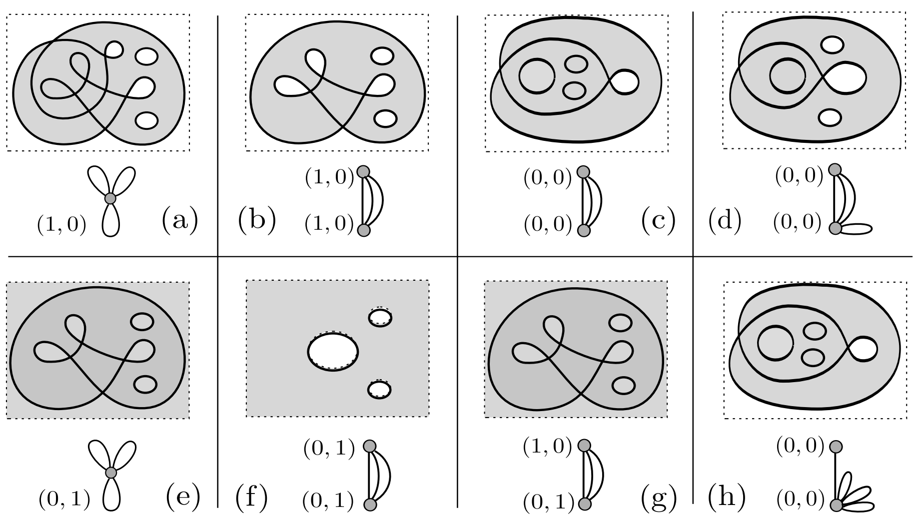

Figure 14: Examples of graphs with one vertex.

1. The graph $\mathcal{G}_{(0,1)}(1, 0)$ can be associated with the identity from projective plane to itself.
2. The graph $\mathcal{G}_{(1,0)}(1, E)$ with $E = 1$ is realized as displayed in Figure 9-(d). In order to implement $E > 1$, we can just apply $E - 1$ convenient vertical surgeries (see Figure 14-(a));
3. The graph $\mathcal{G}_{(0,1)}(1, E)$ with $E = 1$ is realized as displayed in Figure 9-(f). To get $E > 1$, we can apply $E - 1$ vertical surgeries (see Figure 14-(e)).

4. The graph $\mathcal{G}_{(0,0)}(2, E)$ with $E = 1$ is realized through the trivial projection of the 2-sphere into the plane with a unique singular curve, no double points, nor cusps, composed with any embedding of the plane into $\mathbb{P}$. And for $E = 2$ can be realized as in Figure 9-(a). For $E > 2$, we just need to apply $E - 2$ vertical surgeries (see Figure 14-(c) for oriented surfaces or Figures 14-(d) and 14-(h) for non-oriented).
5. The graph $\mathcal{G}_{(2,0)}(2, E)$ with $E = 1$ is realized in Figure 9-(b). For $E > 1$, we can apply convenient $E - 1$ vertical surgeries as illustrated in Figure 14-(b).
6. The graph $\mathcal{G}_{(0,2)}(2, E)$ with $E = 1$ is realized in Figure 2-(a). To realize this graph with $E > 1$ we can apply $E - 1$ vertical surgeries as in Figure 14-(f).
7. The graph $\mathcal{G}_{(1,1)}(2, E)$ with $E = 1$ is realized in Figure 12-(b). The case $E > 1$, is obtained as the result of $E - 1$ convenient vertical surgeries (see Figure 14-(g)).

Now, trough convenient horizontal surgeries between the above maps and those in Figure 11 we can realize any quasibalanced graph of type $\mathcal{G}_{(T,P)}(2, E)$ through a fold map with planar apparent contour, where convenient vertical surgeries may add an aleatory number of edges.

As a consequence we can state the following:

Proposition 4.14. *Any graph of type*

$$\mathcal{G}_{(0,1)}(1,0), \quad \mathcal{G}_{(1,0)}(1,E), \quad \mathcal{G}_{(0,1)}(1,E), \quad \mathcal{G}_{(2,0)}(2,E),$$
$$\mathcal{G}_{(1,1)}(2,E), \quad \mathcal{G}_{(0,2)}(2,E), \quad \textit{or} \quad \mathcal{G}_{(0,0)}(2,E)$$

is the graph of a fold map with planar apparent contour from a closed surface M to $\mathbb{P}$.

Corollary 4.15. *Any quasibalanced graph of type $\mathcal{G}_{(T,P)}(2, E)$ is the graph of a fold map with planar apparent contour from a closed surface M to $\mathbb{P}$.*

Definition 4.16. We recall that a *maximal tree* in a connected graph $\mathcal{G}$ is a subgraph of type tree that contains all the vertices of $\mathcal{G}$.

Remark 4.17. Observe that a bipartite graph $\mathcal{G}_{(T,P)}(V, E)$ is a quasibalanced graph if and only if any one of its maximal tree subgraphs is quasibalanced.

Theorem 4.18. *Any bipartite graph $\mathcal{G}_{(T,P)}(V, E)$ is the graph of a fold map with planar apparent contour from a closed surface M to $\mathbb{P}$ if and only if it is quasibalanced.*

Proof. The proof that any bipartite quasibalanced graph can be the graph of a fold map from a closed surface to $\mathbb{P}$ with planar apparent contour is analogous to that of Lemma 4.12. On the other hand, the proof that any bipartite graph associated to a fold map $f : M \longrightarrow \mathbb{P}$ with planar apparent contour is quasibalanced is analogous to that of Lemma 4.11. □

Theorem 4.19. *Any quasibalanced graph is the graph of a fold map from a closed surface to $\mathbb{P}$ with planar apparent contour.*

Proof. Let $\mathcal{G}_{(T,P)}(V, E)$ be a quasibalanced graph with β cycles and $\mathcal{G}_{(T,P)}(V, E - \beta)$ a quasibalanced maximal tree of $\mathcal{G}_{(T,P)}(V, E)$. By Lemma 4.12, there is a fold map

$f\colon M \longrightarrow \mathbb{P}$ with planar apparent contour, associated to $\mathcal{G}_{(T,P)}(V, E-\beta)$. A fold map associated to the graph $\mathcal{G}_{(T,P)}(V,E)$ can be obtained by β vertical surgeries (convenient) creating the singular curves corresponding to the β removed edges. $\square$

In [11] the authors introduce a computational tool that checks theoretical conditions in order to determine whether a weighted graph, as a topological invariant of stable maps can be associated to fold maps from closed surfaces to $\mathbb{P}$.

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