

THE STABLE HULL OF AN EXACT ∞ -CATEGORY

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Abstract

We construct a left adjoint $\mathcal{H}^{\text{st}}: \mathbf{Ex}_\infty \rightarrow \mathbf{St}_\infty$ to the inclusion $\mathbf{St}_\infty \hookrightarrow \mathbf{Ex}_\infty$ of the ∞ -category of stable ∞ -categories into the ∞ -category of exact ∞ -categories, which we call the stable hull. For every exact ∞ -category \mathcal{E} , the unit functor $\mathcal{E} \rightarrow \mathcal{H}^{\text{st}}(\mathcal{E})$ is fully faithful and preserves and reflects exact sequences. This provides an ∞ -categorical variant of the Gabriel–Quillen embedding for ordinary exact categories. If \mathcal{E} is an ordinary exact category, the stable hull $\mathcal{H}^{\text{st}}(\mathcal{E})$ is equivalent to the bounded derived ∞ -category of \mathcal{E} .

1. Introduction

Every abelian category has a canonical structure of an ordinary exact category given by the class of all short exact sequences. Conversely, every ordinary small exact category admits an embedding into an abelian category with good properties, the Gabriel–Quillen embedding.

Theorem 1.1 ([TT07, Th. A.7.1]). *Let \mathcal{E} be a small exact category. Then, there is an abelian category \mathcal{A} and a fully faithful exact functor $\mathcal{E} \rightarrow \mathcal{A}$ that reflects exact sequences. Moreover, \mathcal{E} is closed under extensions in \mathcal{A} .*

As extension-closed subcategories of abelian categories are exact categories, this gives an alternative description of exact categories as extension-closed subcategories of abelian categories.

From the point of view of derived categories, it is natural to work with the structure of triangulated categories, which do not have good categorical properties. In recent years, Lurie proposed an enhancement for triangulated categories called *stable ∞ -categories* [Lur09b].

Exact ∞ -categories were introduced by Barwick in [Bar15] as a generalization of ordinary exact categories in the sense of Quillen [Qui73]. Small exact ∞ -categories together with exact functors between them can be organized into an ∞ -category, \mathbf{Ex}_∞ . The ∞ -category \mathbf{Ex}_∞ contains as a full subcategory both

- the category of ordinary small exact categories and exact functors between them and

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- the ∞ -category \mathbf{St}_∞ of small stable ∞ -categories and exact functors between them.

In this article we construct a functor

$$\mathcal{H}^{\text{st}}: \mathbf{Ex}_\infty \rightarrow \mathbf{St}_\infty$$

called the *stable hull* functor. The main result of this article is the following theorem, whose proof is given at the end of Section 4.2.

Theorem 1.2. *The functor \mathcal{H}^{st} is left adjoint to the inclusion $\mathbf{St}_\infty \hookrightarrow \mathbf{Ex}_\infty$. Moreover, for every exact ∞ -category \mathcal{E} , the unit functor $\eta_{\mathcal{E}}: \mathcal{E} \rightarrow \mathcal{H}^{\text{st}}(\mathcal{E})$*

1. *is fully faithful,*
2. *preserves and reflects exact sequences and*
3. *the essential image of \mathcal{E} in $\mathcal{H}^{\text{st}}(\mathcal{E})$ is closed under extensions.*

In fact, for every exact ∞ -category \mathcal{E} and every stable ∞ -category \mathcal{C} , restriction along the unit functor $\mathcal{E} \rightarrow \mathcal{H}^{\text{st}}(\mathcal{E})$ induces an equivalence of ∞ -categories

$$\text{Fun}^{\text{ex}}(\mathcal{H}^{\text{st}}(\mathcal{E}), \mathcal{C}) \xrightarrow{\simeq} \text{Fun}^{\text{ex}}(\mathcal{E}, \mathcal{C})$$

between the full subcategories of the functor ∞ -categories spanned by those functors which are exact. The study of a closely related universal property of the bounded derived category of an ordinary exact category was initiated by Keller using *epivalent towers* [Kel91] and later advanced by Porta [Por17] using the language of derivators. Recently, Bunke, Cisinski, Kasprowski and Winges showed that for an ordinary exact category \mathcal{E} , the canonical functor $\mathcal{E} \rightarrow D^b(\mathcal{E})$ into the bounded derived ∞ -category of \mathcal{E} is the universal exact functor into a stable ∞ -category [BCKW19, Cor. 7.59]. Their result readily implies the existence of a canonical equivalence

$$D^b(\mathcal{E}) \xrightarrow{\simeq} \mathcal{H}^{\text{st}}(\mathcal{E}). \tag{1}$$

Note that if \mathcal{E} is moreover abelian, a universal property of $D^b(\mathcal{E})$ as a stable ∞ -category with a t -structure was established by Antieau, Gepner and Heller [AGH19, Prop. 3.26]. In the context of exact categories, t -structures are in general not available.

Theorem 1.2 and the equivalence in (1) show that unit functor $\eta_{\mathcal{E}}: \mathcal{E} \rightarrow \mathcal{H}^{\text{st}}(\mathcal{E})$ provides both

- an ∞ -categorical variant of the Gabriel–Quillen embedding and
- a generalization of the bounded derived ∞ -category of an ordinary exact category to the more general class of exact ∞ -categories.

Furthermore, in light of [Bar15, Ex. 3.5], Theorem 1.2 immediately yields the following characterization of exact ∞ -categories:

Corollary 1.3. *Let \mathcal{E} be a small additive ∞ -category and let $\mathcal{E}_\dagger, \mathcal{E}^\dagger$ be subcategories of \mathcal{E} . The following are equivalent:*

1. *The triple $(\mathcal{E}, \mathcal{E}_\dagger, \mathcal{E}^\dagger)$ is an exact ∞ -category.*
2. *There exists a stable ∞ -category \mathcal{C} and a fully faithful functor $\mathcal{E} \hookrightarrow \mathcal{C}$ such that*
 - (a) *the essential image of \mathcal{E} is closed under extensions in \mathcal{C} ,*
 - (b) *a morphism in \mathcal{E} lies in \mathcal{E}_\dagger if and only if its cofiber in \mathcal{C} lies in the essential image of \mathcal{E} and*
 - (c) *a morphism in \mathcal{E} lies in \mathcal{E}^\dagger if and only if its fiber in \mathcal{C} lies in the essential image of \mathcal{E} .*

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2. Preliminaries

We use freely the language of ∞ -categories as developed in [Lur09a, Lur17].

2.1. Stable and prestable ∞ -categories

First proposed by Lurie [Lur09b], stable ∞ -categories provide an enhancement of triangulated categories in the sense of Verdier [Ver96].

Definition 2.1 ([Lur17, Prop. 1.1.1.9]). An ∞ -category \mathcal{C} is *stable* if it has the following properties:

1. The ∞ -category \mathcal{C} is pointed, that is it admits a zero object.
2. The ∞ -category \mathcal{C} admits finite limits and finite colimits.
3. A square in \mathcal{C} is cocartesian if and only if it is cartesian.

Proposition 2.2 ([Lur17, Th. 1.1.2.14]). *Let \mathcal{C} be a stable ∞ -category. The homotopy category $\mathrm{Ho}(\mathcal{C})$ is additive and has a canonical triangulated structure.*

Definition 2.3 ([Lur17, Def. 1.2.1.4]). Let \mathcal{C} be a stable ∞ -category. A *t-structure* on \mathcal{C} is a t-structure on the homotopy category $\mathrm{Ho}(\mathcal{C})$. More precisely, if \mathcal{C} is equipped with a t-structure, we denote the full subcategories of \mathcal{C} spanned by those objects belonging to $(\mathrm{Ho}(\mathcal{C}))_{\geq n}$ and $(\mathrm{Ho}(\mathcal{C}))_{\leq n}$ by $\mathcal{C}_{\geq n}$ and $\mathcal{C}_{\leq n}$, respectively.

Proposition 2.4 ([Lur17, Prop. 1.2.1.5, Prop. 1.2.1.10, Rem. 1.2.1.1]). *Let \mathcal{C} be a stable ∞ -category with a t-structure. Then the following statements hold:*

1. For all $n \in \mathbb{N}$, the inclusion $\mathcal{C}_{\geq n} \hookrightarrow \mathcal{C}$ admits a right adjoint $\tau_{\geq n}: \mathcal{C} \rightarrow \mathcal{C}_{\geq n}$.
2. For all $n \in \mathbb{N}$, the inclusion $\mathcal{C}_{\leq n} \hookrightarrow \mathcal{C}$ admits a left adjoint $\tau_{\leq n}: \mathcal{C} \rightarrow \mathcal{C}_{\leq n}$.
3. For all $m, n \in \mathbb{N}$, there is a natural equivalence

$$\tau_{\leq m} \circ \tau_{\geq n} \simeq \tau_{\geq n} \circ \tau_{\leq m}.$$

4. The heart $\mathcal{C}^{\heartsuit} := \mathcal{C}_{\leq 0} \cap \mathcal{C}_{\geq 0}$ is equivalent to the nerve of its homotopy category, which is an abelian category.

We denote the composite $\tau_{\leq 0} \circ \tau_{\geq 0}$ by $\pi_0: \mathcal{C} \rightarrow \mathcal{C}^{\heartsuit}$.

Definition 2.5 ([Lur20, Constr. C.1.1.1]). Let \mathcal{D} be a pointed ∞ -category which admits finite colimits and let $\Sigma: \mathcal{D} \rightarrow \mathcal{D}$ be the suspension functor, given on objects by $\Sigma X = 0 \sqcup_X 0$. We denote the direct limit of the tower

$$\mathcal{D} \xrightarrow{\Sigma} \mathcal{D} \xrightarrow{\Sigma} \mathcal{D} \xrightarrow{\Sigma} \dots$$

by $SW(\mathcal{D})$ and refer to it as the *Spanier–Whitehead ∞ -category of \mathcal{D}* . It comes with a canonical functor $\mathcal{D} \rightarrow SW(\mathcal{D})$.

Remark 2.6. As can be seen from the definition, the ∞ -category $\mathcal{SW}(\mathcal{D})$ is generated by the essential image of \mathcal{D} under finite suspension.

Proposition 2.7 ([Lur20, Prop. C.1.1.7]). *Let \mathcal{D} be a pointed ∞ -category which admits finite colimits. The ∞ -category $\mathcal{SW}(\mathcal{D})$ is a stable ∞ -category and, for every stable ∞ -category \mathcal{C} , restriction along the functor $\mathcal{D} \rightarrow \mathcal{SW}(\mathcal{D})$ induces an equivalence*

$$\mathrm{Fun}^{ex}(\mathcal{SW}(\mathcal{D}), \mathcal{C}) \xrightarrow{\simeq} \mathrm{Fun}^{rex}(\mathcal{D}, \mathcal{C})$$

between the full subcategory of $\mathrm{Fun}(\mathcal{SW}(\mathcal{D}), \mathcal{C})$ spanned by the exact functors and the full subcategory of $\mathrm{Fun}(\mathcal{D}, \mathcal{C})$ spanned by the right exact functors.

Definition 2.8 ([Lur20, Def. C.1.2.1, Prop. C.1.2.2]). *Let \mathcal{D} be a pointed ∞ -category admitting finite colimits. The following conditions are equivalent:*

1. *The canonical functor $\mathcal{D} \rightarrow \mathcal{SW}(\mathcal{D})$ is fully faithful and the essential image is closed under extensions.*
2. *The ∞ -category \mathcal{D} is equivalent to a full subcategory of a stable ∞ -category \mathcal{C} which is closed under extensions and finite colimits.*

*If \mathcal{D} has these properties, we call \mathcal{D} a *prestable ∞ -category*.*

Example 2.9. Let \mathcal{D} be a prestable ∞ -category, and let \mathcal{D}' be a full subcategory of \mathcal{D} which is closed under extensions and finite colimits. Then \mathcal{D}' is a prestable ∞ -category and the canonical exact functor $i: \mathcal{SW}(\mathcal{D}') \rightarrow \mathcal{SW}(\mathcal{D})$ is exact. As i restricts to the fully faithful composite $\mathcal{D}' \hookrightarrow \mathcal{D} \hookrightarrow \mathcal{SW}(\mathcal{D})$ and $\mathcal{SW}(\mathcal{D}')$ is generated by \mathcal{D}' under finite suspensions, i is fully faithful.

Definition 2.10. An ∞ -category is *additive* if it has finite products and finite coproducts and its homotopy category is additive.

Definition 2.11. Let \mathcal{A} be a small additive ∞ -category. We write $\mathcal{P}_\Sigma(\mathcal{A})$ for the full subcategory of the presheaf category of spaces spanned by those presheaves which preserve finite products. The ∞ -category $\mathcal{P}_\Sigma(\mathcal{A})$ is often called the *nonabelian derived ∞ -category of \mathcal{A}* .

Notation 2.12. Let \mathcal{B} be a small additive 1-category. We define

$$\mathrm{Mod}(\mathcal{B}) := \mathrm{Fun}^\pi(\mathcal{B}^{op}, \mathbf{Ab}).$$

The following proposition is one of the key inputs of this article.

Proposition 2.13 ([Lur20, Prop. C.1.5.7]). *Let \mathcal{A} be a small additive ∞ -category. Restriction along the infinite loops functor*

$$\Omega^\infty: \mathbf{Sp}^{cn} \rightarrow \mathbf{S}$$

from the ∞ -category of connective spectra to the ∞ -category of spaces induces an equivalence of ∞ -categories

$$\mathrm{Fun}^\pi(\mathcal{A}^{op}, \mathbf{Sp}^{cn}) \xrightarrow{\simeq} \mathrm{Fun}^\pi(\mathcal{A}^{op}, \mathbf{S})$$

between the full subcategories of the functor categories spanned by those functors which preserve finite products.

As a consequence, the ∞ -category $\mathcal{P}_\Sigma(\mathcal{A})$ is prestable and there exists a t -structure on $\mathcal{SW}(\mathcal{P}_\Sigma(\mathcal{A}))$ such that

$\mathcal{SW}(\mathcal{P}_\Sigma(\mathcal{A}))_{\geq 0}$ is the essential image of $\mathcal{P}_\Sigma(\mathcal{A})$ in $\mathcal{SW}(\mathcal{P}_\Sigma(\mathcal{A}))$.

Under this t -structure, the heart of $\mathcal{SW}(\mathcal{P}_\Sigma(\mathcal{A}))$ identifies with $\text{Mod}(\text{Ho}(\mathcal{A}))$. In particular, for all $F \in \mathcal{P}_\Sigma(\mathcal{A})$, $x \in \mathcal{A}$, the homotopy group $\pi_0(F(x))$ is canonically isomorphic to $(\pi_0(F))(x)$.

2.2. Exact ∞ -categories

Definition 2.14 ([Bar15]). An exact ∞ -category is a triple $(\mathcal{E}, \mathcal{E}_\dagger, \mathcal{E}^\dagger)$ where \mathcal{E} is an additive ∞ -category and $\mathcal{E}_\dagger, \mathcal{E}^\dagger$ are subcategories of \mathcal{E} such that the following conditions hold:

1. Every morphism whose domain is a zero object is in \mathcal{E}_\dagger . Every morphism whose codomain is a zero object is in \mathcal{E}^\dagger .
2. Pushouts of morphisms in \mathcal{E}_\dagger exist and every pushout of a morphism in \mathcal{E}_\dagger is in \mathcal{E}_\dagger ; pullbacks of morphisms in \mathcal{E}^\dagger exist and every pullback of a morphism in \mathcal{E}^\dagger is in \mathcal{E}^\dagger .
3. For a square in \mathcal{E} of the form

$$\begin{array}{ccc} x & \xrightarrow{f} & y \\ \downarrow g & & \downarrow g' \\ x' & \xrightarrow{f'} & y' \end{array},$$

the following conditions are equivalent:

- (a) The square is cocartesian, $f \in \mathcal{E}_\dagger$ and $g \in \mathcal{E}^\dagger$.
- (b) The square is cartesian, $f' \in \mathcal{E}_\dagger$ and $g' \in \mathcal{E}^\dagger$.

We call the morphisms in \mathcal{E}_\dagger *cofibrations* and, in diagrams, mark them by tails. We call the morphisms in \mathcal{E}^\dagger *fibrations* and, in diagrams, mark them by double heads.

In the following, we suppress $\mathcal{E}_\dagger, \mathcal{E}^\dagger$ from the notation.

Definition 2.15. Let \mathcal{E} be an exact ∞ -category. An *exact sequence* is a bicartesian square in \mathcal{E} of the form

$$\begin{array}{ccc} x & \xrightarrow{i} & y \\ \downarrow & \square & \downarrow p \\ 0 & \xrightarrow{\quad} & z \end{array} \quad (2)$$

such that i is a cofibration and p is a fibration. We sometimes write $x \xrightarrow{i} y \xrightarrow{p} z$ for such a sequence. We call z the *cofiber of i* and x the *fiber of p* . We sometimes write $\text{cofib}(i)$ for z and $\text{fib}(p)$ for x .

Example 2.16. Let \mathcal{E} be an ordinary exact category in the sense of Quillen. The exact structure on \mathcal{E} endows (the nerve of) \mathcal{E} with the structure of an exact ∞ -category, and vice versa.

Example 2.17. Let \mathcal{C} be an additive ∞ -category. The triple $(\mathcal{C}, \mathcal{C}, \mathcal{C})$ is an exact ∞ -category if and only if \mathcal{C} is stable.

Example 2.18 (cf. [Bar15, Ex. 3.5]). Let \mathcal{D} be a prestable ∞ -category and denote by \mathcal{D}' the subcategory of \mathcal{D} spanned by morphisms p occurring in bicartesian squares of the form (2). Then $(\mathcal{D}, \mathcal{D}, \mathcal{D}')$ is an exact ∞ -category.

3. The ∞ -category of finite additive presheaves

Definition 3.1. Let \mathcal{E} be a small exact ∞ -category. We denote by $\mathcal{P}_{\Sigma, f}(\mathcal{E})$ the smallest subcategory of $\mathcal{P}_{\Sigma}(\mathcal{E})$ which contains \mathcal{E} and is closed under finite colimits.

Proposition 3.2. *The ∞ -category $\mathcal{P}_{\Sigma, f}(\mathcal{E})$ admits finite colimits and, for every ∞ -category \mathcal{D} which admits finite colimits, restriction along the inclusion $\mathcal{E} \hookrightarrow \mathcal{P}_{\Sigma, f}(\mathcal{E})$ induces an equivalence*

$$\mathrm{Fun}^{\mathrm{rex}}(\mathcal{P}_{\Sigma, f}(\mathcal{E}), \mathcal{D}) \xrightarrow{\simeq} \mathrm{Fun}^{\Sigma}(\mathcal{E}, \mathcal{D})$$

between the full subcategory of $\mathrm{Fun}(\mathcal{P}_{\Sigma, f}(\mathcal{E}), \mathcal{D})$ spanned by those functors which are right exact and the full subcategory of $\mathrm{Fun}(\mathcal{E}, \mathcal{D})$ spanned by those functors which preserve finite coproducts.

Proof. By taking the smallest subcategory of $\mathcal{P}_{\Sigma}(\mathcal{E})$ which contains \mathcal{E} and is closed under finite colimits, we follow the construction of the ∞ -category with the claimed universal property given in the proof of [Lur09a, Prop. 5.3.6.2]. \square

Notation 3.3. Let $f: x \rightarrow y$ be a morphism in \mathcal{E} . We denote its image in $\mathcal{P}_{\Sigma, f}(\mathcal{E}) \subseteq \mathcal{P}_{\Sigma}(\mathcal{E})$ under the Yoneda embedding by $\hat{f}: \hat{x} \rightarrow \hat{y}$. Similarly, we denote its image in $\mathrm{Mod}(\mathrm{Ho}(\mathcal{E}))$ under the composite $\mathcal{E} \hookrightarrow \mathcal{P}_{\Sigma}(\mathcal{E}) \xrightarrow{\pi_0} \mathrm{Mod}(\mathrm{Ho}(\mathcal{E}))$ by $\bar{f}: \bar{x} \rightarrow \bar{y}$.

Proposition 3.4. *Let $x \in \mathcal{E}$ and $F \in \mathcal{P}_{\Sigma}(\mathcal{E})$. The homotopy group $\pi_0(\mathrm{Map}(\hat{x}, \Sigma F))$ is trivial.*

Proof. By the Yoneda Lemma, there is an isomorphism of abelian groups

$$\pi_0(\mathrm{Map}(\hat{x}, \Sigma F)) \cong \pi_0((\Sigma F)(x)).$$

The codomain identifies with $\pi_0(\Sigma F)(x)$; this is a trivial group as F is in the aisle of the standard t-structure on $\mathcal{S}\mathcal{W}(\mathcal{P}_{\Sigma}(\mathcal{E}))$. \square

Proposition 3.5. *The ∞ -category $\mathcal{P}_{\Sigma, f}(\mathcal{E})$ is prestable.*

Proof. As, by definition, $\mathcal{P}_{\Sigma, f}(\mathcal{E})$ is closed under finite colimits inside $\mathcal{P}_{\Sigma}(\mathcal{E})$, it suffices to prove that $\mathcal{P}_{\Sigma, f}(\mathcal{E})$ is closed under extensions in $\mathcal{P}_{\Sigma}(\mathcal{E})$. Equivalently, we prove that the cofiber of every map $Z \rightarrow \Sigma X$ with $Z, X \in \mathcal{P}_{\Sigma, f}(\mathcal{E})$ lies in $\Sigma \mathcal{P}_{\Sigma, f}(\mathcal{E}) \subseteq \Sigma \mathcal{P}_{\Sigma}(\mathcal{E})$. We show this by proving the following: For every object $X \in \mathcal{P}_{\Sigma, f}(\mathcal{E})$ the class of objects $Z \in \mathcal{P}_{\Sigma, f}(\mathcal{E})$ satisfying this property contains the essential image of the Yoneda embedding and is closed under finite colimits in $\mathcal{P}_{\Sigma}(\mathcal{E})$. The former condition follows directly from Proposition 3.4.

For the latter, the duals of [Cis19, Th. 7.3.27, Prop. 7.3.28] imply that it is sufficient to show that this collection of objects is closed under pushouts.

Consider a pushout square σ in $\mathcal{P}_{\Sigma,f}(\mathcal{E})$

$$\begin{array}{ccc} Z & \xrightarrow{g} & Z' \\ \downarrow h & & \downarrow h' \\ Z'' & \xrightarrow{g'} & Y \end{array}$$

such that Z , Z' and Z'' have this property. We need to show that the cofiber of every morphism $f: Y \rightarrow \Sigma X$ is in $\Sigma\mathcal{P}_{\Sigma,f}(\mathcal{E})$, so let such a morphism be given. We extend f to a morphism of diagrams $f': \sigma \rightarrow \text{Const}_{\Sigma X}$, where the codomain is the diagram constant in ΣX . Because colimits commute, the pushout of f' , taken in the ∞ -category of squares in $\mathcal{P}_{\Sigma}(\mathcal{E})$, along a morphism with codomain zero is a pushout diagram of the form

$$\begin{array}{ccc} \text{cofib}(f \circ h' \circ g) & \longrightarrow & \text{cofib}(f \circ h') \\ \downarrow & & \downarrow \\ \text{cofib}(f \circ g') & \longrightarrow & \text{cofib}(f). \end{array}$$

As $\mathcal{P}_{\Sigma,f}(\mathcal{E})$ is closed under finite colimits, this implies that $\text{cofib}(f)$ is in $\mathcal{P}_{\Sigma,f}(\mathcal{E})$, finishing the proof. \square

Remark 3.6. We have the following diagram of fully faithful functors, which commutes up to natural equivalence:

$$\begin{array}{ccc} \mathcal{P}_{\Sigma,f}(\mathcal{E}) & \hookrightarrow & \mathcal{P}_{\Sigma}(\mathcal{E}) \\ \downarrow & & \downarrow \\ \mathcal{SW}(\mathcal{P}_{\Sigma,f}(\mathcal{E})) & \hookrightarrow & \mathcal{SW}(\mathcal{P}_{\Sigma}(\mathcal{E})). \end{array}$$

While $\mathcal{P}_{\Sigma}(\mathcal{E})$ embeds into $\mathcal{SW}(\mathcal{P}_{\Sigma}(\mathcal{E}))$ as the aisle of a t-structure, $\mathcal{P}_{\Sigma,f}(\mathcal{E})$ need not be the aisle of a t-structure on $\mathcal{SW}(\mathcal{P}_{\Sigma,f}(\mathcal{E}))$, as $\mathcal{P}_{\Sigma,f}(\mathcal{E})$ might not have finite limits. Furthermore, the heart $\mathcal{SW}(\mathcal{P}_{\Sigma}(\mathcal{E}))^{\heartsuit}$ is not contained in $\mathcal{P}_{\Sigma,f}(\mathcal{E})$. However, we can still consider the homotopy groups of objects in $\mathcal{P}_{\Sigma,f}(\mathcal{E})$ relative to the t-structure on $\mathcal{SW}(\mathcal{P}_{\Sigma}(\mathcal{E}))$ —these might just not lie in $\mathcal{P}_{\Sigma,f}(\mathcal{E})$. Similarly, objects in $\mathcal{P}_{\Sigma,f}(\mathcal{E})$ can be truncated as objects in $\mathcal{P}_{\Sigma}(\mathcal{E})$ —but again, the resulting object does not have to be inside $\mathcal{P}_{\Sigma,f}(\mathcal{E})$.

3.1. Primitive acyclic objects

Definition 3.7. Let $i: x \rightarrow y$ be a morphism in \mathcal{E} and z be an object in \mathcal{E} . A morphism $\text{cofib}(\hat{i}) \rightarrow z$ is called a *primitive quasi-isomorphism* if there exists an exact sequence $x \xrightarrow{i} y \xrightarrow{P} z$ such that f is the canonical morphism induced by the sequence. The colimit in $\mathcal{P}_{\Sigma,f}(\mathcal{E})$ of the diagram

$$\begin{array}{ccccc} \hat{x} & \xrightarrow{i} & \hat{y} & & \\ \downarrow & \searrow & \downarrow & \searrow & \\ & & 0 & \xrightarrow{\hat{p}} & \hat{z} \\ \downarrow & & \downarrow & & \\ 0 & \xrightarrow{\quad} & 0 & & \\ \downarrow & \searrow & \downarrow & & \\ & & 0 & & \end{array} \quad (3)$$

is a *primitive acyclic* object.

Proposition 3.8. *The colimit of a diagram in $\mathcal{P}_{\Sigma,f}(\mathcal{E})$ of the form (3) is canonically equivalent to the cofiber of the primitive quasi-isomorphism $\text{cofib}(\hat{i}) \rightarrow z$ in $\mathcal{P}_{\Sigma,f}(\mathcal{E})$.*

Proof. This is a direct application of [Lur09a, Prop. 4.4.2.2] to the diagram (3) and the decomposition

$$\begin{array}{ccc}
 \begin{array}{c} \bullet \longrightarrow \bullet \\ \searrow \downarrow \\ \bullet \end{array} & \longleftarrow & \begin{array}{c} \bullet \longrightarrow \bullet \\ \searrow \downarrow \\ \bullet \longrightarrow \circ \end{array} \\
 \downarrow & & \\
 \begin{array}{c} \bullet \longrightarrow \bullet \\ \downarrow \searrow \downarrow \\ \circ \longrightarrow \circ \end{array} & & \begin{array}{c} \bullet \longrightarrow \bullet \\ \downarrow \searrow \downarrow \\ \circ \longrightarrow \circ \end{array}
 \end{array}$$

of the cube. □

Definition 3.9. An object $M \in \text{Mod}(\text{Ho}(\mathcal{E}))$ is *effaceable* if it is the cokernel of a morphism $\bar{y} \xrightarrow{\bar{p}} \bar{z}$, where $p: y \rightarrow z$ is a fibration in \mathcal{E} . We denote the full subcategory of $\text{Mod}(\text{Ho}(\mathcal{E}))$ spanned by the effaceable functors by $\text{eff}(\text{Ho}(\mathcal{E}))$.

The next proposition ensures we can do a lot of the work in functor categories between ordinary categories.

Proposition 3.10. *The primitive acyclic objects constitute the essential image of the composition $\text{eff}(\text{Ho}(\mathcal{E})) \hookrightarrow \text{Mod}(\text{Ho}(\mathcal{E})) \simeq \mathcal{SW}(\mathcal{P}_{\Sigma}(\mathcal{E}))^{\heartsuit} \hookrightarrow \mathcal{P}_{\Sigma}$. In particular, this essential image lies in $\mathcal{P}_{\Sigma,f}(\mathcal{E})$.*

Proof. Let $A \in \mathcal{P}_{\Sigma,f}(\mathcal{E})$ be a primitive acyclic object corresponding to an exact sequence $x \xrightarrow{i} y \xrightarrow{p} z$. We claim that A is the effaceable object corresponding to \bar{p} .

Let Z be the cofiber of \hat{i} in $\mathcal{P}_{\Sigma,f}(\mathcal{E})$. We show that the induced map

$$\pi_n(Z) \rightarrow \pi_n(\hat{z})$$

is an isomorphism for $n > 0$, and that in the sequence

$$\bar{x} \xrightarrow{\bar{i}} \bar{y} \xrightarrow{f} \pi_0(Z) \xrightarrow{g} \bar{z},$$

the map f is an epimorphism and the map g is a monomorphism. The statement then follows from the long exact sequence

$$\cdots \rightarrow \pi_{n+1}(A) \rightarrow \pi_n(Z) \rightarrow \pi_n(\hat{z}) \rightarrow \pi_n(A) \rightarrow \pi_{n-1}(Z) \rightarrow \cdots$$

To this end, note that all of these claims can be checked pointwise. Fix an object $t \in \mathcal{E}$. There is a commutative diagram where the rows are the Serre long exact sequences

$$\begin{array}{ccccccc}
 \cdots & \rightarrow & \pi_0(\text{Map}(\hat{t}, \hat{x})) & \rightarrow & \pi_0(\text{Map}(\hat{t}, \hat{y})) & \rightarrow & \pi_0(\text{Map}(\hat{t}, Z)) & \rightarrow & \pi_{-1}(\text{Map}(\hat{t}, \hat{x})) \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 \cdots & \rightarrow & \pi_0(\text{Map}(t, x)) & \rightarrow & \pi_0(\text{Map}(t, y)) & \rightarrow & \pi_0(\text{Map}(t, z)) & \longrightarrow & C
 \end{array}$$

whose bottom row is induced by the fiber sequence $x \xrightarrow{i} y \xrightarrow{p} z$ in \mathcal{E} and whose top row is induced by the fiber sequence $\hat{x} \rightarrow \hat{y} \rightarrow Z$ in $\mathcal{SW}(\mathcal{P}_{\Sigma}(\mathcal{E}))$. Proposition 3.4

implies that $\pi_{-1}(\text{Map}(\hat{t}, \hat{x})) = 0$. The claim follows from the Four Lemma and the Five Lemma. \square

In the next two lemmas, we prove that $\text{eff}(\text{Ho}(\mathcal{E}))$ is *weakly Serre* as a subcategory of $\text{Mod}(\text{Ho}(\mathcal{E}))$.

Lemma 3.11. *The full subcategory of effaceable functors is closed under extensions in $\text{Mod}(\text{Ho}(\mathcal{E}))$.*

Proof. Let $K \rightarrow L \rightarrow M$ be a short exact sequence in $\text{Mod}(\text{Ho}(\mathcal{E}))$ such that K and M lie in $\text{eff}(\text{Ho}(\mathcal{E}))$. By definition, we can extend this to a diagram with exact columns

$$\begin{array}{ccccccc} & & \overline{y_1} & & \overline{y_2} & & \\ & & \downarrow \overline{p_1} & & \downarrow \overline{p_2} & & \\ & & \overline{z_1} & & \overline{z_2} & & \\ & & \downarrow c_1 & & \downarrow c_2 & & \\ 0 & \longrightarrow & K & \longrightarrow & L & \longrightarrow & M \longrightarrow 0, \end{array}$$

where p_1, p_2 are fibrations in \mathcal{E} and c_1, c_2 are epimorphisms in $\text{Mod}(\text{Ho}(\mathcal{E}))$. Since objects in the essential image of the Yoneda embedding are projective in $\text{Mod}(\text{Ho}(\mathcal{E}))$, the Horseshoe Lemma yields a commutative diagram with exact columns

$$\begin{array}{ccccccc} \overline{y_1} & \longrightarrow & \overline{y_1} \oplus \overline{y_2} & \longrightarrow & \overline{y_2} & & \\ \downarrow \overline{p_1} & & \downarrow \begin{pmatrix} \overline{p_1} & 0 \\ \overline{f} & -\overline{p_2} \end{pmatrix} & & \downarrow \overline{p_2} & & \\ \overline{z_1} & \longrightarrow & \overline{z_1} \oplus \overline{z_2} & \longrightarrow & \overline{z_2} & & \\ \downarrow c_1 & & \downarrow c_3 & & \downarrow c_2 & & \\ 0 & \longrightarrow & K & \longrightarrow & L & \longrightarrow & M \longrightarrow 0, \end{array}$$

where c_3 is an epimorphism. Proposition A.3 guarantees that the morphism indicated by a dashed arrow is a fibration in \mathcal{E} . Hence L is effaceable. \square

Lemma 3.12. *The full subcategory of effaceable functors is closed under kernels and cokernels in $\text{Mod}(\text{Ho}(\mathcal{E}))$.*

Proof. Let $g: L \rightarrow M$ be morphism in $\text{eff}(\text{Ho}(\mathcal{E}))$. We denote the kernel and the cokernel in $\text{Mod}(\text{Ho}(\mathcal{E}))$ by $f: K \rightarrow L$ and $h: M \rightarrow N$, respectively. Again, since the essential image of $\text{Ho}(\mathcal{E})$ in $\text{Mod}(\text{Ho}(\mathcal{E}))$ consists of projective objects, we can construct a commutative diagram with exact columns

$$\begin{array}{ccccccc} & & \overline{y_1} & \xrightarrow{\overline{b}} & \overline{y_2} & & \\ & & \downarrow \overline{p_1} & & \downarrow \overline{p_2} & & \\ & & \overline{z_1} & \xrightarrow{\overline{c}} & \overline{z_2} & & \\ & & \downarrow c_2 & & \downarrow c_3 & & \\ 0 & \longrightarrow & K & \xrightarrow{f} & L & \xrightarrow{g} & M \xrightarrow{h} N \longrightarrow 0, \end{array} \tag{4}$$

where p_1 and p_2 are fibrations in \mathcal{E} , and c_2 and c_3 are epimorphisms in $\text{Mod}(\text{Ho}(\mathcal{E}))$.

Here, projectiveness of \bar{z}_2 and the fact that c_3 is an epimorphism means that $g \circ c_2$ factors through c_3 as \bar{c} . The composition $\bar{c} \circ \bar{p}_1$ factors through the kernel of c_3 , so by exactness of the right column and projectiveness of \bar{y}_2 it factors through \bar{p}_2 as \bar{b} . The commutative square

$$\begin{array}{ccc} y_1 & \xrightarrow{b} & y_2 \\ \downarrow p_1 & & \downarrow p_2 \\ z_1 & \xrightarrow{c} & z_2 \end{array}$$

in $\text{Ho}(\mathcal{E})$ can be lifted to a commutative square in \mathcal{E} . By Proposition A.2, this square can be completed further to a commutative diagram

$$\begin{array}{ccccc} x_1 & \xrightarrow{a} & x_2 & \xlongequal{\quad} & x_2 \\ \downarrow i_1 & \square & \downarrow i_3 & & \downarrow i_2 \\ y_1 & \xrightarrow{b_1} & d & \xrightarrow{b_2} & y_2 \\ \downarrow p_1 & & \downarrow p_3 & \square & \downarrow p_2 \\ z_1 & \xlongequal{\quad} & z_1 & \xrightarrow{c} & z_2, \end{array}$$

where all columns are exact sequences and the marked squares are bicartesian. The commutativity of the bottom right square implies that the composition $c_2 \circ \bar{p}_3$ in $\text{Mod}(\text{Ho}(\mathcal{E}))$ factors through f , say as $f \circ c_1$. Hence we can extend the diagram (4) to a larger diagram

$$\begin{array}{ccccccc} \bar{y}_1 \oplus \bar{x}_2 & \xrightarrow{(id \ 0)} & \bar{y}_1 & \xrightarrow{\bar{b}} & \bar{y}_2 & \xrightarrow{\begin{pmatrix} 0 \\ id \end{pmatrix}} & \bar{z}_1 \oplus \bar{y}_2 \\ \downarrow \begin{pmatrix} \bar{b}_1 & \bar{i}_3 \end{pmatrix} & & \downarrow \bar{p}_1 & & \downarrow \bar{p}_2 & & \downarrow \begin{pmatrix} \bar{c} & \bar{p}_2 \end{pmatrix} \\ \bar{d} & \xrightarrow{\bar{p}_3} & \bar{z}_1 & \xrightarrow{\bar{c}} & \bar{z}_2 & \xlongequal{\quad} & \bar{z}_2 \\ \downarrow c_1 & & \downarrow c_2 & & \downarrow c_3 & & \downarrow h \circ c_3 \\ 0 & \longrightarrow & K & \xrightarrow{f} & L & \xrightarrow{g} & M & \xrightarrow{h} & N & \longrightarrow & 0. \end{array} \tag{5}$$

Exactness of the right column is immediate and $h \circ c_3$ is an epimorphism as it is the composition of two epimorphisms. Note that for every exact sequence $p \rightarrow q \twoheadrightarrow r$ in \mathcal{E} , the corresponding Serre sequence implies that the sequence

$$\bar{p} \rightarrow \bar{q} \rightarrow \bar{r}$$

is exact at \bar{q} . Applying this observation to the exact sequence $x_2 \xrightarrow{i_3} d \xrightarrow{p_3} z_1$, a short diagram chase shows that the sequence

$$\bar{y}_1 \oplus \bar{x}_2 \xrightarrow{\begin{pmatrix} \bar{b}_1 & \bar{i}_3 \end{pmatrix}} \bar{d} \rightarrow \text{coker}(\bar{p}_1)$$

is exact at \bar{d} , which shows that the left column in (5) is exact. Note that [Bar15,

Lem. 4.7] implies that there exists an exact sequence in \mathcal{E} of the form

$$d \xrightarrow{\begin{pmatrix} b_2 \\ -p_3 \end{pmatrix}} z_1 \oplus y_2 \xrightarrow{\begin{pmatrix} c & p_2 \end{pmatrix}} z_2.$$

Again, a short diagram chase shows that the existence of the above sequence implies that the sequence

$$\bar{d} \xrightarrow{p_3} \bar{z}_1 \rightarrow \text{coker}(\bar{p}_2)$$

is exact at \bar{z}_1 , which shows that c_1 is an epimorphism in $\text{Mod}(\text{Ho}(\mathcal{E}))$.

By [Bar15, Lem. 4.7], the morphisms $(b_1 \ i_3)$ and $(c \ p_2)$ are fibrations in \mathcal{E} . Hence the left and the right column show that K and N are effaceable, respectively. \square

The following corollary is a standard result for weakly Serre subcategories.

Corollary 3.13. *Let*

$$M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow M_4 \rightarrow M_5$$

be an exact sequence in $\text{Mod}(\text{Ho}(\mathcal{E}))$ such that all objects but M_3 are effaceable. Then M_3 is effaceable as well.

Proof. This sequence decomposes as

$$\begin{array}{ccccccc} & & & & N_2 & & \\ & & & & \nearrow & \searrow & \\ M_1 & \rightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & M_4 \rightarrow M_5, \\ & & \searrow & \nearrow & & & \\ & & & N_1 & & & \end{array}$$

where N_1 is the cokernel of $M_1 \rightarrow M_2$, N_2 is the kernel of $M_4 \rightarrow M_5$ and the sequence

$$N_1 \rightarrow M_3 \rightarrow N_2$$

is short exact in $\text{Mod}(\text{Ho}(\mathcal{E}))$. Propositions 3.11 and 3.12 imply that M_3 is effaceable. \square

3.2. Comparison with Krause’s derived Auslander formula

Let \mathcal{A} be a small abelian category. Krause [Kra15] proves that acyclic complexes in \mathcal{A} of the form

$$\cdots \rightarrow 0 \rightarrow X_2 \rightarrow X_1 \rightarrow X_0 \rightarrow 0 \rightarrow \cdots$$

constitute the essential image of the composition

$$\text{eff}(\mathcal{A}) \hookrightarrow \text{mod}(\mathcal{A}) \simeq D^b(\text{mod}(\mathcal{A}))^\heartsuit \hookrightarrow D^b(\text{mod}(\mathcal{A}))$$

under the canonical equivalence

$$D^b(\text{mod}(\mathcal{A})) \simeq \mathcal{K}^b(\mathcal{A}).$$

Proposition 3.10 is a generalization of this observation: There exists a diagram of ∞ -categories which commutes up to natural equivalence

$$\begin{array}{ccccc}
 \mathcal{SW}(\mathcal{P}_{\Sigma,f}(\mathcal{A})) & \xleftarrow{\sim} & & \xrightarrow{\sim} & \mathcal{K}^b(\mathcal{A}) \\
 \downarrow & & \swarrow & & \searrow \\
 & & \mathcal{A} & & \\
 & & \downarrow & & \downarrow \wr \\
 \text{eff}(\mathcal{A}) & \hookrightarrow & \text{mod}(\mathcal{A}) & \xrightarrow{\heartsuit} & D^b(\text{mod}(\mathcal{A})) \\
 & & \downarrow & & \downarrow \\
 & & \text{Mod}(\mathcal{A}) & & \\
 \swarrow \heartsuit & & & & \searrow \heartsuit \\
 \mathcal{SW}(\mathcal{P}_{\Sigma}(\mathcal{A})) & \xleftarrow{\sim} & & \xrightarrow{\sim} & D^b(\text{Mod}(\mathcal{A})),
 \end{array}$$

where all functors are fully faithful. Note that [BCKW19, Prop. 7.55] together with [Lur09a, Prop. 1.3.3.14] implies the existence of the horizontal equivalences and the commutativity up to natural equivalence.

3.3. Acyclic objects

Definition 3.14. The objects in the stable closure of the primitive acyclic objects in $\mathcal{SW}(\mathcal{P}_{\Sigma,f}(\mathcal{E}))$ —equivalently, the objects in $\mathcal{SW}(\mathcal{P}_{\Sigma,f}(\mathcal{E}))$ arising from the primitive acyclic objects by positive and negative suspensions and cofibers—are *acyclic* objects.

Proposition 3.15. *Let $X \in \mathcal{SW}(\mathcal{P}_{\Sigma,f}(\mathcal{E}))$ be an acyclic object. For all $n \in \mathbb{N}$, the homotopy group $\pi_n(X)$ is effaceable.*

Proof. As $\mathcal{SW}(\mathcal{P}_{\Sigma,f}(\mathcal{E}))$ is a subcategory of $\mathcal{SW}(\mathcal{P}_{\Sigma}(\mathcal{E}))$ which is closed under positive and negative suspensions and cofibers, it is sufficient to show that the subcategory of $\mathcal{SW}(\mathcal{P}_{\Sigma}(\mathcal{E}))$ spanned by the objects with this property is closed under positive and negative suspensions and cofibers, and contains the primitive acyclic objects. That this subcategory is closed under positive and negative suspensions is clear. The fact that for a primitive acyclic object X , the homotopy groups are effaceable is shown in Proposition 3.10 (note that $\pi_n(X) = 0$ for $n \neq 0$ in this case). To prove that for a map $f: X \rightarrow Y$ where X, Y have effaceable homotopy groups, the cofiber has this property as well, consider the corresponding long exact sequence of homotopy groups and apply Corollary 3.13. \square

Proposition 3.16. *Let $X \in \mathcal{SW}(\mathcal{P}_{\Sigma,f}(\mathcal{E}))$ be an acyclic object. The truncation of X , $\tau_{\geq 0}X \in \mathcal{P}_{\Sigma}(\mathcal{E})$, is acyclic and lies in $\mathcal{P}_{\Sigma,f}(\mathcal{E})$.*

Proof. The homotopy groups of $\tau_{\geq 0}X \in \mathcal{P}_{\Sigma}(\mathcal{E})$ agree with those of X in positive degrees and are zero in negative degrees. By Proposition 3.15, its homotopy groups are all effaceable. Consider the homotopy groups of $\tau_{\geq 0}X$ as objects in $\mathcal{P}_{\Sigma}(\mathcal{E})$ via the embedding $\text{Mod}(\text{Ho}(\mathcal{E})) \xrightarrow{\heartsuit} \mathcal{P}_{\Sigma}(\mathcal{E})$. Because it is bounded and connective, $\tau_{\geq 0}X$ is generated under repeated extensions of positive suspensions of its homotopy groups. As $\mathcal{P}_{\Sigma,f}(\mathcal{E})$ is closed in $\mathcal{P}_{\Sigma}(\mathcal{E})$ under colimits and extensions, Proposition 3.10 finishes the proof. \square

4. The stable hull

Definition 4.1. A morphism in $\mathcal{SW}(\mathcal{P}_{\Sigma,f}(\mathcal{E}))$ is a *quasi-isomorphism* if its cofiber is an acyclic object. We denote the Dwyer-Kan localization of $\mathcal{SW}(\mathcal{P}_{\Sigma,f}(\mathcal{E}))$ at the class of quasi-isomorphisms by $\mathcal{H}^{\text{st}}(\mathcal{E})$. Similarly, we denote the Dwyer-Kan localization of $\mathcal{P}_{\Sigma,f}(\mathcal{E})$ at the class of quasi-isomorphisms in $\mathcal{P}_{\Sigma,f}(\mathcal{E})$ by $\mathcal{H}_{\geq 0}^{\text{st}}(\mathcal{E})$.

Remark 4.2. The Dwyer-Kan localization of ∞ -categories, sometimes also just called *∞ -categorical localization*, is the universal functor sending a class of edges to equivalences. This is not equivalent to the concept described by Lurie in [Lur09a, Sec. 5.2.7]. For a very thorough treatment of the Dwyer-Kan localization, see [Cis19, Sec. 7.1].

Remark 4.3. The functor $\mathcal{SW}(\mathcal{P}_{\Sigma,f}(\mathcal{E})) \rightarrow \mathcal{H}^{\text{st}}(\mathcal{E})$ is the Verdier quotient of the source $\mathcal{SW}(\mathcal{P}_{\Sigma,f}(\mathcal{E}))$ by its full subcategory of acyclic objects. It is well known that Verdier quotients of triangulated categories allow for a simple description of their hom-spaces. A very explicit description of Verdier quotients of stable ∞ -categories has been found by Nikolaus and Scholze in [NS18, Th. I.3.3], which we will quickly repeat here for $\mathcal{H}^{\text{st}}(\mathcal{E})$: For two objects $X, Y \in \mathcal{SW}(\mathcal{P}_{\Sigma,f}(\mathcal{E}))$ with images \bar{X}, \bar{Y} in $\mathcal{H}^{\text{st}}(\mathcal{E})$, the mapping space is given by the filtered colimit

$$\text{Map}(\bar{X}, \bar{Y}) \simeq \text{colim}_f: Z \rightarrow Y \text{ a quasi-isomorphism} (X, Z).$$

The mapping spaces of Verdier quotients of stable ∞ -categories are described by Nikolaus and Scholze in [NS18, Th. I.3.3].

Proposition 4.4. *The quasi-isomorphisms endow $\mathcal{SW}(\mathcal{P}_{\Sigma,f}(\mathcal{E}))$ with the structure of an ∞ -category of cofibrant objects, as in the dual of [Cis19, Def. 7.5.7], where all morphisms are cofibrations. The quasi-isomorphisms which lie in $\mathcal{P}_{\Sigma,f}(\mathcal{E})$ endow $\mathcal{P}_{\Sigma,f}(\mathcal{E})$ with the structure of an ∞ -category of cofibrant objects, where all morphisms are cofibrations.*

Proof. That the axioms given in [Cis19, Def. 7.4.12] are fulfilled follows readily from the following three properties of quasi-isomorphisms:

- *Equivalences are quasi-isomorphisms:* The cofibers of equivalences are the zero objects.
- *Quasi-isomorphisms are stable under pushouts:* Every morphism has the same cofiber as its pushout along every other morphism.
- *Quasi-isomorphisms have the 2-out-of-3-property:* This can be checked on the homotopy categories, where it follows from the octahedral axiom: if f and g are composable, then the cofiber of $g \circ f$ is an extension of the cofiber of f and the cofiber of g . \square

We obtain the following corollary as an application of [Cis19, Prop. 7.5.11].

Corollary 4.5. *Let \mathcal{D} be an ∞ -category with finite colimits. Restrictions along the functors $\mathcal{SW}(\mathcal{P}_{\Sigma,f}(\mathcal{E})) \rightarrow \mathcal{H}^{\text{st}}(\mathcal{E})$ and $\mathcal{P}_{\Sigma,f}(\mathcal{E}) \rightarrow \mathcal{H}_{\geq 0}^{\text{st}}(\mathcal{E})$ induce the following equivalences of ∞ -categories:*

$$\begin{aligned} \text{Fun}^{\text{rex}}(\mathcal{H}^{\text{st}}(\mathcal{E}), \mathcal{D}) &\simeq \text{Fun}_{q-i}^{\text{rex}}(\mathcal{SW}(\mathcal{P}_{\Sigma,f}(\mathcal{E})), \mathcal{D}), \\ \text{Fun}^{\text{rex}}(\mathcal{H}_{\geq 0}^{\text{st}}(\mathcal{E}), \mathcal{D}) &\simeq \text{Fun}_{q-i}^{\text{rex}}(\mathcal{P}_{\Sigma,f}(\mathcal{E}), \mathcal{D}). \end{aligned}$$

Here, the subscript indicates that we restrict further to the full subcategory of those

functors which send quasi-isomorphisms to equivalences.

Proposition 4.6. *Let $X \in \mathcal{P}_{\Sigma,f}(\mathcal{E})$, $Y \in \mathcal{SW}(\mathcal{P}_{\Sigma,f}(\mathcal{E}))$, and $w: X \rightarrow Y$ be a quasi-isomorphism. Using the conventions of Remark 3.6, the truncation $\tau_{\geq 0}Y \in \mathcal{P}_{\Sigma}(\mathcal{E})$ lies in $\mathcal{P}_{\Sigma,f}(\mathcal{E})$ and the induced map $X \rightarrow \tau_{\geq 0}Y$ is a quasi-isomorphism.*

Proof. In the following proof, we view w as a morphism in $\mathcal{SW}(\mathcal{P}_{\Sigma}(\mathcal{E}))$ by means of the fully faithful functor $\mathcal{SW}(\mathcal{P}_{\Sigma,f}(\mathcal{E})) \hookrightarrow \mathcal{SW}(\mathcal{P}_{\Sigma}(\mathcal{E}))$. Note that by the 2-out-of-3 property of quasi-isomorphisms and since the counit $\tau_{\geq 0}X \rightarrow X$ is an equivalence, it suffices to prove the claim for $\tau_{\geq 0}w$. Applying $\tau_{\geq 0}$ to the cofiber diagram of w yields a commutative diagram

$$\begin{array}{ccccc} \tau_{\geq 0}X & \xrightarrow{\tau_{\geq 0}w} & \tau_{\geq 0}Y & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \tau_{\geq 0}Z & \longrightarrow & \tau_{\geq 0}\Sigma X, \end{array}$$

where Z —and hence by Proposition 3.16, $\tau_{\geq 0}Z$ as well—is acyclic by assumption. Both the right and the outer square are bicartesian before truncating, hence they are pullback diagrams in $\mathcal{P}_{\Sigma}(\mathcal{E})$, as truncation preserves limits. Because $\mathcal{P}_{\Sigma}(\mathcal{E})$ is prestable and $\tau_{\geq 0}\Sigma X$ is equivalent to ΣX , they are bicartesian in $\mathcal{P}_{\Sigma}(\mathcal{E})$ and hence bicartesian in $\mathcal{SW}(\mathcal{P}_{\Sigma}(\mathcal{E}))$ as well. By the Pasting Law for cartesian squares [Lur09a, Lem. 4.4.2.1], the left square is bicartesian as well. Since $\mathcal{P}_{\Sigma,f}(\mathcal{E})$ is closed in $\mathcal{P}_{\Sigma}(\mathcal{E})$ under extensions, this finishes the proof. \square

Proposition 4.7. *The functor $\mathcal{H}_{\geq 0}^{st}(\mathcal{E}) \rightarrow \mathcal{H}^{st}(\mathcal{E})$, which is induced by the composite $\mathcal{P}_{\Sigma,f}(\mathcal{E}) \hookrightarrow \mathcal{SW}(\mathcal{P}_{\Sigma,f}(\mathcal{E})) \rightarrow \mathcal{H}^{st}(\mathcal{E})$, is fully faithful.*

Proof. By Corollary 4.5, the induced functor is right exact. Hence, by the dual of [Cis19, Th. 7.6.10], it is sufficient to check that the induced functor on homotopy categories is fully faithful. Note that the homotopy category of the ∞ -categorical Dwyer-Kan localization agrees with the 1-categorical localization of the homotopy category and that the morphism sets of the localization can be calculated using left calculus of fractions. We need to prove that, for objects $X, Y \in \mathcal{P}_{\Sigma,f}(\mathcal{E})$, the induced map

$$\mathrm{hom}_{\mathrm{Ho}(\mathcal{H}_{\geq 0}^{st}(\mathcal{E}))}(X, Y) \rightarrow \mathrm{hom}_{\mathrm{Ho}(\mathcal{H}^{st}(\mathcal{E}))}(X, Y)$$

is an isomorphism. To prove that this map is injective, note that any two roofs in $\mathrm{Ho}(\mathcal{P}_{\Sigma,f}(\mathcal{E}))$ which become equivalent in $\mathrm{Ho}(\mathcal{H}^{st}(\mathcal{E}))$ yield a commutative diagram in $\mathrm{Ho}(\mathcal{SW}(\mathcal{P}_{\Sigma,f}(\mathcal{E})))$ of the form

$$\begin{array}{ccccc} & & Y' & & \\ & \nearrow & \downarrow \wr & \nwarrow \sim & \\ X & \longrightarrow & \tilde{Y} & \longleftarrow \sim & Y \\ & \searrow & \uparrow \wr & \swarrow \sim & \\ & & Y'' & & \end{array}$$

where all the marked morphisms are quasi-isomorphisms and all the objects but \tilde{Y} are in $\mathrm{Ho}(\mathcal{P}_{\Sigma,f}(\mathcal{E}))$. Replacing \tilde{Y} with $\tau_{\geq 0}\tilde{Y}$ and using Proposition 4.6, we can form

an analogous diagram in $\mathrm{Ho}(\mathcal{P}_{\Sigma,f}(\mathcal{E}))$, showing the roofs are equivalent as maps in $\mathrm{Ho}(\mathcal{H}_{\geq 0}^{\mathrm{st}}(\mathcal{E}))$. To show that this map is surjective, using the same argument, every roof in $\mathrm{Ho}(\mathcal{SW}(\mathcal{P}_{\Sigma,f}(\mathcal{E})))$ is equivalent to a roof in $\mathrm{Ho}(\mathcal{P}_{\Sigma,f}(\mathcal{E}))$. \square

Corollary 4.8. *The ∞ -category $\mathcal{H}_{\geq 0}^{\mathrm{st}}(\mathcal{E})$ is prestable and the induced functor is an equivalence $\mathcal{SW}(\mathcal{H}_{\geq 0}^{\mathrm{st}}(\mathcal{E})) \rightarrow \mathcal{H}^{\mathrm{st}}(\mathcal{E})$.*

Proof. The ∞ -category $\mathcal{H}^{\mathrm{st}}(\mathcal{E})$ is stable by [NS18, Th. I.3.3]. By Corollary 4.5, the embedding $\mathcal{H}_{\geq 0}^{\mathrm{st}}(\mathcal{E}) \hookrightarrow \mathcal{H}^{\mathrm{st}}(\mathcal{E})$ is right exact. We first show that $\mathcal{H}_{\geq 0}^{\mathrm{st}}(\mathcal{E})$ is closed in $\mathcal{H}^{\mathrm{st}}(\mathcal{E})$ under extensions, which proves both that $\mathcal{H}_{\geq 0}^{\mathrm{st}}(\mathcal{E})$ is prestable and that the functor $\mathcal{SW}(\mathcal{H}_{\geq 0}^{\mathrm{st}}(\mathcal{E})) \rightarrow \mathcal{H}^{\mathrm{st}}(\mathcal{E})$ is fully faithful, see Example 2.9.

Let $X \rightarrow Y \rightarrow Z$ be an exact sequence in $\mathcal{H}^{\mathrm{st}}(\mathcal{E})$ with X, Z in the essential image of $\mathcal{H}_{\geq 0}^{\mathrm{st}}(\mathcal{E})$. After shifting, we are left to show that the cofiber of the induced map $Z \rightarrow \Sigma X$ lies in the essential image of $\Sigma\mathcal{H}_{\geq 0}^{\mathrm{st}}(\mathcal{E}) \subseteq \mathcal{H}_{\geq 0}^{\mathrm{st}}(\mathcal{E})$. Since $\mathcal{H}_{\geq 0}^{\mathrm{st}}(\mathcal{E}) \rightarrow \mathcal{H}^{\mathrm{st}}(\mathcal{E})$ is fully faithful, we can assume the map $Z \rightarrow \Sigma X$ to be in $\mathcal{H}_{\geq 0}^{\mathrm{st}}(\mathcal{E})$. Using the left calculus of fractions, such a map comes up to equivalence from a map in $\mathcal{P}_{\Sigma,f}(\mathcal{E})$. The claim now follows from the facts that $\mathcal{P}_{\Sigma,f}(\mathcal{E})$ is a prestable ∞ -category and that the functor $\mathcal{P}_{\Sigma,f}(\mathcal{E}) \rightarrow \mathcal{H}_{\geq 0}^{\mathrm{st}}(\mathcal{E})$ is right exact.

It remains to show that the functor $\mathcal{SW}(\mathcal{H}_{\geq 0}^{\mathrm{st}}(\mathcal{E})) \rightarrow \mathcal{H}^{\mathrm{st}}(\mathcal{E})$ is essentially surjective. For this, note that for every object $X \in \mathcal{H}^{\mathrm{st}}(\mathcal{E})$, there exists some $n \in \mathbb{Z}$ such that $\Sigma^n X$ is equivalent to the image of some $Y \in \mathcal{P}_{\Sigma,f}(\mathcal{E})$ under the composition $\mathcal{P}_{\Sigma,f}(\mathcal{E}) \rightarrow \mathcal{SW}(\mathcal{P}_{\Sigma,f}(\mathcal{E})) \rightarrow \mathcal{H}^{\mathrm{st}}(\mathcal{E})$. But then X is equivalent to the image of $\Sigma^{-n} Y \in \mathcal{SW}(\mathcal{H}_{\geq 0}^{\mathrm{st}}(\mathcal{E}))$. \square

4.1. Size considerations

As passing from \mathcal{E} to $\mathcal{P}_{\Sigma}(\mathcal{E})$ removes any smallness assumptions \mathcal{E} might have, it is *a priori* not clear which smallness properties are preserved by the construction $\mathcal{E} \rightarrow \mathcal{H}^{\mathrm{st}}(\mathcal{E})$. In this section, we see that $\mathcal{H}^{\mathrm{st}}(\mathcal{E})$ is essentially \mathcal{V} -small.

Lemma 4.9. *Let $\mathcal{U} \subseteq \mathcal{V}$ be a Grothendieck universe such that \mathcal{E} is a locally \mathcal{U} -small ∞ -category. Then $\mathcal{SW}(\mathcal{P}_{\Sigma,f}(\mathcal{E}))$ is locally \mathcal{U} -small.*

Proof. By virtue of [Cis19, Cor. 5.7.9] and because

$$\pi_n(\mathrm{hom}(X, Y)) \cong \pi_0(\mathrm{hom}(\Sigma^n X, Y)),$$

it is sufficient to prove that, for every pair of objects $X, Y \in \mathcal{SW}(\mathcal{P}_{\Sigma,f}(\mathcal{E}))$, the set $\pi_0(\mathrm{hom}(X, Y))$ is \mathcal{U} -small. Fix $x \in \mathcal{E}$ and consider the full subcategory of $\mathcal{SW}(\mathcal{P}_{\Sigma,f}(\mathcal{E}))$ spanned by those objects Y for which $\pi_0(\mathrm{hom}(\hat{x}, \Sigma^n Y))$ is \mathcal{U} -small for all $n \in \mathbb{Z}$. This subcategory is clearly closed under suspensions and extensions and contains \mathcal{E} by assumption and Proposition 3.4. As $\mathcal{SW}(\mathcal{P}_{\Sigma,f}(\mathcal{E}))$ is generated by \mathcal{E} under suspensions and extensions, this shows that $\pi_0(\mathrm{hom}(\hat{x}, Y))$ is \mathcal{U} -small for all $x \in \mathcal{E}, Y \in \mathcal{P}_{\Sigma,f}(\mathcal{E})$.

Consider the full subcategory of $\mathcal{P}_{\Sigma,f}(\mathcal{E})$ spanned by those objects X for which $\pi_0(\mathrm{hom}(X, Y))$ is \mathcal{U} -small for all $Y \in \mathcal{P}_{\Sigma,f}(\mathcal{E})$. By an analogous argument, this subcategory is $\mathcal{P}_{\Sigma,f}(\mathcal{E})$ itself, which finishes the proof. \square

Lemma 4.10. *The Spanier–Whitehead ∞ -category $\mathcal{SW}(\mathcal{P}_{\Sigma,f}(\mathcal{E}))$ is essentially \mathcal{V} -small.*

Proof. By [Cis19, Prop. 5.7.6] and Lemma 4.9, it suffices to show that the set of isomorphism classes is \mathcal{V} -small. This is clear, as $\mathcal{SW}(\mathcal{P}_{\Sigma,f}(\mathcal{E}))$ is generated by \mathcal{E} under finite suspensions and extensions. \square

Proposition 4.11. *The stable ∞ -category $\mathcal{H}^{\text{st}}(\mathcal{E})$ is essentially \mathcal{V} -small.*

Proof. As $\pi_n(\text{hom}(X, Y)) \cong \pi_0(\text{hom}(\Sigma^n X, Y))$, this may be checked on the homotopy category, which is the Verdier quotient of $\text{Ho}(\mathcal{SW}(\mathcal{P}_{\Sigma,f}(\mathcal{E})))$ by the acyclics. As the Verdier quotient of a small triangulated category is small [Nee01, Prop. 2.2.1], this follows directly from Lemma 4.10. \square

Remark 4.12. The construction $\mathcal{E} \rightarrow \mathcal{H}^{\text{st}}(\mathcal{E})$ does *not* preserve local smallness. Indeed, in Corollary 4.29 we show that in the case of ordinary exact categories, there is a canonical equivalence $D^b(\mathcal{E}) \xrightarrow{\simeq} \mathcal{H}^{\text{st}}(\mathcal{E})$ and the construction $\mathcal{E} \rightarrow D^b(\mathcal{E})$ does not preserve smallness [Fre64, Ex. 6.2.A].

4.2. Properties of the stable hull

First, we establish the universal property of $\mathcal{H}^{\text{st}}(\mathcal{E})$, which implies that the construction $\mathcal{E} \rightarrow \mathcal{H}^{\text{st}}(\mathcal{E})$ can be organized into a left adjoint to the inclusion functor $\mathbf{St}_{\infty} \hookrightarrow \mathbf{Ex}_{\infty}$.

Lemma 4.13. *Every object $T \in \mathcal{P}_{\Sigma}(\mathcal{E})$ which is local to the primitive quasi-isomorphisms is local to all quasi-isomorphisms.*

Proof. Let $T \in \mathcal{P}_{\Sigma}(\mathcal{E})$ be local to the primitive quasi-isomorphisms. Then, the mapping space $\text{Map}(A, \Sigma T)$ is trivial for every primitive acyclic object A : Let $x \xrightarrow{i} y \xrightarrow{p} z$ be the exact sequence in \mathcal{E} corresponding to a fixed primitive acyclic object A . Then there is a Serre long exact sequence

$$\cdots \rightarrow \pi_n(\text{Map}(\hat{z}, \Sigma T)) \rightarrow \pi_n(\text{cofib}(i), \Sigma T) \rightarrow \pi_{n-1}(\text{Map}(A, \Sigma T)) \rightarrow \cdots$$

The morphism $\pi_n(\text{Map}(\hat{z}, \Sigma T)) \rightarrow \pi_n(\text{Map}(\text{cofib}(i), \Sigma T))$ is an isomorphism for $n > 0$ by assumption, while $\pi_0(\text{Map}(\hat{z}, \Sigma T))$ is trivial by Proposition 3.4.

Now using

- Proposition 3.15,
- the fact that in $\mathcal{P}_{\Sigma}(\mathcal{E}) \subseteq \mathcal{SW}(\mathcal{P}_{\Sigma}(\mathcal{E}))$, every bounded object arises as an iterated extension of its homotopy groups, and
- that the class of objects $Y \in \mathcal{P}_{\Sigma}(\mathcal{E})$ for which $\text{Map}(Y, \Sigma X)$ is trivial is closed under extensions,

we conclude that $\text{Map}(X, \Sigma T)$ is trivial for every acyclic object $X \in \mathcal{P}_{\Sigma,f}(\mathcal{E})$. As the cofiber of every quasi-isomorphism $w: Y \rightarrow Z$ is acyclic, the claim follows from the corresponding Serre long exact sequence

$$\cdots \rightarrow \pi_n(\text{Map}(Z, T)) \rightarrow \pi_n(\text{Map}(Y, T)) \rightarrow \pi_{n-1}(\text{Map}(\text{cofib}(w), T)) \rightarrow \cdots \quad \square$$

Lemma 4.14. *Let \mathcal{D} be an ∞ -category which admits small colimits. A cocontinuous functor $\mathcal{P}_{\Sigma}(\mathcal{E}) \rightarrow \mathcal{D}$ sends primitive quasi-isomorphisms to equivalences if and only if it sends all quasi-isomorphisms to equivalences.*

Proof. Let

$$\mathcal{P}_\Sigma(\mathcal{E}) \xrightleftharpoons[i]{\quad} L_q(\mathcal{P}_\Sigma(\mathcal{E}))$$

be the cocontinuous localization of $\mathcal{P}_\Sigma(\mathcal{E})$ along the quasi-isomorphisms. As $\mathcal{P}_\Sigma(\mathcal{E})$ is presentable, i is the inclusion of the full subcategory of $\mathcal{P}_\Sigma(\mathcal{E})$ spanned by those objects which are local to quasi-isomorphisms. Hence, by Lemma 4.13, this localization is also the cocontinuous localization of $\mathcal{P}_\Sigma(\mathcal{E})$ along the class of the primitive quasi-isomorphisms. Therefore, a cocontinuous functor $\mathcal{P}_\Sigma(\mathcal{E}) \rightarrow \mathcal{D}$ sends quasi-isomorphisms to equivalences if and only if it is equivalent to the restriction of some cocontinuous functor $L_q(\mathcal{P}_\Sigma(\mathcal{E})) \rightarrow \mathcal{D}$ if and only if it sends primitive quasi-isomorphisms to equivalences. \square

Proposition 4.15. *Let \mathcal{D} be an ∞ -category which admits finite colimits. A right exact functor $\mathcal{P}_{\Sigma,f}(\mathcal{E}) \rightarrow \mathcal{D}$ sends primitive quasi-isomorphisms to equivalences if and only if it sends all quasi-isomorphisms to equivalences.*

Proof. By embedding \mathcal{D} into $\mathcal{P}(\mathcal{D}^{op})^{op}$, we can assume \mathcal{D} to admit all small colimits. Then, both right exact functors $\mathcal{P}_{\Sigma,f}(\mathcal{E}) \rightarrow \mathcal{D}$ and colimit-preserving functors $\mathcal{P}_\Sigma(\mathcal{E}) \rightarrow \mathcal{D}$ correspond to functors $\mathcal{E} \rightarrow \mathcal{D}$ which preserve finite coproducts. Therefore, the inclusion $\mathcal{P}_{\Sigma,f}(\mathcal{E}) \hookrightarrow \mathcal{P}_\Sigma(\mathcal{E})$ induces an equivalence

$$\mathrm{Fun}^!(\mathcal{P}_\Sigma(\mathcal{E}), \mathcal{D}) \xrightarrow{\simeq} \mathrm{Fun}^{\mathrm{rex}}(\mathcal{P}_{\Sigma,f}(\mathcal{E}), \mathcal{D}).$$

Here $\mathrm{Fun}^!(\mathcal{P}_\Sigma(\mathcal{E}), \mathcal{D})$ denotes the full subcategory of $\mathrm{Fun}(\mathcal{P}_\Sigma(\mathcal{E}), \mathcal{D})$ spanned by the cocontinuous functors. The conclusion now follows directly from Lemma 4.14. \square

Proposition 4.16. *Let \mathcal{D} be an ∞ -category which admits finite colimits. We write $\mathrm{Fun}^{\Sigma, \mathrm{seq}}(\mathcal{E}, \mathcal{D})$ for the full subcategory of $\mathrm{Fun}(\mathcal{E}, \mathcal{D})$ spanned by those functors which preserve finite coproducts and send exact sequences to pushout squares. Then, restriction along the composition*

$$\mathcal{E} \hookrightarrow \mathcal{P}_{\Sigma,f}(\mathcal{E}) \rightarrow \mathcal{H}_{\geq 0}^{\mathrm{st}}(\mathcal{E})$$

induces an equivalence of ∞ -categories

$$\mathrm{Fun}^{\mathrm{rex}}(\mathcal{H}_{\geq 0}^{\mathrm{st}}(\mathcal{E}), \mathcal{D}) \xrightarrow{\simeq} \mathrm{Fun}^{\Sigma, \mathrm{seq}}(\mathcal{E}, \mathcal{D}).$$

Proof. By Corollary 4.5, restriction along the functor $\mathcal{P}_{\Sigma,f}(\mathcal{E}) \rightarrow \mathcal{H}_{\geq 0}^{\mathrm{st}}(\mathcal{E})$ induces an equivalence of ∞ -categories

$$\mathrm{Fun}^{\mathrm{rex}}(\mathcal{H}_{\geq 0}^{\mathrm{st}}(\mathcal{E}), \mathcal{D}) \xrightarrow{\simeq} \mathrm{Fun}_{\mathrm{q-i}}^{\mathrm{rex}}(\mathcal{P}_{\Sigma,f}(\mathcal{E}), \mathcal{D}).$$

As we have shown in Proposition 4.15, the codomain of this equivalence agrees with $\mathrm{Fun}_{\mathrm{p-q-i}}^{\mathrm{rex}}(\mathcal{P}_{\Sigma,f}(\mathcal{E}), \mathcal{D})$, the full subcategory of $\mathrm{Fun}(\mathcal{P}_{\Sigma,f}(\mathcal{E}), \mathcal{D})$ spanned by the right exact functors which send primitive quasi-isomorphisms to equivalences. Also, the equivalence from Proposition 3.1 restricts to an equivalence

$$\mathrm{Fun}_{\mathrm{p-q-i}}^{\mathrm{rex}}(\mathcal{P}_{\Sigma,f}(\mathcal{E}), \mathcal{D}) \xrightarrow{\simeq} \mathrm{Fun}^{\Sigma, \mathrm{seq}}(\mathcal{E}, \mathcal{D}).$$

Composing these equivalences yields the result. \square

Proposition 4.17. *The functor $\mathcal{E} \rightarrow \mathcal{H}^{\mathrm{st}}(\mathcal{E})$ is fully faithful.*

Proof. Let \mathcal{K} be the class of all finite simplicial sets, and let \mathcal{R} be the union of

- the class of exact sequences in \mathcal{E} and
- the class of diagrams in \mathcal{E} exhibiting finite coproducts.

Proposition 4.16 shows that there is an equivalence $\mathcal{H}_{\geq 0}^{\text{st}}(\mathcal{E}) \simeq \mathcal{P}_{\mathcal{R}}^{\mathcal{K}}$ commuting with the inclusion of \mathcal{E} , where $\mathcal{E} \rightarrow \mathcal{P}_{\mathcal{R}}^{\mathcal{K}}$ is the universal functor into a ∞ -category with \mathcal{K} -indexed colimits which sends diagrams in \mathcal{R} to colimit diagrams, see [Lur09a, Sec. 5.3.6]. As the functor $\mathcal{E} \rightarrow \mathcal{P}_{\mathcal{R}}^{\mathcal{K}}$ is fully faithful, so are the functor $\mathcal{E} \rightarrow \mathcal{H}_{\geq 0}^{\text{st}}(\mathcal{E})$ and the composition $\mathcal{E} \rightarrow \mathcal{H}_{\geq 0}^{\text{st}}(\mathcal{E}) \hookrightarrow \mathcal{H}^{\text{st}}(\mathcal{E})$. \square

Definition 4.18 ([Bar15, Def. 4.1]). Let \mathcal{E}, \mathcal{F} be exact ∞ -categories. We say a functor $\mathcal{E} \rightarrow \mathcal{F}$ is *exact* if it preserves cofibrations, fibrations, zero objects, pushouts along cofibrations and pullbacks along fibrations.

Remark 4.19. As can be easily checked by considering split exact sequences, an exact functor between exact ∞ -categories preserves biproducts.

Proposition 4.20. *Let \mathcal{E}, \mathcal{F} be exact ∞ -categories. A functor $\mathcal{E} \rightarrow \mathcal{F}$ which preserves finite coproducts is exact if and only if it sends exact sequences to exact sequences.*

Proof. By [Bar15, Prop. 4.8], it is sufficient to prove that a pushout square in \mathcal{E} of the form

$$\begin{array}{ccc} A & \longrightarrow & A' \\ \downarrow & & \downarrow \\ B & \longrightarrow & B' \end{array}$$

is sent to a pushout square in \mathcal{F} . By Proposition A.1, such a pushout square fits into a commutative diagram

$$\begin{array}{ccccc} A & \longrightarrow & A' & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ B & \longrightarrow & B & \twoheadrightarrow & C \end{array}$$

such that both the outer rectangle and the right square are bicartesian. By assumption, after applying the functor $\mathcal{E} \rightarrow \mathcal{F}$ these squares are bicartesian; applying Proposition A.1 again finishes the proof. \square

Corollary 4.21. *The functor $\mathcal{E} \hookrightarrow \mathcal{H}^{\text{st}}(\mathcal{E})$ is exact and the functor $\mathcal{E} \hookrightarrow \mathcal{H}_{\geq 0}^{\text{st}}(\mathcal{E})$ preserves exact sequences.*

Proof. Because the three functors

$$\mathcal{E} \hookrightarrow \mathcal{P}_{\Sigma, f}(\mathcal{E}), \mathcal{P}_{\Sigma, f}(\mathcal{E}) \rightarrow \mathcal{SW}(\mathcal{P}_{\Sigma, f}(\mathcal{E})) \text{ and } \mathcal{SW}(\mathcal{P}_{\Sigma, f}(\mathcal{E})) \rightarrow \mathcal{H}^{\text{st}}(\mathcal{E})$$

preserve finite coproducts, so does their composition. Hence, it suffices to prove that $\mathcal{E} \hookrightarrow \mathcal{H}_{\geq 0}^{\text{st}}(\mathcal{E})$ preserves exact sequences; that the functor $\mathcal{E} \hookrightarrow \mathcal{H}^{\text{st}}(\mathcal{E})$ is exact then follows from Proposition 4.20. We need to show for that for an exact sequence of the

form $x \xrightarrow{i} y \xrightarrow{p} z$, the induced map

$$\text{cofib}(\hat{i}) \rightarrow z$$

in $\mathcal{H}_{\geq 0}^{\text{st}}(\mathcal{E})$ is an equivalence. This map can be chosen to be the image of the corresponding map in $\mathcal{P}_{\Sigma, \mathcal{F}}(\mathcal{E})$, which is a (primitive) quasi-isomorphism. \square

We are ready to establish the universal property of the stable hull.

Proposition 4.22. *Let \mathcal{E} be an exact ∞ -category \mathcal{E} . For every stable ∞ -category \mathcal{C} , restriction along the functor $\mathcal{E} \hookrightarrow \mathcal{H}^{\text{st}}(\mathcal{E})$ induces an equivalence of ∞ -categories*

$$\text{Fun}^{\text{ex}}(\mathcal{H}^{\text{st}}(\mathcal{E}), \mathcal{C}) \rightarrow \text{Fun}^{\text{ex}}(\mathcal{E}, \mathcal{C})$$

between the full subcategories of the functor ∞ -categories spanned by those functors which are exact.

Proof. Using the notation of Proposition 4.16, the codomain of the functor agrees with $\text{Fun}^{\Sigma, \text{seq}}(\mathcal{E}, \mathcal{C})$ by Proposition 4.20. The functor $\mathcal{E} \hookrightarrow \mathcal{H}^{\text{st}}(\mathcal{E})$ induces equivalences of ∞ -categories

$$\text{Fun}^{\text{ex}}(\mathcal{H}^{\text{st}}(\mathcal{E}), \mathcal{C}) \xrightarrow{\simeq} \text{Fun}^{\text{rex}}(\mathcal{H}_{\geq 0}^{\text{st}}(\mathcal{E}), \mathcal{C}) \xrightarrow{\simeq} \text{Fun}^{\Sigma, \text{seq}}(\mathcal{E}, \mathcal{C}).$$

The rightmost equivalence comes from the Proposition 4.16, while the leftmost one comes from Corollary 4.8 together with the universal property of the Spanier–Whitehead construction. \square

The following definition is taken from [Bar15, Not. 4.2].

Definition 4.23. We let $\mathbf{Ex}_{\infty}^{\Delta}$ be the following simplicial category: The objects of $\mathbf{Ex}_{\infty}^{\Delta}$ are small exact ∞ -categories and, for two small exact ∞ -categories \mathcal{E}, \mathcal{F} , we set

$$\mathbf{Ex}_{\infty}^{\Delta}(\mathcal{E}, \mathcal{F}) := \text{core}(\text{Fun}^{\text{ex}}(\mathcal{E}, \mathcal{F})),$$

where $\text{Fun}^{\text{ex}}(\mathcal{E}, \mathcal{F})$ is the full subcategory of $\text{Fun}(\mathcal{E}, \mathcal{F})$ spanned by the exact functors. We let \mathbf{Ex}_{∞} be the homotopy coherent nerve of $\mathbf{Ex}_{\infty}^{\Delta}$ and \mathbf{St}_{∞} be the full subcategory of \mathbf{Ex}_{∞} spanned by the stable ∞ -categories (with every morphism marked both as a fibration and a cofibration as in Example 2.17).

Corollary 4.24. *The embedding $\mathbf{St}_{\infty} \hookrightarrow \mathbf{Ex}_{\infty}$ admits a left adjoint*

$$\mathcal{H}^{\text{st}}: \mathbf{Ex}_{\infty} \rightarrow \mathbf{St}_{\infty}.$$

The unit of this adjunction is, for every exact ∞ -category \mathcal{E} , equivalent to the canonical functor $\mathcal{E} \rightarrow \mathcal{H}^{\text{st}}(\mathcal{E})$.

Proof. By [Lur09a, Prop. 5.2.7.8], it is sufficient to show that for every object $\mathcal{E} \in \mathbf{Ex}_{\infty}$, there exists a morphism $\mathcal{E} \rightarrow \mathcal{H}$ in \mathbf{Ex}_{∞} such that \mathcal{H} is in \mathbf{St}_{∞} and for every stable ∞ -category \mathcal{C} , the induced map

$$\text{Map}_{\mathbf{Ex}_{\infty}}(\mathcal{H}, \mathcal{C}) \rightarrow \text{Map}_{\mathbf{Ex}_{\infty}}(\mathcal{E}, \mathcal{C})$$

is an isomorphism in the homotopy category of spaces. By the construction of \mathbf{Ex}_{∞} , this map is equivalent to the induced map

$$\text{core}(\text{Fun}^{\text{ex}}(\mathcal{H}, \mathcal{C})) \rightarrow \text{core}(\text{Fun}^{\text{ex}}(\mathcal{E}, \mathcal{C})).$$

Proposition 4.22 shows that $\mathcal{E} \hookrightarrow \mathcal{H}^{\text{st}}(\mathcal{E})$ has the desired property. \square

In [Kel90] Keller provides an elementary proof of the Gabriel–Quillen embedding. The proof of the following proposition is an ∞ -categorical analogue of Keller’s own proof that this embedding reflects exact sequences and is closed under exact sequences.

Proposition 4.25. *The functor $\mathcal{E} \hookrightarrow \mathcal{H}^{\text{st}}(\mathcal{E})$ reflects exact sequences and the essential image of this functor is closed under extensions.*

Proof. As the embedding $\mathcal{H}_{\geq 0}^{\text{st}}(\mathcal{E}) \hookrightarrow \mathcal{SW}(\mathcal{H}_{\geq 0}^{\text{st}}(\mathcal{E})) \simeq \mathcal{H}^{\text{st}}(\mathcal{E})$ reflects exact sequences and has an essential image which is closed under extensions, it is sufficient to show that the functor $\mathcal{E} \hookrightarrow \mathcal{H}_{\geq 0}^{\text{st}}(\mathcal{E})$ has these properties as well.

Let $\hat{x} \rightarrow Y' \rightarrow \hat{z}$ be an exact sequence in $\mathcal{H}_{\geq 0}^{\text{st}}(\mathcal{E})$ with $x, z \in \mathcal{E}$. As the functor $\mathcal{E} \hookrightarrow \mathcal{H}_{\geq 0}^{\text{st}}(\mathcal{E})$ is fully faithful, it suffices to show that there exists an exact sequence $x \xrightarrow{i} y \xrightarrow{p} z$ in \mathcal{E} which is equivalent to the given sequence in $\mathcal{H}_{\geq 0}^{\text{st}}(\mathcal{E})$. The hom-sets in $\text{Ho}(\mathcal{H}_{\geq 0}^{\text{st}}(\mathcal{E}))$ can be computed using the left calculus of fractions on $\text{Ho}(\mathcal{P}_{\Sigma, f}(\mathcal{E}))$. Using this, one can form a commuting diagram of the form

$$\begin{array}{ccccc}
 \hat{x} & \xrightarrow{j} & Y & \longrightarrow & \text{cofib}(j) \\
 & \searrow & \uparrow \wr & & f \downarrow \wr \\
 & & Y' & & Z \\
 & & & & \uparrow \wr \\
 & & & & \hat{z}
 \end{array}$$

in $\text{Ho}(\mathcal{H}_{\geq 0}^{\text{st}}(\mathcal{E}))$, where all the solid arrows come from morphisms in $\mathcal{P}_{\Sigma, f}(\mathcal{E})$, the cofiber of j is taken in $\mathcal{P}_{\Sigma, f}(\mathcal{E})$ and the vertical maps are all quasi-isomorphisms in $\mathcal{P}_{\Sigma, f}(\mathcal{E})$.

It now suffices to show that the exact sequence $\hat{x} \rightarrow Y \rightarrow Z$ is equivalent to the image of an exact sequence in \mathcal{E} . Let $d: Z \rightarrow Q$ be the cofiber of f in $\mathcal{P}_{\Sigma, f}(\mathcal{E})$. Note that, as f is a quasi-isomorphism, Q is acyclic. By Proposition 3.15, $\pi_0(Q)$ is effaceable, hence it is the cokernel of some fibration $\bar{q}: \bar{w} \rightarrow \bar{v}$ in $\text{Mod}(\text{Ho}(\mathcal{E}))$. The diagram in $\text{Mod}(\text{Ho}(\mathcal{E}))$

$$\begin{array}{ccccc}
 & & & & \bar{v} \\
 & & & & \downarrow \\
 \bar{z} & \xrightarrow{\pi_0(g)} & \pi_0(Z) & \xrightarrow{\pi_0(d)} & \pi_0(Q)
 \end{array}$$

has a lift, as the vertical map is an epimorphism. We denote the pullback of q along this lift by $p_*: y_* \rightarrow z$. Note that the composition $d \circ g \circ \hat{p}_*$ is the zero-map in $\text{Ho}(\mathcal{P}_{\Sigma, f}(\mathcal{E}))$. Hence the diagram in $\mathcal{P}_{\Sigma, f}(\mathcal{E})$

$$\begin{array}{ccccc}
 & & & & \text{cofib}(j) \\
 & & & & \downarrow \\
 \hat{y}_* & \xrightarrow{\hat{p}_*} & \hat{z} & \xrightarrow{g} & Z
 \end{array}$$

has a lift which can, by Proposition 3.4, be lifted further to a square in $\mathcal{P}_{\Sigma,f}(\mathcal{E})$

$$\begin{array}{ccc} \hat{y}_* & \xrightarrow{p_*} & \hat{z} \\ \downarrow & & \downarrow g \\ Y & \longrightarrow & Z. \end{array}$$

By Proposition A.2, this square can be extended to a morphism of exact sequences of the form

$$\begin{array}{ccccccc} \sigma & & \hat{x}_* & \longrightarrow & \hat{y}_* & \xrightarrow{p_*} & \hat{z} \\ \downarrow & & \downarrow & \square & \downarrow & & \parallel \\ \sigma'' & & \hat{x} & \longrightarrow & \hat{y} & \longrightarrow & \hat{z} \\ \downarrow & & \parallel & & \downarrow & \square & \downarrow \wr \\ \sigma' & & \hat{x} & \xrightarrow{j} & Y & \longrightarrow & Z \end{array}$$

in which the marked squares are bicartesian. The morphism $\sigma'' \rightarrow \sigma'$ is an equivalence. Since the functor $\mathcal{E} \hookrightarrow \mathcal{H}_{\geq 0}^{\text{st}}(\mathcal{E})$ is fully faithful, \mathcal{E} has fibers of fibrations and p_* is a fibration, \hat{x}_* can be chosen to be the image of some $x_* \in \mathcal{E}$. By a similar argument, the middle row can be chosen to be the image of an exact sequence in \mathcal{E} . \square

We are ready to prove Theorem 1.2.

Proof of Theorem 1.2. The fact that \mathcal{H}^{st} is left adjoint to the inclusion $\mathbf{St}_{\infty} \hookrightarrow \mathbf{Ex}_{\infty}$ is established in Corollary 4.24, which also states that the unit of the adjunction is equivalent to the construction of the stable hull. We apply our findings about the stable hull to prove the three claimed properties of the unit functor. For the stable hull, we proved (1) in Proposition 4.17 and (2) and (3) in Proposition 4.25. \square

4.3. The stable hull of an ordinary exact category

If \mathcal{E} is an exact 1-category in the sense of Quillen, its nerve inherits the structure of an exact ∞ -category. In this case, the universal property of the stable hull and the universal property of the bounded derived category proven in [BCKW19, Cor. 7.59] imply the existence of a canonical equivalence

$$D^b(\mathcal{E}) \simeq \mathcal{H}^{\text{st}}(\mathcal{E}).$$

For completeness, we include a slightly different proof which explains our naming of acyclic objects and quasi-isomorphisms.

Proposition 4.26 ([BCKW19, Prop. 7.55]). *Let \mathcal{E} be an exact 1-category. Then there exists a canonical equivalence*

$$\mathcal{K}^b(\mathcal{E}) \simeq \mathcal{SW}(\mathcal{P}_{\Sigma,f}(\mathcal{E}))$$

whose composition with the inclusion $\mathcal{E} \rightarrow \mathcal{K}^b(\mathcal{E})$ is equivalent to the Yoneda embedding $\mathcal{E} \rightarrow \mathcal{P}_{\Sigma,f}(\mathcal{E})$.

Remark 4.27. While the definition of $\mathcal{P}_{\Sigma,f}(\mathcal{E})$ used by Bunke, Cisinski, Kasprzycki and Wings differs from ours, Proposition 3.5 ensures that they are equivalent.

Proposition 4.28. *Let \mathcal{E} be an exact category, considered an exact ∞ -category as in Example 2.16. Then, the equivalence described in Proposition 4.26 sends an object to an acyclic object in the sense of Definition 3.14 if and only if it is an acyclic complex in the usual sense. Hence, this equivalence preserves and reflects quasi-isomorphisms.*

Proof. The full subcategory of acyclic chain complexes in $\mathcal{K}^b(\mathcal{E})$ is the triangulated closure of acyclic complexes of the form

$$\cdots \rightarrow 0 \rightarrow X_2 \xrightarrow{i} X_1 \xrightarrow{p} X_0 \rightarrow 0 \rightarrow \cdots.$$

It suffices to show that, under the equivalence $\mathcal{K}^b(\mathcal{E}) \simeq \mathcal{SW}(\mathcal{P}_{\Sigma, f}(\mathcal{E}))$, an acyclic complex of this form is sent to the primitive acyclic object corresponding to the exact sequence $X_2 \xrightarrow{i} X_1 \xrightarrow{p} X_0$. This follows from right exactness, as both of these objects are equivalent to the cofiber of the induced map $\text{cofib}(i) \rightarrow X_0$, see Proposition 3.8. \square

Corollary 4.29 (cf. [BCKW19, Cor. 7.59]). *Let \mathcal{E} be an exact category, considered an exact ∞ -category as in Example 2.16. Then, there is a canonical equivalence $D^b(\mathcal{E}) \simeq \mathcal{H}^{st}(\mathcal{E})$, whose composition with the inclusion $\mathcal{E} \hookrightarrow D^b(\mathcal{E})$ is equivalent to the inclusion $\mathcal{E} \hookrightarrow \mathcal{H}^{st}(\mathcal{E})$.*

Appendix A. Diagram lemmas in exact ∞ -categories

In this appendix, we include the proofs of three diagram lemmas we use in the article.

Proposition A.1. *Let \mathcal{E} be an exact ∞ -category. Consider a square in \mathcal{E} of the form*

$$\begin{array}{ccc} x & \xrightarrow{i} & y \\ \downarrow f & & \downarrow g \\ x' & \xrightarrow{j} & y' \end{array},$$

where the morphisms i and j are cofibrations. The following conditions are equivalent:

1. The square is bicartesian.
2. The square is a pushout square.
3. There exists a commutative diagram

$$\begin{array}{ccc} x & \xrightarrow{i} & y \\ \downarrow f & & \downarrow g \\ x' & \xrightarrow{j} & y' \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & z \end{array},$$

where both the lower and the outer rectangle are bicartesian.

Proof. The equivalence (1) \Leftrightarrow (2) is proven in [Bar15, Lemma 4.5].

For (2) \Rightarrow (3), one pushes out j along $x' \twoheadrightarrow 0$, and applies the Pasting Lemma for pushouts.

We show (3) \Rightarrow (2). We first reduce to the case where f is the identity by constructing a diagram

$$\begin{array}{ccccc}
 x & \xrightarrow{\quad} & y & \xrightarrow{\quad} & \\
 \downarrow & \xrightarrow{\quad} & \downarrow & \xrightarrow{\quad} & \downarrow \\
 x' & \xrightarrow{\quad} & w & \xrightarrow{\quad} & y' \\
 \downarrow & \xrightarrow{\quad} & \downarrow & \xrightarrow{\quad} & \downarrow \\
 0 & \xrightarrow{\quad} & z' & \xrightarrow{\quad} & z
 \end{array}$$

Here the front face is the original diagram, and the backsides are obtained by pushing out twice (in particular, $w = x' \amalg_x y$). As both the other front and the outer back faces are cocartesian, the morphism $z' \rightarrow z$ is an equivalence [Cis19, Rem. 6.2.4]. Thus in the lower cube, the bottom, the back and the front face are bicartesian, hence the top face of the bottom cube fulfills the requirements of (3). If the claim holds for that diagram, then the morphism $w \rightarrow y'$ is an equivalence, finishing the proof.

So we assume, without loss of generality, that f is the identity. We then extend the original diagram as indicated below. Note that all the new squares are bicartesian.

$$\begin{array}{ccccc}
 x, & \xrightarrow{\quad} & x \oplus x & \xrightarrow{\quad} & x \oplus y \\
 \parallel & \left(\begin{array}{c} id \\ -id \end{array} \right) \square & \left(\begin{array}{c} id \ 0 \\ id \ id \end{array} \right) \downarrow \wr & \left(\begin{array}{c} id \ 0 \\ 0 \ i \end{array} \right) \square & \left(\begin{array}{c} id \ 0 \\ i \ id \end{array} \right) \downarrow \wr \\
 x & \xrightarrow{\quad} & x \oplus x & \xrightarrow{\quad} & x \oplus y \\
 \downarrow & \left(\begin{array}{c} id \\ 0 \end{array} \right) \square & \left(\begin{array}{c} 0 \ id \end{array} \right) \downarrow & \left(\begin{array}{c} id \ 0 \\ 0 \ i \end{array} \right) \square & \left(\begin{array}{c} id \ 0 \\ 0 \ id \end{array} \right) \downarrow \\
 0 & \xrightarrow{\quad} & x & \xrightarrow{\quad i \quad} & y \\
 \parallel & \square & \parallel & & \downarrow g \\
 0 & \xrightarrow{\quad} & x & \xrightarrow{\quad j \quad} & y'
 \end{array} \tag{6}$$

The rightmost vertical morphism $(j \ g) : x \oplus y \rightarrow y'$ is a fibration, since it constitutes the leftmost vertical morphism in the diagram

$$\begin{array}{ccc}
 x \oplus y & \xrightarrow{\quad (0 \ id) \quad} & y \\
 \left(\begin{array}{c} id \ 0 \\ 0 \ g \end{array} \right) \downarrow & \square & \downarrow g \\
 x \oplus y' & \xrightarrow{\quad (0 \ id) \quad} & y' \\
 \left(\begin{array}{c} j \ id \end{array} \right) \downarrow & & \downarrow \\
 y' & \xrightarrow{\quad} & z,
 \end{array}$$

where the rightmost vertical morphism is a fibration by assumption, the top square is readily seen to be bicartesian and the bottom square is cartesian by [Bar15, Lemma 4.6].

As all small squares in the diagram (6) are cartesian, so is the outer square. As the right morphism is a fibration, it is cocartesian as well. The Pasting Lemma guarantees that the bottom right square is cocartesian, finishing the proof. \square

Proposition A.2. *Let \mathcal{E} be an exact ∞ -category. For every square*

$$\begin{array}{ccc} y & \xrightarrow{p} \twoheadrightarrow & z \\ \downarrow f & & \downarrow g \\ y' & \xrightarrow{p'} \twoheadrightarrow & z', \end{array}$$

there exist two morphisms between exact sequences $a: \sigma \rightarrow \sigma''$, $b: \sigma'' \rightarrow \sigma'$ and a composition $c = b \circ a$ such that the restriction along $\Delta^2 \rightarrow \Delta^1 \times \Delta^1$ yields a commutative diagram of the form

$$\begin{array}{ccccc} \sigma & x \twoheadrightarrow y \twoheadrightarrow z & & & \\ \downarrow a & \downarrow \square \downarrow & & & \parallel \\ \sigma'' & x' \twoheadrightarrow w \twoheadrightarrow z & & & \\ \downarrow b & \parallel \downarrow \square \downarrow & & & \downarrow \\ \sigma' & x' \twoheadrightarrow y' \twoheadrightarrow z', & & & \end{array} \quad (7)$$

where the marked squares are bicartesian and the right vertical rectangle is the original square.

Proof. By taking the pullback of p' along g , we obtain a factorization of the original square

$$\begin{array}{ccc} y & \xrightarrow{p} \twoheadrightarrow & z \\ \downarrow & & \parallel \\ w & \xrightarrow{p''} \twoheadrightarrow & z \\ \downarrow & \square \downarrow & \downarrow \\ y' & \xrightarrow{p'} \twoheadrightarrow & z'. \end{array}$$

Every exact sequence in $\text{Fun}(\Delta^1 \times \Delta^1, \mathcal{E})$ is an iterated Kan extension of the fibration it restricts to. Hence, the restriction functor along the right vertical arrow

$$\text{Fun}(\Delta^1 \times \Delta^1, \mathcal{E}) \rightarrow \text{Fun}(\Delta^1, \mathcal{E})$$

restricts, by [Lur09a, Prop. 4.3.2.15], to a trivial Kan fibration from the full subcategory spanned by the exact sequences to the full subcategory spanned by the fibrations. Hence, this diagram can be extended to a diagram of exact sequences of

the form

$$\begin{array}{ccccc}
 x & \xrightarrow{\quad} & y & \xrightarrow{p} & z \\
 \downarrow & & \downarrow & & \parallel \\
 x' & \xrightarrow{\quad} & w & \xrightarrow{p''} & z \\
 \downarrow & & \downarrow & & \downarrow \\
 x'' & \xrightarrow{p'} & y' & \xrightarrow{p'} & z'.
 \end{array}$$

The top left square is bicartesian by Proposition A.1. By the dual of Proposition A.1, the morphism $x' \rightarrow x''$ is an equivalence; identifying x' with x'' along this equivalence finishes the proof. \square

Proposition A.3. *Let \mathcal{E} be an exact ∞ -category, $p_1: y_1 \twoheadrightarrow z_1$ and $p_2: y_2 \twoheadrightarrow z_2$ two fibrations, and $g: y_1 \rightarrow z_2$ a morphism in \mathcal{E} . The morphism $y_1 \oplus y_2 \rightarrow z_1 \oplus z_2$ given by the matrix*

$$\begin{pmatrix} -p_1 & 0 \\ g & p_2 \end{pmatrix}$$

is a fibration.

Proof. The map in question is a composition of the morphisms

$$y_1 \oplus y_2 \xrightarrow{\begin{pmatrix} id & 0 \\ 0 & p_2 \end{pmatrix}} y_1 \oplus z_2 \xrightarrow{\begin{pmatrix} id & 0 \\ g & id \end{pmatrix}} y_1 \oplus z_2 \xrightarrow{\begin{pmatrix} -p_1 & 0 \\ 0 & id \end{pmatrix}} z_1 \oplus z_2,$$

where each map is a fibration: the two outer maps are pullbacks of p_2 and $-p_1$ along projections, and the one in the middle is an equivalence. \square

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