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ON COHOMOLOGY IN SYMMETRIC TENSOR CATEGORIES IN PRIME CHARACTERISTIC

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Abstract

We describe graded commutative Gorenstein algebras $\mathcal{E}_n(p)$ over a field of prime characteristic p, and we conjecture that $\mathrm{Ext}^\bullet_{\mathsf{Ver}_{p^{n+1}}}(\mathbb{1},\mathbb{1}) \cong \mathcal{E}_n(p)$, where $\mathsf{Ver}_{p^{n+1}}$ are the new symmetric tensor categories recently constructed by the current authors, with Ostrik, and also by Coulembier. We investigate the combinatorics of these algebras, and the relationship with Minc's partition function, as well as possible actions of the Steenrod operations on them.

Evidence for the conjecture includes a large number of computations for small values of n. We also provide some theoretical evidence. Namely, we use a Koszul construction to identify a homogeneous system of parameters in $\mathcal{E}_n(p)$ with a homogeneous system of parameters in $\mathrm{Ext}^\bullet_{\mathsf{Ver}_{p^{n+1}}}(\mathbb{1},\mathbb{1})$. These parameters have degrees $2^i - 1$ if $p = 2$ and $2(p^i - 1)$ if p is odd, for $1 \leqslant i \leqslant n$. This at least shows that $\mathrm{Ext}^\bullet_{\mathsf{Ver}_{p^{n+1}}}(\mathbb{1},\mathbb{1})$ is a finitely generated graded commutative algebra with the same Krull dimension as $\mathcal{E}_n(p)$. For $p = 2$ we also show that $\mathrm{Ext}^\bullet_{\mathsf{Ver}_{2^{n+1}}}(\mathbb{1},\mathbb{1})$ has the expected rank $2^{n(n-1)/2}$ as a module over the subalgebra of parameters.

1. Introduction

In our paper [**2**], we introduced a nested sequence of incompressible symmetric tensor abelian categories in characteristic two. These were very recently generalised to all primes in our work with Ostrik [**3**] and simultaneously by Coulembier [**8**]. These categories, Ver_{p^n} and $\mathsf{Ver}^+_{p^n}$, seem to be new fundamental objects deserving further study.

Here, our primary aim is to state a conjecture describing the ring structure of $\mathrm{Ext}^\bullet_{\mathsf{Ver}_{p^{n+1}}}(\mathbb{1},\mathbb{1})$. We have made large numbers of computations using the computer algebra system Magma [**5**], and we conjecture that the answer should be the graded

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commutative $\mathbf{k}$-algebra $\mathcal{E}_n(p)$ introduced below, where $\mathbf{k}$ is a field of characteristic p. After defining these algebras, we prove the following.

Theorem 1.1. *For $n \geqslant 0$, the algebra $\mathcal{E}_n(p)$ is a graded commutative finitely generated Gorenstein $\mathbf{k}$-algebra of Krull dimension n. If $p = 2$ then it is an integral domain, while for p odd it has nilpotent elements. The Poincaré series $f(q) = \sum_{d \geqslant 0} q^d \dim \mathcal{E}_n(p)_d$ is a rational function of q satisfying $f(1/q) = (-q)^n f(q)$.*

There are natural inclusion maps $\mathcal{E}_{n-1}(p) \to \mathcal{E}_n(p)$, and in each degree the sequence

$$\mathbf{k} = \mathcal{E}_0(p) \to \mathcal{E}_1(p) \to \cdots \to \mathcal{E}_{n-1}(p) \to \mathcal{E}_n(p) \to \cdots$$

stabilises at some finite stage. So it makes sense to examine the colimit $\mathcal{E}_\infty(p) = \varinjlim_n \mathcal{E}_n(p)$. The Poincaré series of this algebra in the case $p = 2$ is Minc's partition function [**19**]. We adapt Andrews' proof of a Rogers–Ramanujan style formula [**1**] for the reciprocal of the generating function for this partition function so that it gives us the Poincaré series for $\mathcal{E}_n(p)$ for all $n \geqslant 0$ and all primes p.

Theorem 1.2. *The dimension of $\mathcal{E}_n(p)_d$ is equal to $\sum_{m=1}^{n} N_p(m,d)$, and the dimension of $\mathcal{E}_\infty(p)_d$ is equal to $\sum_{m=1}^{\infty} N_p(m,d)$, where $\sum_{m,d=0}^{\infty} N_p(m,d) t^m q^d = \dfrac{1}{\sum_{i=0}^{\infty} (-1)^i t^i \ell_{i,p}(q)}$ and*

$$\ell_{i,p}(q) = \begin{cases} \displaystyle\prod_{j=1}^{i} \frac{q^{2^j-1}}{1-q^{2^j-1}} & p = 2, \\ \displaystyle\prod_{j=1}^{i} \frac{q^{2p^{j-1}(p-1)-1} + q^{2(p^j-1)}}{1-q^{2(p^j-1)}} & p \text{ odd.} \end{cases}$$

The relationship with the symmetric tensor abelian categories constructed in [**2, 3, 8**] is as follows. Since the subcategory $\mathsf{Ver}^+_{p^n} \subset \mathsf{Ver}_{p^n}$ is a direct summand, this inclusion induces an isomorphism $\mathrm{Ext}^\bullet_{\mathsf{Ver}_{p^n}}(\mathbb{1},\mathbb{1}) \cong \mathrm{Ext}^\bullet_{\mathsf{Ver}^+_{p^n}}(\mathbb{1},\mathbb{1})$ and so we only consider Ver_{p^n}.

Conjecture 1.3. *The graded commutative $\mathbf{k}$-algebra $\mathrm{Ext}^\bullet_{\mathsf{Ver}_{p^{n+1}}}(\mathbb{1},\mathbb{1})$ is isomorphic to $\mathcal{E}_n(p)$. The inclusion $\mathsf{Ver}_{p^n} \subset \mathsf{Ver}_{p^{n+1}}$ induces the inclusion map $\mathcal{E}_{n-1}(p) \to \mathcal{E}_n(p)$.*

We have the following computational evidence for this conjecture. In all characteristics, this is true for $n \leqslant 1$. In characteristic two, we have checked both the dimensions and the algebra structure for $n = 2$ in all degrees, for $n = 3$ up to degree 40, and for $n = 4$ up to degree 26. For $p = 3$, $n = 2$, 3, and for $p = 5$, $n = 2$, we have checked the dimensions and algebra structure up to degree 40. These computations were carried out using the computer algebra package Magma.

In a symmetric tensor abelian category, the Steenrod operations act on $\mathrm{Ext}^\bullet(\mathbb{1},\mathbb{1})$ and satisfy the Cartan formula and unstable condition, as well as the homogeneous form of the Adem relations in which it is not assumed that the operation Sq^0 ($p = 2$), respectively $\mathcal{P}^0$ (p odd) acts as the identity (see [**18**]; the construction there extends to the setting of symmetric tensor categories). We investigate the possibilities for their action on $\mathcal{E}_n(p)$. Our conclusions are cleanest when $p = 2$. In that case, we show that the only possible action of the Steenrod operations on $\mathcal{E}_n(2)$ compatible with the inclusions is that all $\mathsf{Sq}^i = 0$ except for the mandatory $\mathsf{Sq}^{|x|}(x) = x^2$. This makes the action much more like that on the cohomology of a p-restricted Lie algebra than like that on the cohomology of a finite group. In the case p odd, the existence of nilpotent elements interferes with the arguments, and we can only prove a weaker statement.

In the final sections, we provide some theoretical evidence for Conjecture 1.3, and some tools that may help prove it. Namely, we first consider the Koszul complex of the generating object V of $\mathsf{Ver}_{p^{n+1}}$ and compute its cohomology. Then we use the Koszul complex to express $\mathrm{Ext}^\bullet_{\mathsf{Ver}_{p^{n+1}}}(\mathbb{1}, X)$ as the cohomology of an explicit complex of vector spaces. While we cannot yet compute this cohomology in general, this construction explains the conjectural shape of the answer and provides upper bounds for dimensions of the individual Ext spaces. In particular, it implies the existence of the subalgebra of parameters, $\mathbf{k}[y_1, \dots, y_n] \subset \mathrm{Ext}^\bullet_{\mathsf{Ver}_{p^{n+1}}}(\mathbb{1}, \mathbb{1})$, where $\deg(y_i)$ equals $2^i - 1$ if $p = 2$ and $2(p^i - 1)$ if $p > 2$. We show that $\mathrm{Ext}^\bullet_{\mathsf{Ver}_{p^{n+1}}}(\mathbb{1}, \mathbb{1})$ is module-finite over this subalgebra, and for $p = 2$ show that the rank of this module is $2^{n(n-1)/2}$, as predicted by Conjecture 1.3.

More generally, we at least show the following.

Theorem 1.4. *The graded commutative* $\mathbf{k}$*-algebra* $\mathrm{Ext}^\bullet_{\mathsf{Ver}_{p^{n+1}}}(\mathbb{1}, \mathbb{1})$ *is finitely generated, with Krull dimension* n*. Moreover, for any* $X \in \mathsf{Ver}_{p^{n+1}}$*,* $\mathrm{Ext}^\bullet_{\mathsf{Ver}_{p^{n+1}}}(\mathbb{1}, X)$ *is a finitely generated module over this algebra.*

This confirms Conjecture 2.18 of [**14**] for the categories $\mathsf{Ver}_{p^{n+1}}$.

Once the Ext algebra is better understood, this will be the starting point for applying support theory to the categories $\mathsf{Ver}_{p^{n+1}}$, along the lines of the theory for finite groups, developed by Carlson and others [**7**]. For example, one might hope that $\mathrm{Ext}^\bullet_{\mathsf{Ver}_{p^{n+1}}}(\mathbb{1}, \mathbb{1})$ stratifies the stable category of $\mathsf{Ver}_{p^{n+1}}$ as a tensor triangulated category, in the sense of Benson, Iyengar and Krause [**4**]. This would give a classification of the tensor ideal thick subcategories, as well as the tensor ideal localising subcategories of the stable category of the ind-completion. If Conjecture 1.3 holds, then the inclusion of the subalgebra of parameters $\mathbf{k}[y_1, \dots, y_n] \hookrightarrow \mathrm{Ext}^\bullet_{\mathsf{Ver}_{p^{n+1}}}(\mathbb{1}, \mathbb{1})$ is an inseparable isogeny. This implies that it induces a bijection on homogeneous prime ideals, and so $\mathsf{Proj}\,\mathrm{Ext}^\bullet_{\mathsf{Ver}_{p^{n+1}}}(\mathbb{1}, \mathbb{1})$ is a weighted projective space.

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2. Graded algebras

For a prime p let $\mathbb{Z}[\frac{1}{p}]$ denote the ring of integers with p inverted. An element of $\mathbb{Z}[\frac{1}{p}]$ is a rational number $r = m/n$ where $m, n \in \mathbb{Z}$ and n is a power of p. We say that such an element r is *even* if $r/2$ is also in $\mathbb{Z}[\frac{1}{p}]$ and *odd* otherwise. So for $p = 2$, every element is even. If $a \in \mathbb{Z}[\frac{1}{p}]$, we write $(-1)^a$ to denote $+1$ if a is even and -1 if a is odd.

We consider $\mathbb{Z}[\frac{1}{p}]$-graded algebras R over a field $\mathbf{k}$ of characteristic p. If x is a homogeneous element of R, we write $|x|$ for the degree of x. We say that such an algebra is *graded commutative* if it satisfies $yx = (-1)^{|x||y|}xy$.

If R is a $\mathbb{Z}[\frac{1}{p}]$-graded $\mathbf{k}$-algebra, we write $\mathsf{Int}(R)$ for the $\mathbb{Z}$-graded algebra derived from R by means of the inclusion of $\mathbb{Z}$ in $\mathbb{Z}[\frac{1}{p}]$. So for $m \in \mathbb{Z}$, the homogeneous part of degree m is given by $\mathsf{Int}(R)_m = R_m$.

Example 2.1. Let $\mathbf{k}$ be a field of characteristic two, and let $\mathbf{k}[X^{2^*}]$ be the algebra generated by the elements X^{2^n} with $n \in \mathbb{Z}$, with the obvious relations $(X^{2^n})^2 = X^{2^{n+1}}$. This is a $\mathbb{Z}[\frac{1}{2}]$-graded commutative $\mathbf{k}$-algebra, with $|X^{2^n}| = 2^n$. We have $\mathsf{Int}(\mathbf{k}[X^{2^*}]) = \mathbf{k}[X]$.

Example 2.2. Let $\mathbf{k}$ be a field of odd characteristic p, and let $\mathbf{k}[X^{p^*}] \otimes \Lambda(Y)$ be the algebra generated by elements X^{p^n} with $n \in \mathbb{Z}$ and Y with the relations $(X^{p^n})^p = X^{p^{n+1}}$, $Y^2 = 0$, $XY = YX$. This is a $\mathbb{Z}[\frac{1}{p}]$-graded commutative $\mathbf{k}$-algebra, with $|X^{p^n}| = 2p^n$ and $|Y| = 1$. We have $\mathsf{Int}(\mathbf{k}[X^{p^*}] \otimes \Lambda(Y)) = \mathbf{k}[X] \otimes \Lambda(Y)$.

Definition 2.3. We define the *Reynolds operator* $\rho\colon R \to \mathsf{Int}(R)$ to be the map which is the identity on elements of $\mathsf{Int}(R)$ and zero on homogeneous elements of R whose degree is not an integer.

Lemma 2.4. *The map ρ is an $\mathsf{Int}(R)$-module homomorphism.*

Proof. Multiplication by elements of $\mathsf{Int}(R)$ preserves whether or not the degree of an element is an integer. □

Proposition 2.5. *If R is a Cohen–Macaulay $\mathbf{k}$-algebra then so is $\mathsf{Int}(R)$.*

Proof. For every element of R, some power is an element of $\mathsf{Int}(R)$. So R is an integral extension of $\mathsf{Int}(R)$. By Lemma 2.4, the Reynolds operator $\rho\colon R \to \mathsf{Int}(R)$ is an $\mathsf{Int}(R)$-module homomorphism. The proposition now follows from Proposition 12 of Hochster and Eagon [**16**]. □

3. The algebra $\mathcal{E}_n(p)$

We treat separately the cases $p = 2$ and p odd.

3.1. The algebra $\mathcal{E}_n(2)$

In this section, we examine the case $p = 2$, and we let $\mathbf{k}$ be a field of characteristic two.

Definition 3.1. Let $R = R(n, 2)$ be the $\mathbb{Z}[\frac{1}{2}]$-graded commutative polynomial algebra $\mathbf{k}[x_1, \dots, x_n]$ with $|x_i| = \frac{2^i-1}{2^i}$, and let $\mathcal{E}_n(2) = \mathsf{Int}(R)$.

Example 3.2. If $n = 1$, we have $R = \mathbf{k}[x_1]$ with $|x_1| = \frac{1}{2}$. The algebra $\mathsf{Int}(R)$ is generated by $u = x_1^2$, so $\mathcal{E}_1(2) = \mathbf{k}[u]$.

Example 3.3. If $n = 2$, we have $R = \mathbf{k}[x_1, x_2]$ with $|x_1| = \frac{1}{2}$, $|x_2| = \frac{3}{4}$. The algebra $\mathsf{Int}(R)$ is generated by $u = x_1^2$, $v = x_1x_2^2$, $w = x_2^4$. Then

$$\mathcal{E}_2(2) = \mathsf{Int}(R) = \mathbf{k}[u, v, w]/(uw + v^2)$$

with $|u| = 1$, $|v| = 2$, $|w| = 3$.

Example 3.4. If $n = 3$, we have $R = \mathbf{k}[x_1, x_2, x_3]$ with $|x_1| = \frac{1}{2}$, $|x_2| = \frac{3}{4}$, $|x_3| = \frac{7}{8}$. Then $\mathcal{E}_3(2) = \mathsf{Int}(R)$ has a homogeneous system of parameters $y_1 = x_1^2$, $y_2 = x_2^4$, $y_3 = x_3^8$, of degrees $1, 3, 7$. The quotient by these parameters has the following basis.

deg	0	1	2	3	4	5	6	7	8
elt	1		$x_1x_2^2$	$x_1x_2x_3^2$	$x_1x_3^4$ $x_2^3x_3^2$	$x_2^2x_3^4$	$x_2x_3^6$		$x_1x_2^3x_3^6$

The Poincaré series of $\mathcal{E}_3(2)$ is therefore given by

$$\sum_{d\geqslant 0} q^d \dim \mathcal{E}_3(2)_d = \frac{1 + q^2 + q^3 + 2q^4 + q^5 + q^6 + q^8}{(1 - q)(1 - q^3)(1 - q^7)}.$$

Theorem 3.5. *The algebra $\mathcal{E}_n(2)$ is a Gorenstein integral domain. It has a regular homogeneous sequence of parameters $y_1 = x_1^2$, $y_2 = x_2^4$, $y_3 = x_3^8, \dots,$ $y_n = x_n^{2^n}$ of degrees $1, 3, 7, \dots, 2^n - 1$. Modulo this regular sequence, we get a graded Frobenius algebra of dimension $2^{\frac{n(n-1)}{2}}$ with dualising element $\alpha = x_1x_2^3x_3^7 \dots x_{n-1}^{2^{n-1}-1}x_n^{2^n-2}$ in degree $2^{n+1} - 2n - 2$. The Poincaré series $f(q) = \sum_{d\geqslant 0} q^d \dim \mathcal{E}_n(2)_d$ is a rational function of q satisfying $f(1/q) = (-q)^n f(q)$.*

Proof. It follows from Proposition 2.5 that $\mathcal{E}_n(2) = \mathsf{Int}(\mathbf{k}[x_1, \dots, x_n])$ is a Cohen–Macaulay integral domain. So the homogeneous sequence of parameters $y_1, y_2, y_3, \dots,$ y_n is a regular sequence.

If $x_1^{a_1} \dots x_n^{a_n}$ is a monomial in $\mathcal{E}_n(2)$ then a_n is even. If such a monomial is not divisible by any of the parameters then $a_i \leqslant 2^i - 1$ for $1 \leqslant i < n$, and $a_n \leqslant 2^n - 2$. The monomial $x_1^{1-a_1}x_2^{3-a_2}x_3^{7-a_3} \dots x_{n-1}^{2^{n-1}-1-a_{n-1}}x_n^{2^n-2-a_n}$ is also a basis element of $\mathcal{E}_n(2)$ and the product of this with $x_1^{a_1} \dots x_n^{a_n}$ is equal to α. So $\mathcal{E}_n(2)/(x_1^2, x_2^4, x_3^8, \dots, x_n^{2^n})$ is a Frobenius algebra with a basis consisting of these monomials, and with dualising element α.

It is easy to verify using the Frobenius property that $f(1/q) = (-q)^n f(q)$. It then follows by Theorem 4.4 of Stanley [**22**] that $\mathcal{E}_n(2)$ is a Gorenstein algebra. Alternatively, it is shown in Eisenbud [**10**, §21.3] that the Gorenstein property holds for a graded Cohen–Macaulay ring if and only if the quotient by a regular sequence of parameters is a Frobenius algebra. □

There is a natural inclusion map of algebras $R(n - 1, 2) \to R(n, 2)$ given by sending each x_i in $R(n - 1, 2)$ to the element with the same name in $R(n, 2)$. It is easy to check that in each degree the sequence

$$R(1, 2) \to \cdots \to R(n - 1, 2) \to R(n, 2) \to \cdots$$

stabilises at some finite stage. So we take the colimit $R(\infty, 2) = \varinjlim_n R(n, 2)$. Applying Int, we obtain inclusion maps $\mathcal{E}_1(2) \to \cdots \to \mathcal{E}_{n-1}(2) \to \mathcal{E}_n(2) \to \cdots$ whose colimit we denote $\mathcal{E}_\infty(2) = \mathsf{Int}(R(\infty, 2))$.

3.2. The algebra $\mathcal{E}_n(p)$, $p > 2$

For odd primes, we should double the degrees of the polynomial generators and introduce new exterior generators of degree one smaller.

Let p be an odd prime and let $\mathbf{k}$ be a field of characteristic p. Let $R = R(n,p)$ be the $\mathbb{Z}[\frac{1}{p}]$-graded commutative algebra $\mathbf{k}[x_1,\dots,x_n]\otimes\Lambda(\xi_1,\dots,\xi_n)$ with $|x_i| = \frac{2(p^i-1)}{p^i}$ and $|\xi_i| = |x_i| - 1 = \frac{p^i-2}{p^i}$. Note that $|x_i|$ is even and $|\xi_i|$ is odd. We define $\mathcal{E}_n(p) = \mathsf{Int}(R(n,p))$.

Example 3.6. If $n = 1$, we have $R = \mathbf{k}[x_1]\otimes\Lambda(\xi_1)$ with $|x_1| = \frac{2(p-1)}{p}$ and $|\xi_1| = \frac{p-2}{p}$. In this case, the algebra $\mathcal{E}_1(p) = \mathsf{Int}(R)$ is generated by the elements $y = x_1^p$ and $\eta = x_1^{p-1}\xi_1$ with $|y| = 2p-2$, $|\eta| = 2p-3$, namely, $\mathcal{E}_1(p) = \mathbf{k}[y]\otimes\Lambda(\eta)$.

Example 3.7. If $p = 3$ and $n = 2$, we have $R = \mathbf{k}[x_1,x_2]\otimes\Lambda(\xi_1,\xi_2)$ with $|x_1| = \frac{4}{3}$, $|x_2| = \frac{16}{9}$, $|\xi_1| = \frac{1}{3}$, $|\xi_2| = \frac{7}{9}$. In this case, the algebra $\mathcal{E}_2(3) = \mathsf{Int}(R)$ is generated by the following elements:

element	degree	element	degree	element	degree
$x_1^2\xi_1$	3	$x_1^2x_2^2\xi_2$	7	$x_2^6\xi_1$	11
x_1^3	4	$x_1^2x_2^3$	8	$x_1x_2^6$	12
$x_1x_2^2\xi_1\xi_2$	6	$x_2^5\xi_1\xi_2$	10	$x_2^8\xi_2$	15
$x_1x_2^3\xi_1$	7	$x_1x_2^5\xi_2$	11	x_2^9	16

A regular homogeneous system of parameters is given by $y_1 = x_1^3$ and $y_2 = x_2^9$, and the quotient by these parameters is a graded Frobenius algebra with dualising element $x_1^2x_2^8\xi_1\xi_2$ in degree 18. We have

$$\sum_{d\geqslant 0} q^d \dim\mathcal{E}_2(3)_d = \frac{1+q^3+q^6+2q^7+q^8+q^{10}+2q^{11}+q^{12}+q^{15}+q^{18}}{(1-q^4)(1-q^{16})}.$$

Theorem 3.8. *The ring $\mathcal{E}_n(p)$ is Gorenstein. It has a homogeneous system of parameters $y_1 = x_1^p, y_2 = x_2^{p^2},\dots,y_n = x_n^{p^n}$ of degrees $2(p-1)$, $2(p^2-1)$, $\dots$, $2(p^n-1)$. Modulo this regular sequence, we get a graded Frobenius algebra of dimension $2^np^{\frac{n(n-1)}{2}}$ with dualising element $\alpha = x_1^{p-1}x_2^{p^2-1}\cdots x_n^{p^n-1}\xi_1\xi_2\cdots\xi_n$, an element which lies in degree $2\big(\frac{p^{n+1}-1}{p-1}\big) - 3n - 2$. The Poincaré series $f(q) = \sum_{d\geqslant 0} q^d\dim\mathcal{E}_n(p)_d$ is a rational function of q satisfying $f(1/q) = (-q)^nf(q)$.*

Proof. It follows from Proposition 2.5 that $\mathcal{E}_n(p)$ is Cohen–Macaulay. Since $y_1, y_2, \dots, y_n$ are elements of $\mathcal{E}_n(p)$ which form a regular sequence of parameters in $R(n,p)$, they also form a regular sequence of parameters in $\mathcal{E}_n(p)$. If $x_1^{a_1}\cdots x_n^{a_n}\xi_1^{\varepsilon_1}\cdots\xi_n^{\varepsilon_n}$ ($\varepsilon_i\in\{0,1\}$ for $1\leqslant i\leqslant n$) is a monomial in $\mathcal{E}_n(p)$ which is not divisible by any of the parameters then $a_i\leqslant p^i-1$ for $1\leqslant i\leqslant n$. The monomial

$$x_1^{p-1-a_1}x_2^{p^2-1-a_2}\cdots x_n^{p^n-1-a_n}\xi_1^{1-\varepsilon_1}\cdots\xi_n^{1-\varepsilon_n}$$

is also a basis element of $\mathcal{E}_n(p)$ and its product with $x_1^{a_1}\cdots x_n^{a_n}\xi_1^{\varepsilon_1}\cdots\xi_n^{\varepsilon_n}$ is α.

Again it is easy to verify using the Frobenius property that $f(1/q) = (-q)^nf(q)$. But this time, we cannot show the Gorenstein property as in the proof of Theorem 3.5, using Theorem 4.4 of [**22**], because $\mathcal{E}_n(p)$ is not an integral domain. However, the alternative argument using §21.3 of [**10**] still shows that $\mathcal{E}_n(p)$ is Gorenstein. □

Remark 3.9. Recall that there is an action of the multiplicative group $\mathbb{G}_m$ on the algebras $\mathbf{k}[x_1,\dots,x_n]$ and $\mathbf{k}[x_1,\dots,x_n]\otimes\Lambda(\xi_1,\dots,\xi_n)$ defined by their $\mathbb{Z}$-grading (the fractional degrees multiplied by p^n). Also we have the semisimple infinitesimal subgroup scheme $\mu_{p^n}\subset\mathbb{G}_m$ defined by the equation $a^{p^n}=1$ (i.e., $\mu_{p^n}=(\mathbb{G}_m)_{(n)}$, the n-th Frobenius kernel of $\mathbb{G}_m$). For $p>2$, $\mathcal{E}_n(p)=(\mathbf{k}[x_1,\dots,x_n]\otimes\Lambda(\xi_1,\dots,\xi_n))^{\mu_{p^n}}$, the subring of invariants, and for $p=2$ we similarly have the subring of invariants $\mathcal{E}_n(2)=\mathbf{k}[x_1,\dots,x_{n-1},x_n]^{\mu_{2^n}}=\mathbf{k}[x_1,\dots,x_{n-1},x_n^2]^{\mu_{2^n}}$. Since $\sum_i\deg(x_i)-\sum_i\deg(\xi_i)$ is an integer, the action of μ_{p^n} on the super-space spanned by the variables x_i,ξ_i for $p>2$ has Berezinian equal to 1 (recall that degrees of odd variables should be counted with a minus sign). Similarly, for $p=2$ the action of μ_{2^n} on the variables $x_1,\dots,x_{n-1},x_n^2$ has determinant equal to 1, as $\sum_{i=1}^{n-1}\deg x_i+2\deg x_n$ is an integer. This is related to the fact that the ring $\mathcal{E}_n(p)$ is Gorenstein. For example, for $p=2$ this follows from a group scheme generalization of Watanabe's theorem: the algebra of invariants for a homogeneous unimodular action of a finite semisimple group scheme on a polynomial algebra is Gorenstein. This is a special case of [**17**, Theorem 0.1].

4. Generating functions

4.1. Generating functions, $p=2$

For an integer $d\geqslant 0$, the degree d part of $\mathcal{E}_n(2)$ has a basis consisting of the monomials $x_1^{a_1}x_2^{a_2}\dots x_n^{a_n}$ such that the a_j are non-negative integers, and

$$d=\tfrac{1}{2}a_1+\tfrac{3}{4}a_2+\cdots+\tfrac{2^n-1}{2^n}a_n.$$

The smallest integer degree of a term with $a_j>0$ is j, which occurs for the monomial $x_1x_2\dots x_{j-1}x_j^2$. So for d an integer, we must have $a_j=0$ for $j>d$. It follows that the maps of vector spaces $\mathcal{E}_1(2)_d\to\mathcal{E}_2(2)_d\to\cdots$ are eventually isomorphisms, and $\mathcal{E}_\infty(2)_d$ is a finite dimensional vector space, spanned by the monomials $x_1^{a_1}x_2^{a_2}\cdots$ with

$$d=\tfrac{1}{2}a_1+\tfrac{3}{4}a_2+\tfrac{7}{8}a_3+\cdots.$$

Such an expression is a *partition* of d into parts $\frac{1}{2},\frac{3}{4},\frac{7}{8},\cdots$. These are enumerated in sequence A002843 of the On-line Encyclopedia of Integer Sequences (which is sequence 405 of Sloane's Handbook [**21**]). This sequence has been studied by Minc [**19**], Andrews [**1**], and Flajolet and Prodinger [**15**]; see also Nguyen, Schwartz and Tran [**20**] for a context in algebraic topology. The first few terms are

$$1,1,2,4,7,13,24,43,78,141,253,456,820,1472,2645,4749,8523,15299,27456,49267,$$
$$88407,158630,284622,510683,916271,1643963,2949570,5292027,9494758,\dots$$

A few more terms can be found at https://oeis.org/A002843/b002843.txt. This sequence grows like $C\lambda^n$, where

$$C:=0.74040259366730734\dots,\qquad\lambda:=1.79414718754168546\dots\tag{4.1}$$

Our analysis of the generating function $\sum_{d=0}^{\infty}q^d\dim\mathcal{E}_n(2)_d$ follows Andrews [**1**]. Since there are many misprints in the relevant section of [**1**], and we are doing something slightly different, we choose to repeat the argument in our context. The analogous argument for p odd, which we carry out later, is not dealt with in [**1**].

Let $N(m,d)$ be the number of monomials of degree d in $x_1,\dots,x_m$ with $a_m>0$. Thus the dimension of $\mathcal{E}_n(2)_d$ is $\sum_{m=1}^{n} N(m,d)$, and the dimension of $\mathcal{E}_\infty(2)_d$ is $\sum_{m=1}^{\infty} N(m,d)$.

We can rewrite these monomials in terms of new variables $z_1,z_2,\dots$ as follows. Set $z_1=x_1^2$, and $z_i=x_{i-1}^{-1}x_i^2$ for $i\geqslant 2$. These variables z_i are degree one elements of the larger $\mathbb{Z}[\frac{1}{2}]$-graded ring of Laurent polynomials $\mathbf{k}[x_1,x_1^{-1},x_2,x_2^{-1},\dots]$. Then we have $x_1^{a_1}x_2^{a_2}\cdots=z_1^{b_1}z_2^{b_2}\dots$ where $a_i=2b_i-b_{i+1}$. The constraints $a_i\geqslant 0$ translate to $2b_i\geqslant b_{i+1}$ for $i\geqslant 1$, and since the b_i are eventually zero, they are all non-negative. Thus $N(m,d)$ is the number of sequences $(b_1,\dots,b_m)$ of nonnegative integers with $\sum_{i=1}^{m} b_i=d$, and $2b_i\geqslant b_{i+1}$ for $1\leqslant i<m$.

Set $\mu_m(q)=\sum_{d=0}^{\infty} N(m,d)q^d$, and $\mu_0(q)=1$. We would like to compute $\mu_m(q)$.

In fact, we will compute a more general generating function, taking into account the degrees with respect to all z_i. Introduce auxiliary variables $q_1,q_2,\dots$ corresponding to the statistics $b_1,b_2,\dots$; i.e., we define the multivariate Poincaré series of $\mathcal{E}_n(2)$

$$\mu_m(q_1,\dots,q_m):=\sum_{b_1,\dots,b_m:2b_i\geqslant b_{i+1}}^{\infty} q_1^{b_1}\cdots q_m^{b_m},$$

so that the usual Poincaré series of this algebra is $\mu_m(q)=\mu_m(q,\dots,q)$.

Thus we have

$$\mu_m(q_1,\dots,q_m)=\sum_{b_1=1}^{\infty}\sum_{b_2=1}^{2b_1}\cdots\sum_{b_m=1}^{2b_{m-1}} q_1^{b_1}\dots q_m^{b_m}.$$

For the last sum we have $\displaystyle\sum_{b_m=1}^{2b_{m-1}} q_m^{b_m}=\frac{q_m}{1-q_m}(1-q_m^{2b_{m-1}})$ and so we obtain

$$\mu_m=\frac{q_m}{1-q_m}\left(\mu_{m-1}-\sum_{b_1=1}^{\infty}\sum_{b_2=1}^{2b_1}\cdots\sum_{b_{m-1}=1}^{2b_{m-2}} q_1^{b_1}\cdots q_{m-2}^{b_{m-2}}(q_{m-1}q_m^2)^{b_{m-1}}\right).$$

Now for the last sum we have

$$\sum_{b_{m-1}=1}^{2b_{m-2}}(q_{m-1}q_m^2)^{b_{m-1}}=\frac{q_{m-1}q_m^2}{1-q_{m-1}q_m^2}(1-(q_{m-1}q_m^2)^{2b_{m-2}})$$

and so we obtain

$$\mu_m=\frac{q_m}{1-q_m}\Big(\mu_{m-1}-\frac{q_{m-1}q_m^2}{1-q_{m-1}q_m^2}\Big(\mu_{m-2}-\sum_{b_1=1}^{\infty}\sum_{b_2=1}^{2b_1}\cdots\sum_{b_{m-2}=1}^{2b_{m-3}} q_1^{b_1}\cdots q_{m-3}^{b_{m-3}}(q_{m-2}q_{m-1}^2q_m^4)^{b_{m-2}}\Big)\Big).$$

We continue this way, using induction. At the end, we use $\mu_0=1$. We obtain

$$\sum_{i=1}^{m}(-1)^i\mu_{m-i}\Big(\frac{q_m}{1-q_m}\Big)\Big(\frac{q_{m-1}q_m^2}{1-q_{m-1}q_m^2}\Big)\cdots\Big(\frac{q_{m-i}\cdots q_m^{2^i}}{1-q_{m-i}\cdots q_m^{2^i}}\Big)=\begin{cases}0 & m>0,\\ 1 & m=0.\end{cases}$$

So we set

$$\ell_m(q_1,\dots,q_m) = \frac{q_1 q_2^3 q_3^7 \cdots q_m^{2^m-1}}{(1-q_m)(1-q_{m-1}q_m^2)\cdots(1-q_1q_2^2\dots q_m^{2^{m-1}})},$$

and we have $\sum_{i=0}^m (-1)^i \mu_{m-i}\ell_i = \begin{cases} 0 & m>0, \\ 1 & m=0. \end{cases}$

Now we introduce another variable t, so $\displaystyle\sum_{m=0}^{\infty}\sum_{i=0}^{m} t^{m-i}\mu_{m-i}\cdot(-1)^i t^i \ell_i = 1$. Setting $j=m-i$ and

$$\mu(t,\mathbf{q}) := \sum_{m=0}^{\infty}\mu_m(q_1,\dots,q_m)t^m,\ \ \mu(t,q) := \mu(t,q,q,\dots) = \sum_{m=0}^{\infty}\mu_m(q)t^m,$$

we rewrite this as

$$\mu(t,\mathbf{q})g(t,\mathbf{q}) = 1,\ \ g(t,\mathbf{q}) := \sum_{i=0}^{\infty}(-1)^i t^i \ell_i(q_1,\dots,q_i). \tag{4.2}$$

This yields $\mu(t,\mathbf{q}) = \frac{1}{g(t,\mathbf{q})}$. In particular, $\mu(t,q) = \frac{1}{g(t,q)}$, where $g(t,q) := g(t,q,q,\dots)$. Thus we obtain the following result.

Theorem 4.1. *We have*

$$\mu(t,\mathbf{q}) = \left(\sum_{m=0}^{\infty}\frac{(-1)^m t^m q_1 q_2^3 q_3^7\cdots q_m^{2^m-1}}{(1-q_m)(1-q_{m-1}q_m^2)\cdots(1-q_1q_2^2\dots q_m^{2^{m-1}})}\right)^{-1}.$$

In particular,

$$\sum_{m,d=0}^{\infty} N(m,d)t^m q^d \ = \ 1 \Bigg/ \sum_{i=0}^{\infty}\frac{(-1)^i t^i q^{1+3+7+\dots+(2^i-1)}}{(1-q)(1-q^3)(1-q^7)\cdots(1-q^{2^i-1})}\ .$$

Note that $1+3+7+\dots+(2^i-1) = 2^{i+1}-i-2$.

Expanding this out, the reciprocal of the generating function for $N(m,d)$ is

$$1 - \frac{tq}{1-q} + \frac{t^2q^4}{(1-q)(1-q^3)} - \frac{t^3q^{11}}{(1-q)(1-q^3)(1-q^7)} + \cdots$$

which tabulates as follows:

	1	q	q^2	q^3	q^4	q^5	q^6	q^7	q^8	q^9	q^{10}	q^{11}	q^{12}	q^{13}	q^{14}	q^{15}	q^{16}	q^{17}	q^{18}
1	1																		
t		−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
t^2					1	1	1	2	2	2	3	3	3	4	4	4	5	5	5
t^3												−1	−1	−1	−2	−2	−2	−3	−4

Taking the reciprocal, we obtain the table of coefficients $N(m,d)$:

	1	q	q^2	q^3	q^4	q^5	q^6	q^7	q^8	q^9	q^{10}	q^{11}	q^{12}	q^{13}	q^{14}	q^{15}	q^{16}	q^{17}	q^{18}
1	1																		
t		1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
t^2			1	2	2	3	4	4	5	6	6	7	8	8	9	10	10	11	12
t^3				1	3	4	6	9	11	14	18	22	26	31	36	41	47	53	60
t^4					1	4	7	11	18	25	33	45	59	74	94	116	139	168	199
t^5						1	5	11	19	33	51	72	102	141	187	246	319	403	504
t^6							1	6	16	31	57	96	146	216	313	436	595	802	1056
t^7								1	7	22	48	94	170	278	432	654	954	1353	1888
t^8									1	8	29	71	149	287	502	822	1299	1979	2918
t^9										1	9	37	101	228	466	867	1497	2470	3922
t^{10}											1	10	46	139	338	732	1442	2623	4520
t^{11}												1	11	56	186	487	1117	2322	4442
t^{12}													1	12	67	243	684	1661	3635
t^{13}														1	13	79	311	939	2413
t^{14}															1	14	92	391	1263
t^{15}																1	15	106	484
t^{16}																	1	16	121
t^{17}																		1	17
t^{18}																			1

The coefficients of the Poincaré series for $\mathcal{E}_n(2)$ are given by adding the first n rows of this table, while the coefficients of the Poincaré series for $\mathcal{E}_\infty(2)$ are given by adding all the rows; in other words by setting $t=1$. Thus, setting $N(d) := \sum_{m\geqslant 0} N(m,d)$, we get

$$\sum_{d=0}^{\infty} N(d)q^d = \frac{1}{\phi(q)}, \quad \phi(q) := \sum_{i=0}^{\infty} \frac{(-1)^i q^{1+3+7+\cdots+(2^i-1)}}{(1-q)(1-q^3)(1-q^7)\cdots(1-q^{2^i-1})}.$$

Note that the series $\phi(q)$ defines an analytic function in the disk $|q|<1$, and that the numbers C,λ in (4.1) are determined as follows: $\lambda = \frac{1}{\alpha}$, where α is the smallest positive zero of $\phi(q)$, while $C = -\frac{1}{\alpha\phi'(\alpha)}$.

It is easy to see from this computation that the reciprocal of the generating function is much easier to compute than the generating function itself, and has much smaller coefficients. The same will be true for p odd.

Remark 4.2. Recall [**2**] that the category $\mathsf{Ver}^+_{2^{n+1}}$ is the category of modules in Ver_{2^n} over the algebra $A := \Lambda V$, where $V = X_{n-1}$ is the generating object of Ver_{2^n}. Thus the group $\mathbb{G}_m$ acts on A by scaling V. This action gives rise to an action of $\mathbb{G}_m$ on $\mathrm{Ext}^\bullet_{\mathsf{Ver}_{2^{n+1}}}(\mathbb{1},\mathbb{1})$, i.e., a $\mathbb{Z}$-grading on each cohomology group. We expect that on $\mathcal{E}_n(2)$, this grading is given by the degree with respect to the variable z_n. In other words, we expect that the 2-variable Poincaré series of $\mathcal{E}_n(2)$ taking into account this grading is $\mu_m(q,\dots,q,qv)$.

So let us compute the generating function $\mu(t,q,v) := \sum_{m=0}^\infty \mu_m(q,\dots,q,qv)t^m$. Arguing as above, we get $\mu(t,q,v) - \mu(t,q) + \mu(t,q)g(t,q,v) = 1$, where

$$g(t,q,v) := \sum_{i=0}^{\infty} \frac{(-1)^i t^i q^{2^{i+1}-i-2} v^{2^i-1}}{(1-qv)(1-q^3v^2)(1-q^7v^4)\cdots(1-q^{2^i-1}v^{2^{i-1}})}.$$

Thus, we have

$$\mu(t,q,v) = 1 + \frac{1-g(t,q,v)}{g(t,q)}.$$

4.2. Generating functions, $p>2$

The details for p odd are similar to those for $p=2$, but are quite a bit harder to keep straight. So we have chosen to write out the computation again in full.

For an integer $d \geqslant 0$, the degree d part of $\mathcal{E}_p(n)$ has a basis consisting of the monomials $x_1^{a_1} \dots x_n^{a_n} \xi_1^{\varepsilon_1} \dots \xi_n^{\varepsilon_n}$ such that the a_j are non-negative integers, each ε_j is zero or one, and

$$d = \tfrac{2p-2}{p} a_1 + \tfrac{2p^2-2}{p^2} a_2 + \cdots + \tfrac{2p^n-2}{p^n} a_n \ + \ \tfrac{p-2}{p} \varepsilon_1 + \tfrac{p^2-2}{p^2} \varepsilon_2 + \cdots + \tfrac{p^n-2}{p^n} \varepsilon_n.$$

Let $N_p(m,d)$ be the number of such monomials in degree d with $a_m + \varepsilon_m > 0$. Thus the dimension of $\mathcal{E}_n(p)_d$ is $\sum_{m=1}^n N_p(m,d)$, and the dimension of $\mathcal{E}_\infty(p)_d$ is $\sum_{m=1}^\infty N_p(m,d)$.

Set $z_1 = x_1^p$, $\zeta_1 = x_1^{p-1}\xi_1$, and $z_i = x_{i-1}^{-1} x_i^p$, $\zeta_i = x_{i-1}^{-1} x_i^{p-1} \xi_i$ for $i \geqslant 2$. Then we have $|z_i| = 2p-2$, $|\zeta_i| = 2p-3$ $(1 \leqslant i \leqslant n)$ and

$$(x_1^{a_1} x_2^{a_2} \cdots)(\xi_1^{\varepsilon_1} \xi_2^{\varepsilon_2} \cdots) = (z_1^{b_1} z_2^{b_2} \cdots)(\zeta_1^{\varepsilon_1} \zeta_2^{\varepsilon_2} \cdots),$$

where $a_i = pb_i + (p-1)\varepsilon_i - b_{i+1} - \varepsilon_{i+1}$. Then the conditions on the b_i and the ε_i are that b_i are non-negative integers, $\varepsilon_i = 0$ or 1, and $pb_i + (p-1)\varepsilon_i \geqslant b_{i+1} + \varepsilon_{i+1}$ for $i \geqslant 1$.

Set $\mu_m(q) = \sum_{d=0}^\infty N_p(m,d) q^d$, and $\mu_0(q) = 1$. Then we have

$$\mu_m(q) = \sum_{b_1+\varepsilon_1=1}^{\infty} \ \sum_{b_2+\varepsilon_2=1}^{pb_1+(p-1)\varepsilon_1} \cdots \sum_{b_m+\varepsilon_m=1}^{pb_{m-1}+(p-1)\varepsilon_{m-1}} q^{(2p-2)(b_1+\cdots+b_m)+(2p-3)(\varepsilon_1+\cdots+\varepsilon_m)}.$$

We would like to compute $\mu_m(q)$. As in the case $p=2$, we introduce auxiliary variables $q_1, q_2, \dots, w_1, w_2, \dots$ corresponding to the statistics $b_1, b_2, \dots, \varepsilon_1, \varepsilon_2, \dots$; i.e., we define the multivariate Poincaré series of $\mathcal{E}_n(p)$

$$\mu_m(q_1,\dots,q_m; w_1,\dots,w_m) := \sum_{b_1+\varepsilon_1=1}^{\infty} \ \sum_{b_2+\varepsilon_2=1}^{pb_1+(p-1)\varepsilon_1} \cdots \sum_{b_m+\varepsilon_m=1}^{pb_{m-1}+(p-1)\varepsilon_{m-1}} q_1^{b_1} \cdots q_m^{b_m} w_1^{\varepsilon_1} \cdots w_m^{\varepsilon_m}$$

so that the usual Poincaré series of this algebra is

$$\mu_m(q) = \mu_m(q^{2p-2}, \dots, q^{2p-2}; q^{2p-3}, \dots, q^{2p-3}).$$

We have

$$\sum_{b+\varepsilon=1}^{s} q^b w^\varepsilon = \frac{(w+q)(1-q^s)}{1-q}. \tag{4.3}$$

So, summing over b_m, ε_m, we get

$$\mu_m = \frac{w_m + q_m}{1 - q_m} \Bigg(\mu_{m-1} - \sum_{b_1+\varepsilon_1=1}^{\infty} \ \sum_{b_2+\varepsilon_2=1}^{pb_1+(p-1)\varepsilon_1} \cdots$$
$$\cdots \sum_{b_{m-1}+\varepsilon_{m-1}=1}^{pb_{m-2}+(p-1)\varepsilon_{m-2}} q_1^{b_1} \cdots q_{m-2}^{b_{m-2}} w_1^{\varepsilon_1} \cdots w_{m-2}^{\varepsilon_{m-2}} (q_{m-1} q_m^p)^{b_{m-1}} (w_{m-1} q_m^{p-1})^{\varepsilon_{m-1}} \Bigg).$$

Thus, summing over $b_{m-1}, \varepsilon_{m-1}$ and using (4.3) again, we have

$$\mu_m = \frac{w_m + q_m}{1-q_m}\left(\mu_{m-1} - \frac{w_{m-1}q_m^{p-1} + q_{m-1}q_m^p}{1-q_{m-1}q_m^p}\left(\mu_{m-2} - \sum_{b_1+\varepsilon_1=1}^{\infty}\sum_{b_2+\varepsilon_2=1}^{pb_1+(p-1)\varepsilon_1}\dots\right.\right.$$
$$\left.\left.\dots\sum_{b_{m-2}+\varepsilon_{m-2}=1}^{pb_{m-3}+(p-1)\varepsilon_{m-3}} q_1^{b_1}\dots q_{m-3}^{b_{m-3}}w_1^{\varepsilon_1}\dots w_{m-3}^{\varepsilon_{m-3}}(q_{m-2}q_{m-1}^p q_m^{p^2})^{b_{m-2}}(w_{m-2}q_{m-1}^{p-1}q_m^{p^2-p})^{\varepsilon_{m-2}}\right)\right).$$

Continuing inductively and using that $\mu_0 = 1$, we obtain

$$\sum_{i=0}^{m}(-1)^i\mu_{m-i}\ell_{i,p} = \begin{cases} 0 & m > 0, \\ 1 & m = 0, \end{cases}$$

where

$$\ell_{i,p}(q) = \left(\frac{w_m+q_m}{1-q_m}\right)\left(\frac{w_{m-1}q_m^{p-1}+q_{m-1}q_m^p}{1-q_{m-1}q_m^p}\right)\left(\frac{w_{m-2}q_{m-1}^{p-1}q_m^{p^2-p}+q_{m-2}q_{m-1}^p q_m^{p^2}}{1-q_{m-2}q_{m-1}^p q_m^{p^2}}\right)$$
$$\cdots\left(\frac{w_{m-i+1}q_{m-i+2}^{p-1}\cdots q_m^{p^{i-1}-p^{i-2}} + q_{m-i+1}q_{m-i+2}^p q_m^{p^{i-1}}}{1-q_{m-i+1}q_{m-i+2}^p\cdots q_m^{p^{i-1}}}\right).$$

Introducing a new variable t, we rewrite this as $\left(\sum_{j=0}^{\infty} t^j\mu_j\right)\left(\sum_{i=0}^{\infty}(-1)^i t^i \ell_{i,p}\right) = 1$, so μ_j can be determined from the generating function

$$\sum_{j=0}^{\infty} t^j \mu_j = \frac{1}{\sum_{i=0}^{\infty}(-1)^i t^i \ell_{i,p}}.$$

In particular, setting $w_i = q^{2p-3}$, $q_i = q^{2p-2}$, we get

$$\ell_{i,p}(q) = q^{(2p-2)(p^i-1)-i}\frac{(1+q)(1+q^{2p-1})\cdots(1+q^{2p^{i-1}-1})}{(1-q^{2p-2})(1-q^{2p^2-2})\cdots(1-q^{2p^i-2})}.$$

Thus we obtain the following result.

Theorem 4.3. *We have*

$$\sum_{m,d=0}^{\infty} N_p(m,d)t^m = \left(\sum_{i=0}^{\infty}(-1)^i t^i q^{(2p-2)(p^i-1)-i}\frac{(1+q)(1+q^{2p-1})\cdots(1+q^{2p^{i-1}-1})}{(1-q^{2p-2})(1-q^{2p^2-2})\cdots(1-q^{2p^i-2})}\right)^{-1}.$$

Remark 4.4. Recall [3, Subsection 4.14] that the principal block of the category $\mathsf{Ver}^+_{p^{n+1}}$ (i.e., the block of the unit object) is equivalent to the category of modules in Ver_{p^n} over the algebra $A := \Lambda V$, where $V = \mathbb{T}_1$ is the generating object of Ver_{p^n}. Thus the group $\mathbb{G}_m$ acts on A by scaling V. This action gives rise to an action of $\mathbb{G}_m$ on $\mathrm{Ext}^\bullet_{\mathsf{Ver}_{p^{n+1}}}(\mathbb{1},\mathbb{1})$, i.e., a $\mathbb{Z}$-grading on each cohomology group. We expect that on $\mathcal{E}_n(p)$, this grading is given by the degree with respect to the variables z_n and ζ_n. In other words, we expect that the 2-variable Poincaré series of $\mathcal{E}_n(p)$ taking into account this grading is $\mu_m(q^{2p-2},\dots,q^{2p-2},(qv)^{2p-2};q^{2p-3},\dots,q^{2p-3},(qv)^{2p-3})$.

So let us compute the generating function

$$\mu(t,q,v) := \sum_{m=0}^{\infty} \mu_m(q^{2p-2},\dots,q^{2p-2},(qv)^{2p-2};q^{2p-3},\dots,q^{2p-3},(qv)^{2p-3})t^m.$$

Arguing as above, we get $\mu(t,q,v) - \mu(t,q) + \mu(t,q)g(t,q,v) = 1$, where

$$g(t,q,v) :=$$
$$\sum_{i=0}^{\infty} \frac{(-1)^i t^i q^{(2p-2)(p^i-1)-i} v^{(2p-2)(p^{i-1}-1)-1}(1+qv)(1+q^{2p-1}v^{2p-2})\dots(1+q^{2p^{i-1}-1}v^{(2p-2)p^{i-2}})}{(1-q^{2p-2}v^{2p-2})(1-q^{2p^2-2}v^{(2p-2)p})\dots(1-q^{2p^i-2}v^{(2p-2)p^{i-1}})}.$$

Thus, we have

$$\mu(t,q,v) = 1 + \frac{1-g(t,q,v)}{g(t,q)},$$

where $g(t,q) := g(t,q,1)$.

Here is a table of the coefficients in the reciprocal of the generating function for $N_p(m,d)$ with $p=3$.

	1	q	q^2	q^3	q^4	q^5	q^6	q^7	q^8	q^9	q^{10}	q^{11}	q^{12}	q^{13}	q^{14}	q^{15}	q^{16}	q^{17}	q^{18}
1	1																		
t				−1	−1			−1	−1			−1	−1			−1	−1		
t^2															1	1			1
t^3																			

	q^{19}	q^{20}	q^{21}	q^{22}	q^{23}	q^{24}	q^{25}	q^{26}	q^{27}	q^{28}	q^{29}	q^{30}	q^{31}	q^{32}	q^{33}	q^{34}	q^{35}	q^{36}
1																		
t	−1	−1			−1	−1			−1	−1			−1	−1			−1	−1
t^2	2	1		1	2	1		1	2	1		2	3	1		2	4	2
t^3																	−1	−1

Reciprocating, we obtain the table of coefficients $N_3(m,d)$. These tables become sparser as the prime increases.

5. Action of the Steenrod operations

In this section, we examine possible actions of the Steenrod operations on the algebra $\mathcal{E}_\infty(p)$.

5.1. Steenrod operations for $p=2$

We begin with the easier case $p=2$.

Theorem 5.1. *There is only one possibility for the action of the Steenrod operations on $\mathcal{E}_\infty(2)$ in such a way that the Cartan formula*

$$\mathsf{Sq}^n(xy) = \sum_{i+j=n} \mathsf{Sq}^i(x)\mathsf{Sq}^j(y)$$

and the unstable conditions $\mathsf{Sq}^i(x) = x^2$ for $i = |x|$ and $\mathsf{Sq}^i(x) = 0$ for $i > |x|$ hold. Namely for $x \in \mathcal{E}_\infty(2)$, we have $\mathsf{Sq}^{|x|}(x) = x^2$, and $\mathsf{Sq}^i(x) = 0$ for $i \neq |x|$. In particular, if x has degree greater than zero then $\mathsf{Sq}^0(x) = 0$.

Proof. We begin by examining the elements $x_n^{2^n}$ of degree $2^n - 1$, and we show by induction on n that $\mathsf{Sq}^i(x_n^{2^n}) = 0$ for $i < 2^n - 1$. Let $T = \mathsf{Sq}^0 + \mathsf{Sq}^1 + \mathsf{Sq}^2 + \cdots$ be the total Steenrod operation, which by the Cartan formula is a ring homomorphism. In particular, note that Sq^i of a 2^nth power vanishes when i is not divisible by 2^n. Our goal is to show that $T(x_n^{2^n}) = (x_n^{2^n})^2$ for all $n \geqslant 1$.

We begin with $n = 1$. We have $(x_1^2)(x_2^4) = (x_1x_2^2)^2$. Applying Sq^3 to this relation, we obtain

$$\mathsf{Sq}^0(x_1^2)(x_2^4)^2 + (x_1^2)^2\mathsf{Sq}^2(x_2^4) = \mathsf{Sq}^3((x_1x_2^2)^2) = 0.$$

Therefore $\mathsf{Sq}^0(x_1^2)$ is divisible by $(x_1^2)^2$, and is hence zero, and so $T(x_1^2) = (x_1^2)^2$.

Now for the inductive step. Assume that $T(x_{n-1}^{2^{n-1}}) = (x_{n-1}^{2^{n-1}})^2$. We have the relation

$$(x_{n-1}^{2^{n-1}})^{2^{n-1}-1}(x_n^{2^n}) = (x_{n-1}^{2^{n-1}-1}x_n^2)^{2^{n-1}}$$

in $\mathcal{E}_\infty(2)$. Applying T, we get

$$(x_{n-1}^{2^{n-1}})^{2^n-2}\,T(x_n^{2^n}) = (T(x_{n-1}^{2^{n-1}-1}x_n^2))^{2^{n-1}}.$$

The right hand side is zero in degrees not divisible by 2^{n-1}. It follows that $T(x_n^{2^n})$ is zero in degrees not congruent to minus two modulo 2^{n-1}. So the only possibilities for non-zero Steenrod operations on $x_n^{2^n}$ are Sq^{2^n-1} and $\mathsf{Sq}^{2^{n-1}-1}$.

We also have the relation

$$(x_n^{2^n})(x_{n+1}^{2^{n+1}})^{2^n-1} = (x_nx_{n+1}^{2^{n+1}-2})^{2^n}$$

in $\mathcal{E}_\infty(2)$. Applying T, we get

$$T(x_n^{2^n})(T(x_{n+1}^{2^{n+1}}))^{2^n-1} = (T(x_nx_{n+1}^{2^{n+1}-2}))^{2^n}.$$

The right hand side is zero in degrees not divisible by 2^n. So in particular, examining the term in degree $2^{n+1}(2^n-1) - 2^{n-1}$, we have

$$(x_n^{2^n})^2\mathsf{Sq}^{(2^{n+1}-1)(2^n-1)-2^{n-1}}((x_{n+1}^{2^{n+1}})^{2^n-1}) + (\mathsf{Sq}^{2^{n-1}-1}(x_n^{2^n}))(x_{n+1}^{2^{n+1}})^{2^{n+1}-2} = 0.$$

So $\mathsf{Sq}^{2^{n-1}-1}(x_n^{2^n})$ is divisible by $(x_n^{2^n})^2$, and is hence zero. Hence $T(x_n^{2^n}) = (x_n^{2^n})^2$, and the inductive step is complete.

Finally, given any monomial $x = x_1^{a_1}\cdots x_n^{a_n} \in \mathcal{E}_\infty(2)$, we raise it to the 2^nth power to obtain an element of the subring generated by $x_1^2, x_2^4, x_3^8, \dots$. Then $T(x)^{2^n} = T(x^{2^n}) = (x^{2^n})^2 = (x^2)^{2^n}$, and since we are in an integral domain of characteristic two, this implies that $T(x) = x^2$. $\square$

5.2. Steenrod operations for $p > 2$

Next, we examine possible actions of the Steenrod operations on the algebra $\mathcal{E}_\infty(p)$ for p odd. Our conclusions are weaker than in the case $p = 2$, because of the existence of nilpotent elements.

Theorem 5.2. *Suppose that the Steenrod operations act on $\mathcal{E}_\infty(p)$ with p odd in such a way that the Cartan formula and unstable conditions hold. Then on the subring spanned by the monomials not involving any of the ξ_i, we have $\mathcal{P}^m(x) = x^p$ and $\mathcal{P}^i(x) = 0$ for $i \neq n$, where $|x| = 2m$.*

Proof. Let T be the total Steenrod operation $\mathcal{P}^0 + \mathcal{P}^1 + \cdots$. The argument to show that $T(x_n^{p^n}) = (x_n^{p^n})^p$ for p odd is similar to the case $p = 2$, but involves one more induction. We therefore write it out in full.

Our first task is to show that $T(x_1^p) = (x_1^p)^p$. We begin as before with

$$(x_1^p)(x_2^{p^2})^{p-1} = (x_1 x_2^{p(p-1)})^p,$$

a relation of degree $2p^2(p-1)$. Applying $\mathcal{P}^{p^2(p-1)-1}$ to this, we get

$$\mathcal{P}^{p-2}(x_1^p)\,(x_2^{p^2})^{p(p-1)} + (x_1^p)^p\,\mathcal{P}^{(p^2-1)(p-1)-1}((x_2^{p^2})^{p-1}) = 0.$$

Therefore $\mathcal{P}^{p-2}(x_1^p)$ is divisible by $(x_1^p)^p$, and hence it is zero. We work downwards in degree by induction. Suppose we have shown that $\mathcal{P}^{p-i}(x_1^p), \dots, \mathcal{P}^{p-2}(x_1^p)$ are all zero. Then applying $\mathcal{P}^{p^2(p-1)-i}$ to the above relation, we get

$$\mathcal{P}^{p-i-1}(x_1^p)\,(x_2^{p^2})^{p(p-1)} + (x_1^p)^p\,\mathcal{P}^{(p^2-1)(p-1)-i-1}((x_2^{p^2})^{p-1}) = 0.$$

Therefore $\mathcal{P}^{p-i-1}(x_1^p)$ is divisible by $(x_1^p)^p$, and hence it is zero. Once we reach $i = p-1$, we have completed the proof that $T(x_1^p) = (x_1^p)^p$.

Next, we suppose that we have already shown that $T(x_{n-1}^{p^{n-1}}) = (x_{n-1}^{p^{n-1}})^p$. We have the relation

$$(x_{n-1}^{p^{n-1}})^{p^{n-1}-1}(x_n^{p^n}) = (x_{n-1}^{p^{n-1}-1} x_n^p)^{p^{n-1}}$$

in $\mathcal{E}_\infty(p)$. Applying T, we get

$$(x_{n-1}^{p^{n-1}})^{p^n-p}\,T(x_n^{p^n}) = (T(x_{n-1}^{p^{n-1}-1} x_n^p))^{p^{n-1}}.$$

The right hand side is zero in degrees not divisible by p^{n-1}. So the only possibilities for non-zero Steenrod operations on $x_n^{p^n}$ are $\mathcal{P}^{p^n-ip^{n-1}-1}$ for $0 \leqslant i \leqslant p-1$.

We also have the relation

$$(x_n^{p^n})(x_{n+1}^{p^{n+1}})^{p^n-1} = (x_n x_{n+1}^{p^{n+1}-p})^{p^n}$$

in $\mathcal{E}_\infty(p)$. Applying T, we get

$$T(x_n^{p^n})(T(x_{n+1}^{p^{n+1}}))^{p^n-1} = (T(x_n x_{n+1}^{p^{n+1}-p}))^{p^n}.$$

The right hand side is zero in degrees not divisible by p^n. We show by induction on i that $\mathcal{P}^{p^n-ip^{n-1}-1}(x_n^{p^n}) = 0$ for $1 \leqslant i \leqslant p-1$. If we have proved this for smaller values of i, then we get

$$\mathcal{P}^{p^n-ip^{n-1}-1}(x_n^{p^n})\,(x_{n+1}^{p^{n+1}})^{p^{n+1}-p} + (x_n^{p^n})^p\,\mathcal{P}^{(p^{n+1}-1)(p^n-1)-ip^{n-1}}((x_{n+1}^{p^{n+1}})^{p^n-1}) = 0.$$

So $\mathcal{P}^{p^n-ip^{n-1}-1}(x_n^{p^n})$ is divisible by $(x_n^{p^n})^p$, and is hence zero. This completes the proof that $T(x_n^{p^n}) = (x_n^{p^n})^p$. □

6. The Koszul complex

We assume that $p^n > 3$. We will consider the symmetric tensor categories Ver_{p^n} over $\mathbf{k}$ defined in [**3**]. Namely, let $\mathcal{T}_p := \mathrm{Tilt}SL_2(\mathbf{k})$ be the category of tilting modules over $SL_2(\mathbf{k})$. Let $T_i \in \mathcal{T}_p$ be the tilting module for $SL_2(\mathbf{k})$ with highest weight i. The module T_{p^n-1} generates a tensor ideal $\mathcal{I}_n \subset \mathcal{T}_p$ spanned by T_i for $i \geqslant p^n - 1$. We define $\mathcal{T}_{n,p}$ to be the quotient category $\mathcal{T}_p/\mathcal{I}_n$. Then Ver_{p^n} is the abelian envelope of $\mathcal{T}_{n,p}$, i.e., the unique abelian symmetric tensor category containing $\mathcal{T}_{n,p}$ such that faithful symmetric monoidal functors out of $\mathcal{T}_{n,p}$ into abelian symmetric tensor categories uniquely factor through Ver_{p^n}.

More concretely, Ver_{p^n} is the category $R-$mod of finite dimensional modules over the algebra $R := \mathrm{End}(\oplus_{i=p^{n-1}-1}^{p^n-2} T_i)$,[1] realized as the homotopy category of projective resolutions $P^\bullet$ in $R-$mod with the usual tensor product. Namely, it turns out that the tensor product of resolutions is a resolution (i.e., acyclic in negative degrees), there is a unit object, and the corresponding tensor category is rigid (with $T_i^* \cong T_i$) and equipped with a natural faithful symmetric monoidal functor $\mathcal{T}_{n,p} \to \mathsf{Ver}_{p^n}$ given by $P^\bullet \mapsto H^0(P^\bullet)$.

Let $\mathbb{T}_i$ be the image of T_i in Ver_{p^n}. In particular, we let $V = \mathbb{T}_1$ be the image of the 2-dimensional irreducible representation T_1 of $SL_2(\mathbf{k})$, also denoted by V (these of course depend on n but to lighten the notation we do not indicate this explicitly). Note that in both categories $\Lambda^2 V$ is the unit object and $\Lambda^i V = 0$ for $i \geqslant 3$. Recall [**11, 13**] that we have the Koszul complex $K^\bullet := S^\bullet V \otimes \Lambda V$ in Ver_{p^n} (i.e., we use the symmetric power superscript as the cohomological degree). This complex may also be graded by total degree, which is preserved by the differential. So it splits into a direct sum of complexes $K_m^\bullet$, $m \geqslant 0$:

$$0 \to S^{m-2}V \to S^{m-1}V \otimes V \to S^m V \to 0$$

(where we agree that $S^j V = 0$ if $j < 0$). The map $S^{m-1}V \otimes V \to S^m V$ in this complex is induced by the multiplication map of the algebra SV, so it is surjective when $m \neq 0$.

Proposition 6.1. *If $1 \leqslant m \leqslant p^n - 2$ then the complex $K_m^\bullet$ is exact.*

Proof. It suffices to show that for any $i \in [p^{n-1} - 1, p^n - 2]$ the sequence

$$0 \to \mathrm{Hom}_{\mathsf{Ver}_{p^n}}(S^m V, \mathbb{T}_i) \to \mathrm{Hom}_{\mathsf{Ver}_{p^n}}(S^{m-1}V \otimes V, \mathbb{T}_i) \to \mathrm{Hom}_{\mathsf{Ver}_{p^n}}(S^{m-2}V, \mathbb{T}_i) \to 0 \tag{6.1}$$

is exact. This sequence can be rewritten as

$$\begin{aligned} 0 \to \mathrm{Hom}_{\mathsf{Ver}_{p^n}}(V^{\otimes m}, \mathbb{T}_i)^{S_m} &\to \mathrm{Hom}_{\mathsf{Ver}_{p^n}}(V^{\otimes m}, \mathbb{T}_i)^{S_{m-1}} \\ &\to \mathrm{Hom}_{\mathsf{Ver}_{p^n}}(V^{\otimes m-2}, \mathbb{T}_i)^{S_{m-2}} \to 0. \end{aligned} \tag{6.2}$$

By Theorem 4.2 of [**3**], sequence (6.2) can be rewritten as

$$\begin{aligned} 0 \to \mathrm{Hom}_{\mathcal{T}_{n,p}}(V^{\otimes m}, T_i)^{S_m} &\to \mathrm{Hom}_{\mathcal{T}_{n,p}}(V^{\otimes m}, T_i)^{S_{m-1}} \\ &\to \mathrm{Hom}_{\mathcal{T}_{n,p}}(V^{\otimes m-2}, T_i)^{S_{m-2}} \to 0. \end{aligned} \tag{6.3}$$

Now, if $1 \leqslant m \leqslant p^n - 2$, then by Proposition 3.5 of [**3**], sequence (6.3) can be rewritten as follows:

$$\begin{aligned} 0 \to \mathrm{Hom}_{\mathcal{T}_p}(V^{\otimes m}, T_i)^{S_m} &\to \mathrm{Hom}_{\mathcal{T}_p}(V^{\otimes m}, T_i)^{S_{m-1}} \\ &\to \mathrm{Hom}_{\mathcal{T}_p}(V^{\otimes m-2}, T_i)^{S_{m-2}} \to 0, \end{aligned} \tag{6.4}$$

where V now denotes the 2-dimensional irreducible representation of $SL_2(\mathbf{k})$. The Hom spaces in this sequence are just Homs between representations of $SL_2(\mathbf{k})$. Thus

[1] It does not matter whether to take endomorphisms in $\mathcal{T}_p$ or $\mathcal{T}_{n,p}$ – the corresponding natural map of endomorphism rings is an isomorphism.

sequence (6.4) can be written as

$$\begin{aligned}0 &\to \mathrm{Hom}_{SL_2(\mathbf{k})}(S^mV, T_i) \to \mathrm{Hom}_{SL_2(\mathbf{k})}(S^{m-1}V \otimes V, T_i)\\ &\to \mathrm{Hom}_{SL_2(\mathbf{k})}(S^{m-2}V, T_i) \to 0.\end{aligned} \tag{6.5}$$

We will now use the following lemma.

Lemma 6.2. *For $m \leqslant p^n - 1$ one has $\mathrm{Ext}^1_{SL_2(\mathbf{k})}(S^mV, T_i) = 0$.*

Proof. Since $i \geqslant p^{n-1} - 1$, it suffices to show that for any j,

$$\mathrm{Ext}^1_{SL_2(\mathbf{k})}(S^mV, \mathrm{St}_{n-1} \otimes T_j) = 0,$$

where $\mathrm{St}_{n-1} := T_{p^{n-1}-1}$ is the $(n-1)$st Steinberg module (note that it is self-dual). We have

$$\mathrm{Ext}^1_{SL_2(\mathbf{k})}(S^mV, \mathrm{St}_{n-1} \otimes T_j) = \mathrm{Ext}^1_{SL_2(\mathbf{k})}(S^mV \otimes \mathrm{St}_{n-1}, T_j).$$

By [**3**, Lemma 3.3], $S^mV \otimes \mathrm{St}_{n-1}$ has a filtration whose successive quotients are tilting modules. Thus, since $\mathrm{Ext}^1(T_l, T_j) = 0$, $S^mV \otimes \mathrm{St}_{n-1}$ is a direct sum of T_i, i.e., a tilting module. This implies the statement, using again that $\mathrm{Ext}^1(T_l, T_j) = 0$. □

Now the exactness of (6.5) follows from the fact that the sequence of $SL_2(\mathbf{k})$-representations

$$0 \to S^{m-2}V \to S^{m-1}V \otimes V \to S^mV \to 0$$

is exact (being a homogeneous part of the ordinary Koszul complex) and Lemma 6.2. This completes the proof of Proposition 6.1. □

Let $\mathrm{q} = e^{\pi i/p^n}$.

Corollary 6.3.

(i) *For $m \leqslant p^n - 2$ we have*

$$\mathrm{FPdim}(S^mV) = [m+1]_{\mathrm{q}} := \frac{\mathrm{q}^{m+1} - \mathrm{q}^{-m-1}}{\mathrm{q} - \mathrm{q}^{-1}} \in \mathbb{R}$$

and $\dim(S^mV) = m + 1 \in \mathbf{k}$.

(ii) *The Jordan–Hölder multiplicities of the objects S^mV are the decomposition numbers of tilting modules into Weyl modules computed in [**23**] (see [**3**, Proposition 4.17]).*

Proof. (i) It follows from Proposition 6.1 that

$$\begin{aligned}\mathrm{FPdim}(S^mV) &= (\mathrm{q} + \mathrm{q}^{-1})\mathrm{FPdim}(S^{m-1}V) - \mathrm{FPdim}(S^{m-2}V),\\ \dim(S^mV) &= 2\dim(S^{m-1}V) - \dim(S^{m-2}V).\end{aligned}$$

Thus the statement follows by induction, using that $S^0V = \mathbb{1},\ S^1V = V$.

(ii) This follows from (i), using [**3**, Theorem 4.5(iv) and Propositions 4.12, 4.16]. □

Recall [**3**] that Ver_{p^n} has exactly two invertible objects up to isomorphism for $p > 2$ and exactly one (the unit) for $p = 2$. For $p > 2$ let ψ be the unique non-trivial invertible object of Ver_{p^n} (generating the category of supervector spaces). If $p = 2$, we agree that $\psi = \mathbb{1}$.

Corollary 6.4.

(i) $S^{p^n-2}V = \psi$.

(ii) $S^{p^n-3}V = V \otimes \psi$.

(iii) $S^j V = 0$ *for all* $j > p^n - 2$.

Proof. (i) By Corollary 6.3, we have $\mathrm{FPdim}(S^{p^n-2}V) = 1$, which implies that $S^{p^n-2}V$ is invertible (see [**12**, Ex. 4.5.9]). For $p = 2$ this implies that $S^{2^n-2}V = \mathbb{1}$, and for $p > 2$ that $S^{p^n-2}V = \psi$ (as $S^{p^n-2}V \in \mathsf{Ver}_{p^n}^-$ since $p^n - 2$ is odd).

(ii) Similarly, by Corollary 6.3, $\mathrm{FPdim}(S^{p^n-3}V) = \mathrm{q} + \mathrm{q}^{-1} < 2$, so $S^{p^n-3}V$ is simple. But by the results of [**3**], the only object $X \in \mathsf{Ver}_{p^n}$ of Frobenius–Perron dimension $\mathrm{q} + \mathrm{q}^{-1}$ such that ψ is a quotient of $X \otimes V$ is $X \cong V \otimes \psi$. Thus $S^{p^n-3}V \cong V \otimes \psi$.

(iii) The map $S^{p^n-3}V \to S^{p^n-2}V \otimes V$ corresponds by adjunction to the surjective map $S^{p^n-3}V \otimes V \to S^{p^n-2}V$, which is nonzero by (i). Hence by (ii) it is an isomorphism. Thus the morphism $S^{p^n-2}V \otimes V \to S^{p^n-1}V$ must be 0 (as $K^\bullet_{p^n-1}$ is a complex). But this map is surjective, so $S^{p^n-1}V = 0$. This implies the statement. □

Remark 6.5. In particular, this implies that

$$\sum_{m=0}^{\infty} \dim(S^m V) z^m = (1-z)^{p^n-2} \in \mathbf{k}[[z]].$$

Also we clearly have

$$\sum_{m=0}^{\infty} \dim(\Lambda^m V) z^m = 1 + 2z + z^2 = (1+z)^2 \in \mathbf{k}[[z]].$$

Thus the p-adic dimensions of V defined in [**13**] are as follows:

$$\mathrm{Dim}_-(V) = 2 \in \mathbb{Z}_p, \ \mathrm{Dim}_+(V) = 2 - p^n \in \mathbb{Z}_p.$$

Similarly, we get

$$\sum_{m=0}^{\infty} \mathrm{FPdim}(S^m V) z^m = \frac{1 + z^{p^n}}{(1-\mathrm{q}z)(1-\mathrm{q}^{-1}z)}. \tag{6.6}$$

We also obtain

Corollary 6.6.

(i) *The Koszul complex* $K^\bullet$ *is exact in all degrees except* 0 *and* $p^n - 2$. *Moreover* $H^0(K^\bullet) = \mathbb{1}$ *sitting in total degree* 0 *and* $H^{p^n-2}(K^\bullet) = \psi$ *sitting in total degree* p^n.

(ii) *The algebra* SV *is* $(p^n-2, 2)$*-Koszul and the algebra* ΛV *is* $(2, p^n-2)$*-Koszul in the sense of Brenner, Butler and King [**6**] (see [**11**, Definition 5.3]).*

Corollary 6.7. *The algebra* SV *in* Ver_{p^n} *is Frobenius.*

Proof. Assume the contrary, and let k be the largest integer such that the left kernel of the pairing $S^k V \otimes S^{p^n-2-k}V \to S^{p^n-2}V = \psi$ is nonzero. Denote this kernel by N. Then the composite map $N \otimes V \to S^k V \otimes V \to S^{k+1}V$ is zero. Thus the composite map $N \to S^k V \to S^{k+1}V \otimes V$ is zero. But by Proposition 6.1, the map $S^k V \to S^{k+1} V \otimes V$ is injective. Thus $N = 0$, a contradiction. □

Remark 6.8. Recall [**3**, Subsection 4.4] that the category $\mathsf{Ver}_{p^n} = \mathsf{Ver}_{p^n}(\mathbf{k})$ lifts to a semisimple braided (non-symmetric) category $\mathsf{Ver}_{p^n}(\mathbf{K})$ over a field $\mathbf{K}$ of characteristic zero, corresponding to the quantum group $SL_2^{-\mathrm{q}}$ where q is a primitive root of unity of order $2p^n$ in $\mathbf{K}$. In $\mathsf{Ver}_{p^n}(\mathbf{K})$ we have the quantum symmetric algebra $S_{-\mathrm{q}}V$, which is a lift of SV over $\mathbf{K}$ and is also Frobenius $(p^n-2,2)$-almost Koszul (see [**11**, Subsection 5.5]). In particular, we have the quantum Koszul complex $S^\bullet_{-\mathrm{q}}V \otimes \Lambda_{-\mathrm{q}}V$ in $\mathsf{Ver}_{p^n}(\mathbf{K})$ which is a flat deformation of the Koszul complex $S^\bullet V \otimes \Lambda V$ and has the cohomology as described in Corollary 6.6.

Corollary 6.6 allows us to construct an injective resolution $Q_\bullet$:

$$Q_0 \to Q_1 \to Q_2 \to \cdots$$

of the augmentation ΛV-module $\mathbb{1}$ by free ΛV-modules, which is periodic with period 2^n-1 for $p=2$ and antiperiodic with period p^n-1 for $p>2$ (where antiperiodic means that it multiplies by ψ when shifted by this period; in particular, this is $2(p^n-1)$-periodic). Namely, for $0 \leqslant i \leqslant p^n-2$ we have $Q_{2r(p^n-1)+m} = S^mV \otimes \Lambda V$, and $Q_{(2r+1)(p^n-1)+m} = S^mV \otimes \psi \otimes \Lambda V$.

Remark 6.9. If $p^n=2$ (i.e., $p=2$, $n=1$) then $V=0$, so the Koszul complex reduces to $\mathbb{1}$ sitting in degree 0 and hence does not fit the above general pattern; but we will not consider this trivial case. If $p^n=3$ (i.e., $p=3, n=1$) then $V=\psi$, so $\Lambda^3\psi \neq 0$ and hence the Koszul complex $S^\bullet V \otimes \Lambda V$ still does not fit the general pattern (in fact, in this case $\mathsf{Ver}_{p^n} = \mathrm{Supervec}$, so the Koszul complex is exact except in degree 0). However, now this can be remedied by a slight modification of the definition. Namely, let $\Lambda_{\mathrm{tr}}V$ be the quotient of ΛV by $\Lambda^3 V$ (forcing the desired equality $\Lambda^3V=0$). Then we have the truncated Koszul complex $K^\bullet_{\mathrm{tr}} := S^\bullet V \otimes \Lambda_{\mathrm{tr}}V$ which is easily shown to have the same properties as the usual Koszul complex $K^\bullet$ for $p^n>3$. Thus if $p^n=3$ then, abusing terminology and notation, by ΛV we will mean $\Lambda_{\mathrm{tr}}V$, and by the Koszul complex the truncated Koszul complex; then the above results will also apply to this case.

As an application let us compute the multiplicities of the unit object in the symmetric powers of V for $p=2$.

Proposition 6.10. *If $p=2$ then $[S^mV:\mathbb{1}]=0$ if m is odd and $[S^mV:\mathbb{1}]=1$ if m is even. Thus $[SV:\mathbb{1}]=2^{n-1}$.*

Proof. Notice that for $X \in \mathsf{Ver}_{2^n}$ the multiplicity $[X:\mathbb{1}]$ of $\mathbb{1}$ in X is equal to $\mathrm{Tr}(\mathrm{FPdim}(X))/2^{n-1}$ (the trace of the algebraic number in the field $\mathbb{Q}(\mathrm{q}+\mathrm{q}^{-1})$ where $\mathrm{q} := e^{\pi i/2^n}$) ; this follows since by [**2**], $\mathrm{TrFPdim}(X)=0$ for any nontrivial simple $X \in \mathsf{Ver}_{2^{n+1}}$. So we have

$$\sum_m [S^mV:\mathbb{1}]z^m = \frac{1}{2^{n-1}}\mathrm{Tr}\left(\frac{1+z^{2^n}}{(1-\mathrm{q}z)(1-\mathrm{q}^{-1}z)}\right).$$

Thus the result follows from the following lemma.

Lemma 6.11. $\dfrac{1}{2^{n-1}}\mathrm{Tr}\left(\dfrac{1+z^{2^n}}{(1-\mathrm{q}z)(1-\mathrm{q}^{-1}z)}\right) = \dfrac{1-z^{2^n}}{1-z^2} = \displaystyle\sum_{j=0}^{2^{n-1}-1} z^{2j}.$

Proof. We have

$$\frac{1}{2^{n-1}}\mathrm{Tr}\left(\frac{1+z^{2^n}}{(1-\mathrm{q}z)(1-\mathrm{q}^{-1}z)}\right)=\frac{1}{2^{n-1}}\sum_{k=1}^{2^{n-1}}\frac{1+z^{2^n}}{(1-\mathrm{q}^{2k-1}z)(1-\mathrm{q}^{-2k+1}z)}.$$

This is the unique polynomial $h(z)\in\mathbb{Q}[z]$ of degree 2^n-2 such that $h(\mathrm{q}^j)=\frac{2}{1-\mathrm{q}^{2j}}$ for any odd number j. But the polynomial $\frac{1-z^{2^n}}{1-z^2}$ satisfies these conditions, hence the result. □

This completes the proof of Proposition 6.10. □

Remark 6.12. Another proof of Proposition 6.10 is obtained by applying Proposition 4.16 and Theorem 4.42 of [**3**]. Namely, $[S^iV:\mathbb{1}]$ is an entry of the decomposition matrix of Ver_{2^n}, so it is 0 if i is odd and 1 if i is even. This follows since the descendants of the number 2^n-1 are exactly all the odd numbers between 1 and 2^n-1.

7. Ext computations

7.1. Ext computations for $p=2$

Consider now the case $p=2$. In this case, we can use the resolution $Q_\bullet$ to give the following recursive procedure of computation of the additive structure of the cohomology $\mathrm{Ext}^\bullet_{\mathsf{Ver}_{2^{n+1}}}(\mathbb{1},X)$ (for indecomposable X).

We will denote the generating object of $\mathsf{Ver}_{2^{k+1}}$ by X_k and recall that $\mathsf{Ver}^+_{2^{k+2}}$ is the category of ΛX_k-modules in $\mathsf{Ver}_{2^{k+1}}$. Also the resolution $Q_\bullet$ in $\mathsf{Ver}_{2^{k+1}}$ will be denoted by $S^\bullet X_k[y_{k+1}]\otimes\Lambda X_k$, where y_{k+1} is a variable of degree $2^{k+1}-1$ for $k\geqslant 0$. This is justified by this resolution being periodic with period $2^{k+1}-1$. Also if $Y^\bullet,Z^\bullet$ are complexes in an abelian category $\mathcal{A}$ then by $\mathrm{Ext}^m(Z^\bullet,Y^\bullet)$ we will mean $\mathrm{Hom}(Z^\bullet,Y^\bullet[m])=\mathrm{Hom}(Z^\bullet,Y^{\bullet+m})$ with Hom taken in the derived category $D(\mathcal{A})$.

Recall that $\mathsf{Ver}_{2^{n+1}}=\mathsf{Ver}^+_{2^{n+1}}\oplus\mathsf{Ver}^-_{2^{n+1}}$. If $X\in\mathsf{Ver}^-_{2^{n+1}}$, then $\mathrm{Ext}^\bullet_{\mathsf{Ver}_{2^{n+1}}}(\mathbb{1},X)$ is zero. Thus, it suffices to compute $\mathrm{Ext}^\bullet_{\mathsf{Ver}^+_{2^{n+1}}}(\mathbb{1},X)$ for $X\in\mathsf{Ver}^+_{2^{n+1}}$. In that case, we have

$$\begin{aligned}\mathrm{Ext}^\bullet_{\mathsf{Ver}^+_{2^{n+1}}}(\mathbb{1},X)&\cong\mathrm{Ext}^\bullet_{\Lambda X_{n-1}}(\mathbb{1},X)\cong\mathrm{Ext}^\bullet_{\Lambda X_{n-1}}(\mathbb{1},Q_\bullet\otimes X)\\&\cong\mathrm{Ext}^\bullet_{\Lambda X_{n-1}}(\mathbb{1},S^\bullet X_{n-1}[y_n]\otimes\Lambda X_{n-1}\otimes X)\\&\cong\mathrm{Ext}^\bullet_{\mathsf{Ver}_{2^n}}(\mathbb{1},S^\bullet X_{n-1}[y_n]\otimes X)\\&\cong\mathrm{Ext}^\bullet_{\mathsf{Ver}^+_{2^n}}(\mathbb{1},(S^\bullet X_{n-1}[y_n]\otimes X)^+),\end{aligned}\tag{7.1}$$

where the superscript $+$ means that we are taking the part lying in $\mathsf{Ver}^+_{2^n}$, and in the last two expressions X is regarded as an object of Ver_{2^n} using the corresponding forgetful functor $\mathsf{Ver}^+_{2^{n+1}}\to\mathsf{Ver}_{2^n}$. Here for the penultimate isomorphism we invoked the Shapiro lemma, using that the ΛX_{n-1}-module $S^kX_{n-1}[y_n]\otimes\Lambda X_{n-1}\otimes X$ is free and therefore coinduced (as ΛX_{n-1} is a Frobenius algebra).

Thus we get a recursion expressing of $\mathrm{Ext}^\bullet_{\mathsf{Ver}^+_{2^{n+1}}}(\mathbb{1},X)$ in terms of $\mathrm{Ext}^\bullet_{\mathsf{Ver}^+_{2^n}}(\mathbb{1},X')$. While this is a good news, unfortunately X' is not an object any more but rather a complex of objects finite in the negative direction. Luckily, the same calculation applies if X is such a complex, i.e., an object of the derived category $D^+(\mathsf{Ver}_{2^{n+1}}^+)$ of $\mathsf{Ver}_{2^{n+1}}^+$, which allows us to iterate this construction. Namely, for an object $X \in D^+(\mathsf{Ver}_{2^{n+1}}^+)$, let

$$E_n(X) := \underline{\mathrm{Hom}}_{\Lambda X_{n-1}}(\mathbb{1}, S^\bullet X_{n-1}[y_n] \otimes \Lambda X_{n-1} \otimes X)^+ = (S^\bullet X_{n-1}[y_n] \otimes X)^+$$

(the internal Hom taken in the category Ver_{2^n}). This gives an additive functor

$$E_n \colon D^+(\mathsf{Ver}_{2^{n+1}}^+) \to D^+(\mathsf{Ver}_{2^n}^+).$$

Lemma 7.1. *If $X \in \mathsf{Ver}_{2^n}^+$ (i.e., a trivial ΛX_{n-1}-module) then the differential in the complex $E_n(X)$ is zero.*

Proof. It is easy to see that for a finite dimensional vector space V over $\mathbf{k}$, the differential on $\mathrm{Hom}_{\Lambda V}(\mathbf{k}, S^\bullet V \otimes \Lambda V) = S^\bullet V$ induced by the Koszul differential on $S^\bullet V \otimes \Lambda V$ is zero. The lemma is a straightforward generalization of this fact. □

Corollary 7.2. *Suppose $X \in \mathsf{Ver}_{2^n}$. Then we have an isomorphism*

$$\begin{aligned}\mathrm{Ext}^\bullet_{\mathsf{Ver}_{2^{n+1}}^+}(\mathbb{1}, X) &\cong \bigoplus_{i\geqslant 0} \mathrm{Ext}^{\bullet - i}_{\mathsf{Ver}_{2^n}}(\mathbb{1}, S^i X_{n-1}[y_n] \otimes X) \\ &= \bigoplus_{i\geqslant 0} \mathrm{Ext}^{\bullet - i}_{\mathsf{Ver}_{2^n}^+}(\mathbb{1}, (S^i X_{n-1}[y_n] \otimes X)^+).\end{aligned}$$

This isomorphism maps the grading induced by the grading on ΛX_{n-1} to the grading defined by $\deg(X_{n-1}) = 1$, $\deg(y_n) = 2^n - 1$ *(i.e., it coincides with the cohomological grading).*

Proof. Follows immediately from Lemma 7.1. □

Remark 7.3. Corollary 7.2 does not quite give a recursion to compute the Ext groups, since the object $(S^i X_{n-1} \otimes X)^+$ may not belong to $\mathsf{Ver}_{2^{n-1}}$ (i.e., it may carry a nontrivial action of ΛX_{n-2}). However, it has some useful consequences given below.

Now recall that $\mathsf{Ver}_2 = \mathrm{Vec}$ and $\mathsf{Ver}_{2^2}^+$ is the category of $\mathbf{k}[\xi]$-modules where $\xi^2 = 0$. Define a functor $E_1 \colon D^+(\mathsf{Ver}_{2^2}^+) \to D^+(\mathrm{Vec})$ by

$$E_1(X) := \mathrm{Hom}_{\mathbf{k}[\xi]}(\mathbf{k}, \mathbf{k}[y_1, \xi] \otimes X) = \mathbf{k}[y_1] \otimes X,$$

with the differential $d(y_1^m \otimes x) = y_1^{m+1} \otimes \xi x + y_1^m \otimes dx$. We thus obtain the following proposition.

Proposition 7.4. *We have a natural isomorphism*

$$\mathrm{Ext}^\bullet_{D^+(\mathsf{Ver}_{2^{n+1}}^+)}(\mathbb{1}, X) \cong \mathrm{Ext}^\bullet_{D^+(\mathsf{Ver}_{2^n}^+)}(\mathbb{1}, E_n(X))$$

for $n \geqslant 2$, and

$$\mathrm{Ext}^\bullet_{D^+(\mathsf{Ver}_{2^2}^+)}(\mathbb{1}, X) \cong \mathrm{Ext}^\bullet_{D^+(\mathrm{Vec})}(\mathbb{1}, E_1(X)).$$

This implies the following corollary. Let

$$E = E_1 \circ \cdots \circ E_n \colon D^+(\mathsf{Ver}_{2^{n+1}}^+) \to D^+(\mathrm{Vec}).$$

Corollary 7.5. *We have a linear natural isomorphism*

$$\mathrm{Ext}^\bullet_{D^+(\mathsf{Ver}^+_{2^{n+1}})}(\mathbb{1}, X) = H^\bullet(E(X)).$$

The complex of vector spaces $E(X)$ has the following structure:

$$E(X) = (S^\bullet X_1 \otimes \cdots \otimes (S^\bullet X_{n-2} \otimes (S^\bullet X_{n-1} \otimes X)^+)^+ \dots)^+[y_1, y_2, \dots, y_n], \quad (7.2)$$

and it is easy to see that the differential is linear over $\mathbf{k}[y_1, \dots, y_n]$, since multiplication by y_i is induced by the shift in the corresponding periodic resolution. Thus we get

Proposition 7.6. *For any $X \in \mathsf{Ver}_{2^{n+1}}$, $\mathrm{Ext}^\bullet_{\mathsf{Ver}_{2^{n+1}}}(\mathbb{1}, X)$ is a graded finitely generated module over $\mathbf{k}[y_1, \dots, y_n]$.*

In particular, we get that

$$\mathrm{Ext}^\bullet_{\mathsf{Ver}_{2^{n+1}}}(\mathbb{1}, \mathbb{1}) = \mathrm{Ext}^\bullet_{\mathsf{Ver}^+_{2^{n+1}}}(\mathbb{1}, \mathbb{1}) = H^\bullet(E(\mathbb{1})),$$

where

$$E(\mathbb{1}) = (S^\bullet X_1 \otimes \cdots \otimes (S^\bullet X_{n-2} \otimes (S^\bullet X_{n-1})^+)^+ \dots)^+[y_1, y_2, \dots, y_n].$$

Note that we have $1 \in E(\mathbb{1})$ and $d(1) = 0$, so we obtain a natural linear map

$$\phi\colon \mathbf{k}[y_1, \dots, y_n] \to \mathrm{Ext}^\bullet_{\mathsf{Ver}_{2^{n+1}}}(\mathbb{1}, \mathbb{1}).$$

Proposition 7.7. *For $1 \leqslant i \leqslant n$ multiplication by y_i on $\mathrm{Ext}^\bullet_{\mathsf{Ver}_{2^{n+1}}}(\mathbb{1}, X)$ coincides with the cup product with $\phi(y_i)$. In particular, ϕ is an algebra homomorphism.*

Proof. The proof is by induction in n. For $i < n$ the statement follows from the inductive assumption. For $i = n$, we see that the cup product with $\phi(y_n)$ can be realised as the Yoneda product (= concatenation) with the Koszul complex $K^\bullet$, which represents the class $\phi(y_n)$ in the Yoneda realization of Ext. This proves the first statement. The second statement then follows since

$$\phi(ab) = (ab) \cdot 1 = a \cdot (b \cdot 1) = a \cdot \phi(b) = \phi(a)\phi(b). \qquad \square$$

Proposition 7.8. *For $X \in \mathsf{Ver}^+_{2^n}$ the natural map*

$$\mathrm{Ext}^\bullet_{\mathsf{Ver}_{2^n}}(\mathbb{1}, X)[y_n] \to \mathrm{Ext}^\bullet_{\mathsf{Ver}_{2^{n+1}}}(\mathbb{1}, X)$$

is an injective morphism of $\mathbf{k}[y_1, \dots, y_n]$-modules which is also a morphism of algebras for $X = \mathbb{1}$.

Proof. This follows from the isomorphism

$$\mathrm{Ext}^\bullet_{\mathsf{Ver}_{2^{n+1}}}(\mathbb{1}, X) \cong \mathrm{Ext}^\bullet_{\mathsf{Ver}_{2^n}}(\mathbb{1}, (S^{\mathrm{even}} X_{n-1} \otimes X)[y_n])$$

since $\mathbb{1}$ is a direct summand of $S^{\mathrm{even}} X_{n-1}$. $\square$

Proposition 7.9. *Let $U \in \mathsf{Ver}_{2^n}$ and $X := U \otimes \Lambda X_{n-1} \in \mathsf{Ver}^+_{2^{n+1}}$ be a free ΛX_{n-1}-module. Then y_n acts on $\mathrm{Ext}^\bullet_{\mathsf{Ver}_{2^{n+1}}}(\mathbb{1}, X)$ by zero.*

Proof. By the Shapiro lemma we have

$$\mathrm{Ext}^\bullet_{\mathsf{Ver}_{2^{n+1}}}(\mathbb{1}, X) \cong \mathrm{Ext}^\bullet_{\Lambda X_{n-1}}(\mathbb{1}, X) \cong \mathrm{Ext}^\bullet_{\mathsf{Ver}_{2^n}}(\mathbb{1}, U).$$

Therefore, the group $\mathbb{G}_m$ scaling X_{n-1} acts trivially on $\mathrm{Ext}^\bullet_{\mathsf{Ver}_{2^{n+1}}}(\mathbb{1}, X)$. So the statement follows, as y_n has degree $2^n - 1$ with respect to this action. $\square$

Let $S \subset \{1,\dots,n-1\}$ and $X_S := \bigotimes_{i\in S} X_i$ be the simple object of Ver_{2^n} attached to S in [**2**].

Proposition 7.10.

(i) *If $i \in S$ and $Y \in \mathsf{Ver}^+_{2^{n+1}}$ then multiplication by y_i acts by zero on the space $\mathrm{Ext}^\bullet_{\mathsf{Ver}^+_{2^{n+1}}}(Y, X_S)$. Hence $\mathrm{Ext}^\bullet_{\mathsf{Ver}^+_{2^{n+1}}}(Y, X_S)$ is a torsion module over the ring $\mathbf{k}[y_1,\dots,y_n]$ unless $S = \varnothing$ (i.e., $X_S = \mathbb{1}$).*

(ii) *The annihilator of $\mathrm{Ext}^\bullet_{\mathsf{Ver}^+_{2^{n+1}}}(X_S, X_S)$ in $\mathbf{k}[y_1,\dots,y_n]$ is generated by y_i with $i \in S$.*

Proof. (i) The proof is by induction in n. The base is clear, so we just have to justify the induction step. We have

$$\mathrm{Ext}^\bullet_{\mathsf{Ver}^+_{2^{n+1}}}(Y, X_S) \cong \mathrm{Ext}^\bullet_{\mathsf{Ver}^+_{2^n}}(\mathbb{1}, (X_S \otimes S^\bullet X_{n-1} \otimes Y^*)^+[y_n]).$$

If $n-1 \notin S$ then $X_S \in \mathsf{Ver}^+_{2^n}$ so this can be written as

$$\mathrm{Ext}^\bullet_{\mathsf{Ver}^+_{2^{n+1}}}(Y, X_S) \cong \mathrm{Ext}^\bullet_{\mathsf{Ver}^+_{2^n}}((Y \otimes S^\bullet X_{n-1}[y_n]^*)^+, X_S)$$

and the statement follows from the inductive assumption. On the other hand, if $n-1 \in S$ then setting $S' = S \setminus \{n-1\}$, we have $X_S = X_{S'} \otimes X_{n-1}$. So we get

$$\mathrm{Ext}^\bullet_{\mathsf{Ver}^+_{2^{n+1}}}(Y, X_S) \cong \mathrm{Ext}^\bullet_{\mathsf{Ver}^+_{2^n}}(\mathbb{1}, X_{S'} \otimes X_{n-1} \otimes (S^\bullet X_{n-1} \otimes Y^*)^-[y_n]),$$

where the superscript minus sign means that we are taking the part lying in $\mathsf{Ver}^-_{2^n}$. But $(S^\bullet X_{n-1} \otimes Y^*)^- = X_{n-1} \otimes W^\bullet$ for some $W^\bullet \in \mathsf{Ver}^+_{2^n}$, and $X_{n-1} \otimes X_{n-1} = \Lambda X_{n-2}$ (and some differential on the tensor product whose exact form is not important for this argument). Thus we get

$$\begin{aligned}\mathrm{Ext}^\bullet_{\mathsf{Ver}^+_{2^{n+1}}}(Y, X_S) &\cong \mathrm{Ext}^\bullet_{\mathsf{Ver}^+_{2^n}}(\mathbb{1}, X_{S'} \otimes X_{n-1} \otimes X_{n-1} \otimes W^\bullet[y_n]) \\ &\cong \mathrm{Ext}^\bullet_{\Lambda X_{n-2}}(\mathbb{1}, X_{S'} \otimes \Lambda X_{n-2} \otimes W^\bullet[y_n]) \\ &\cong \mathrm{Ext}^\bullet_{\mathsf{Ver}_{2^{n-1}}}(W^\bullet[y_n]^*, X_{S'}).\end{aligned}$$

So by Proposition 7.9 the element y_{n-1} acts on this space by zero, and the statement again follows from the inductive assumption.

(ii) By (i) the annihilator is at least as big as claimed, and we only need to show that it is not bigger. This is shown again by induction in n. The base is again easy so we only need to do the induction step. If $n-1 \notin S$ then by Proposition 7.8 we have an inclusion $\mathrm{Ext}^\bullet_{\mathsf{Ver}^+_{2^n}}(X_S, X_S)[y_n] \to \mathrm{Ext}^\bullet_{\mathsf{Ver}^+_{2^{n+1}}}(X_S, X_S)$, so the result follows from the inductive assumption for $n-1$. On the other hand, if $n-1 \in S$ then $X_S = X_{S'} \otimes X_{n-1}$ so

$$\begin{aligned}\mathrm{Ext}^\bullet_{\mathsf{Ver}^+_{2^{n+1}}}(X_S, X_S) &\cong \mathrm{Ext}^\bullet_{\Lambda X_{n-1}}(X_{S'}, X_{S'} \otimes \Lambda X_{n-2}) \\ &\cong \mathrm{Ext}^\bullet_{\Lambda X_{n-2}}(X_{S'}, X_{S'} \otimes \Lambda X_{n-2} \otimes S^{\mathrm{even}} X_{n-1}) \\ &\cong \mathrm{Ext}^\bullet_{\mathsf{Ver}_{2^{n-1}}}(X_{S'}, X_{S'} \otimes S^{\mathrm{even}} X_{n-1}),\end{aligned}$$

which contains $\mathrm{Ext}^\bullet_{\mathsf{Ver}_{2^{n-1}}}(X_{S'}, X_{S'}) = \mathrm{Ext}^\bullet_{\mathsf{Ver}^+_{2^{n-1}}}(X_{S'}, X_{S'})$ as a direct summand as $S^{\mathrm{even}} X_{n-1}$ contains $\mathbb{1}$ as a direct summand. Thus the result again follows from the inductive assumption (this time for $n-2$). $\square$

Corollary 7.11. *The rank r_n of the module $\mathrm{Ext}^\bullet_{\mathsf{Ver}_{2^{n+1}}}(\mathbb{1},\mathbb{1})$ over $\mathbf{k}[y_1,\dots,y_n]$ satisfies the equality $r_n = r_{n-1}[SX_{n-1} : \mathbb{1}]$.*

Proof. In view of the isomorphism (7.1) applied to $X = \mathbb{1}$, this follows from Proposition 7.10 (i). □

Corollary 7.12. *We have $r_n = 2^{\frac{n(n-1)}{2}}$.*

Proof. This follows from Corollary 7.11 and Proposition 6.10, using that $r_1 = 1$. □

Recall that the algebra $\mathrm{Ext}^\bullet_{\mathsf{Ver}_{2^{n+1}}}(\mathbb{1},\mathbb{1})$ has a $\mathbb{Z}$-grading coming from the grading on ΛX_{n-1}, where y_n has degree 2^n-1. Define the field $F_n := \mathbf{k}(y_1,\dots,y_{n-1})$, and let $r_n(v)$ be the Poincaré polynomial of $\mathrm{Ext}^\bullet_{\mathsf{Ver}_{2^{n+1}}}(\mathbb{1},\mathbb{1}) \otimes_{\mathbf{k}[y_1,\dots,y_{n-1}]} F_n$ as a module over the algebra $F_n[y_n]$. Then the above arguments yield

Corollary 7.13. $\mathrm{Ext}^\bullet_{\mathsf{Ver}_{2^{n+1}}}(\mathbb{1},\mathbb{1}) \otimes_{\mathbf{k}[y_1,\dots,y_{n-1}]} F_n$ *is a free $F_n[y_n]$-module, and for $n \geqslant 2$*

$$r_n(v) = 2^{\frac{(n-1)(n-2)}{2}} \frac{1-v^{2^n}}{1-v^2} = 2^{\frac{(n-1)(n-2)}{2}} \sum_{j=0}^{2^{n-1}-1} v^{2j}.$$

This agrees with Conjecture 1.3. Also the formula $r_n = 2^{\frac{n(n-1)}{2}}$ is now obtained by evaluating $r_n(v)$ at $v=1$.

Remark 7.14. As stated in Conjecture 1.3, we expect that moreover $\mathrm{Ext}^\bullet_{\mathsf{Ver}_{2^{n+1}}}(\mathbb{1},\mathbb{1})$ is a free $\mathbf{k}[y_1,\dots,y_n]$-module (even without localization in $y_1,\dots,y_{n-1}$).

More generally, for every object $X \in \mathsf{Ver}^+_{2^{n+1}}$ we obtain upper bounds for the Poincaré polynomials of generators of $\mathrm{Ext}^\bullet_{\mathsf{Ver}_{2^{n+1}}}(\mathbb{1},X)$,

$$r_n(X,z,v) := \sum_{i,j=0}^{\infty} z^i v^j \dim\left(\mathrm{Ext}^\bullet_{\mathsf{Ver}_{2^{n+1}}}(\mathbb{1},X)\Big/\textstyle\sum_{k=1}^n \mathsf{Im}(y_k)\right)^{i,j},$$

where i is the cohomological degree and j is the v-degree.

Proposition 7.15. *For $n \geqslant 2$ we have $r_n(X,z,v) \leqslant r_n^*(X,z,v)$, in the sense that each coefficient on the left is less than or equal to the corresponding coefficient on the right, where*

$$r_n^*(X,z,v) := \frac{1}{2^{n-1}} \mathrm{Tr}\left(\mathrm{FPdim}(X) \frac{1+(zv)^{2^n}}{(1-\mathrm{q}zv)(1-\mathrm{q}^{-1}zv)} \prod_{j=2}^{n-1} \frac{1+z^{2^j}}{(1-\mathrm{q}^{2^{j-1}}z)(1-\mathrm{q}^{-2^{j-1}}z)}\right).$$

In particular, all generators have degree $\leqslant 2^{n+1}-2(n+1)$.

Proof. The bound for $r_n(X,z,v)$ follows from the form of $E(X)$ given in (7.2) and formula (6.6) by a direct computation. This implies the bound on the degree of generators, since the degree of r_n^* with respect to z is $2^{n+1}-2(n+1)$. □

In particular, for $X = \mathbb{1}$ we get

Corollary 7.16. $\mathrm{Ext}^\bullet_{\mathsf{Ver}_{2^{n+1}}}(\mathbb{1},\mathbb{1})$ *is a finitely generated module over* $\mathbf{k}[y_1,\dots,y_n]$ *with Poincaré polynomial of generators* $r_n(z,v)\leqslant r_n^*(z,v)$*, where*

$$r_n^*(z,v):=\frac{1}{2^{n-1}}\mathrm{Tr}\left(\frac{1+(zv)^{2^n}}{(1-\mathrm{q}zv)(1-\mathrm{q}^{-1}zv)}\prod_{j=2}^{n-1}\frac{1+z^{2^j}}{(1-\mathrm{q}^{2^{j-1}}z)(1-\mathrm{q}^{-2^{j-1}}z)}\right).$$

In particular, all the generators have degree at most $2^{n+1}-2(n+1)$*, and there is exactly one generator of that degree. Moreover, the Poincaré polynomial of generators is palindromic, i.e., satisfies the equation* $P(z)=z^{2^{n+1}-n-1}P(z^{-1})$*.*

Proof. It only remains to show that the Poincaré polynomial of the generators is palindromic, which follows from the fact that the complex $E(\mathbb{1})$ is self-dual. $\square$

Remark 7.17. Note that according to Conjecture 1.3, the degree bound of Corollary 7.16 is expected to be sharp: the largest degree of a generator is expected to equal $2^{n+1}-2(n+1)$, with exactly one generator in that degree. On the other hand, the bound $r_n\leqslant r_n^*$ is rather poor: we have $\log_2(r_n^*(1,1))\sim n^2$ as $n\to\infty$, while $\log_2(r_n(1,1))=\frac{n(n-1)}{2}$. This is not surprising, as this bound does not take into account the fact that the complex $E(\mathbb{1})$ has a nontrivial differential for $n\geqslant 3$ and shrinks drastically when we compute its cohomology.

Example 7.18. 1. Let $n=2$. Then we have $E(\mathbb{1})=(SX_1)^+[y_1,y_2]$. But $S^0X_1=S^2X_1=\mathbb{1}$, $S^1X_1=X_1$, and all the other symmetric powers are zero. Thus, $(SX_1)^+=\mathbf{k}\oplus\mathbf{k}w$, where w has cohomological degree 2. Also in this case it is easy to see that the differential in $E(\mathbb{1})$ is zero (so the bound $r_2^*(z,v)=1+(zv)^2$ is sharp). Thus $\mathrm{Ext}^\bullet_{\mathsf{Ver}_{2^3}}(\mathbb{1},\mathbb{1})$ is a free $\mathbf{k}[y_1,y_2]$ module of rank 2 with generators of degree 0 and 2, which agrees with the result of [**2**].

2. Let $n=3$. Let $S^i:=S^iX_2$. Then one can show by a direct computation that

$$S^0=\mathbb{1},\ S^1=X_2,\ S^2=[\mathbb{1},X_1],\ S^3=X_1\otimes X_2,\ S^4=[X_1,\mathbb{1}],\ S^5=X_2,\ S^6=\mathbb{1},$$

and all the other symmetric powers are zero (where $Y=[Y_1,\dots,Y_m]$ means that Y is a uniserial object with composition series $Y_1,\dots,Y_m$, with head Y_1 and socle Y_m). Thus $\mathrm{Ext}^\bullet_{\mathsf{Ver}_{2^4}}(\mathbb{1},\mathbb{1})$ is isomorphic to

$$\mathrm{Ext}^\bullet_{\mathsf{Ver}_{2^3}}(\mathbb{1},\mathbb{1})[0]\oplus\mathrm{Ext}^\bullet_{\mathsf{Ver}_{2^3}}(\mathbb{1},[\mathbb{1},X])[2]\oplus\mathrm{Ext}^\bullet_{\mathsf{Ver}_{2^3}}(\mathbb{1},[X,\mathbb{1}])[4]\oplus\mathrm{Ext}^\bullet_{\mathsf{Ver}_{2^3}}(\mathbb{1},\mathbb{1})[6],$$

where $X=X_1$ and the numbers in square brackets are degree shifts. Now, consider the portion of the long exact sequence

$$\mathrm{Hom}(\mathbb{1},X)\to\mathrm{Ext}^1(\mathbb{1},\mathbb{1})\to\mathrm{Ext}^1(\mathbb{1},[X,\mathbb{1}])\to\mathrm{Ext}^1(\mathbb{1},X)\to\mathrm{Ext}^2(\mathbb{1},\mathbb{1}),\tag{7.3}$$

where Ext groups are taken in Ver_{2^3}. It was shown in [**2**] that the Poincaré series of $\mathrm{Ext}^\bullet(\mathbb{1},X)$ is $\frac{z}{1-z^3}$. Also we have $\dim\mathrm{Ext}^1(\mathbb{1},[X,\mathbb{1}])\geqslant 2$ since we have two different nontrivial extensions of $\mathbb{1}$ by $[X,\mathbb{1}]$, namely $[\mathbb{1}\oplus X,\mathbb{1}]$ and $[\mathbb{1},X,\mathbb{1}]$ (both indecomposable quotients of the projective cover of $\mathbb{1}$ in Ver_{2^3}). Thus the dimension of $\mathrm{Ext}^1(\mathbb{1},[X,\mathbb{1}])$ is two, and the sequence (7.3) looks like $0\to\mathbf{k}\to\mathbf{k}^2\to\mathbf{k}\to\mathbf{k}$. This implies that the last map in this sequence (the connecting homomorphism $\mathrm{Ext}^1(\mathbb{1},X)\to\mathrm{Ext}^2(\mathbb{1},\mathbb{1})$) is zero. Since the map $\mathrm{Ext}^\bullet(\mathbb{1},X)\to\mathrm{Ext}^{\bullet+1}(\mathbb{1},\mathbb{1})$ is linear over $\mathbf{k}[y_2]$, we see that this map is zero in all degrees (as $\mathrm{Ext}^\bullet(\mathbb{1},X)$ is a free

$\mathbf{k}[y_2]$-module on one generator in degree 1). Thus, $\mathrm{Ext}^\bullet(\mathbb{1},[X,\mathbb{1}]) \cong \mathrm{Ext}^\bullet(\mathbb{1},X) \oplus \mathrm{Ext}^\bullet(\mathbb{1},\mathbb{1})$, so the Poincaré series of $\mathrm{Ext}^\bullet(\mathbb{1},[X,\mathbb{1}])$ is $\frac{1+z}{(1-z)(1-z^3)}$.

Now, the object $[X,\mathbb{1},\mathbb{1},X]$ is the projective cover of X. This implies that

$$\mathrm{Ext}^\bullet(\mathbb{1},[\mathbb{1},X]) \cong \mathrm{Ext}^{\bullet+1}(\mathbb{1},[X,\mathbb{1}]).$$

Thus the Poincaré series of $\mathrm{Ext}^\bullet(\mathbb{1},[\mathbb{1},X])$ is $\frac{z+z^2}{(1-z)(1-z^3)}$. Altogether we obtain that the Poincaré series of $\mathrm{Ext}^\bullet_{\mathsf{Ver}_{2^4}}(\mathbb{1},\mathbb{1})$ is given by the formula

$$\begin{aligned} h(z,v) &= \frac{(1+(vz)^6)(1+z^2)+(vz)^2(z+z^2)+(vz)^4(1+z)}{(1-z)(1-z^3)(1-(vz)^7)} \\ &= \frac{1+z^2+v^2z^3+(v^2+v^4)z^4+v^4z^5+v^6z^6+v^6z^8}{(1-z)(1-z^3)(1-(vz)^7)}. \end{aligned}$$

One can check directly that $\mathrm{Ext}^\bullet_{\mathsf{Ver}_{2^4}}(\mathbb{1},\mathbb{1})$ is a free module over $\mathbf{k}[y_1,y_2,y_3]$. Thus the Poincaré polynomial of its generators is

$$r_3(z,v) = 1+z^2+v^2z^3+(v^2+v^4)z^4+v^4z^5+v^6z^6+v^6z^8.$$

On the other hand, it is easy to compute that

$$\begin{aligned} r_3^*(z,v) &= 1+(1+v^2)z^2+2v^2z^3+(v^2+v^4)z^4+2v^4z^5+(v^4+v^6)z^6+v^6z^8 \\ &= r_3(z,v)+v^2(z^2+z^3)+v^4(z^5+z^6). \end{aligned}$$

This means that the differential in the complex $E(\mathbb{1})/(y_1,y_2,y_3)$ acts as a rank 1 operator between degrees $2\to 3$ and $5\to 6$ and otherwise acts by zero. In other words, when computing the cohomology of this complex, we kill two elements of cohomological degrees $2,3$ in v-degree 2 and two elements of cohomological degrees $5,6$ in v-degree 4.

It is instructive to write down the complex $E(\mathbb{1})$ explicitly. We have

$$E(\mathbb{1}) = M^+[y_1,y_2,y_3],\ M := SX_1 \otimes (SX_2)^+.$$

The components of M are as follows (with $X := X_1$):

$$\begin{gathered} M^0=\mathbb{1},\quad M^1=X,\quad M^2=\mathbb{1}\oplus[\mathbb{1},X],\quad M^3=X\otimes[\mathbb{1},X]=[X,\mathbb{1},\mathbb{1}], \\ M^4=[\mathbb{1},X]\oplus[X,\mathbb{1}],\quad M^5=X\otimes[X,\mathbb{1}]=[\mathbb{1},\mathbb{1},X], \\ M^6=\mathbb{1}\oplus[X,\mathbb{1}],\quad M^7=X,\quad M^8=\mathbb{1}. \end{gathered}$$

Thus, $E(\mathbb{1})$ has the following components (as $\Lambda\mathbb{1}$-modules):

$$\begin{gathered} E^0=\mathbb{1},\quad E^1=0,\quad E^2=\mathbb{1}\oplus\mathbb{1},\quad E^3=[\mathbb{1},\mathbb{1}],\quad E^4=\mathbb{1}\oplus\mathbb{1}, \\ E^5=[\mathbb{1},\mathbb{1}],\quad E^6=\mathbb{1}\oplus\mathbb{1},\quad E^7=0,\quad E^8=\mathbb{1}. \end{gathered}$$

The differential maps $E^2=\mathbb{1}\oplus\mathbb{1}\to E^3=[\mathbb{1},\mathbb{1}]$, $E^5=[\mathbb{1},\mathbb{1}]\to E^6=\mathbb{1}\oplus\mathbb{1}$, both by rank 1 operators, and is zero in other degrees.

7.2. Ext computations for $p>2$

In this section we would like to generalise some of the results of the previous section to the case $p>2$. The constructions and formulas are very similar to the case $p=2$ but not exactly the same due to presence of the invertible object ψ and some other differences, so we chose to repeat them.

As in the case $p=2$, we can use the resolution $Q_\bullet$ to give the following recursive procedure for the computation of the additive structure of the cohomology $\mathrm{Ext}^\bullet_{\mathsf{Ver}_{p^{n+1}}}(\mathbb{1},X)$ (for indecomposable X).

For $k\geqslant 1$ we will denote the generating object of Ver_{p^k} by X_{k-1} and recall [**3**, Subsection 4.14] that the principal block $\mathsf{Ver}^0_{p^{k+1}}$ of $\mathsf{Ver}_{p^{k+1}}$ is naturally equivalent to the category of ΛX_{k-1}-modules in Ver_{p^k}. Let us denote this equivalence by F; i.e., for an object $X\in\mathsf{Ver}^0_{p^{k+1}}$ we denote the corresponding ΛX_{k-1}-module by FX.

In the Yoneda realization of Ext, the Koszul complex $K^\bullet=S^\bullet X_{k-1}\otimes\Lambda X_{k-1}$ represents a class $\tau_k\in\mathrm{Ext}^{p^k-1}_{\Lambda X_{k-1}}(\mathbb{1},\psi)$, and the class $y_k:=\tau_k^2$ of degree $2(p^k-1)$ is represented by the concatenation of $S^\bullet X_{k-1}\otimes\Lambda X_{k-1}$ with $S^\bullet X_{k-1}\otimes\Lambda X_{k-1}\otimes\psi$, which we will denote by $S^\bullet X_{k-1}\otimes\Lambda X_{k-1}\otimes S^\bullet\psi_{k-1}$, where ψ_{k-1} is ψ sitting in degree p^k-1. Thus $Q_\bullet=S^\bullet X_{k-1}[y_k]\otimes\Lambda X_{k-1}\otimes S^\bullet\psi_{k-1}$.

If $X\in\mathsf{Ver}_{p^n}$ but $X\notin\mathsf{Ver}^0_{p^n}$ then we have $\mathrm{Ext}^\bullet_{\mathsf{Ver}_{p^n}}(\mathbb{1},X)=0$. So, it suffices to compute $\mathrm{Ext}^\bullet_{\mathsf{Ver}^0_{p^{n+1}}}(\mathbb{1},X)$ for $X\in\mathsf{Ver}^0_{p^{n+1}}$. In that case, we have

$$\begin{aligned}\mathrm{Ext}^\bullet_{\mathsf{Ver}^0_{p^{n+1}}}(\mathbb{1},X)&\cong\mathrm{Ext}^\bullet_{\Lambda X_{n-1}}(\mathbb{1},FX)\cong\mathrm{Ext}^\bullet_{\Lambda X_{n-1}}(\mathbb{1},Q_\bullet\otimes FX)\\&\cong\mathrm{Ext}^\bullet_{\Lambda X_{n-1}}(\mathbb{1},S^\bullet X_{n-1}[y_n]\otimes\Lambda X_{n-1}\otimes S^\bullet\psi_{n-1}\otimes FX)\\&\cong\mathrm{Ext}^\bullet_{\mathsf{Ver}_{p^n}}(\mathbb{1},S^\bullet X_{n-1}[y_n]\otimes S^\bullet\psi_{n-1}\otimes FX)\\&\cong\mathrm{Ext}^\bullet_{\mathsf{Ver}^0_{p^n}}(\mathbb{1},(S^\bullet X_{n-1}[y_n]\otimes S^\bullet\psi_{n-1}\otimes FX)^0),\end{aligned}$$

where the superscript zero means that we are taking the part lying in $\mathsf{Ver}^0_{p^n}$, and in the last two expressions FX is regarded as an object of Ver_{p^n} using the corresponding forgetful functor ΛX_{n-1}-mod $\to\mathsf{Ver}_{p^n}$ forgetting the structure of a ΛX_{n-1}-module.

The same calculation applies if X is a complex, i.e., an object of the derived category $D^+(\mathsf{Ver}^0_{p^{n+1}})$ of $\mathsf{Ver}^0_{p^{n+1}}$. Namely, for an object $X\in D^+(\mathsf{Ver}^0_{p^{n+1}})$, let

$$\begin{aligned}E_n(X)&:=\underline{\mathrm{Hom}}_{\Lambda X_{n-1}}(\mathbb{1},S^\bullet X_{n-1}[y_n]\otimes\Lambda X_{n-1}\otimes S^\bullet\psi_{n-1}\otimes FX)^0\\&=(S^\bullet X_{n-1}[y_n]\otimes S^\bullet\psi_{n-1}\otimes FX)^0.\end{aligned}$$

This gives an additive functor $E_n\colon D^+(\mathsf{Ver}^0_{p^{n+1}})\to D^+(\mathsf{Ver}^0_{p^n})$.

The following lemma is a straightforward analog of Lemma 7.1.

Lemma 7.19. *If $X\in\mathsf{Ver}_{p^n}$ (with trivial action of ΛX_{n-1}) then the differential in the complex $E_n(X)$ is zero.*

Corollary 7.20. *Suppose $X\in\mathsf{Ver}_{p^n}$. Then we have an isomorphism*

$$\begin{aligned}\mathrm{Ext}^\bullet_{\mathsf{Ver}^0_{p^{n+1}}}(\mathbb{1},X)&\cong\bigoplus_{i\geqslant 0}\mathrm{Ext}^\bullet_{\mathsf{Ver}_{p^n}}(\mathbb{1},S^iX_{n-1}[y_n]\otimes S^\bullet\psi_{n-1}\otimes FX)\\&=\bigoplus_{i\geqslant 0}\mathrm{Ext}^\bullet_{\mathsf{Ver}^0_{p^n}}(\mathbb{1},(S^iX_{n-1}[y_n]\otimes S^\bullet\psi_{n-1}\otimes FX)^0).\end{aligned}$$

This isomorphism maps the grading induced by the grading on ΛX_{n-1} to the grading defined by $\deg(X_{n-1})=1$, $\deg(y_n)=2p^n-2$, $\deg(\psi_{n-1})=p^n-1$ *(i.e., it coincides with the cohomological grading).*

As for $p=2$, Corollary 7.20 does not quite give a recursion to compute the Ext groups, since the object $(S^iX_{n-1}\otimes S^\bullet\psi_{n-1}\otimes FX)^0$ may not belong to $\mathsf{Ver}_{p^{n-1}}$ (i.e., may carry a nontrivial action of ΛX_{n-2}). However, it has some useful consequences given below.

Proposition 7.21. *For $n\geqslant 1$ we have*

$$\mathrm{Ext}^\bullet_{D^+(\mathsf{Ver}^0_{p^{n+1}})}(\mathbb{1},X)=\mathrm{Ext}^\bullet_{D^+(\mathsf{Ver}^0_{p^n})}(\mathbb{1},E_n(X)).$$

This implies the following corollary. Let $E=E_1\circ\cdots\circ E_n\colon \mathsf{Ver}^0_{p^{n+1}}\to\mathsf{Ver}^0_p=\mathrm{Vec}$.

Corollary 7.22. *We have a linear isomorphism*

$$\mathrm{Ext}^\bullet_{D^+(\mathsf{Ver}^0_{p^{n+1}})}(\mathbb{1},X)\cong H^\bullet(E(X)).$$

The complex of vector spaces $E(X)$ has the following structure:

$$\begin{aligned}E(X)=(&S^\bullet X_0\otimes S^\bullet\psi_0\otimes F(S^\bullet X_1\otimes S^\bullet\psi_1\otimes\cdots\\ &\cdots\otimes F(S^\bullet X_{n-1}\otimes S^\bullet\psi_{n-1}\otimes FX)^0\ldots)^0)^0[y_1,y_2,\ldots,y_n],\end{aligned}$$

and it is easy to see as in the case $p=2$ that the differential is linear over $\mathbf{k}[y_1,\ldots,y_n]$. Thus we get

Proposition 7.23. *For any $X\in\mathsf{Ver}_{p^{n+1}}$, $\mathrm{Ext}^\bullet_{\mathsf{Ver}_{p^{n+1}}}(\mathbb{1},X)$ is a graded finitely generated module over $\mathbf{k}[y_1,\ldots,y_n]$.*

In particular, we get that

$$\mathrm{Ext}^\bullet_{\mathsf{Ver}_{p^{n+1}}}(\mathbb{1},\mathbb{1})=\mathrm{Ext}^\bullet_{\mathsf{Ver}^0_{p^{n+1}}}(\mathbb{1},\mathbb{1})=H^\bullet(E(\mathbb{1})),$$

where

$$\begin{aligned}E(\mathbb{1})=(&S^\bullet X_0\otimes S^\bullet\psi_0\otimes F(S^\bullet X_1\otimes S^\bullet\psi_1\otimes\cdots\\ &\otimes F(S^\bullet X_{n-1}\otimes S^\bullet\psi_{n-1})^0\ldots)^0)^0[y_1,y_2,\ldots,y_n].\end{aligned}$$

Note that we have $1\in E(\mathbb{1})$ and $d(1)=0$, so we obtain a natural linear map

$$\phi\colon\mathbf{k}[y_1,\ldots,y_n]\to\mathrm{Ext}^\bullet_{\mathsf{Ver}_{p^{n+1}}}(\mathbb{1},\mathbb{1}).$$

Proposition 7.24. *For $1\leqslant i\leqslant n$ multiplication by y_i on $\mathrm{Ext}^\bullet_{\mathsf{Ver}_{p^{n+1}}}(\mathbb{1},X)$ coincides with the cup product with $\phi(y_i)$. In particular, ϕ is an algebra homomorphism.*

Proof. The proof is the same as that of Proposition 7.7, using that $\phi(y_n)$ can be realised as Yoneda product with the complex $K^\bullet\otimes S^\bullet\psi_{n-1}$, where $K^\bullet$ is the Koszul complex. The only difference is the presence of the additional factor $S^\bullet\psi_{n-1}$. □

Proposition 7.25. *For $X\in\mathsf{Ver}^0_{p^n}$ the natural map*

$$\mathrm{Ext}^\bullet_{\mathsf{Ver}_{p^n}}(\mathbb{1},X)[y_n]\to\mathrm{Ext}^\bullet_{\mathsf{Ver}_{p^{n+1}}}(\mathbb{1},X)$$

is an injective morphism of $\mathbf{k}[y_1,\ldots,y_n]$-modules which is also a morphism of algebras for $X=\mathbb{1}$.

Proof. This follows from the isomorphism

$$\mathrm{Ext}^\bullet_{\mathsf{Ver}_{p^{n+1}}}(\mathbb{1},X)\cong\mathrm{Ext}^\bullet_{\mathsf{Ver}_{p^n}}(\mathbb{1},(S^\bullet X_{n-1}\otimes S^\bullet\psi_{n-1})^0\otimes X)[y_n]$$

since $\mathbb{1}$ is a direct summand of $(S^\bullet X_{n-1}\otimes S^\bullet\psi_{n-1})^0$. □

8. Some further computations

In order to search for similar patterns for $\mathrm{Ext}^{\bullet}_{\mathsf{Ver}_{p^{n+1}}}(\mathbb{1}, S)$ with S simple in the principal block, it makes sense to compute a number of examples. For instance, the simplest case $n = 1$ can be computed using the theory of Brauer tree algebras, and the answer is as follows.

Let $X_0, \dots, X_{N-1}$ label the simple modules for a chain-shaped Brauer tree algebra of length N, in the order they occur in the Brauer tree (note that in the case of Verlinde categories $N = p - 1$ and $X_0 = \mathbb{1}$). Then we have

Proposition 8.1 ([9])**.** *The Poincare series of* $\mathrm{Ext}^{\bullet}(X_i, X_j)$ *is given by the formula*

$$\sum_{k=0}^{\infty} t^k \dim \mathrm{Ext}^k(X_i, X_j) = \frac{Q_{ijN}(t) + t^{2N-1} Q_{ijN}(t^{-1})}{1 - t^{2N}},$$

where $Q_{ijN}(t) := t^{|i-j|} + t^{|i-j|+2} + \cdots + t^{N-1-|N-1-i-j|}$.

Example 8.2. If $i = 0$, Proposition 8.1 gives

$$\sum_{k=0}^{\infty} t^k \dim \mathrm{Ext}^k(X_0, X_j) = \frac{t^j + t^{2N-1-j}}{1 - t^{2N}}.$$

We also computed $\mathrm{Ext}^{\bullet}_{\mathsf{Ver}_{p^3}}(\mathbb{1}, S)$ for S simple in the cases $p = 2$ and $p = 3$. For $p = 2$, by the results of [2], we have the following (note that $L_0 = \mathbb{1}$):

$$\sum_{i=0}^{\infty} t^i \dim \mathrm{Ext}^i_{\mathsf{Ver}_{2^3}}(\mathbb{1}, \mathbb{1}) = \frac{1 + t^2}{(1-t)(1-t^3)}, \quad \sum_{i=0}^{\infty} t^i \dim \mathrm{Ext}^i_{\mathsf{Ver}_{2^3}}(\mathbb{1}, L_2) = \frac{t}{1 - t^3}.$$

For $p = 3$, the Poincaré series computed using Magma agree at least up to degree 100 with the following (again $L_0 = \mathbb{1}$):

$$\sum_{i=0}^{\infty} t^i \dim \mathrm{Ext}^i_{\mathsf{Ver}_{3^3}}(\mathbb{1}, \mathbb{1}) = \frac{1 + t^3 + t^6 + 2t^7 + t^8 + t^{10} + 2t^{11} + t^{12} + t^{15} + t^{18}}{(1-t^4)(1-t^{16})},$$

$$\sum_{i=0}^{\infty} t^i \dim \mathrm{Ext}^i_{\mathsf{Ver}_{3^3}}(\mathbb{1}, L_4) = \frac{t + t^4 + t^5 + t^6 + t^8 + 2t^9 + t^{10} + t^{12} + t^{13} + t^{14} + t^{17}}{(1-t^4)(1-t^{16})},$$

$$\sum_{i=0}^{\infty} t^i \dim \mathrm{Ext}^i_{\mathsf{Ver}_{3^3}}(\mathbb{1}, L_6) = \frac{t^2 + t^{13}}{1 - t^{16}},$$

$$\sum_{i=0}^{\infty} t^i \dim \mathrm{Ext}^i_{\mathsf{Ver}_{3^3}}(\mathbb{1}, L_{10}) = \frac{t^2 + 2t^3 + t^4 + t^7 + t^8 + t^{10} + t^{11} + t^{14} + 2t^{15} + t^{16}}{(1-t^4)(1-t^{16})},$$

$$\sum_{i=0}^{\infty} t^i \dim \mathrm{Ext}^i_{\mathsf{Ver}_{3^3}}(\mathbb{1}, L_{12}) = \frac{t + t^2 + t^4 + t^5 + t^6 + 2t^9 + t^{12} + t^{13} + t^{14} + t^{16} + t^{17}}{(1-t^4)(1-t^{16})},$$

$$\sum_{i=0}^{\infty} t^i \dim \mathrm{Ext}^i_{\mathsf{Ver}_{3^3}}(\mathbb{1}, L_{16}) = \frac{t^5 + t^{10}}{1 - t^{16}}.$$

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