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RANKS OF HOMOTOPY AND COHOMOLOGY GROUPS FOR RATIONALLY ELLIPTIC SPACES AND ALGEBRAIC VARIETIES

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Abstract

We discuss inequalities between the values of *homotopical and cohomological Poincaré polynomials* of the self-products of rationally elliptic spaces. For rationally elliptic quasi-projective varieties, we prove inequalities between the values of generating functions for the ranks of the graded pieces of the weight and Hodge filtrations of the canonical mixed Hodge structures on homotopy and cohomology groups. Several examples of such mixed Hodge polynomials and related inequalities for rationally elliptic quasi-projective algebraic varieties are presented. One of the consequences is that the homotopical (resp. cohomological) mixed Hodge polynomial of a rationally elliptic toric manifold is a sum (resp. a product) of polynomials of projective spaces. We introduce an invariant called *stabilization threshold* $\mathtt{pp}(X;\varepsilon)$ for a simply connected rationally elliptic space X and a positive real number ε, and we show that the Hilali conjecture implies that $\mathtt{pp}(X;1)\leqslant 3$.

1. Introduction

A rationally elliptic space is a simply connected topological space X such that

$$\dim\left(\pi_*(X)\otimes\mathbb{Q}\right)<\infty \text{ and } \dim H^*(X;\mathbb{Q})<\infty,$$

where $\pi_*(X)\otimes\mathbb{Q}:=\sum_{i\geqslant 1}\pi_i(X)\otimes\mathbb{Q}$ and $H^*(X;\mathbb{Q}):=\sum_{j\geqslant 0}H^j(X;\mathbb{Q})$. This interesting class of spaces has received considerable attention, but a complete picture of structure, geometry or invariants of spaces in this class appears to be far from clear. Very strong restrictions on the ranks of homotopy group were found a long time ago by J. B. Friedlander and S. Halperin (see [16] and also [14] or [15]). To recall them, let x_i (resp. y_j) denote a basis of $\pi_{\mathrm{odd}}(X)\otimes\mathbb{Q}$ (resp. $\pi_{\mathrm{even}}(X)\otimes\mathbb{Q}$) and let n be the formal dimension of the space X, i.e., the maximal degree n such that $H^n(X;\mathbb{Q})\neq 0$.

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We set

$$\pi_{\mathrm{even}}(X)\otimes\mathbb{Q}:=\sum_{k\geqslant 1}\pi_{2k}(X)\otimes\mathbb{Q},\quad \pi_{\mathrm{odd}}(X)\otimes\mathbb{Q}:=\sum_{k\geqslant 0}\pi_{2k+1}(X)\otimes\mathbb{Q},$$

and

$$H^{\mathrm{even}}(X;\mathbb{Q}):=\sum_{k\geqslant 0}H^{2k}(X;\mathbb{Q}),\quad H^{\mathrm{odd}}(X;\mathbb{Q}):=\sum_{k\geqslant 0}H^{2k+1}(X;\mathbb{Q}).$$

Then we have the following:

(a) $\sum_i \deg x_i \leqslant 2n-1, \sum_j \deg y_j \leqslant n$.
(b) $n=\sum_i \deg x_i - \sum_j(\deg y_j - 1)$.
(c) $\chi^{\pi}(X):=\dim\left(\pi_{\mathrm{even}}(X)\otimes\mathbb{Q}\right)-\dim\left(\pi_{\mathrm{odd}}(X)\otimes\mathbb{Q}\right)\leqslant 0$.
(d) $0\leqslant\chi(X)=\dim H^{\mathrm{even}}(X;\mathbb{Q})-\dim H^{\mathrm{odd}}(X;\mathbb{Q})$.
(e) $\chi(X)>0\Longleftrightarrow\chi^{\pi}(X)=0$.
(f) Betti numbers $b_i=\dim H_i(X;\mathbb{Q})$ of X satisfy Poincaré duality [**14**, §38 Poincaré Duality]. In particular $b_n=1$ and $b_{n-1}=b_1=0$.
(g) Betti numbers satisfy inequalities:[1] $b_m\leqslant\frac{1}{2}\binom{n}{m}, m\neq 0,n$ (cf. [**27**, Corollary to Theorem 1]).

Moreover, the Hilali conjecture [**20**] (also see [**21, 22**]), which is still open, suggests that:

$$\dim\left(\pi_*(X)\otimes\mathbb{Q}\right)\leqslant\dim H^*(X;\mathbb{Q}). \tag{1.1}$$

The present paper, instead of (1.1), shows different types of inequalities between the ranks of homotopy and cohomology groups of rationally elliptic spaces (cf. [**30**]). They are stated in terms of the *cohomological Poincaré polynomial* and the *homotopical Poincaré polynomial*. For a simply connected rationally elliptic space X we put

$$P_X(t):=\sum_{k\geqslant 0}\dim H^k(X;\mathbb{Q})t^k\quad\text{and}\quad P_X^{\pi}(t):=\sum_{k\geqslant 2}\dim(\pi_k(X)\otimes\mathbb{Q})t^k.$$

In [**30**] the second named author showed that there exists a positive integer n_0 such that for all $n>n_0$ one has $P_{X^n}^{\pi}(1)<P_{X^n}(1)$. Here $X^n=\underbrace{X\times\cdots\times X}_{n}$ is the Cartesian product of n copies of X. Below we show the following (announced in [**31**]):

Theorem 1.2. *Let X be a simply connected rationally elliptic space. For any positive real number ε there exists a positive integer $n(\varepsilon)$ such that for all $n\geqslant n(\varepsilon)$ and all $t\geqslant\varepsilon$*

$$P_{X^n}^{\pi}(t)<P_{X^n}(t). \tag{1.3}$$

Remark 1.4. Note that, since X is simply connected, $P_X(t)=1$ implies that X is rationally homotopy equivalent to a point (cf. [**14**, Theorem 8.6]), and hence $P_X^{\pi}=0$. In particular, the inequality (1.3) is satisfied with $n(\varepsilon)=1,\forall\varepsilon>0$. Therefore, in Theorem 1.2 we assume that $P_X(t)>1$. We also note that the formal dimension of a simply connected space is bigger than or equal to 2.

[1] This inequality implies that $\dim H^*(X;\mathbb{Q})\leqslant 2^{n-1}+1$, which is sharper than $\dim H^*(X;\mathbb{Q})\leqslant 2^n$ ([**15**, Theorem 2.75]).

Theorem 1.2 suggests the following invariant of a rationally elliptic homotopy type:

Definition 1.5. *The stabilization threshold* is the smallest integer $n(\varepsilon)$ such that inequality (1.3) takes place for all $n \geqslant n(\varepsilon)$.

We denote the stabilization threshold by $\mathfrak{pp}(X;\varepsilon)$, where $\mathfrak{pp}$ stands for "Poincaré polynomial". For example, for $\varepsilon = 1$ we have

1. $\mathfrak{pp}(S^{2n+1};1) = 1$,
2. $\mathfrak{pp}(S^{2n};1) = 3$,
3. $\mathfrak{pp}(\mathbb{C}P^1,1) = 3$ and $\mathfrak{pp}(\mathbb{C}P^n,1) = 2$ if $n \geqslant 2$.

In terms of this invariant, the inequality of Theorem 1.2 implies the following:

Corollary 1.6. *For any $\varepsilon > 0$ and $r \geqslant \mathfrak{pp}(X;\varepsilon)$ we have*

$$r\left(\sum_{i=2}^{n} \dim\left(\pi_i(X)\otimes\mathbb{Q}\right)\varepsilon^i\right) < \left(1+\sum_{i=2}^{l}\dim H^i(X,\mathbb{Q})\varepsilon^i\right)^r,$$

where n (resp. l) is the degree of homotopical (resp. cohomological) Poincare polynomial.

Note that inequality (1.1) is a special case of Corollary 1.6 for the spaces with stabilization threshold $\mathfrak{pp}(X;1) = 1$, but not for the spaces with $\mathfrak{pp}(X;1) \geqslant 2$. The argument used in the proof of Theorem 1.2 is an elementary calculus observation and based only on the difference in behavior of homotopy groups and cohomology groups in products.

Several results on stabilization threshold and specific values in some examples are presented in Sections 2, 3 and 4 respectively, but let us point out that we have the following result about the upper bound of the stabilization threshold $\mathfrak{pp}(X;1)$:

Theorem 1.7. *Let X be a simply connected rationally elliptic space of formal dimension $n \geqslant 3$. Then*

$$\mathfrak{pp}(X;1) \leqslant n.$$

We also show that the Hilali conjecture implies sharp bound, independent of dimension:

Theorem 1.8. *If a simply connected rationally elliptic space X satisfies the Hilali conjecture, then we have*

$$\mathfrak{pp}(X;1) \leqslant 3.$$

In particular, the question if 3 is an unconditional bound of the threshold $\mathfrak{pp}(X;1)$ is a weakening of the Hilali conjecture. Note (see Corollary 3.6) that the threshold $\mathfrak{pp}(X;1)$ does not exceed 3 if the formal dimension does not exceed 20 since the Hilali conjecture is verified up in this range (see [**7**]). The Hilali conjecture is also valid for formal spaces (see [**21**]), hence the stabilization threshold $\mathfrak{pp}(X;1)$ does not exceed 3 also for, e.g., the following spaces, which are formal:

- compact Kähler manifolds [**12**],
- projective varieties with isolated normal singularities with high connectivity of links [**8**], and

- smooth quasi-projective manifolds with pure Hodge structure (by Dupont's "purity implies formality" theorem [13]).

Now, let X be a quasi-projective algebraic variety. Both the homotopy and the cohomology groups carry mixed Hodge structures ([10], [11], [25], [17], [18], [26]), which are functorial for regular maps. An invariant of these mixed Hodge structures is given by the generating functions for the dimensions of graded pieces of Hodge and weight filtrations as follows:

$$MH_X(t,u,v) := \sum_{k,p,q} \dim\Big(Gr_{F^\bullet}^p Gr_{p+q}^{W_\bullet} H^k(X;\mathbb{C})\Big) t^k u^p v^q, \tag{1.9}$$

where $(W_\bullet, F^\bullet)$ is the mixed Hodge structure of the cohomology groups.

$$MH_X^\pi(t,u,v) := \sum_{k,p,q} \dim\Big(Gr_{\tilde{F}^\bullet}^p Gr_{p+q}^{\tilde{W}_\bullet}((\pi_k(X)\otimes\mathbb{C})^\vee)\Big) t^k u^p v^q,$$

where $(\widetilde{W}_\bullet, \widetilde{F}^\bullet)$ is the mixed Hodge structure of the dual of homotopy groups. They will be called respectively the *cohomological mixed Hodge polynomial* and the *homotopical mixed Hodge polynomial* of X. A refinement of Theorem 1.2 (announced in [31]) for algebraic varieties is as follows:

Theorem 1.10. *Let ε and r be positive real numbers such that $\varepsilon < r$ and define $\mathscr{C}_{\varepsilon,r} := [\varepsilon, r]\times[\varepsilon, r]\times[\varepsilon, r] \subset (\mathbb{R}_{\geqslant 0})^3$ to be the cube of size $r-\varepsilon$. Let X be a rationally elliptic quasi-projective variety. Then there exists a positive integer $n_{\varepsilon,r}$ such that for all $n \geqslant n_{\varepsilon,r}$ the following strict inequality holds:*

$$MH_{X^n}^\pi(t,u,v) < MH_{X^n}(t,u,v)$$

for $\forall (t,u,v) \in \mathscr{C}_{\varepsilon,r}$.

Similarly to $\mathfrak{pp}(X;1)$, we can consider the smallest integer n_0 such that for $\forall n \geqslant n_0$ the following holds

$$MH_{X^n}^\pi(t,u,v) < MH_{X^n}(t,u,v) \quad \forall t \geqslant a, \forall u \geqslant b, \forall v \geqslant c.$$

We denote it by $\mathfrak{mhp}(X;a,b,c)$, where $\mathfrak{mhp}$ stands for "mixed Hodge polynomial".

Actual calculations of homotopy and cohomology groups of rationally elliptic quasi-projective varieties are rather sparse with the main focus being on low dimensional cases (e.g., see [1], [3] and [19] where such rationally elliptic spaces are identified) and even less is known about their mixed Hodge theory refinements. Therefore, besides inequalities, we include several examples, in particular toric varieties and arrangements of linear subspaces and calculate the stabilization thresholds for them.

It would be interesting to find non-trivial[2] examples of singular algebraic varieties which are rationally elliptic and study for their mixed homotopy and homology polynomials and their stabilization thresholds.

In §2 we prove Theorems 1.2 and 1.10 and several results on stabilization thresholds. Theorems 1.7 and 1.8 are proven in §3. In the final §4 we give explicit calculations of the homotopical and cohomological mixed Hodge polynomials of several compact

[2]A trivial example is $X \times C$ where X is any rationally elliptic smooth or singular variety and C is a rational cuspidal curve (which is homeomorphic to S^2).

and open manifolds, including some toric varieties and complement to arrangements of linear subspaces in affine space. In this section we also introduce and discuss homotopical E-function which is an analog of classical cohomological E-function.

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2. Proofs of the main results

The isomorphism $\pi_i(X \times Y) = \pi_i(X) \oplus \pi_i(Y)$ along with the Künneth isomorphism $H^n(X \times Y, \mathbb{Q}) = \sum_{i+j=n} H^i(X; \mathbb{Q}) \otimes H^j(Y; \mathbb{Q})$ imply that the homotopical Poincaré polynomial $P^\pi_X(t)$ and the cohomological Poincaré polynomial $P_X(t)$ are respectively additive and multiplicative, i.e.,

$$P^\pi_{X\times Y}(t) = P^\pi_X(t) + P^\pi_Y(t) \quad \text{and} \quad P_{X\times Y}(t) = P_X(t) \times P_Y(t),$$

which imply that Theorem 1.2 is an immediate consequence of the following:

Lemma 2.1. *Let ε be a positive real number. Let $P(x)$ and $Q(x)$ be two polynomials of the following types:*

$$P(x) = \sum_{k=2}^{p} a_k x^k,\ a_k \geqslant 0, \qquad Q(x) = 1 + \sum_{k=2}^{q} b_k x^k, \quad b_k \geqslant 0,\ b_q \neq 0.$$

Then there exists a positive integer n_0 such that for $\forall n \geqslant n_0$

$$nP(x) < Q(x)^n \quad (\forall x \geqslant \varepsilon). \tag{2.2}$$

Remark 2.3. For our purpose it is sufficient to consider $b_q = 1$, but we do not assume it.

Proof of Lemma 2.1. To begin, select a positive integer N_0 such that $\deg\left(Q(x)^{N_0}\right) > \deg\left(N_0 P(x)\right)$ and take $s_0 > 1, s_0 \in \mathbb{R}$ such that $Q(x)^{N_0} > N_0 P(x)$ for any $x > s_0$. Then $R(s, r)$ defined by $R(s, r) := Q(s)^r - rP(s)$ we have the following for all $r \geqslant N_0$ and all $s > s_0$:

$$\frac{\partial R(s,r)}{\partial r} = \log Q(s) \cdot Q(s)^r - P(s) > \log Q(s) \cdot N_0 P(s) - P(s),$$

which is positive for all $s > \max(s_0, e)$ since $\log Q(s) > 1$, because $Q(s) \geqslant Q(e) = 1 + \sum_{k=2}^{q} b_k e^k > e$ since $b_q \neq 0$. Thus for all $s > \max(s_0, e)$ the function $R(s, r)$ is increasing with respect to r and $R(s, N_0) = Q(s)^{N_0} - N_0 P(s) > 0$, thus, in particular $R(x, n) = Q(x)^n - nP(x) > 0$ for all $x > \max(s_0, e)$ and for all $n \geqslant N_0$. Therefore we have that

$$nP(x) < Q(x)^n \text{ for all } x > \max(s_0, e) \text{ and for all } n \geqslant N_0.$$

Now, we have the following

$$\lim_{n\to\infty} \frac{nP(\xi)}{Q(\xi)^n} = P(\xi) \lim_{n\to\infty} \frac{n}{Q(\xi)^n} = 0$$

for any fixed $\xi \in [\varepsilon, s_0]$, since $Q(\xi) > 1$ for $\xi > 0$. Therefore, we see that there exists

an integer $n(\xi)$ such that for all $n > n(\xi)$ one has $nP(\xi) < Q(\xi)^n$. Having such an integer $n(\xi)$ for each ξ, we can find δ_ξ such that for $|x-\xi| < \delta_\xi$ and $n > n(\xi)$ one has $nP(\xi) < Q(\xi)^n$. Selecting a finite set of ξ_i such that the intervals of length δ_{ξ_i} centered at ξ_i cover $[\varepsilon, s_0]$, we see that for $N \geqslant \max\{n(\xi_i), N_0\}$ one has (2.2) for all $x \geqslant \varepsilon$. □

Remark 2.4. Let $\mathfrak{n}(\varepsilon, P, Q)$ be the smallest integer n_0 satisfying conditions of Lemma 2.1. We can find an upper bound $\mathfrak{u}$ of the threshold $\mathfrak{n}(\varepsilon, P, Q)$, i.e., $\mathfrak{n}(\varepsilon, P, Q) \leqslant \mathfrak{u}$, as follows.

(A) First we consider the case when $0 < \varepsilon \leqslant 1$: Let m be the number of the monomials $a_{l_i}x^{l_i} (1 \leqslant i \leqslant m)$ in $P(x)$ and $b_q x^q$ be the top degree term of $Q(x)$. Let $\mathfrak{u}_i$ be an upper bound of the stabilization threshold $\mathfrak{n}(\varepsilon, ma_{l_i}x^{l_i}, 1 + b_q x^q)$, i.e., $\mathfrak{n}(\varepsilon, ma_{l_i}x^{l_i}, 1 + b_q x^q) \leqslant \mathfrak{u}_i$, and let $\mathfrak{u} := \max\{\mathfrak{u}_1, \cdots, \mathfrak{u}_m\}$. Then for $\forall n \geqslant \mathfrak{u}$ we have for all i:

$$n(ma_{l_i}x^{l_i}) < (1 + b_q x^q)^n \quad \forall x \geqslant \varepsilon$$

and hence

$$nP(x) = \frac{n}{m}\sum_{i=1}^{m} ma_{l_i}x^{l_i} < \frac{1}{m}\sum_{1}^{m}(1 + b_q x^q)^n < \left(1 + b_k x^k + \cdots b_q x^q\right)^n = Q(x)^n.$$

Therefore we get that $\mathfrak{n}(\varepsilon, P, Q) \leqslant \mathfrak{u}$.

Now, each upper bound $\mathfrak{u}_i$ of the threshold $\mathfrak{n}(\varepsilon, ma_{l_i}x^{l_i}, 1 + b_q x^q)$ is obtained as follows, by considering the inequality $n(ma_i x^{l_i}) < (1 + b_q x^q)^n$ for each of the two cases (a) $x > 1$ and (b) $\varepsilon \leqslant x \leqslant 1$:
(a) $x > 1$:

1. Find an integer s such that $sq > \ell_i$ and $s \geqslant 2$ (condition used in the next step),
2. Find an integer $\widehat{n}_0$ (depending on a_{l_i}, b_q, l_i, s) such that $\dfrac{ma_{l_i}}{b_q^s} \leqslant \dfrac{1}{\widehat{n}_0}\dbinom{\widehat{n}_0}{s}$ for $\widehat{n}_0 \geqslant s$, which implies that $\dfrac{ma_{l_i}}{b_q^s} \leqslant \dfrac{1}{n}\dbinom{n}{s}$ for $\forall n \geqslant \widehat{n}_0 \geqslant s$. (If $s = 1$, then $\frac{1}{n}\binom{n}{s} = 1$ for $\forall n$, in which case there might not exist such an integer $\widehat{n}_0$, depending on the integers m, a_{l_i}, b_q.)

Then, for $\forall n \geqslant \widehat{n}_0$:

$$n(ma_{l_i}x^{l_i}) \leqslant \binom{n}{s} b_q^s x^{\ell_i} < \binom{n}{s} b_q^s x^{qs} = \binom{n}{s}(b_q x^q)^s < (1 + b_q x^q)^n \quad \text{for } x > 1.$$

(b) $\varepsilon \leqslant x \leqslant 1$:

First we observe that $x^{l_i} \leqslant 1$ for $\varepsilon \leqslant x \leqslant 1$, hence it suffices to consider the inequality $n(ma_{l_i}) < (1 + b_q x^q)^n$, which implies that $n(ma_{l_i}x^{l_i}) < (1 + b_q x^q)^n$.

3. Find a positive integer $\widetilde{n}_0$ which is larger than the largest of the roots of the following equation:

$$(ma_{l_i})y = (1 + b_q \varepsilon^q)^y.$$

In order to show the inequality $n(ma_{l_i}) < (1 + b_q x^q)^n$ for $\forall n \geqslant \widetilde{n}_0$ and for $x \in [\varepsilon, 1]$, for a fixed u we consider the line $z = e\log(u)y$, which as direct calculation readily shows, is tangent to the curve $z = u^y$ at the point $y_*(u) = \frac{1}{\log(u)}$. Any other line

through the origin of (z,y)-plain either does not intersect $z=u^y$ or intersects it at two points. Taking $u=1+b_qx^q$, we conclude that if $ma_{l_i}<e\log(1+b_qx^q)$, then $(ma_{l_i})y<(1+b_qx^q)^y$ for $\forall y$, in particular, $n(ma_{l_i})<(1+b_qx^q)^n$ for $\forall n$. Otherwise, $n(ma_{l_i})<(1+b_qx^q)^n$ is satisfied for $\forall n\geqslant y_0(x)$ where $y_0(x)$ is the largest coordinate of intersection of the line $z=(ma_{l_i})y$ and exponential curve $z=(1+b_qx^q)^y$. To get an upper bound on $y_0(x), x\in[\varepsilon,1]$, note that the largest y-coordinate of the intersection of the line $z=(ma_{l_i})y$ with the exponential curve $z=u^y$, is increasing when u is getting smaller and its minimal value is $(1+b_q\varepsilon^q)$,i.e. for $x=\varepsilon$. Hence the upper bound of $y_0(x)$ is the largest of the roots of the equation $(ma_{l_i})y=(1+b_q\varepsilon^q)^y$. Therefore, we have that $n(ma_{l_i})<(1+b_qx^q)^n$ for $\forall n\geqslant\widehat{n}_0$., i.e.,

$$n(ma_{l_i}x^{l_i})<(1+b_qx^q)^n \quad \text{for } \forall x\in[\varepsilon,1].$$

Finally, we let $\mathfrak{u}_i=\max\{\widehat{n}_0,\widetilde{n}_0\}$, then for $\forall n\geqslant\mathfrak{u}_i$ we have

$$n(ma_{l_i}x^{l_i})<(1+b_qx^q)^n \quad \text{for } \forall x\geqslant\varepsilon.$$

(B) In the case when $\varepsilon>1$: We do the same thing as in (a), just by replacing $x>1$ by $x\geqslant\varepsilon$. Then we let $\mathfrak{u}_i:=\widehat{n}_0$.

Remark 2.5. We have the following inequality for the stabilization thresholds:

$$\mathfrak{pp}(X\times Y;\varepsilon)\leqslant\max\{\mathfrak{pp}(X;\varepsilon),\mathfrak{pp}(Y;\varepsilon)\} \tag{2.6}$$

for a positive real number ε such that $P_X(\varepsilon)\geqslant 2$ and $P_Y(\varepsilon)\geqslant 2$. Indeed, we let $\mathfrak{pp}(X;\varepsilon):=n_X$ and $\mathfrak{pp}(Y;\varepsilon):=n_Y$, then we have

$$nP_X^\pi(t)<P_X(t)^n \quad \forall n\geqslant n_X,\forall t\geqslant\varepsilon,$$

and

$$nP_Y^\pi(t)<P_Y(t)^n, \quad \forall n\geqslant n_Y,\forall t\geqslant\varepsilon.$$

Then for $\forall n\geqslant\max\{n_X,n_Y\}$ and $\forall t\geqslant\varepsilon$ we have

$$n(P_X^\pi(t)+P_Y^\pi(t))<P_X(t)^n+P_Y(t)^n. \tag{2.7}$$

Since $P_X(t)$ and $P_Y(t)$ are increasing functions and $P_X(\varepsilon)\geqslant 2$ and $P_Y(\varepsilon)\geqslant 2$, $P_X(t)\geqslant 2$ and $P_Y(t)\geqslant 2$ for $\forall t\geqslant\varepsilon$. Hence we have

$$P_X(t)^n+P_Y(t)^n\leqslant P_X(t)^n\cdot P_Y(t)^n=\left(P_X(t)\cdot P_Y(t)\right)^n. \tag{2.8}$$

$P_X(t)^n+P_Y(t)^n\leqslant P_X(t)^n\cdot P_Y(t)^n$ follows from that

$$P_X(t)^n\cdot P_Y(t)^n-P_X(t)^n-P_Y(t)^n=\left(P_X(t)^n-1\right)\left(P_Y(t)^n-1\right)-1\geqslant 0$$

because $P_X(t)^n-1\geqslant 1$ and $P_Y(t)^n-1\geqslant 1$ for $\forall t\geqslant\varepsilon$. Therefore it follows from (2.7) and (2.8) that $nP_{X\times Y}^\pi(t)<P_{X\times Y}(t)^n$ for $\forall n\geqslant\max\{n_X,n_Y\}$ and $\forall t\geqslant\varepsilon$. Therefore we get $\mathfrak{pp}(X\times Y;\varepsilon)\leqslant\max\{\mathfrak{pp}(X;\varepsilon),\mathfrak{pp}(Y;\varepsilon)\}$. However, note that in general one has inequality, $\mathfrak{pp}(X\times Y;\varepsilon)\neq\max\{\mathfrak{pp}(X;\varepsilon),\mathfrak{pp}(Y;\varepsilon)\}$. For example, we can see that $\mathfrak{pp}(S^{2n};1)=3$, but $\mathfrak{pp}(S^{2n}\times S^{2n};1)=2$.

Now we will turn to comparison of the homotopical and cohomological mixed Hodge polynomials.

In fact the cohomological mixed Hodge polynomial is also multiplicative just like the (cohomological) Poincaré polynomial $P_X(t)$

$$MH_{X\times Y}(t,u,v) = MH_X(t,u,v) \times MH_Y(t,u,v),$$

which follows from the fact that the mixed Hodge structure is compatible with the tensor product (e.g., see [**28**]). On the other hand the homotopical mixed Hodge polynomial is additive just like the homotopical Poincaré polynomial $P^\pi_X(t)$

$$MH^\pi_{X\times Y}(t,u,v) = MH^\pi_X(t,u,v) + MH^\pi_Y(t,u,v)$$

since $\pi_*(X\times Y) = \pi_*(X)\oplus\pi_*(Y)$ and the category of mixed Hodge structures is abelian and the direct sum of a mixed Hodge structure is also a mixed Hodge structure. In this paper the following special multiplicativity and additivity are sufficient:

$$MH_{X^n}(t,u,v) = \{MH_X(t,u,v)\}^n, \tag{2.9}$$

and

$$MH^\pi_{X^n}(t,u,v) = nMH^\pi_X(t,u,v). \tag{2.10}$$

In fact, in a similar way to that of Theorem 2.1, using multiplicativity and additivity relations (2.9) and (2.10), we can show the following proposition. Let $\mathbb{R}_{>0}$ be the set of positive real numbers.

Proposition 2.11. *Let $(s,a,b)\in(\mathbb{R}_{>0})^3$. Let X be a rationally elliptic quasi-projective variety. Then there exists a positive integer $n_{(s,a,b)}$ such that for $\forall n \geqslant n_{(s,a,b)}$ the following strict inequality holds*

$$MH^\pi_{X^n}(t,u,v) < MH_{X^n}(t,u,v)$$

for $|t-s|\ll 1, |u-a|\ll 1, |v-b|\ll 1$.

Proof. For the sake of completeness and/or the sake of the reader, we give a proof, which is similar to the proof of Lemma 2.1. We set

$$MH^\pi_X(t,u,v) = \sum_{k\geqslant 2, p\geqslant 0, q\geqslant 0} a_{k,p,q}t^k u^p v^q,$$

and

$$MH_X(t,u,v) = 1 + \sum_{k\geqslant 1, p\geqslant 0, q\geqslant 0} b_{k,p,q}t^k u^p v^q.$$

If all the coefficients $b_{k,p,q}=0$, then $H^*(X;\mathbb{Q}) = \mathbb{Q} = H^*(pt;\mathbb{Q})$, which implies (as in Remark 1.4) that X is rationally homotopy equivalent to the point, hence $\pi_*(X)=0$. The above strict inequality automatically holds. So we can assume that $b_{k_0,p_0,q_0}\neq 0$ for some (k_0,p_0,q_0). Then for $(s,a,b)\in(\mathbb{R}_{>0})^3$ we have

$$MH_X(s,a,b) = 1 + \sum_{k\geqslant 1, p\geqslant 0, q\geqslant 0} b_{k,p,q}s^k a^p b^q \geqslant 1 + b_{k_0,p_0,q_0}s^{k_0}a^{p_0}b^{q_0} > 1.$$

Therefore whatever the value of $MH^\pi_X(s,a,b)$ is, by the same argument as in the

proof of Theorem 2.1,

$$\lim_{n\to\infty}\frac{n(MH_X^{\pi}(s,a,b))}{(MH_X(s,a,b))^n}=MH_X^{\pi}(s,a,b)\lim_{n\to\infty}\frac{n}{(MH_X(s,a,b))^n}=0.$$

Hence there exists a positive integer $\widehat{N_{(s,a,b)}}$ such that for $\forall n\geqslant\widehat{N_{(s,a,b)}}$

$$\frac{n(MH_X^{\pi}(s,a,b))}{(MH_X(s,a,b))^n}<1. \tag{2.12}$$

Equivalently, we have for $\forall n\geqslant\widehat{N_{(s,a,b)}}$

$$nMH_X^{\pi}(s,a,b)<MH_X(s,a,b)^n.$$

Now the proof is concluded as the proof of Lemma 2.1 using openness of condition (2.12). □

The following theorem follows from the above proposition and the compactness of the cube $\mathscr{C}_{\varepsilon,r}$.

Theorem 2.13. *Let ε and r be positive real numbers such that $\varepsilon<r$ and define $\mathscr{C}_{\varepsilon,r}:=[\varepsilon,r]\times[\varepsilon,r]\times[\varepsilon,r]\subset(\mathbb{R}_{\geqslant 0})^3$. Let X be a rationally elliptic quasi-projective variety. Then there exists a positive integer $n_{\varepsilon,r}$ such that for all $n\geqslant n_{\varepsilon,r}$ the following strict inequality holds:*

$$MH_{X^n}^{\pi}(t,u,v)<MH_{X^n}(t,u,v)$$

for $\forall(t,u,v)\in\mathscr{C}_{\varepsilon,r}$.

Remark 2.14. In a similar manner to the proof of (2.6) in Remark 2.5, we can see the following inequality as to the threshold $\mathfrak{mhp}$:

$$\mathfrak{mhp}(X\times Y;a,b,c)\leqslant\max\{\mathfrak{mph}(X;a,b,c),\mathfrak{mhp}(Y;a,b,c)\}$$

for positive real numbers a,b,c such that $MH_X(a,b,c)\geqslant 2$ and $MH_Y(a,b,c)\geqslant 2$.

Remark 2.15. We defined in Introduction the stabilization threshold $\mathfrak{pp}(X;\varepsilon)$ as the smallest integer n_0 such that for all $n\geqslant n_0$ the following inequality (1.3) holds: $P_{X^n}^{\pi}(t)<P_{X^n}(t)(\forall t\geqslant\varepsilon)$. In particular, it takes place for the product space $X^{\mathfrak{pp}(X;\varepsilon)}$. On the other hand this inequality is equivalent to $nP_X^{\pi}(t)<(P_X(t))^n\,(\forall t\geqslant\varepsilon)$, study of which is a key ingredient for our results. This inequality can be considered without assuming that n is an integer, but for n being a positive real number. The same applies to the stabilization threshold $\mathfrak{mph}(X;a,b,c)$. Thus we can consider the *real stabilization thresholds* $\mathfrak{pp}_{\mathbb{R}}(X;\varepsilon)$ and $\mathfrak{mph}_{\mathbb{R}}(X;a,b,c)$, which are more subtle invariants than the integral ones and are more difficult to analyze. For details on properties and calculation of these invariants of pairs of polynomials, rational elliptic homotopy types and quasi-projective varieties, we refer to [**23**].

3. Bounds for stabilization thresholds

We will start with a conditional result, which yields unconditional bound in small dimensions.

Theorem 3.1. *If a simply connected rationally elliptic space X satisfies the Hilali conjecture, then we have $\mathfrak{pp}(X;1) \leqslant 3$.*

Proof. Let X be a simply connected rationally elliptic space of formal dimension n. Let the homotopical and cohomological Poincaré polynomials of X be

$$P_X^\pi(t) = a_2t^2 + \cdots + a_it^i + \cdots + a_\ell t^\ell,$$

and

$$P_X(t) = 1 + b_2t^2 + \cdots + b_kt^k + \cdots + t^n.$$

(Note that $a_2 = b_2$ by the Hurewicz theorem and recall that $b_n = 1$ and $b_{n-1} = b_1 = 0$.)

First we observe that in order to prove that for a positive integer $\mathfrak{m} \geqslant 2$

$$\mathfrak{pp}(X;1) \leqslant \mathfrak{m},$$

it suffices to show that

$$\mathfrak{m}P_X^\pi(t) < P_X(t)^{\mathfrak{m}} \quad (\forall t \geqslant 1). \tag{3.2}$$

Which implies that

$$(\mathfrak{m}+1)P_X^\pi(t) < P_X(t)^{\mathfrak{m}+1} \quad (\forall t \geqslant 1) \tag{3.3}$$

and by induction we get $mP_X^\pi(t) < P_X(t)^m$ $(\forall t \geqslant 1)$ for $\forall m \geqslant \mathfrak{m}$. Indeed, the inequality (3.2) implies

$$(\mathfrak{m}+1)P_X^\pi(t) < (\mathfrak{m}+1)\left(\frac{1}{\mathfrak{m}}P_X(t)^{\mathfrak{m}}\right). \tag{3.4}$$

Now

$$\begin{aligned} P_X(t)^{\mathfrak{m}+1} - (\mathfrak{m}+1)\left(\frac{1}{\mathfrak{m}}P_X(t)^{\mathfrak{m}}\right) &= P_X(t)^{\mathfrak{m}}\left(P_X(t) - \frac{\mathfrak{m}+1}{\mathfrak{m}}\right) \\ &= P_X(t)^{\mathfrak{m}}\left(1 + b_2t^2 + \cdots t^n - 1 - \frac{1}{\mathfrak{m}}\right) \\ &\geqslant P_X(t)^{\mathfrak{m}}\left(t^n - \frac{1}{\mathfrak{m}}\right) \\ &\geqslant P_X(t)^{\mathfrak{m}}\left(1 - \frac{1}{\mathfrak{m}}\right) \\ &> 0 \quad (\text{since } \mathfrak{m} \geqslant 2). \end{aligned}$$

Hence we obtain (3.3) by the inequality (3.4).

Now, we show that

$$3P_X^\pi(t) < P_X(t)^3 \quad \forall t \geqslant 1.$$

First, we need to observe that it follows from [**14**, Theorem 32.15] that we have the following bound for the degree ℓ of $P_X^\pi(t)$:

$$\ell \leqslant 2n - 1. \tag{3.5}$$

$$\begin{aligned}
&(P_X(t))^3 - 3P_X^{\pi}(t)\\
&= (t^n + b_{n-2}t^{n-2} + \cdots + b_2t^2 + 1)^3 - 3(a_\ell t^\ell + \cdots + a_2t^2)\\
&\geqslant (t^n + b_{n-2}t^{n-2} + \cdots + b_2t^2 + 1)^3 - 3t^\ell(a_\ell + \cdots + a_2)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\text{(since } t^j \geqslant t^2 (j \geqslant 2) \text{ for } \forall t \geqslant 1)\\
&\geqslant (t^n + b_{n-2}t^{n-2} + \cdots + b_2t^2 + 1)^3 - 3t^{2n-1}(a_\ell + \cdots + a_2) \qquad \text{(by (3.5))}.
\end{aligned}$$

The Hilali conjecture is $\dim(\pi_*(X) \otimes \mathbb{Q}) \leqslant \dim H_*(X;\mathbb{Q})$, i.e. $P_X^{\pi}(1) \leqslant P_X(1)$, or

$$a_\ell + \cdots + a_2 \leqslant 1 + b_{n-2} + \cdots + b_2 + 1.$$

Before going furthermore, for the presentation below we point out the following about $1 + b_{n-2} + \cdots + b_2 + 1$:

1. If $n = 2$, $P_X(t) = 1 + t^2$ (thus, $P_X^{\pi}(t) = t^2 + \cdots$). As a consequence, it follows that $1 + b_{n-2} + \cdots + b_2 + 1 = 1 + 1$, thus, the part $b_{n-2} + \cdots + b_2 = 0$.
2. If $n = 3$, then $P_X(t) = 1 + t^3$ (thus, $P_X^{\pi}(t) = t^3 + \cdots$), since it follows from the Poincaré duality of Betti numbers (see (f) in Introduction) that $b_2 = b_1 = 0$. Hence $1 + b_{n-2} + \cdots + b_2 + 1 = 1 + 1$, thus, the part $b_{n-2} + \cdots + b_2 = 0$.
3. If $n = 4$, then $P_X(t) = 1 + b_2t^2 + t^4$, since $b_3 = b_1 = 0$. Hence, it follows that $1 + b_{n-2} + \cdots + b_2 + 1 = 1 + b_2 + 1$, thus, the part $b_{n-1} + \cdots + b_2 = b_2$.

With the part $b_{n-2} + \cdots + b_2$ in the cases when $n = 2, 3, 4$ being understood as above, the above sequence of inequalities continues as follows:

$$\begin{aligned}
&\geqslant (t^n + b_{n-2}t^{n-2} + \cdots + b_2t^2 + 1)^3 - 3t^{2n-1}(1 + b_{n-2} + \cdots + b_2 + 1)\\
&= \left\{(t^n + 1) + (b_{n-2}t^{n-2} + \cdots + b_2t^2)\right\}^3 - 3t^{2n-1}\left\{2 + (b_{n-2} + \cdots + b_2)\right\}\\
&\geqslant (t^n + 1)^3 + 3(t^n + 1)^2(b_{n-2}t^{n-2} + \cdots + b_2t^2) - 6t^{2n-1} - 3t^{2n-1}(b_{n-2} + \cdots + b_2)\\
&\geqslant (t^n + 1)^3 - 6t^{2n-1} + 3(t^n + 1)^2(b_{n-2} + \cdots + b_2) - 3t^{2n-1}(b_{n-2} + \cdots + b_2)\\
&\geqslant (t^n + 1)^3 - 6t^{2n-1} + 3t^{2n}(b_{n-2} + \cdots + b_2) - 3t^{2n-1}(b_{n-2} + \cdots + b_2)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{(using } (t^n + 1)^2 \geqslant t^{2n})\\
&= (t^n + 1)^3 - 6t^{2n-1} + 3(t^{2n} - t^{2n-1})(b_{n-2} + \cdots + b_2)\\
&\geqslant (t^n + 1)^3 - 6t^{2n-1} \qquad\qquad\qquad \text{(since } t^{2n} - t^{2n-1} = t^{2n-1}(t - 1) \geqslant 0)\\
&\geqslant (t^n + 1)^3 - 6t^{2n} \qquad\qquad\qquad\quad \text{(again, since } t^{2n} \geqslant t^{2n-1} \text{ for } t \geqslant 1)\\
&= (t^n)^3 - 3(t^n)^2 + 3t^n + 1\\
&= (t^n - 1)^3 + 2\\
&> 0.
\end{aligned}$$

Therefore, $3P_X^{\pi}(t) < P_X(t)^3 \quad \forall t \geqslant 1$. □

Combining Theorem 3.1 with the result of [7] we obtain:

Corollary 3.6. *For a rationally elliptic space X of homological dimension not exceeding 20, the stabilization threshold $\mathfrak{pp}(X;1)$ is at most 3.*

The next proposition gives unconditional bound on the stabilization threshold, depending, however, on the homological dimension.

Proposition 3.7. *Let X be a simply connected rationally elliptic space of formal dimension $n \geqslant 3$. Then we have*

$$\mathfrak{pp}(X;1) \leqslant n.$$

The argument below uses, in addition to (3.5), the following bound (cf. [**14**, Theorem 32.15]):

$$P_X^{\pi}(1) = a_2 + a_3 + \cdots + a_\ell \leqslant n. \tag{3.8}$$

Proof. In order to prove the proposition, it suffices to show that

$$nP_X^{\pi}(t) < P_X(t)^n \quad \forall t \geqslant 1.$$

$$\begin{aligned}
&(P_X(t))^n - nP_X^{\pi}(t)\\
&\geqslant (t^n+1)^n - n(a_\ell t^\ell + \cdots + a_2 t^2)\\
&\geqslant (t^n+1)^n - nt^\ell(a_\ell + \cdots + a_2)\\
&\geqslant (t^n+1)^n - n^2 t^{2n-1} \qquad \text{(by (3.5) and (3.8))}\\
&\geqslant (t^n+1)^n - n^2 t^{2n} \qquad \text{(since } t^{2n} \geqslant t^{2n-1} \text{ for } t \geqslant 1)\\
&= \sum_{k=0}^{n} \binom{n}{k} t^{nk} - n^2 t^{2n}\\
&> \sum_{k=2}^{n} \binom{n}{k} t^{nk} - n^2 t^{2n}\\
&\geqslant t^{2n} \sum_{k=2}^{n} \binom{n}{k} - n^2 t^{2n} \qquad \text{(since } t^{nk} \geqslant t^{2n} \text{ for } k \geqslant 2 \text{ and } t \geqslant 1)\\
&= t^{2n} \left\{ \sum_{k=2}^{n} \binom{n}{k} - n^2 \right\}\\
&= t^{2n} \left\{ \sum_{k=0}^{n} \binom{n}{k} - \binom{n}{1} - \binom{n}{0} - n^2 \right\}\\
&= t^{2n} \left\{ 2^n - (n^2+n+1) \right\}\\
&> 0
\end{aligned}$$

assuming that $n \geqslant 5$. For $n = 3, 4$ the claim follows from Corollary 3.6 above. □

Remark 3.9. For $n = 2$ the above proposition does not hold since the formal dimension of $\mathbb{C}P^1$ is 2, but $\mathfrak{pp}(\mathbb{C}P^1;1) = 3$.

We conclude this section with the question on "mixed Hodge polynomial" version of Theorem 3.1 and Proposition 3.7. More precisely:

1. Does there exist a fixed integer $\mathfrak{a}(\geqslant 3)$ such that $\mathfrak{mhp}(X;1,1,1) \leqslant \mathfrak{a}$ for any rationally elliptic quasi-projective variety X satisfying the Hilali conjecture?
2. Does there exist an integer $\mathfrak{a}(n)(\geqslant n)$ such that $\mathfrak{mhp}(X;1,1,1) \leqslant \mathfrak{a}(n)$ for any rationally elliptic quasi-projective variety X of formal dimension n?

4. Examples and concluding remarks

Here we present several explicit calculations of thresholds and introduce and discuss some property of homotopical E-function which is an analog of classical cohomological E-function.

4.1. Examples

The purpose of this section is to provide examples of calculations of exact values of stabilization thresholds.

4.1.1. $\mathbb{C}^{n+1} \setminus 0$

Here $n > 0$. This is a smooth quasi-projective variety, for which the mixed Hodge structures on cohomology and homotopy can be constructed using log-forms (cf. [10] and [25] resp.). Since this space can be retracted on S^{2n+1} and the Hurewicz isomorphism preserves the Hodge structure (cf. [17]) and calculating the mixed Hodge structure on $H_n(\mathbb{C}^{n+1} \setminus 0)$ (for example using Gysin exact sequence for the homology of the complement to smooth divisor on the blow up of $\mathbb{P}^{n+1}$ at a point) we obtain:

$$MH_{\mathbb{C}^{n+1}\setminus\{0\}}(t,u,v) = 1 + t^{2n+1}(uv)^{n+1},$$

and

$$MH^{\pi}_{\mathbb{C}^{n+1}\setminus\{0\}}(t,u,v) = t^{2n+1}(uv)^{n+1}.$$

Hence we have

$$MH_{\mathbb{C}^{n+1}\setminus\{0\}}(t,u,v) = 1 + MH^{\pi}_{\mathbb{C}^{n+1}\setminus\{0\}}(t,u,v).$$

4.1.2. Projective spaces

Example 4.1. We start with $X = \mathbb{C}P^n$. We have

$$P_{\mathbb{C}P^n}(t) = 1 + t^2 + \cdots + t^{2n} \quad \text{and} \quad P^{\pi}_{\mathbb{C}P^n}(t) = t^2 + t^{2n+1}.$$

One easily verifies that

$$\mathfrak{pp}(\mathbb{C}P^n; 1) = \begin{cases} 3 & if \quad n = 1, \\ 2 & if \quad n \geqslant 2. \end{cases}$$

The mixed Hodge polynomials are as follows:

$$MH_{\mathbb{C}P^n}(t,u,v) = 1 + t^2uv + t^4(uv)^2 + \cdots + t^{2i}(uv)^i + \cdots + t^{2n}(uv)^n,$$

and

$$MH^{\pi}_{\mathbb{C}P^n}(t,u,v) = t^2uv + t^{2n+1}(uv)^{n+1}.$$

The cohomological case is trivial and the claim in the homotopical case follows using the Hurewicz isomorphism for π_2 and for higher homotopy groups the locally trivial fibration $\mathbb{C}^{\times} \hookrightarrow \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}P^n$, the calculation in §4.1.1 and the corresponding exact sequence

$$\cdots \to \pi_{2n+1}(\mathbb{C}^{\times}) \to \pi_{2n+1}(\mathbb{C}^{n+1} \setminus \{0\}) \to \pi_{2n+1}(\mathbb{C}P^n) \to \pi_{2n}(\mathbb{C}^{\times}) \to \cdots,$$

which is an exact sequence of mixed Hodge structures [17, Theorem 4.3.4].

One easily verifies that:

1. $\mathfrak{mhp}(\mathbb{C}P^1;1,1,1)=3$.
2. If $n\geqslant 2$, then $\mathfrak{mhp}(\mathbb{C}P^n;1,1,1)=2$. In fact, this can be made to the following a bit sharper statement: for $\forall m\geqslant 2$

$$MH^{\pi}_{(\mathbb{C}P^n)^m}(t,u,v)<MH_{(\mathbb{C}P^n)^m}(t,u,v)\quad \text{for } \forall t\geqslant 1, \forall (u,v) \text{ such that } uv\geqslant 1.$$

4.1.3. Compact toric manifolds

In [3, Theorem 3.3] I. Biswas, V. Muñoz and A. Murillo show that the homological Poincaré polynomial of a rationally elliptic toric manifold coincides with that of a product of complex projective spaces. Below, using a recent result due to M. Wiemeler [29] we show that the same thing holds for the homotopical Poincaré polynomial, in fact, for the homotopical mixed Hodge polynomial, and furthermore we also show that the homological mixed Hodge polynomial of a rationally elliptic toric manifold coincides with that of a product of complex projective spaces, which is a stronger version of the above result of Biswas–Muñoz–Murillo:

Theorem 4.2. *The homotopical and cohomological mixed Hodge polynomials of a rationally elliptic toric manifold of complex dimension n coincides with those of a product of complex projective spaces. To be more precise, if X is the quotient of*

$$\prod_{i=1}^{k}(\mathbb{C}^{n_i+1}\setminus\{0\})$$

by a free action of commutative algebraic groups, i.e., $(\mathbb{C}^{\times})^k$. Here $n=\sum_{i=1}^k n_i$. Then we have

1. $MH^{\pi}_X(t,u,v)=MH^{\pi}_{\prod_i^k \mathbb{C}P^{n_i}}(t,u,v)=\sum_{i=1}^k MH^{\pi}_{\mathbb{C}P^{n_i}}(t,u,v)$, *i.e.,*

$$MH^{\pi}_X(t,u,v)=\sum_{i=1}^{k}\Big(t^2uv+t^{2n_i+1}(uv)^{n_i+1}\Big)=kt^2uv+\sum_{i=1}^{k}t^{2n_i+1}(uv)^{n_i+1}.$$

2. $MH_X(t,u,v)=MH_{\prod_i^k \mathbb{C}P^{n_i}}(t,u,v)=\prod_{i=1}^k MH_{\mathbb{C}P^{n_i}}(t,u,v)$, *i.e.,*

$$MH_X(t,u,v)=\prod_{i=1}^{k}\Big(1+t^2uv+\cdots+t^{2j}(uv)^j+\cdots+t^{2n_i}(uv)^{n_i}\Big).$$

Proof. In [29] M. Wiemeler shows that there is an algebraic isomorphism $X\cong X'$ where X' is the quotient described above:

$$X'=\Big(\prod_{i=1}^{k}(\mathbb{C}^{n_i+1}\setminus\{0\})\Big)/(\mathbb{C}^{\times})^k. \tag{4.3}$$

(1) First we observe that

$$\pi_j\Big(\prod_{i=1}^{k}(\mathbb{C}^{n_i+1}\setminus\{0\})\Big)\otimes\mathbb{Q}=\begin{cases}\underbrace{\mathbb{Q}\oplus\cdots\oplus\mathbb{Q}}_{a} & j=2n_i+1,\\ 0 & j\neq 2n_i+1.\end{cases}$$

Here a is the number of the same integer n_i.

$$\pi_j\Big((\mathbb{C}^\times)^k\Big)\otimes\mathbb{Q}=\begin{cases}\underbrace{\mathbb{Q}\oplus\cdots\oplus\mathbb{Q}}_{k} & j=1,\\ 0 & j\neq 1.\end{cases}$$

Hence, since each $2n_i+1\geqslant 3$, it follows from the long exact sequences of homotopy groups that there is an isomorphism of mixed Hodge structures:

$$\pi_j(X)\otimes\mathbb{Q}\cong\begin{cases}\pi_j\Big(\prod_{i=1}^k(\mathbb{C}^{n_i+1}\setminus\{0\})\Big)\otimes\mathbb{Q} & j=2n_i+1,\\ \pi_1\Big((\mathbb{C}^\times)^k\Big)\otimes\mathbb{Q}=\underbrace{\mathbb{Q}\oplus\cdots\oplus\mathbb{Q}}_{k}, & j=2,\\ 0 & j\neq 2, j=2n_i+1.\end{cases}$$

Then it follows from the proof in the above Example 4.1 that we have the isomorphism of mixed Hodge structures

$$\pi_j(X)\otimes\mathbb{Q}\cong\begin{cases}\pi_j\Big(\prod_{i=1}^k\mathbb{C}P^{n_i}\Big)\otimes\mathbb{Q} & j=2,2n_i+1,\\ 0 & j\neq 2, j=2n_i+1.\end{cases}$$

Therefore we have

$$MH^\pi_X(t,u,v)=MH^\pi_{\prod_i^k\mathbb{C}P^{n_i}}(t,u,v)=\sum_{i=1}^k MH^\pi_{\mathbb{C}P^{n_i}}(t,u,v).$$

(2) It follows from [**29**] that X' is a so-called Bott manifold, i.e., there is a sequence of fiber bundles over complex projective spaces with a complex projective space as a fiber:

$$X'=B_k\xrightarrow{p_k}B_{k-1}\to\cdots\to B_i\xrightarrow{p_i}B_{i-1}\to\cdots B_2\xrightarrow{p_1}B_1\to B_0=\{pt\},$$

where $p_1\colon B_1=\mathbb{C}^{n_1+1}\to B_0=\{pt\}$ and each $p_i\colon\mathbb{P}(\mathbb{C}^{n_i+1}\times B_{i-1})\to B_{i-1}$ is the projection map of the projectivization $\mathbb{P}(\mathbb{C}^{n_i+1}\times B_{i-1})$ of the product $\mathbb{C}^{n_i+1}\times B_{i-1}$ or a Whitney sum of trivial complex line bundles over B_{i-1}. This sequence is sometimes called a Bott tower. Note that the fiber space of p_i is nothing but the complex projective space $\mathbb{C}P^{n_i}$. Then it follows from Deligne's degeneration of Leray spectral sequence (see [**28**]) that for each projection map $p_i\colon B_i\to B_{i-1}$ the cohomology of B_i with mixed Hodge structure is the tensor product of the cohomology of the base B_{i-1} and the fiber $\mathbb{C}P^{n_i}$ with mixed Hodge structures. Therefore the mixed Hodge polynomial $MH_X(t,u,v)$ coincides with that of the product of these complex projective spaces:

$$MH_X(t,u,v)=MH_{\prod_i^k\mathbb{C}P^{n_i}}(t,u,v)=\prod_{i=1}^k MH_{\mathbb{C}P^{n_i}}(t,u,v).$$

□

It follows from the above Theorem 4.2 that the cohomological and homotopical Poincaré polynomials of a rationally elliptic toric manifold are the same as those of a product of complex projective spaces, thus as explained in the introduction we get the following:

Corollary 4.4. *The Hilali conjecture holds for rationally elliptic toric manifolds.*

Proof. Since a rationally elliptic toric manifold is formal, thus it follows that the Hilali conjecture holds (see [**3**]). Here we give another simple direct proof, using the above calculation. Let X be a rationally elliptic toric manifold described as (4.3). Then it follows from the above Theorem 4.2 that we have

$$P_X^\pi(1) = MH_X^\pi(1,1,1) = \sum_{i=1}^{k}(1+1) = 2k, \quad P_X(1) = MH_X(1,1,1) = \prod_{i=1}^{k}(1+n_i).$$

Since each $n_i \geqslant 1$, we have

$$\prod_{i=1}^{k}(1+n_i) \geqslant 2^k.$$

1. If $k=1$, then $2k = 2 = 2^1 \leqslant 1+n_1$, thus $P_X^\pi(1) \leqslant P_X(1)$.
2. If $k=2$, then $2k = 4 = 2^2 \leqslant \prod_{i=1}^{2}(1+n_i)$, thus $P_X^\pi(1) \leqslant P_X(1)$.
3. If $k \geqslant 3$, then $2k < 2^k \leqslant \prod_{i=1}^{k}(1+n_i)$, thus $P_X^\pi(1) < P_X(1)$.

Therefore, in any case we do have $P_X^\pi(1) \leqslant P_X(1)$. □

Corollary 4.5. *Let X be a rationally elliptic toric manifold and let*

$$MH_X(t,u,v) = MH_{\prod_i^k \mathbb{C}P^{n_i}}(t,u,v), \qquad MH_X^\pi(t,u,v) = MH^\pi_{\prod_i^k \mathbb{C}P^{n_i}}(t,u,v).$$

If each $n_i \geqslant 2$, then $\mathfrak{mhp}(X;1,1,1) = 2$, and if $n_i = 1$ for some i, then it will be the case that $\mathfrak{mhp}(X;1,1,1) = 3$.

Proof.

$$MH_X(t,u,v) = MH_{\prod_i^k \mathbb{C}P^{n_i}}(t,u,v) = \prod_{i=1}^{k} MH_{\mathbb{C}P^{n_i}}(t,u,v),$$

and

$$MH_X^\pi(t,u,v) = MH^\pi_{\prod_i^k \mathbb{C}P^{n_i}}(t,u,v) = \sum_{i=1}^{k} MH^\pi_{\mathbb{C}P^{n_i}}(t,u,v).$$

(1) If each $n_i \geqslant 2$, then it follows from Example 4.1 that

$$2MH^\pi_{\mathbb{C}P^{n_i}}(t,u,v) < \Big(MH_{\mathbb{C}P^{n_i}}(t,u,v)\Big)^2,$$

hence we have

$$2\Big(\sum_{i=1}^{k} MH^\pi_{\mathbb{C}P^{n_i}}(t,u,v)\Big) = \sum_{i=1}^{k} 2MH^\pi_{\mathbb{C}P^{n_i}}(t,u,v) \ < \sum_{i=1}^{k}\Big(MH_{\mathbb{C}P^{n_i}}(t,u,v)\Big)^2.$$

Now, for $\forall t \geqslant 1, \forall u \geqslant 1$ and $\forall v \geqslant 1$ we have

$$\sum_{i=1}^{k}\Big(MH_{\mathbb{C}P^{n_i}}(t,u,v)\Big)^2 < \prod_{i=1}^{k}\Big(MH_{\mathbb{C}P^{n_i}}(t,u,v)\Big)^2.$$

To show this, first we note that each $MH_{\mathbb{C}P^{n_i}}(t,u,v) \geqslant 2$ for $\forall t \geqslant 1, \forall u \geqslant 1$ and

$\forall v \geqslant 1$. Then it suffices to show that if each $d_i \geqslant 4 (1 \leqslant i \leqslant k)$, then

$$d_1 + d_2 + \cdots + d_k < d_1 d_2 \cdots d_k.$$

Indeed, this follows by induction. Clearly $a_1 + a_2 < a_1 a_2$ since

$$a_1 a_2 - (a_1 + a_2) = (a_1 - 1)(a_2 - 1) - 1 \geqslant 3 \times 3 - 1 > 0.$$

Suppose that $d_1 + d_2 + \cdots + d_{k-1} < d_1 d_2 \cdots d_{k-1}$. Then

$$\begin{aligned} d_1 + d_2 + \cdots + d_{k-1} + d_k &= (d_1 + d_2 + \cdots + d_{k-1}) + d_k \\ &< d_1 d_2 \cdots d_{k-1} + d_k \\ &< d_1 d_2 \cdots d_{k-1} d_k \,. \end{aligned}$$

Therefore we have

$$2\Big(\sum_{i=1}^{k} MH^{\pi}_{\mathbb{C}P^{n_i}}(t,u,v)\Big) < \Big(\prod_{i=1}^{k} MH_{\mathbb{C}P^{n_i}}(t,u,v)\Big)^2.$$

(2) If $n_i = 1$ for some i, then it follows from Example 4.1 that $\mathfrak{mhp}(\mathbb{C}P^1; 1, 1, 1) = 3$, i.e., $3MH^{\pi}_{\mathbb{C}P^1}(t,u,v) < \big(MH_{\mathbb{C}P^1}(t,u,v)\big)^3$. Surely for the other ones we have

$$3MH^{\pi}_{\mathbb{C}P^{n_j}}(t,u,v) < \big(MH_{\mathbb{C}P^{n_j}}(t,u,v)\big)^3.$$

Hence by the same argument as above we have

$$3\Big(\sum_{i=1}^{k} MH^{\pi}_{\mathbb{C}P^{n_i}}(t,u,v)\Big) < \Big(\prod_{i=1}^{k} MH_{\mathbb{C}P^{n_i}}(t,u,v)\Big)^3.$$

Hence $\mathfrak{mhp}(X; 1, 1, 1) = 3$. □

Remark 4.6. Even if we fix $u = 1$ and $v = 1$ in the above proof of Corollary 4.5, we have the same proof, therefore we have that if each $n_i \geqslant 2$, then $\mathfrak{pp}(X; 1) = 2$, and if $n_i = 1$ for some i, then $\mathfrak{pp}(X; 1) = 3$.

4.1.4. Arrangements of linear subspaces

G. Debongnie (cf. [9]) described the structure of arrangements of subspaces in $\mathbb{C}^n$ which complements are rationally elliptic. If follows that such complements are products of $\prod_i \big(\mathbb{C}^{n_i+1} \setminus 0\big)$. Combining this with calculation in §4.1.1, we obtain:

Theorem 4.7. *The homotopical and cohomological mixed Hodge polynomials of a simply connected rationally elliptic complement X of an arrangement of linear subspaces are as follows:*

1. $MH^{\pi}_X(t,u,v) = MH^{\pi}_{\prod_i^k (\mathbb{C}^{n_i+1}\setminus 0)}(t,u,v) = \sum_{i=1}^{k} MH^{\pi}_{\mathbb{C}^{n_i+1}\setminus 0}(t,u,v)$, *i.e.,*

$$MH^{\pi}_X(t,u,v) = \sum_{i=1}^{k} t^{2n_i+1}(uv)^{n_i+1}.$$

2. $MH_X(t,u,v) = MH_{\prod_i^k (\mathbb{C}^{n_i+1}\setminus 0)}(t,u,v) = \prod_{i=1}^{k} MH_{\mathbb{C}^{n_i+1}\setminus 0}(t,u,v)$, *i.e.,*

$$MH_X(t,u,v) = \prod_{i=1}^{k} \Big(1 + t^{2n_i+1}(uv)^{n_i+1}\Big).$$

In particular, we obtain:

Corollary 4.8. *In notations of Theorem 4.7, we have*

$$\mathfrak{pp}(X;1)=1 \quad \textit{and} \quad \mathfrak{mhp}(X;1,1,1)=1.$$

4.2. Homotopical E-function

Specialization $t=-1$ of the homotopical Poincare polynomial $MH^{\pi}_{X}(t,u,v)$ is a homotopical analog of E-functions (cf. [2]) and is well behaved in several constructions described below.

Definition 4.9. The homotopical E-polynomial $E^{\pi}(X,u,v)$ of a complex algebraic variety X which is rational elliptic is defined as follows:

$$E^{\pi}(X,u,v):=MH^{\pi}_{X}(-1,u,v).$$

Recall that the homological E-function is defined as $E(X,u,v):=MH^{c}_{X}(-1,u,v)$, where one uses in (1.9) the compactly supported cohomology. $E(X,u,v)$ satisfies the additivity relation: for an algebraic subvariety $Y\subset X$ one has (cf. [2])

$$E(X,u,v)=E(Y,u,v)+E(X\setminus Y,u,v). \tag{4.10}$$

This follows from the long exact sequence of compactly supported cohomology groups:

$$\cdots\to H^{k}_{c}(X\setminus Y)\to H^{k}_{c}(X)\to H^{k}_{c}(Y)\to H^{k+1}_{c}(X\setminus Y)\to\cdots.$$

Additivity relation for the homotopical E-polynomials comes in the context of locally trivial fibrations

$$F\hookrightarrow E\to B \tag{4.11}$$

of pointed complex algebraic varieties of rationally elliptic E,F,B, which induces a long exact sequence of homotopy groups with mixed Hodge structures (see [17, Theorem 4.3.4]):

$$\cdots\to\pi_{k}(F)\to\pi_{k}(E)\to\pi_{k}(B)\to\pi_{k-1}(F)\to\cdots. \tag{4.12}$$

The sequence (4.12) yields the following:

Proposition 4.13. *Let E,F,B be simply connected pointed complex algebraic varieties forming a locally trivial fibration (4.11) such that any two of them are rationally elliptic. Then we have*

$$E^{\pi}(E,u,v)=E^{\pi}(F,u,v)+E^{\pi}(B,u,v).$$

In the case of homological E-polynomials one has multiplicativity in the case of locally trivial fibrations (4.11).[3]

Theorem 4.14 (see [4, 5, 6, 24])**.** *Let $F\hookrightarrow E\to B$ be a smooth complex algebraic fiber bundles. If the fundamental group $\pi_1(B)$ of the base space B acts trivially on the cohomology $H^{*}(F;\mathbb{Q})$ of the fiber space F, then,*

$$E(E,u,v)=E(F,u,v)\cdot E(B,u,v).$$

[3]For which one does not need to assume that spaces are rationally elliptic.

This is a reformulation of the relation between the Euler characteristics of bigraded components:

$$e^{p,q}(X) = \sum_k (-1)^k \dim\Big(Gr_{F^\bullet}^p Gr_{p+q}^{W_\bullet} H^k(X;\mathbb{C})\Big)$$

discussed on p. 935 of [6] (the multiplicativity relation is stated in this paper for χ_y-genus which is a specialization of $E(X,u,v)$).

Finally we note that a homotopy theoretical analog of additivity relation (4.10) holds. To state it, recall (cf. [17], [18]) that if (X,Y) is a pair of pointed complex algebraic varieties, the homotopy groups support a mixed Hodge structure such that the homotopy exact sequence of the pair (X,Y) is an exact sequence of the mixed Hodge structures. This sequence implies that for rationally elliptic spaces X and Y such that $\pi_i(X,Y)=0$ for large i

$$\begin{aligned} E^\pi(X,Y,u,v) &:= MH^\pi_{(X,Y)}(-1,u,v) \\ &= \sum_k (-1)^k \dim\Big(Gr_{\tilde{F}^\bullet}^p Gr_{p+q}^{\tilde{W}_\bullet}((\pi_k(X,Y)\otimes\mathbb{C})^\vee)\Big) u^p v^q \end{aligned}$$

is well-defined and the following additivity relation holds:

$$E^\pi(X,u,v) = E^\pi(Y,u,v) + E^\pi(X,Y,u,v). \tag{4.15}$$

Let X be a compact complex algebraic variety and Y be a closed subvariety of X such that $X\setminus Y$ is smooth, then for homological E-functions one has:

$$E(X\setminus Y,u,v) = E(X,Y,u,v),$$

which shows that additivity (4.15) corresponds to (4.10).

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