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ALGEBRAICALLY COFIBRANT AND FIBRANT OBJECTS REVISITED

JOHN BOURKE AND SIMON HENRY

(*communicated by Brooke Shipley*)

Abstract

We extend all known results about transferred model structures on algebraically cofibrant and fibrant objects by working with weak model categories. We show that for an accessible weak model category there are always Quillen equivalent transferred weak model structures on both the categories of algebraically cofibrant and algebraically fibrant objects. Under additional assumptions, these transferred weak model structures are shown to be left, right or Quillen model structures. By combining both constructions, we show that each combinatorial weak model category is connected, via a chain of Quillen equivalences, to a combinatorial Quillen model category in which all objects are fibrant.

1. Introduction

Given a combinatorial Quillen model category $\mathcal{C}$, one can construct a Quillen equivalent combinatorial model category in which all objects are fibrant [**8, 25**]. This is achieved by considering the category T-**Alg** of *algebraically fibrant* objects in $\mathcal{C}$, which contains objects of $\mathcal{C}$ equipped with chosen lifts against all generating trivial cofibrations and morphisms preserving the chosen lifts. The category T-**Alg** is itself locally presentable and comes equipped with an adjunction:

$$F\colon \mathcal{C} \leftrightarrows \mathsf{T}\text{-}\mathbf{Alg}\colon U,$$

where U is the forgetful functor and F its left adjoint — the Quillen equivalent model structure on T-**Alg** is obtained by right transfer along U. In the original paper of Nikolaus [**25**], this was done under the mild assumption that every trivial cofibration in $\mathcal{C}$ is a monomorphism, but this assumption was later removed by the first named author in [**8**].

A dual version of this result was developed by Ching and Riehl in [**11**]. This uses the notion of algebraic weak factorisation system, which endows the cofibrant replacement construction with the structure of a comonad Q, whose coalgebras are

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the *algebraically cofibrant* objects. (The T of the previous example is the corresponding fibrant replacement monad.) Again, Q-**Coalg** is locally presentable and comes equipped with an adjunction

$$U\colon \mathsf{Q}\text{-}\mathbf{Coalg} \leftrightarrows \mathcal{C}\colon G,$$

where U is the forgetful functor and G its right adjoint, and one can try to left transfer the model structure along U. Ching and Riehl showed that if $\mathcal{C}$ is a combinatorial *simplicial* model category, and if Q is the simplicial comonad obtained by running the *enriched* algebraic small object argument, then the left transferred model structure exists and is Quillen equivalent to that on $\mathcal{C}$.

There are, however, some asymmetries in the above. Firstly, there exist combinatorial model structures for which Q-**Coalg** does not admit the left-transferred model structure — a simple example was described to us by Alexander Campbell [**10**] — whereas T-**Alg** always admits the right transferred model structure. We will see, however, that Q-**Coalg** is always a right semi-model category. On the other hand, even if $\mathcal{C}$ is only a combinatorial right semi-model category we will see that T-**Alg** is a genuine Quillen model category. This result captures, for instance, algebraic Kan complexes on *semi-simplicial sets* as well as Nikolaus' guiding example of algebraic Kan complexes on simplicial sets.

One of our goals in the present paper is to understand the dualisable aspects of the above results. Evidently, this should include results about semi-model categories. We will work using the framework of *weak model categories*, recently introduced by the second named author [**18, 19**], which includes both left and right semi-model categories. Roughly, weak model categories are like Quillen model categories except that some properties fail to hold when one considers maps that are not from a cofibrant to a fibrant object. However, as these are the only maps that are really homotopically meaningful in a model category, this weakening does not affect the homotopy theoretic properties of model categories in an essential way.

A key tool in our approach will be the left and right transfer of weak factorisation systems along adjunctions. To employ the existing transfer results of [**14, 20**] we will also assume that our categories are locally presentable and that the weak factorisation systems are accessible. This leads us to work in the setting of *accessible weak model categories*, which cover a very broad range of examples.

Two of our main results, Theorem 5.3 of Section 5 and Theorem 6.4 of Section 6, are dual. They respectively assert that an accessible weak model structure on $\mathcal{C}$ always left transfers to Q-**Coalg** and right transfers to T-**Alg**. If the base $\mathcal{C}$ is core left saturated — for instance, if $\mathcal{C}$ is a left or right semi-model category — the weak model structure on Q-**Coalg** is always a right semi-model category, with the dual applying to T-**Alg**. Finally, we identify, in Theorems' 5.6 and 6.5, stronger conditions — concerning lifting of cylinder and path objects respectively — that ensure Q-**Coalg** and T-**Alg** are genuine model categories. Using these results, we recover and extend the known results on model structures on algebraically cofibrant objects [**11**] and on algebraically fibrant objects [**8, 25**].

In Section 7 we apply the above to obtain the following rigidification results for weak model categories.

- Each accessible weak model category is connected, via a zigzag of Quillen equivalences, to an accessible right semi-model category in which all objects are cofibrant.
- Similarly, each accessible weak model category is connected, via a zigzag of Quillen equivalences, to an accessible left semi-model category in which all objects are fibrant.
- Moreover, each *combinatorial* weak model category is connected, via a zigzag of Quillen equivalences, to a combinatorial Quillen model category in which all objects are fibrant.

This last result enables one to transport known results about Quillen model structures to semi-model structures or weak model structures as long as these results can be transported along a Quillen equivalence. As an example of this, Lo Monaco has used our result in his recent preprint [**22**] to show that the ∞-category associated to any combinatorial weak model category via Dwyer–Kan localisation is a locally presentable ∞-category.

Another application of our results is to facilitate the construction of zig-zags of Quillen equivalences between various model categories. Generally speaking, model categories where all objects are cofibrant and model categories where all objects are fibrant have rather distinct properties, and it is often easier to compare two model structures of the former kind or two of the latter kind than two arbitrary model structures. For example, our results here are used in a key way in [**16**] where a variant of the model structure for semi-simplicial algebraic Kan complexes from Example 6.8 is used as an intermediate step in a zig-zag between a model structure for n-groupoids (whose objects are all fibrant) and the semi-simplicial model structure (whose objects are all cofibrant).

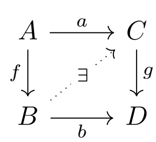
Finally, let us mention that a key tool in the proof of all of these results are new transfer theorems — see Theorem 4.1 and the dual Theorem 4.4 — of independent interest. These give necessary and sufficient conditions for the existence of a transferred weak model structure Quillen equivalent to the original one.

2. Preliminaries on weak model categories

In the present section we review the necessary background on weak model categories from [**18**] and [**19**].

2.1. Weak factorisation systems, premodel categories and Quillen model categories

Let us begin by recalling the well known concept of a weak factorisation system. Given morphisms $f\colon A \to B$ and $g\colon C \to D$ in a category $\mathcal{C}$ we say that f has the *left lifting property* with respect to g if given a commutative square as on the outside of the diagram below



there exists a diagonal morphism making both triangles commute. We say that g has the *right lifting property* with respect to f.

Given a class J of morphisms in $\mathcal{C}$ we sometimes write $LLP(J)$ for the class of morphisms having the left lifting property with respect to each morphism in J and $RLP(J)$ for those having the right lifting property to each morphism of J.

A weak factorisation system $(\mathcal{L}, \mathcal{R})$ on a category $\mathcal{C}$ consists of two classes of maps $\mathcal{L}$ and $\mathcal{R}$ such that

- each morphism $f \in \mathcal{C}$ can be factored as $f = me$ where $e \in \mathcal{L}$ and $m \in \mathcal{R}$;
- $\mathcal{L} = LLP(\mathcal{R})$ and $\mathcal{R} = RLP(\mathcal{L})$.

Definition 2.1. A *premodel category* is a category $\mathcal{C}$ such that:

1. $\mathcal{C}$ is equipped with two weak factorisation systems (*cofibration*/*anodyne fibration*) and (*anodyne cofibration*/*fibration*);
2. Each anodyne cofibration is a cofibration, or equivalently, each anodyne fibration is a fibration;
3. $\mathcal{C}$ is both cocomplete and complete.

The notion of premodel category was introduced independently in [**2**] and [**18**].

Definition 2.2. A Quillen model category consists of a premodel category $\mathcal{C}$ together with a class of morphisms W, called *weak equivalences*, such that

1. W contains the isomorphisms and satisfies 2-out-of-3.
2. A cofibration is an anodyne cofibration if and only if it is a weak equivalence.
3. A fibration is an anodyne fibration if and only if it is a weak equivalence.

2.2. Left and right semi-model categories

Next we turn to left and right semi-model categories, which slightly generalise model categories by restricting certain of the above axioms to core cofibrations and core fibrations respectively. Let us recall the definitions of these two classes of morphisms now.

Notation 2.3. In a premodel category $\mathcal{C}$:

- A *core cofibration* is a cofibration between cofibrant objects. Dually, a *core fibration* is a fibration between fibrant objects.
- A cofibration is said to be *acyclic* if it has the left lifting property against all core fibrations. In particular each anodyne cofibration is acyclic. The acyclic fibrations are defined dually and, as before, we note that each anodyne fibration is acyclic.

The original definitions of left and right semi-model category [**3, 31**] do not require the existence of two fully formed weak factorisation systems. Since we are only interested in the case where such factorisations do exist, we give the following simplified definition. Note that the definition, in either case, simply weakens one of the 3 axioms for a model category.

Definition 2.4. A left semi-model category is a premodel category $\mathcal{C}$ equipped with a class of weak equivalences W such that

1. W contains the isomorphisms and satisfies 2-out-of-3.

2*. A core cofibration is an anodyne cofibration if and only if it is a weak equivalence.

3. A fibration is an anodyne fibration if and only if it is a weak equivalence.

A right semi-model category is a left-semi model category structure on $\mathcal{C}^{op}$ — thus a Quillen model category except that (3) above is weakened to (3*), which asserts that a core fibration is an anodyne fibration if and only if it is a weak equivalence.

Examples 2.5.

(i) The left Bousfield localisation of a combinatorial model category at a set of maps is known to exist, as a model category, only when the original model category is simplicial and left proper [**3**]. On the other hand, it always produces a left semi-model category — see Theorem 4.2 of [**5**].

(ii) The original motivation for introducing left semi-model structures in [**31**] was that for a monoidal model category $\mathcal{C}$, the category of $\mathcal{C}$-operads (and also the category of algebras in $\mathcal{C}$ for a given Σ-cofibrant operad) in general has only a left semi-model structure. These are known to be Quillen model structures only when certain stronger assumptions such as the so-called "monoid axiom" hold. Indeed, it is shown in [**4**, Example 2.9] that if $\mathcal{C} = \mathrm{Ch}(\mathbb{F}_2)$ is the monoidal category of chain complexes over $\mathbb{F}_2$ (or more generally over $\mathbb{F}_p$) with its projective model structure, then the transferred left semi-model structure on the category of operads in $\mathcal{C}$ is not a Quillen model category.

(iii) A simple example of a right semi-model structure, to be used again in Example 5.5, is the right semi-model structure on the category **Gph** of directed graphs constructed in Section 4.1 of [**19**]. Here the cofibrations are the monomorphisms whilst the weak equivalences are those maps $f\colon X \to Y$ inducing a bijection $\Pi_0(X) \to \Pi_0(Y)$ on path components. The fibrations are the maps that have the right lifting property against all cofibrations that are also weak equivalences. Furthermore, one can show that the fibrant objects are the setoids and the fibrations between fibrant objects are the maps that satisfy the "edge lifting property", i.e. those $f\colon X \to Y$ such that any edge $y \to f(x)$ in Y is the image by f of an edge $y_0 \to x$ in X.
To see that the right semi model structure is not a Quillen model structure, let $\bullet$ denote the graph with a single object but no edges. Clearly, the codiagonal $\bullet \coprod \bullet \to \bullet$ is not a weak equivalence but it is, in fact, an anodyne fibration. Indeed, in any square of the form:

$$\begin{array}{ccc} X & \longrightarrow & \bullet \coprod \bullet \\ {\scriptstyle i}\big\downarrow & & \big\downarrow \\ Y & \longrightarrow & \bullet \end{array}$$

neither graph X nor Y can have an edge — therefore i is just an injection of sets, so that there is a diagonal filler as in the category of sets, where injections have the left lifting property with respect to surjections.

(iv) A higher-dimensional version of the preceding example is the right semi-model category **ssSet** of semi-simplicial sets. This was first described as a weak model category in [19], and complete details of the right semi-model structure are in [30]. A semi-simplicial set is a presheaf on the category Δ_+ of finite non-empty ordinals and *injective* order preserving maps. Its cofibrations are the monomorphisms, whilst a map $f\colon X \to Y$ is a weak equivalence if it induces a homotopy equivalence between the geometric realisation of X and Y. The fibrations are the maps with the right lifting property against all cofibrations that are also weak equivalences. Furthermore, the fibrant objects and fibrations between fibrant objects can be characterised by the Kan lifting conditions exactly as for maps of simplicial sets.

As in the case of **Gph**, it is not a Quillen model category. Indeed, if $\Delta_+[0]$ denotes the semi-simplicial set with just a single cell in dimension 0 and no higher-dimensional cells, then the map $\Delta_+[0] \coprod \Delta_+[0] \to \Delta_+[0]$ is an anodyne fibration but not a weak equivalence.

Remark 2.6. A left semi-model category in which each object is cofibrant is a Quillen model category, whilst a right semi-model category in which each object is fibrant is a Quillen model category.

2.3. Weak model categories

We now turn to weak model categories, which include both left and right semi-model categories. Rather than involving weak equivalences, the usual definition involves cylinder and path objects.

In a premodel category, a relative cylinder object for a cofibration $A \hookrightarrow B$ is a factorisation:

$$B \coprod_A B \overset{i}{\hookrightarrow} I_A B \to B$$

of the codiagonal map, such that i is a cofibration and the composite $B \hookrightarrow B \coprod_A B \hookrightarrow I_A B$ with the first pushout coprojection is an acyclic cofibration.[1]

Dually, a relative path object for a fibration $Y \twoheadrightarrow X$ is a factorisation:

$$Y \to P_X Y \overset{p}{\twoheadrightarrow} Y \times_X Y$$

of the diagonal map such that p is a fibration and the composite $P_X Y \twoheadrightarrow Y \times_X Y \twoheadrightarrow Y$ with the first pullback projection is an acyclic fibration.

Definition 2.7. A (factorisation[2]) weak model category is a premodel category which satisfies the following conditions:

- *Cylinder object axiom* — Each cofibration $A \hookrightarrow B$ with A cofibrant and B fibrant admits a relative cylinder object.
- *Path object axiom* — Each fibration $Y \twoheadrightarrow X$ with Y cofibrant and X fibrant admits a relative path object.

[1] It does not matter whether one takes the first or second coprojection; the axioms of a weak model category ultimately imply that both are acyclic cofibrations — see [19].

[2] The word "factorisation" refers to the fact that the notion of weak model category in [19] is slightly more general. There, the existence of the weak factorisation systems is also weakened. Here we work in the framework of accessible and combinatorial weak model categories from [18] in which these are always present.

Remark 2.8. Because the cylinder and path object axioms are restricted to maps from cofibrant to fibrant objects, they only imply the existence of cylinder objects and path objects for bifibrant objects. Indeed, a cylinder object $X \coprod X \hookrightarrow IX \to X$ for a cofibrant object X is a relative cylinder object for the cofibration $\emptyset \hookrightarrow X$. Hence we need X to be both fibrant and cofibrant in order to obtain a cylinder object from the cylinder object axiom as stated.

Although for a general core cofibration $A \hookrightarrow B$ we cannot construct a relative cylinder object, it was observed in [**19**] (see Definition 2.1.12 and Remark 2.1.13) that we can construct a weakening of it called a "weak relative cylinder object", which is still sufficient to define the homotopy relation. A weak relative cylinder object is a diagram of the form:

$$\begin{array}{ccc} B \coprod_A B & \hookrightarrow & I_A B \\ \downarrow & & \downarrow \\ B & \overset{\sim}{\hookrightarrow} & D_A B \end{array}$$

in which the first of the two maps $B \hookrightarrow I_A B$ is an acyclic cofibration. For instance $D_A B$ might be a fibrant replacement of B. Note that if B is fibrant then the acyclic cofibration $B \overset{\sim}{\hookrightarrow} D_A B$ has a section so that one recovers a "strong" cylinder object.

The general theory of weak model categories can be found in [**19**], but we recommend the reader to start with the preliminary section of [**18**] for a shorter overview of the basic theory. Below we highlight a few of basic facts.

- Using the path (or, equivalently, cylinder objects) one can define the *homotopy relation* for maps from cofibrant to fibrant objects. This is an equivalence relation having the usual properties. In particular, one can construct the homotopy category $Ho(\mathcal{C})$ of $\mathcal{C}$ by taking homotopy classes of maps between bifibrant objects.
- One can show that $Ho(\mathcal{C})$ is equivalent to the localisation of the full subcategory $\mathcal{C}_{c\cup f}$ of $\mathcal{C}$ of objects that are either fibrant or cofibrant at the union of the classes of core acyclic cofibrations and core acyclic fibrations. A weak equivalence is defined to a morphism that is inverted by $\mathcal{C}_{c\cup f} \to Ho(\mathcal{C})$. Of course, the notion of weak equivalence therefore makes sense only for arrows between objects that are either fibrant or cofibrant.

Example 2.9. One motivation for introducing weak model categories is that they provide a self-dual axiomatic framework that includes both left and right semi-model categories as special cases. Moreover, it is often much easier to verify the weak model category axioms than those for left and right semi-model categories.

Examples of weak model structures which are neither left nor right semi-model structures are the weak model structures on regular or non-unital strict ∞-categories constructed in [**17**]. These fail to be left semi-model categories by the same argument as for graphs and semi-simplicial sets described above. Moreover, they fail to be right semi-model categories as they are not "left saturated" (see subsection 2.4 and Proposition 2.13 below).

However, the left saturation issue can be easily fixed by taking the left saturation construction of [18], but we do not know whether or not the resulting weak model categories underlie right semi-model categories.

Remark 2.10. The present paper will not be self-contained, relying on a number of results from [18, 19]. For the purposes of readability, we now list the various results about weak model categories that we need. Each of these has a dual version.

(i) In a weak model category $\mathcal{C}$ a core cofibration is acyclic if and only if it is a weak equivalence. This is Proposition 2.2.9(iv) of [19].

(ii) Since weak equivalences satisfy 2-out-of-3, it follows that the acyclic cofibrations satisfy 2-out-of-3 amongst core cofibrations.

(iii) Conversely, Proposition 2.3.3 of [19] asserts that if a premodel category $\mathcal{C}$ satisfies the cylinder object axiom and the acyclic cofibrations between bifibrant objects satisfy the right cancellation property — given core cofibrations $j\colon A \hookrightarrow B$ and $i\colon B \hookrightarrow C$ such that i and $i \circ j$ are acyclic, then i is acyclic too. — then $\mathcal{C}$ is a weak model category.

(iv) The third item above is useful in recognising weak model structures, but the condition there on relative cylinder objects can in fact be further relaxed. Indeed, to construct relative cylinder objects it is enough that:

- given a core cofibration $A \hookrightarrow B$ with B fibrant there exists a cofibration $B \coprod_A B \hookrightarrow I_A B$ such that both maps $B \hookrightarrow I_A B$ are acyclic cofibrations.

The point here is that there is a "trick" to construct a weak relative cylinder object for $A \hookrightarrow B$ by considering the pushout $I_A B \coprod_B I_A B$, and from this one can deduce the cylinder object axiom. (See Lemma 2.3.6 and Remark 2.3.7 of [19].)

2.4. Saturation conditions and semi-model categories as weak model categories

A premodel category is said to be

- *left saturated* if the class of anodyne cofibrations and acyclic cofibrations coincide.
- *core left saturated* if the classes of anodyne cofibrations and acyclic cofibrations between cofibrant objects coincide.
- *right saturated* or *core right saturated* if the dual conditions hold.

In Section 4 of [18] it is shown that any accessible premodel category can be made left and right saturated without changing its classes of core cofibrations and core fibrations, so that one can always assume these conditions are satisfied without affecting the homotopy theoretic information.

Remark 2.11. Let $\mathcal{C}$ be a premodel category.

(i) If every object in $\mathcal{C}$ is fibrant then the class of acyclic and anodyne cofibrations coincide — that is, the premodel category is left saturated.

(ii) More generally, each acyclic cofibration in $\mathcal{C}$ with fibrant target is an anodyne cofibration — this is established in Lemma 2.17 of [18].

Remark 2.12. In relating semi-model categories and weak model categories, the obvious distinction is that the former involve a class of weak equivalences whereas the latter do not.

However, the axioms for a (left) semi-model category ensure that the class of weak equivalences above is uniquely determined — it is the 2-out-of-3 closure of the anodyne fibrations and core anodyne cofibrations. The dual remark applies to right semi-model categories.

In particular, being a left or right semi-model category (or indeed a Quillen model category) is a property of a premodel category rather than additional structure.

Proposition 2.13. *Amongst premodel categories, the left semi-model categories are precisely those factorisation weak model categories which are right saturated, core left saturated and for which each cofibrant object X admits a cylinder object $X \coprod X \hookrightarrow IX \overset{\sim}{\twoheadrightarrow} X$. The dual properties characterise right semi-model categories as weak model categories.*

Proof. See Section 3 of [18]. □

2.5. Accessible weak factorisation systems and weak model categories

A premodel category (or a weak model category) is said to be accessible (resp. combinatorial) if its underlying category is locally presentable and its weak factorisation systems are accessible (resp. cofibrantly generated). The notion of cofibrantly generated weak factorisation system is well known, but let us remind the reader what accessibility means in this context.

To this end, recall that a *functorial factorisation* on a category $\mathcal{C}$ is a functor

$$\mathcal{C}^{\mathbf{2}} \xrightarrow{(L,E,R)} \mathcal{C}^{\mathbf{3}} : \qquad X \xrightarrow{f} Y \qquad \longmapsto \qquad X \xrightarrow{Lf} Ef \xrightarrow{Rf} Y \quad (2.1)$$

from the category of arrows to that of composable pairs which is a section of the composition functor $\mathcal{C}^{\mathbf{3}} \to \mathcal{C}^{\mathbf{2}}$.

We say that a weak factorisation system admits a functorial factorisation if such a functor exists for which the morphisms Lf and Rf always belong to the left and right class of the given weak factorisation system — furthermore, if $\mathcal{C}$ is locally presentable and the functorial factorisation $(L, E, R)\colon \mathcal{C}^{\mathbf{2}} \to \mathcal{C}^{\mathbf{3}}$ can be chosen so as to be an *accessible functor*, then we say that the weak factorisation system is accessible.

2.6. Quillen equivalences between weak model categories

A *Quillen adjunction* between premodel categories is an adjunction:

$$F\colon \mathcal{C} \leftrightarrows \mathcal{A}\colon U$$

such that F preserves cofibrations and U preserves fibrations. By adjointness, these conditions amount to asking that F preserves cofibrations and anodyne cofibrations, or equally that U preserves fibrations and anodyne fibrations. Note that since F then preserves core cofibrations and U preserves core fibrations it follows also — see Lemma 2.22 of [18] of — that F preserves acyclic cofibrations and that U preserves acyclic fibrations.

Now, by Proposition 2.4.3 of [**19**], if the Quillen adjunction

$$F\colon \mathcal{C} \leftrightarrows \mathcal{A}\colon U \tag{2.2}$$

is between weak model categories then it induces an adjunction

$$Ho(F)\colon Ho(\mathcal{C}) \leftrightarrows Ho(\mathcal{A})\colon Ho(U) \tag{2.3}$$

between homotopy categories.

Definition 2.14. A Quillen adjunction between weak model categories as in (2.2) is said to be a *Quillen equivalence* just when the adjunction (2.3) between homotopy categories is an equivalence of categories.

Remark 2.15. The following are a few characterisations of Quillen equivalences that we will need later, given in Proposition 2.4.5 of [**19**].

- (i) For $X \in \mathcal{C}$ cofibrant and $Y \in \mathcal{A}$ fibrant a map $FX \to Y$ is a weak equivalence if and only if its adjoint $X \to UY$ is.
- (ii) At $Y \in \mathcal{A}$ fibrant the derived counit $FQUY \to FUY \to Y$ is a weak equivalence where $QUY \to UY$ is a cofibrant replacement and F reflects weak equivalences between cofibrant objects.
- (iii) As in (ii) but asking only that F reflects weak equivalences between bifibrant objects.

3. Preliminaries on algebraic weak factorisation systems

We now describe the necessary background on algebraic weak factorisation systems — see, for instance [**7, 13, 15, 27**]. This background material is mostly a summary of material in Section 2 of [**7**], to which we refer the reader for a more detailed account.

3.1. Algebraic weak factorisation systems

A functorial factorisation $(L, E, R)\colon \mathcal{C}^{\mathbf{2}} \to \mathcal{C}^{\mathbf{3}}$ as in (2.1) induces endofunctors $L, R\colon \mathcal{C}^{\mathbf{2}} \to \mathcal{C}^{\mathbf{2}}$ of the arrow category and natural transformations $\epsilon\colon L \to 1$ and $\eta\colon 1 \to R$ with components as below.

$$\begin{array}{ccc} A & \xrightarrow{1} & A \\ {\scriptstyle Lf}\downarrow & & \downarrow{\scriptstyle f} \\ Ef & \xrightarrow{Rf} & B \end{array} \qquad\qquad \begin{array}{ccc} A & \xrightarrow{Lf} & Ef \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Rf} \\ B & \xrightarrow{1} & B\ . \end{array}$$

An algebraic weak factorisation system consists of a comonad (L, ϵ, Δ) and monad (R, η, μ) extending the above, together with a distributive law relating L and R whose details will not be required.

Associated to the algebraic weak factorisation system (L, R) we have the categories L-**Coalg** and R-**Alg** of coalgebras for the comonad L and algebras for the monad R — these are thought of as the left and right classes of the AWFS. We denote a coalgebra by $\boldsymbol{f} = (f, s)\colon A \to B$ where $f\colon A \to B$ is the underlying morphism in $\mathcal{C}$ and $s\colon f \to Lf$ the structure map for the coalgebra; likewise, we write $\boldsymbol{g} = (g, r)\colon A \to B$ for an R-algebra where $r\colon Rg \to g$ is the structure map.

3.2. Double categorical aspects

A morphism in L-**Coalg** is a commutative square as below which preserves the L-coalgebra structure.

$$\begin{array}{ccc} A & \xrightarrow{a} & C \\ {\scriptstyle \boldsymbol{f}=(f,s)}\downarrow & & \downarrow{\scriptstyle \boldsymbol{g}=(g,r)} \\ C & \xrightarrow{b} & D. \end{array}$$

Composition in L-**Coalg** witnesses that these squares can be composed horizontally. There is also a vertical composition operation — given L-coalgebras $\boldsymbol{f}\colon A \to B$ and $\boldsymbol{g}\colon B \to C$ the composite morphism $g \circ f$ in $\mathcal{C}$ admits an extension to a coalgebra $\boldsymbol{g} \cdot \boldsymbol{f}\colon A \to C$. This composition is associative and unital when the identity $1_A\colon A \to A$ is equipped with its unique coalgebra structure $\mathbf{1_A}\colon A \to A$. This *vertical composition* respects L-coalgebra homomorphisms and, in total, makes L-**Coalg** into a *double category* whose objects and horizontal morphisms are as in $\mathcal{C}$, whose vertical morphisms are the L-coalgebras and whose squares are the L-coalgebra morphisms. This double categorical structure is used in the proof of Proposition 5.1.

3.3. Algebraic weak factorisation systems versus weak factorisation systems

If f admits the structure of an L-coalgebra and g admits the structure of an R-algebra, then f has the left lifting property with respect to g. It follows that each algebraic weak factorisation has an *underlying weak factorisation system*, whose left and right classes consist of the retract closures of the L-coalgebras and R-algebras respectively.

In the case that a weak factorisation system underlies an algebraic weak factorisation system, we call the algebraic weak factorisation system an *algebraic realisation* of the weak factorisation system.

3.4. Accessibility of algebraic weak factorisation systems

In the case that $\mathcal{C}$ is a locally presentable category the algebraic weak factorisation is said to be *accessible* if its functorial factorisation (L, E, R) is accessible. In fact, each accessible weak factorisation system admits an accessible algebraic realisation — see Proposition 3.5 of [**14**].

3.5. Algebraic cofibrant and fibrant replacement

The counit of the comonad L has component $\epsilon_f = (1_A, Rf)\colon Lf \to f$ — in particular, has identity domain component 1_A. It follows that the comultiplication Δ_f has the same property whence the comonad restricts to a comonad on each coslice $A/\mathcal{C}$. If $\mathcal{C}$ has an initial object $\emptyset$, we thereby obtain a comonad Q on $\emptyset/\mathcal{C} \cong \mathcal{C}$ which is often called the *cofibrant replacement comonad*. Dually, if $\mathcal{C}$ has a terminal object 1 then the monad R restricts to a monad T on the slice $\mathcal{C}/1 \cong \mathcal{C}$, often called the *fibrant replacement monad*.

3.6. Cofibrant generation

A set J of morphisms in a locally presentable category $\mathcal{C}$ cofibrantly (or freely) generates an accessible algebraic weak factorisation system (L, R). An R-algebra consists of a morphism $g\colon X \to Y$ together with a lifting function ϕ, which assigns to each commutative square as on the outside left below

$$\begin{array}{ccc} A & \xrightarrow{a} & X \\ {\scriptstyle j\in J}\big\downarrow & \phi(j,a,b) \nearrow & \big\downarrow{\scriptstyle g} \\ B & \xrightarrow[b]{} & Y \end{array} \qquad\qquad \begin{array}{ccc} A & \xrightarrow{a} & X \\ {\scriptstyle j\in J}\big\downarrow & \nearrow {\scriptstyle \boldsymbol{x}(j,a)} & \\ B & & \end{array}$$

a chosen diagonal filler, as depicted. The morphisms of R-**Alg** are commutative squares which preserve the chosen fillers.

The algebraically fibrant objects — that is, the T-algebras — are the *algebraically J-fibrant objects*: these are objects $X \in \mathcal{C}$ together with a lifting function $\boldsymbol{x}$, which assigns to the solid part of each diagram as on the right above a filler. Again morphisms of T-**Alg** preserve the fillers. Of course, the category of algebraically J-fibrant objects makes sense for any set J and category $\mathcal{C}$, regardless of whether the cofibrantly generated AWFS exists, and we denote it by **J-Fib**. In particular when (L, R) does exist, we have T-**Alg** $\cong$ **J-Fib**.

4. The equivalence transfer theorem

The main aim of the present section is to prove Theorem 4.1, which captures precisely when a weak model category structure can be transferred to a Quillen equivalent weak model category along a Quillen adjunction between premodel categories. We then establish a minor variant of this result, Proposition 4.3, which we will use in Section 5 to transfer a weak model structure to the category of algebraically cofibrant objects. The dual result will be used in Section 6 to handle the category of algebraically fibrant objects.

Theorem 4.1. *Let $\mathcal{C}$ and $\mathcal{A}$ be two premodel categories together with a Quillen adjunction:*

$$F\colon \mathcal{C} \leftrightarrows \mathcal{A}\colon U$$

and suppose that $\mathcal{A}$ is a weak model category. Then $\mathcal{C}$ is a weak model category such that the above adjunction is a Quillen equivalence if and only if the following two conditions are satisfied:

(i) Given a cofibrant object $X \in \mathcal{C}$, a fibrant object $Y \in \mathcal{A}$ and a morphism:

$$FX \xrightarrow{i} Y$$

there exists a cofibration $j\colon X \hookrightarrow Z$ in $\mathcal{C}$ and a factorisation:

$$\begin{array}{ccc} FX & \xrightarrow{i} & Y \\ {\scriptstyle Fj}\big\downarrow & \overset{w}{\nearrow}{\scriptstyle \sim} & \\ FZ & & \end{array}$$

such that w is a weak equivalence.

(ii) If i is a core cofibration in $\mathcal{C}$ and Fi is an acyclic cofibration, then i is an acyclic cofibration.

Proof. We first prove that:

- *if $\mathcal{C}$ is a weak model category and (F, U) is a Quillen equivalence then (i) and (ii) above are satisfied.*

To prove (i) consider a map $FX \to Y$ with X cofibrant and Y fibrant. It corresponds to a morphism $X \to UY$, which we can factor as a cofibration followed by an anodyne fibration:

$$X \hookrightarrow Z \overset{\sim}{\twoheadrightarrow} UY.$$

Since the anodyne fibration $Z \overset{\sim}{\twoheadrightarrow} UY$ is an acyclic fibration between fibrant objects and $\mathcal{C}$ is a weak model category, it is a weak equivalence by the dual of Remark 2.10(i). As (F, U) is a Quillen equivalence, Z is cofibrant and Y is fibrant it follows from Remark 2.15(i) that its adjoint map $FZ \to Y$ is a weak equivalence in $\mathcal{A}$, which gives the desired factorisation establishing (i).

With regards (ii), observe that by Remark 2.15(ii) the left Quillen equivalence F detects weak equivalences between cofibrant objects. Therefore if i is a core cofibration such that Fi is an acyclic cofibration (and in particular a weak equivalence) it follows that i is a weak equivalence, and so an acyclic cofibration as in Remark 2.10(ii). This proves (ii).

Conversely suppose that (i) and (ii) hold. We begin by showing that:

- *$\mathcal{C}$ is also a weak model category.*

To this end, first suppose that in $\mathcal{C}$ one has core cofibrations $A \overset{j}{\hookrightarrow} B \overset{i}{\hookrightarrow} C$ such that j and $i \circ j$ are acyclic. We will show that i is acyclic too. Indeed, since F is left Quillen both Fj and $F(i \circ j)$ are acyclic cofibrations so that, since $\mathcal{A}$ is a weak model category, Fi is an acyclic cofibration too. Therefore, by (ii) of Theorem 4.1, i is acyclic.

In order to show that $\mathcal{C}$ is a weak model category, it suffices by Remark 2.10(iv) to show that given a core cofibration $A \hookrightarrow B$, one can construct a cofibration $B \coprod_A B \hookrightarrow I_A B$ such that both maps $B \hookrightarrow I_A B$ are acyclic.

To this end, consider the codiagonal map $B \coprod_A B \to B$, and let $FB \overset{\sim}{\hookrightarrow} Y \twoheadrightarrow 1$ be a fibrant replacement of FB in $\mathcal{A}$. Applying the factorisation of (i) to the map $F(B \coprod_A B) \to Y$ we obtain a cofibration $B \coprod_A B \hookrightarrow I_A B$ in $\mathcal{C}$ and a weak equivalence $F(I_A B) \to Y$ in $\mathcal{A}$ factoring the map above. In particular, by 2-out-of-3 the maps $F(B) \to F(I_A B)$ are weak equivalences, hence acyclic cofibrations, so that by (ii) the maps $B \hookrightarrow I_A B$ are acyclic cofibrations.

Finally, we show that

- *(F, U) is a Quillen equivalence between $\mathcal{C}$ and $\mathcal{A}$.*

Indeed, let Y be a fibrant object in $\mathcal{A}$ and $u \colon X \overset{\sim}{\twoheadrightarrow} UY$ be a cofibrant replacement of UY in $\mathcal{C}$. The derived counit of the adjunction $F \dashv U$ at Y is then given by the adjoint map $\overline{u} \colon FX \to Y$ — we first show this to be a weak equivalence in $\mathcal{A}$. Indeed using (i) we can factor it as:

$$FX \overset{Fi}{\hookrightarrow} FZ \underset{\sim}{\overset{w}{\to}} Y.$$

By adjointness we obtain a commutative diagram in $\mathcal{C}$ as on the outside of the square below

$$\begin{array}{ccc} X & = & X \\ {\scriptstyle i}\big\downarrow & \overset{t}{\nearrow} & \big\downarrow{\scriptstyle \overline{u}} \\ Z & \xrightarrow{w^*} & UY, \end{array}$$

where w^* denotes the adjoint transpose of w. Since i is a cofibration and u an anodyne fibration we obtain a diagonal filler t as above. Thus u fits into a retract diagram as below left

$$\begin{array}{ccccc} X & \xrightarrow{i} & Z & \xrightarrow{t} & X \\ {\scriptstyle u}\big\downarrow & & \big\downarrow{\scriptstyle w^*} & & \big\downarrow{\scriptstyle u} \\ UY & = & UY & = & UY \end{array} \qquad \begin{array}{ccccc} FX & \xrightarrow{Fi} & FZ & \xrightarrow{Ft} & FX \\ {\scriptstyle \overline{u}}\big\downarrow & & \big\downarrow{\scriptstyle w} & & \big\downarrow{\scriptstyle \overline{u}} \\ Y & = & Y & = & Y \end{array}$$

so that, by adjointness, we obtain a retract diagram as above right. Then $\overline{u}$ is a retract of the weak equivalence w, and so a weak equivalence itself.

To conclude the proof that (F, U) is a Quillen equivalence it suffices, by Remark 2.15(iii), to show that F detects weak equivalences between bifibrant objects. Thus let $f\colon X \to Y$ be a map between bifibrant objects such that Ff is a weak equivalence, and consider a (cofibration/anodyne fibration)-factorisation

$$X \overset{i}{\hookrightarrow} Z \underset{\sim}{\overset{w}{\to}} Y$$

of f. Then w a weak equivalence between cofibrant objects and so is sent by F to a weak equivalence, so that Fi is also a weak equivalence by 2-out-of-3, and hence is an acyclic cofibration. It therefore follows by our assumption (ii) that i is an acyclic cofibration between cofibrant objects and so a weak equivalence. Hence $f = w \circ i$ is a weak equivalence too. □

Remark 4.2. In practice, one typically starts with a weak model structure on $\mathcal{A}$, and an adjunction $F\colon \mathcal{C} \leftrightarrows \mathcal{A}\colon U$ which satisfies Condition (i) of Theorem 4.1 but with "a cofibration in $\mathcal{A}$" replaced by "a map i in $\mathcal{A}$ such that Fi is a cofibration". Given this, one then forms the left transferred weak factorisation systems on $\mathcal{A}$ — that is, such that the (anodyne) cofibrations in $\mathcal{A}$ are the preimage under F of the (anodyne) cofibrations in $\mathcal{C}$. Unfortunately, in order to apply Theorem 4.1 one requires that a map in $\mathcal{A}$ is an acyclic cofibration if and only if Fi is an acyclic cofibration, and this does not appear to be automatic in the situation above.

To avoid this problem there are two solutions. Firstly, one can assume that $\mathcal{A}$ is core left saturated — that is, all core acyclic cofibrations in $\mathcal{A}$ are anodyne cofibrations — or replace $\mathcal{A}$ by its core left saturation in which this property holds — see Section 4 of [**18**]. Alternatively, one can use the following modification of Theorem 4.1 in which Condition (ii) is restricted to anodyne cofibrations, but Condition (i) is strengthened with the assumption that FZ is fibrant. We will use this modification in Theorem 5.3, which establishes the weak model structure on algebraically cofibrant objects. In that case, the fact that FZ is fibrant is ensured by the construction of $FZ \to Y$ as an anodyne fibration.

Proposition 4.3. *Let $\mathcal{C}$ and $\mathcal{A}$ be two premodel categories with a Quillen adjunction:*

$$F\colon \mathcal{C} \leftrightarrows \mathcal{A} \colon U.$$

Suppose that $\mathcal{A}$ is a weak model category and that the adjunction satisfies the following two conditions:

(i) Given a cofibrant object $X \in \mathcal{C}$, a fibrant object $Y \in \mathcal{A}$ and a morphism:

$$FX \xrightarrow{i} Y$$

there exists a cofibration $j\colon X \hookrightarrow Z$ in $\mathcal{C}$ and a factorisation:

$$\begin{array}{ccc} FX & \xrightarrow{i} & Y \\ {\scriptstyle Fj}\big\downarrow & \nearrow_{\sim}^{w} & \\ FZ & & \end{array}$$

such that w is a weak equivalence and FZ is fibrant in $\mathcal{A}$.

(ii) If i is a core cofibration in $\mathcal{C}$ and Fi is an anodyne cofibration, then i is an acyclic cofibration.

Then $\mathcal{C}$ is a weak model category and (F, U) is a Quillen equivalence.

Proof. By Theorem 4.1 we need only show that if $i\colon X \hookrightarrow Y$ is a core cofibration in $\mathcal{C}$ such that Fi is acyclic then i is acyclic. To this end, consider a fibrant replacement:

$$FX \hookrightarrow FY \overset{\sim}{\hookrightarrow} Z.$$

By Condition (i) there is a cofibration $Y \hookrightarrow Z'$ in $\mathcal{C}$ and factorisation

$$FY \hookrightarrow FZ' \overset{\sim}{\to} Z$$

of the map to the fibrant replacement such that FZ' is fibrant and such that the map $FZ' \to Z$ is a weak equivalence. In particular the composite $X \hookrightarrow Y \hookrightarrow Z'$ is a core cofibration whose image by F is an acyclic cofibration with fibrant target. Hence its image under F is an anodyne cofibration (see Remark 2.11(ii)), so the map $X \hookrightarrow Z'$ is acyclic in $\mathcal{C}$ by assumption. A similar argument shows that the map $Y \hookrightarrow Z'$ is also acyclic in $\mathcal{C}$. Hence, by the elimination property of acyclic cofibrations (Remark 2.10.(ii)), the map $X \hookrightarrow Y$ is an acyclic cofibration too. □

We also state explicitly the dual version of Theorem 4.1:

Theorem 4.4. *Let $\mathcal{C}$ and $\mathcal{A}$ be two premodel categories together with a Quillen adjunction:*

$$F\colon \mathcal{C} \leftrightarrows \mathcal{A} \colon U$$

and suppose that $\mathcal{C}$ is a weak model category. Then $\mathcal{A}$ is a weak model category and the adjunction above is a Quillen equivalence if and only if the following two conditions are satisfied:

(i) Given a cofibrant object $X \in \mathcal{C}$, a fibrant object $Y \in \mathcal{A}$ and a morphism:

$$X \xrightarrow{i} UY$$

there exists a fibration $p\colon Z \twoheadrightarrow Y$ in $\mathcal{A}$ and a factorisation:

$$\begin{array}{ccc} & & UZ \\ & {}^{w}\nearrow^{\sim} & \downarrow\!\!\downarrow{\scriptstyle Up} \\ X & \xrightarrow[i]{} & UY \end{array}$$

such that w is a weak equivalence in $\mathcal{C}$.

(ii) If i is a core fibration in $\mathcal{A}$ and Ui is an acyclic fibration in $\mathcal{C}$, then i is an acyclic fibration in $\mathcal{A}$.

It is proved by applying Theorem 4.1 to the Quillen adjunction:

$$U^{op}\colon \mathcal{A}^{op} \leftrightarrows \mathcal{C}^{op}\colon F^{op}.$$

Remark 4.5. Dualizing Proposition 4.3, if we can further show that in Condition (i) the object Z can be chosen so that UZ is cofibrant in $\mathcal{C}$, then Condition (ii) can be weakened to the requirement that if i is a core fibration and Ui is an anodyne fibration then i is an acyclic fibration.

5. Model structures on algebraically cofibrant objects

In the present section, we prove our main result, Theorem 5.3, about model structures on algebraically cofibrant objects. This theorem builds on the following result.

Proposition 5.1. *Let $\mathcal{A}$ be an accessible weak model category endowed with an accessible algebraic factorisation system (L, R) such that if $f\colon X \to Y$ is an arrow in $\mathcal{A}$ from a cofibrant object to a fibrant object then it can be factored as a cofibration that admits an L-coalgebra structure followed by an acyclic fibration. Let Q be the comonad on $\mathcal{A}$ associated to (L, R). Then there is an accessible weak model structure on* Q-**Coalg** *such that:*

1. *A map in* Q-**Coalg** *is a cofibration or anodyne cofibration if and only if its underlying map in $\mathcal{A}$ is.*
2. *The adjunction:*

   $$U\colon \mathsf{Q}\text{-}\mathbf{Coalg} \leftrightarrows \mathcal{A}\colon G,$$

   where U is the forgetful functor is a Quillen equivalence.
3. *A map between cofibrant objects in* Q-**Coalg** *is a weak equivalence or an acyclic cofibration if and only if its underlying map in $\mathcal{A}$ is.*
4. *If $\mathcal{A}$ is combinatorial then so is* Q-**Coalg**.

Proof. Since the comonad L is accessible, its restriction Q is also accessible. Since Q is an accessible comonad on a locally presentable category, and the category of coalgebras for Q can be constructed using pie limits, it follows from the limit theorem of Makkai and Pare — Theorem 5.1.6 of [**23**] — that Q-**Coalg** is locally presentable. For a detailed proof of this claim, we refer the reader to Proposition A.1 of [**11**].

Since the forgetful functor U is a left adjoint between locally presentable categories, by Theorem 2.6 of [**14**], there exist accessible weak factorisation systems on Q-**Coalg** whose left classes consist of those maps f for which Uf is a cofibration or anodyne cofibration respectively. We thus have a Quillen adjunction of premodel categories for which (1) holds by definition. To prove that this a weak model structure on Q-**Coalg** satisfying (2) above, it therefore suffices to verify Condition (i) of Proposition 4.3.

To this end, let $(X, \boldsymbol{x})$ be a cofibrant Q-coalgebra — such is specified by a cofibrant object $X \in \mathcal{A}$ together with an L-coalgebra structure $\boldsymbol{x} \colon \emptyset \to X$. As in Section 3 we use boldface to indicate that $\boldsymbol{x}$ is a morphism $x \colon \emptyset \to X$ equipped with additional structure. Consider a morphism $f \colon X \to Y$ with Y fibrant in $\mathcal{A}$. By our assumptions, there exists a factorisation of f

$$X \xrightarrow{i} E \overset{p}{\twoheadrightarrow} Y$$

as a cofibration i admitting an L-coalgebra structure $\boldsymbol{i}$ followed by an acyclic fibration p. We then consider the composite

$$\begin{array}{ccc}
\emptyset & \xrightarrow{1} & \emptyset \\
{\scriptstyle \boldsymbol{x}}\downarrow & & \downarrow{\scriptstyle \boldsymbol{x}} \\
X & \xrightarrow{1} & X \\
{\scriptstyle \boldsymbol{1}}\downarrow & & \downarrow{\scriptstyle \boldsymbol{i}} \\
X & \xrightarrow[i]{} & E
\end{array}$$

in which the upper square is the identity L-coalgebra map. The lower square is an L-coalgebra morphism by the coalgebra dual of point (c) of Proposition 5 of [**7**]. Now since L-**Coalg** is, as described in Section 3.2, a double category, the composite square is a morphism of L-coalgebras $\boldsymbol{x} \to \boldsymbol{i} \circ \boldsymbol{x}$. This equips the fibrant object E with the structure of a Q-coalgebra such that $i \colon (X, \boldsymbol{x}) \to (E, \boldsymbol{i} \circ \boldsymbol{x})$ is a morphism of Q-coalgebras. In particular, it is a cofibration in Q-**Coalg** whose image under U is i, therefore verifying Condition (i) of Proposition 4.3, as required.

Then for (3) since U is a left Quillen equivalence it preserves and detects weak equivalences between cofibrant objects, as acyclic core cofibration are the same as core cofibration that are weak equivalences by Remark 2.10(i), this immediately implies the result.

Condition (4) holds by Remark 3.8 in [**24**], which proves that combinatorial weak factorisation systems between locally presentable categories can be left transferred along a left adjoint and, moreover, the transferred weak factorisation system is again combinatorial. □

Proposition 5.2. *Under the same assumptions as in Proposition 5.1, suppose further that $\mathcal{A}$ is core left saturated and that all objects admitting a Q-coalgebra structure are cofibrant. Then* Q-**Coalg** *is a right semi-model category with all objects cofibrant.*

Proof. If every object admitting a Q-coalgebra structure is cofibrant then every object in the weak model category Q-**Coalg** from Proposition 5.1 is cofibrant. In particular, every object of Q-**Coalg** admits a strong path object. Hence Proposition 2.13 implies that it is a right semi-model category if and only if it is left saturated and core right saturated. As every object of Q-**Coalg** is cofibrant it is automatically right saturated (see Remark 2.11(i)). For left saturation, we must show that each acyclic cofibration j in Q-**Coalg** is anodyne. By Proposition 5.1, U reflects the property of being an anodyne cofibration or an acyclic cofibration between cofibrant objects. Since all objects in Q-**Coalg** are cofibrant, it follows that we only we need to show that Uj is anodyne. But this follows since Uj is an acyclic cofibration between cofibrant objects and $\mathcal{A}$ is core left saturated. □

One can also phrase these two results in the following theorem:

Theorem 5.3. *Let $\mathcal{A}$ be an accessible weak model category endowed with (L, R) an accessible algebraic realisation of the (cofibration, anodyne fibration)-weak factorisation system. Let Q be the associated cofibrant replacement comonad on $\mathcal{A}$.*

Then there is an accessible weak model structure on Q-**Coalg** *such that:*

(i) A map in Q-**Coalg** *is a cofibration or anodyne cofibration if and only if its underlying map in $\mathcal{A}$ is.*
(ii) The adjunction:

$$U\colon \mathsf{Q}\text{-}\mathbf{Coalg} \leftrightarrows \mathcal{A}\colon G,$$

where U is the forgetful functor, is a Quillen equivalence.
(iii) Each object is cofibrant in Q-**Coalg**.
(iv) An arrow in Q-**Coalg** *is a weak equivalence or an acyclic cofibration if and only if its underlying map in $\mathcal{A}$ is.*
(v) If $\mathcal{A}$ is core left saturated then Q-**Coalg** *is an accessible right semi-model category.*
(vi) If $\mathcal{A}$ is combinatorial then so is Q-**Coalg**.

Proof. In this case, the (L, R)-factorisation of a map f produces a factorisation in a cofibration that is a L-coalgebra followed by an anodyne (hence acyclic) fibration. Hence applying Proposition 5.1 gives us points (i), (ii), (vi) and the restriction of (iv) to maps between cofibrant objects. Moreover, every L-coalgebra is a cofibration, so it follows that every object in Q-**Coalg** is cofibrant, this gives (iii) and (iv) for all objects. Finally, (v) is exactly Proposition 5.2. □

Remark 5.4. We can also formulate a version of Propositions 5.1 and 5.2 and Theorem 5.3 that does not involve an algebraic weak factorisation system but merely an accessible weak factorisation system, with a chosen functorial factorisation (L, R). The only interest in this is that it avoids the theory of algebraic weak factorisation systems. In this case L is not a comonad but merely a copointed endofunctor, whilst R is a pointed endofunctor. Nonetheless we can consider L-coalgebras (for the copointed endofunctor) and R-algebras respectively and we require either that every arrow from a cofibrant object to a fibrant object can be factored as in Proposition 5.1 or, more simply that the underlying weak factorisation system of (L, R) is the (cofibration, anodyne fibration) weak factorisation system as in Theorem 5.3.

Restricting L to the coslice under the initial object, we obtain the copointed endofunctor Q as before and the accessibility assumption ensures that the category Q-**Coalg** of coalgebras for the copointed endofunctor is locally presentable and that

$U\colon \mathsf{Q}\text{-}\mathbf{Coalg} \to \mathcal{A}$ has a left adjoint. We claim that the conclusions of Propositions 5.1 and 5.2 and Theorem 5.3 hold in this modified setting with largely the same proof. The main difference is that in the proof of (the analogue of) Proposition 5.1, given $(X, \boldsymbol{x})$ a cofibrant Q-coalgebra and $X \to Z \to Y$ a factorisation as a cofibration with an L-coalgebra structure followed by an acyclic fibration, we use the argument of Lemma 2.12 of [**14**] to put an L-coalgebra structure on $\emptyset \to Z$ that makes $X \to Z$ a morphism (and hence a cofibration) of Q-coalgebras instead of Proposition 5 of [**7**].

Example 5.5. In the present example we describe a combinatorial model structure on the category of small categories **Cat** for which the induced right model structure on the category of algebraically cofibrant objects — simply the category of graphs — is not a full Quillen model structure. This is a modification of an example that we learned from Alexander Campbell, and which will appear in [**10**].

To begin with, we equip **Cat** with the Quillen model structure in which:

- A functor $f\colon X \to Y$ is a weak equivalence if and only if the induced function $\pi_0(f)\colon \pi_0(X) \to \pi_0(Y)$ between sets of path components is a bijection.
- The cofibrations are generated by the two inclusions $J = \{\emptyset \to \bullet, (\bullet \quad \bullet) \to (\bullet \to \bullet)\}$. In particular the anodyne fibrations are functors that are full and surjective on objects.

In order to verify that this is a Quillen model structure, let us check that the two classes satisfy the conditions of Smith's theorem, Theorem 1.7 of [**6**]. Certainly each anodyne fibration is a weak equivalence. The latter class, being the preimage under $\Pi_0\colon \mathbf{Cat} \to \mathbf{Set}$ of the isomorphisms, satisfies 2-out-of-3 and is retract stable. Since Π_0 is cocontinuous, having right adjoint the discrete category functor, it follows that the weak equivalences are stable under pushout and, moreover, accessibly embedded in the category of morphisms $\mathbf{Cat}^{\mathbf{2}}$. Since cofibrations are always pushout stable it follows that the intersection of the cofibrations and weak equivalences is also pushout stable. Therefore the conditions of Smith's theorem are met — the model structure exists.

Next consider the forgetful functor $U\colon \mathbf{Cat} \to \mathbf{Gph}$ to the category of graphs and F its left adjoint with $\eta\colon 1 \to UF$ and $\epsilon\colon FU \to 1$ denoting the unit and counit. We will prove that FU is the algebraic cofibrant replacement comonad on **Cat** induced by the set of generating cofibrations J. Letting (L, R) denote the cofibrantly generated algebraic weak factorisation system an R-map is specified by a functor $f\colon Y \to Z$ equipped with (1) for each $z \in Z$ an object $\phi_z \in Y$ with $f\phi_z = z$ and (2) for each $(y, \alpha\colon fy \to fz, z)$ a morphism $\phi_\alpha\colon y \to z$ such that $f\phi_\alpha = \alpha$. Observe that if f is bijective on objects, this amounts to giving a section of Uf in **Gph**. In particular, since $\epsilon\colon FUX \to X$ is the identity on objects the unit component $\eta_{UX}\colon UX \to UFUX$ equips it with the structure of an R-map. It is easy to say that this is the free R-map on $\emptyset \to X$, in the sense that if $f\colon Y \to Z$ is an R-map and $g\colon Y \to Z$ arbitrary, then there exists a unique morphism $\overline{g}\colon FUX \to Y$ making the square below left

$$\begin{array}{ccc} FUX & \xrightarrow{\overline{g}} & Y \\ {\scriptstyle \epsilon_X}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{g} & Z \end{array} \qquad\qquad \begin{array}{ccc} FUX & \xrightarrow{\Delta_X} & FUFUX \\ {\scriptstyle \epsilon_X}\downarrow & & \downarrow{\scriptstyle \epsilon_X \circ \epsilon_{FUX}} \\ X & \xrightarrow{1} & X \end{array}$$

into a morphism of R-maps. It then follows, by definition of the algebraic cofibrant replacement comonad, that its underlying copointed endofunctor is $\epsilon\colon FU \to 1$. By Equation 2.11 of [**7**] the comultiplication $\Delta_X\colon FUX \to FUFUX$ has the property that the square above right is an R-map morphism where the right hand side uses the composition of R-maps. In our setting, this commutativity amounts to commutativity with sections at the level of underlying graphs — that is, to the equation

$$U\Delta_X \circ \eta_{UX} = \eta_{UFUX} \circ \eta_{UX}.$$

Now this equation is satisfied by $\Delta_X = F\eta_{UX}$ and, by the universal property of FUX, by $F\eta_{UX}$ alone. Therefore the cofibrant replacement comonad is given by $(FU, \epsilon, F\eta_U)$ as claimed.

Now since the monad $T = UF$ is parametric right adjoint — see, for instance Example 2.5 of [**32**] — it preserves all connected limits. Since U creates limits it follows that F preserves connected limits — in particular, coreflexive equalisers. As is well known, the monad T is also cartesian so that, in particular, its unit $1 \to UF$ is a cartesian natural transformation. Since the pullback of an isomorphism is an isomorphism, it follows that F reflects isomorphisms. Therefore, by the comonad dual of Beck's monadicity theorem, the adjunction $F \dashv U$ is comonadic and so **Gph** $\simeq$ FU-**Coalg** is the category of algebraically cofibrant objects.

Applying Theorem 5.3 we left transfer the model structure on **Cat** to a right semi-model structure on **Gph** in which all objects are cofibrant. Let us examine, firstly, its cofibrations and weak equivalences. Let $f\colon X \to Y \in$ **Gph**. To say that $Ff\colon FX \to FY$ induces a bijection on path components is equally to say that $f\colon X \to Y$ induces a bijection on path components of the graphs. By the simple nature of the generating cofibrations J in **Cat** it is also straightforward to see that $Ff\colon FX \to FY$ is a cofibration just when $f\colon X \to Y$ is a monomorphism — in fact, the J-cellular maps $g\colon FX \to FY$ are precisely those of the form $g = Ff$ for $f\colon X \to Y$ a monomorphism. Therefore, this is the right semi-model structure for setoids of Example 2.5(iii) and, in particular, it is not a Quillen model structure.

The following theorem refines Theorem 5.3, showing that under additional hypotheses, one obtains a genuine Quillen model structure on Q-**Coalg**.

Theorem 5.6. *Under the assumptions of Theorem 5.3 suppose further that $\mathcal{A}$ is a left semi-model category and that cylinder objects lift along $U\colon$ Q-**Coalg** $\to \mathcal{A}$ — in the sense that if $(X, \boldsymbol{x})$ is a Q-coalgebra, then there exists a cylinder object factorisation $X \coprod X \to IX \to X \in \mathcal{A}$ that lifts to a factorisation of $(X, \boldsymbol{x}) \coprod (X, \boldsymbol{x}) \to (X, \boldsymbol{x}) \in \mathcal{A}$. Then Q-**Coalg** is a Quillen model category.*

Proof. As $\mathcal{A}$ is a left semi-model category, its core acyclic cofibrations are anodyne cofibrations. Now by Theorem 5.3, U reflects anodyne and acyclic cofibrations. Since all objects are cofibrant in Q-**Coalg** it follows that it is left saturated — that is, its acyclic and anodyne cofibrations coincide. Since all objects are cofibrant it is automatically right saturated by Remark 2.11(i). Furthermore, since cylinder objects lift along U, Q-**Coalg** has cylinder objects for all cofibrant objects. Therefore, by Proposition 2.13, Q-**Coalg** is a left semi-model category. But a left semi-model category in which all objects are cofibrant is a Quillen model category, so this concludes the proof. ∎

In the main theorem of [**11**], Ching and Riehl use simplicial enrichment to lift the model structure on a combinatorial simplicial model category to its category of algebraically cofibrant objects. We now give a generalisation of this result with a somewhat different proof, since our argument uses liftings of cylinder objects as opposed to a bar construction.

To begin with, let us suppose that $\mathcal{V}$ is an accessible monoidal model category with cofibrant unit and that $\mathcal{A}$, to begin with, is a combinatorial left semi-model $\mathcal{V}$-category — here, by a left semi-model $\mathcal{V}$-category, we mean one in which the tensoring functor $-.-\colon \mathcal{V}\times\mathcal{A}\to\mathcal{A}$ is a left Quillen bifunctor in the usual sense [**21**].[3] We need, in this setting, to suppose further that for each object X of $\mathcal{V}$ the tensor functor $X\otimes -\colon \mathcal{A}\to\mathcal{A}$ preserves cofibrations. Then — as described in Corollary 13.2.4 of [**28**] — the generating set of cofibrations in $\mathcal{A}$ permits the enriched algebraic small object argument. In particular, one obtains an accessible enriched cofibrant replacement comonad Q on $\mathcal{A}$ so that Q-**Coalg** has the structure of a $\mathcal{V}$-category.

Theorem 5.7. *Let $\mathcal{V}$ be an accessible monoidal model category with cofibrant unit and $\mathcal{A}$ a combinatorial left semi-model $\mathcal{V}$-category such that for each object X of $\mathcal{V}$ the tensor functor $X\otimes -\colon \mathcal{A}\to\mathcal{A}$ preserves cofibrations. Let Q be the $\mathcal{V}$-enriched cofibrant replacement comonad on Q. Then the transferred weak model structure on* Q-**Coalg** *is in fact an enriched Quillen model structure and the Quillen equivalence with $\mathcal{A}$ an enriched Quillen equivalence.*

Proof. Since Q is an enriched comonad the category of Q-coalgebras is an enriched category and the adjunction Q-**Coalg** $\leftrightarrows \mathcal{A}$ an enriched adjunction. Now since U creates colimits, Q-**Coalg** has tensors by objects of $\mathcal{V}$ and these are preserved by U. Since U creates anodyne cofibrations and cofibrations as well as tensors, it follows that $-.-\colon \mathcal{V}\times$ Q-**Coalg** $\to$ Q-**Coalg** is a left Quillen bifunctor. By Theorem 5.6, it remains to check that cylinder objects lift along U.

Consider a (cofibration/anodyne fibration)-factorisation $I\coprod I\to J\to I$ in $\mathcal{V}$ where I is the monoidal unit. Since I is cofibrant the coproduct inclusion $I\to I+I$ is a cofibration, so that the composite $I\to I\coprod I\to J$ is too. By 2 from 3 it is therefore an anodyne cofibration, and so a cylinder object for I in the sense of Section 2.3. Now $X\in\mathcal{A}$ is cofibrant, then $-.X\colon \mathcal{V}\to\mathcal{A}$ is left Quillen, so that applying it to the above produces a cylinder object

$$X\coprod X \longrightarrow J.X \longrightarrow X$$

for X. Furthermore, if $(X,\boldsymbol{x})$ is a coalgebra, then since U creates both tensors and coproducts, the factorisation lifts to a factorisation of the codiagonal on $(X,\boldsymbol{x})$ as required. □

As an instance of the above, we obtain

Theorem 5.8 (Ching–Riehl [**11**])**.** *Let $\mathcal{A}$ be a combinatorial simplicial model category and Q the simplicially enriched cofibrant replacement comonad on $\mathcal{A}$. Then the transferred weak model structure on* Q-**Coalg** *is in fact a simplicial Quillen model structure and the Quillen equivalence with $\mathcal{A}$ an enriched Quillen equivalence.*

[3] In fact, for our purposes it suffices that the two properties for $-.-\colon \mathcal{V}\times\mathcal{A}$ to be a left Quillen bifunctor are required only for morphisms having cofibrant domains and codomains.

6. Model structures on algebraically fibrant objects

The following proposition is dual in form to Proposition 5.1. The only subtlety here is that it involves the notions of locally presentable category and accessible functor, which are not self-dual, so that we cannot appeal to duality to prove the parts of the result that involve these concepts.

Proposition 6.1. *Let $\mathcal{A}$ be an accessible weak model category endowed with an accessible algebraic weak factorisation system (L, R) such that if $f\colon X \to Y$ is an arrow in $\mathcal{A}$ from a cofibrant object to a fibrant object then it can be factored as an acyclic fibration followed by a fibration that admits an R-algebra structure. Let T be the monad on $\mathcal{A}$ associated to (L, R). Then there is an accessible weak model structure on* T-**Alg** *such that:*

1. *A map in* T-**Alg** *is a fibration or anodyne fibration if and only if its underlying map in $\mathcal{A}$ is.*
2. *The adjunction:*
$$F\colon \mathcal{A} \leftrightarrows \mathsf{T}\text{-}\mathbf{Alg}\colon U,$$
where U is the forgetful functor is a Quillen equivalence.
3. *A map between fibrant objects in* T-**Alg** *is a weak equivalence or an acyclic fibration if and only if its underlying map in $\mathcal{A}$ is.*
4. *If $\mathcal{A}$ is combinatorial then so is* T-**Alg**.

Proof. Since the monad R is accessible, its restriction T is also accessible. Since T is an accessible monad on a locally presentable category, it again follows from the limit theorem of Makkai and Pare that T-**Alg** is locally presentable. A self-contained proof of this claim is in Theorem 2.78 of [**1**].

Since the forgetful functor U is a right adjoint between locally presentable categories, by Theorem 2.6 of [**14**], there exist accessible weak factorisation systems on T-**Alg** whose right classes consist of those maps f for which Uf is a fibration or anodyne fibration respectively. The existence of right transfer of combinatorial weak factorisation systems is standard and follows directly from the small object argument.

The remainder of the proof is the exact dual of the proof of Proposition 5.1. □

Exactly as in Proposition 5.2, we also have:

Proposition 6.2. *Under the same assumptions as in Proposition 6.1, if in addition $\mathcal{A}$ is core right saturated and if all objects admitting a T-algebra structure are fibrant, then* T-**Alg** *is a left semi-model category in which all objects are fibrant.*

Example 6.3. Let $\mathcal{C}$ be an accessible weak model category and J a (not necessarily generating) set of acyclic cofibrations. We consider the algebraic weak factorisation system (L, R) cofibrantly generated by J, so that R-algebras are the J-algebraic fibrations and the T-algebras are the algebraically J-fibrant objects. This algebraic weak factorisation system always satisfies the assumptions of Proposition 6.1: indeed, an arrow from a cofibrant object to a fibrant object can be factored as an acyclic cofibration followed by a (core) fibration, and every core fibration is a J-fibration and hence admits an R-algebra structure. Therefore we obtain a Quillen equivalent weak model structure on the category of algebraically J-fibrant objects.

If $\mathcal{C}$ is core right saturated and the lifting property against arrows in J is sufficient to characterise the fibrant objects, then Proposition 6.2 implies that this weak model structure is a left semi-model structure in which every object is fibrant.

Combining Propositions 6.1 and 6.2 we have:

Theorem 6.4. *Let $\mathcal{C}$ be an accessible weak model category endowed with an accessible algebraic realisation (L, R) of the (anodyne cofibration, fibration)-weak factorisation system, and let T be the corresponding fibrant replacement monad.*

*Then there is an accessible weak model structure on T-**Alg** such that:*

- *(i) A map in T-**Alg** is a fibration or anodyne fibration if and only if its underlying map in $\mathcal{C}$ is.*
- *(ii) The adjunction:*

$$F\colon \mathcal{C} \leftrightarrows \mathsf{T}\text{-}\mathbf{Alg}\colon U,$$

  where U is the forgetful functor, is a Quillen equivalence.
- *(iii) All objects of T-**Alg** are fibrant.*
- *(iv) An arrow in T-**Alg** is a weak equivalence or an acyclic fibration if and only if its underlying map in $\mathcal{C}$ is.*
- *(v) If $\mathcal{C}$ is core right saturated then T-**Alg** is an accessible left semi-model category.*
- *(vi) If $\mathcal{C}$ is combinatorial then so is T-**Alg**.*

By a dual argument to Theorem 5.6 we also have.

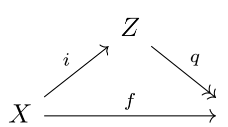
Theorem 6.5. *Under the assumptions of Theorem 6.4 suppose further that $\mathcal{C}$ is a right semi-model category and that path objects lift along $U\colon \mathsf{T}\text{-}\mathbf{Alg} \to \mathcal{C}$ — in the sense that if $(X, \boldsymbol{x})$ is a T-algebra, then there exists a cylinder object factorisation $X \to PX \to X \times X \in \mathcal{C}$ that lifts to a factorisation of $(X, \boldsymbol{x}) \to (X, \boldsymbol{x}) \times (X, \boldsymbol{x}) \in \mathsf{T}$-**Alg**. Then T-**Alg** is a Quillen model category.*

Up to this point the situation is perfectly symmetric. However, even if $\mathcal{C}$ is combinatorial the model structure on algebraically cofibrant objects can produce right semi-model structures that are not Quillen model categories (see Example 5.5). On the other hand it is known from Theorem 19 of [**8**], which builds on [**25**], that if $\mathcal{C}$ is a combinatorial Quillen model category then the weak model structure on algebraically fibrant objects is again a genuine Quillen model category.

In Theorem 6.7 below we generalise this result of [**8**] to the setting of an algebraic weak factorisation system cofibrantly generated by a set of morphisms. The key is the following technical lemma, which is an abstraction of the path object obstruction given in Theorem 19 of [**8**].

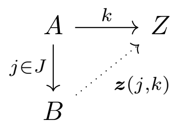
Lemma 6.6. *Let J be a set of morphisms in $\mathcal{C}$ and consider the category **J-Fib** of algebraically J-fibrant objects.*

*Let $f\colon X \to Y$ be an arrow in **J-Fib** which admits a factorisation in $\mathcal{C}$:*



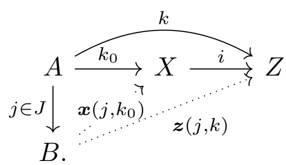
such that i is a monomorphism and q is a J-fibration. Then Z can be equipped with *the structure of an algebraically J-fibrant object with respect to which i and q are morphisms of* **J-Fib**.

Proof. The structure of an algebraically J-fibrant object on Z is specified by the choice of a dotted lifting:

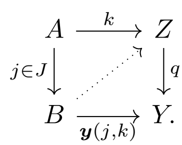


for all $j \in J$ and k as above. We specify this choice in two stages.

1. Firstly, if the map $k = i \circ k_0$ factors through $i \colon X \to Z$, then the factorisation k_0 is unique and we define $\boldsymbol{z}(j,k) = i \circ \boldsymbol{x}(j, k_0)$ as below.



2. On the other hand, if k does not factor through X then firstly we use the **J-Fib** structure on Y to obtain a chosen lift as in the outer square below.



Since q is a J-fibration there exists a dotted lifting, and we define $\boldsymbol{z}(k,j)$ to be one such lifting — the particular choice proves to be irrelevant.

It remains to check that for this choice of structure on Z, both i and q are morphisms of **J-Fib**, which is to say that they preserve the chosen liftings. Firstly, we consider $i \colon X \to Z$ and a map $l \colon A \to X$. Then $i \circ l \colon A \to X \to Z$ factors uniquely through X as $l = (i \circ l)_0$ so that the lifting $\boldsymbol{z}(j, i \circ l)$ is constructed according to the first rule above — the first diagram above interpreted at $k = i \circ l$ then states exactly that $i \colon X \to Z$ is a morphism of **J-Fib**.

For q, we need to consider the different two cases. If $k \colon A \to Z$ factors through $i \colon X \to Z$ as k_0 then we are in the first situation above, so that $\boldsymbol{z}(j,k) = i \circ \boldsymbol{x}(j,k_0)$. Then

$$q \circ \boldsymbol{z}(j,k) = q \circ i \circ \boldsymbol{x}(j,k_0) = f \circ \boldsymbol{x}(j,k_0) = \boldsymbol{y}(j, f \circ k_0) = \boldsymbol{y}(j, q \circ k),$$

where in the second last equation, we use that $f \colon X \to Z$ is a morphism of **J-Fib**.

In the second case, where $k \colon A \to Z$ does not factor through X, q preserves the lifting $\boldsymbol{y}$ by construction. □

The following result improves Theorem 19 of [8] by weakening the hypothesis that $\mathcal{C}$ is a model category to merely a right semi-model category, and we are grateful to Martin Bidlingmaier for a remark that led us to further sharpen the hypotheses on J.

Theorem 6.7. *Let $\mathcal{C}$ be an accessible right semi-model category with a set J of anodyne cofibrations such that every J-fibrant object is fibrant. Then there is a Quillen model structure on the category of algebraically J-fibrant objects* **J-Fib** *and a Quillen equivalence*

$$F\colon \mathcal{C} \leftrightarrows \mathbf{J\text{-}Fib}\colon U,$$

where U is the forgetful functor. The fibrations, weak equivalences, anodyne fibrations and acyclic fibrations in **J-Fib** *are the maps f such that Uf has the corresponding property in $\mathcal{C}$.*

Proof. We consider the algebraic weak factorisation system (L, R) cofibrantly generated by the set J. Since each member of J is an anodyne cofibration and the morphisms of J suffice to characterise the fibrant objects, by Example 6.3, we obtain a transferred left semi-model structure on **J-Fib** in which all objects are fibrant.

Now consider an object $(X, \boldsymbol{x})$ of **J-Fib** and the diagonal $(X, \boldsymbol{x}) \to (X, \boldsymbol{x})^2$. Since $\mathcal{C}$ is a right semi-model structure its underlying map admits a path object as below.

$$X \xrightarrow{\delta} PX \twoheadrightarrow X \times X.$$

Since δ is a (split) monomorphism and the fibration $PX \twoheadrightarrow X \times X$ belongs to $RLP(J)$, Lemma 6.6 ensures that the factorisation lifts along U to **J-Fib**. Therefore path objects lift along U, so that by Theorem 6.5, we obtain the full model structure on **J-Fib**. □

Example 6.8. Consider again the right semi-model structure on semi-simplicial sets described in Example 2.5.(iv), whose fibrations between fibrant objects are those morphisms with the Kan lifting property. Let J be the set of semi-simplicial horn inclusions, so that **J-Fib** is the category **AlgKan** of semi-simplicial algebraic Kan complexes — semi-simplicial sets equipped with chosen Kan fillers. The maps in J are anodyne cofibrations and although J is not quite a generating set of anodyne cofibrations, the fibrant semi-simplicial sets are precisely the objects with the right lifting property against all maps in J, or semi-simplicial Kan complexes. Therefore we can apply Theorem 6.7 and the right semi-model structure of Example 2.5.(iv) transfers along the adjunction

$$F : \mathbf{ssSet} \leftrightarrows \mathbf{AlgKan} : U$$

to a Quillen equivalent Quillen model structure on **AlgKan**. The fibrations and weak equivalences are the maps that are fibrations or weak equivalences on underlying semi-simplicial sets.

7. Replacing weak model categories by genuine ones

The main goal of this short section is to apply the results established so far to show that each combinatorial weak model category is connected, via a zigzag of Quillen equivalences, to a genuine model category. We begin by recalling from Section 4 of [**18**] the core left and right saturation of an accessible weak model category.

Let $\mathcal{C}$ be an accessible weak model category. Then $\mathcal{C}$ admits a second accessible weak model structure called its *core left saturation*, $L^c\mathcal{C}$, with the same cofibrations, anodyne fibrations and core fibrations as $\mathcal{C}$. Furthermore $L^c\mathcal{C}$ is core left saturated and the identity functor $\mathcal{C} \to L^c\mathcal{C}$ is a left Quillen equivalence — in fact it induces an equivalence between the ordinary categories of bifibrant objects preserving all the relevant structure.

Similarly, there is the core right saturation $R^c\mathcal{C}$, for which the identity functor $R^c\mathcal{C} \to \mathcal{C}$ is a right Quillen equivalence. Again this is an accessible weak model category.

Theorem 7.1. *Let $\mathcal{C}$ be an accessible weak model category. Then $\mathcal{C}$ is connected, via a zigzag of Quillen equivalences, to an accessible right semi-model category in which all objects are cofibrant, and also to an accessible left semi-model category in which all objects are fibrant.*

Proof. In the first case, we consider the zigzag $\mathcal{C} \to L^c\mathcal{C} \leftarrow \mathsf{Q}\text{-}\mathbf{Coalg}$ in which Q is the cofibrant replacement comonad for an algebraic realisation of the (cofibration, anodyne fibration) weak factorisation system on $L^c\mathcal{C}$. In fact, this is the same Q as on $\mathcal{C}$ itself, but the weak model structure might differ.

The first component is a left Quillen equivalence as mentioned above, and by Theorem 5.3 the second component is a left Quillen equivalence. Moreover since $L^c\mathcal{C}$ is core left saturated, by Theorem 5.3 the second component is a right semi-model category with all objects cofibrant.

The proof of the second claim proceeds in the same manner, but now using the core right saturation and Theorem 6.4. □

Theorem 7.2. *Let $\mathcal{C}$ be a combinatorial weak model category. Then $\mathcal{C}$ is connected, via a zigzag of Quillen equivalences, to a combinatorial model category in which all objects are fibrant.*

Proof. We consider the zigzag of left Quillen equivalences $\mathcal{C} \to L^c\mathcal{C} \leftarrow \mathsf{Q}\text{-}\mathbf{Coalg}$ of Theorem 7.1. Since $\mathcal{C}$ is combinatorial, so too are $L^c\mathcal{C}$ and $\mathsf{Q}\text{-}\mathbf{Coalg}$. Then letting T denotes the induced fibrant replacement monad for the combinatorial right semi-model structure $\mathsf{Q}\text{-}\mathbf{Coalg}$, it follows from Theorem 6.7 that $\mathsf{T}\text{-}\mathbf{Alg}$ is a combinatorial model category with all objects fibrant and the free functor $\mathcal{C} \to \mathsf{T}\text{-}\mathbf{Alg}$ a left Quillen equivalence, so that we have a zigzag of left Quillen equivalences as below

$$\mathcal{C} \to L^c\mathcal{C} \leftarrow \mathsf{Q}\text{-}\mathbf{Coalg} \to \mathsf{T}\text{-}\mathbf{Alg}.$$ □

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John Bourke bourkej@math.muni.cz

Department of Mathematics and Statistics, Masaryk University, Kotlářská 2, Brno 61137, Czech Republic

Simon Henry shenry2@uottawa.ca

Department of Mathematics and Statistics, University of Ottawa, STEM complex, 150 Louis-Pasteur, K1N 6N5, Ottawa, ON, Canada