Homology, Homotopy and Applications, vol. 24(1), 2022, pp.245–269

STABILITY OF LODAY CONSTRUCTIONS

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(*communicated by J.P.C. Greenlees*)

Abstract

We study the question for which commutative ring spectra A the tensor of a simplicial set X with A, $X \otimes A$, is a stable invariant in the sense that it depends only on the homotopy type of ΣX. We prove several structural properties about different notions of stability, corresponding to different levels of invariance required of $X \otimes A$. We establish stability in important cases, such as complex and real periodic topological K-theory, KU and KO.

1. Introduction

For any simplicial set X and any commutative ring spectrum A one can form the tensor of X with A, $X \otimes A$. An important special case of this construction is the topological Hochschild homology of A, $\mathsf{THH}(A)$, which is $S^1 \otimes A$. In the following we will often work with commutative R-algebras for some commutative ring spectrum R. We will sometimes take coefficients in a commutative A-algebra C, which requires working with *pointed* simplicial sets X; we denote the corresponding object (whose definition we recall in Section 1.1 below) by $\mathcal{L}^R_X(A;C)$. Working over the sphere spectrum S with $C = A$, $\mathcal{L}^S_X(A;A)$ is just $X \otimes A$. When the space is the circle, $\mathcal{L}^R_{S^1}(A;C)$ is $\mathsf{THH}^R(A;C)$.

As topological Hochschild homology is the target of a trace map from algebraic K-theory

$$K(A) \to \mathsf{THH}(A) \tag{1.1}$$

it has been calculated in many cases. Higher order topological Hochschild homology, which is $\mathcal{L}^R_{S^n}(A;C)$, has also been determined in many important classes of examples, see for instance [**3, 8, 13, 22, 24**]. In [**3**] we develop several tools for calculating $\mathcal{L}^R_{\Sigma X}(A;C)$. However, if we want to determine the homotopy type of $\mathcal{L}^R_X(A;C)$ and X doesn't happen to be a suspension, then the range of methods is much sparser.

Rognes' redshift conjecture [**1**] predicts that applying algebraic K-theory raises chromatic level by one in good cases. In particular, higher chromatic phenomena could be detected by iterated algebraic K-theory of rings. If A is a commutative ring spectrum, then so are $K(A)$ and $\mathsf{THH}(A)$, and as the trace map is a map of

Received April 28, 2020, revised May 6, 2021; published on April 13, 2022.
2010 Mathematics Subject Classification: 18G60, 55P43.
Key words and phrases: Loday construction, stability, topological Hochschild homology, commutative ring spectrum.
Article available at `http://dx.doi.org/10.4310/HHA.2022.v24.n1.a13`

commutative ring spectra, one can iterate the trace map from (1.1) to obtain

$$K(K(A)) \to \mathsf{THH}(\mathsf{THH}(A))$$

and one doesn't have to stop at two-fold iterations. As $X \otimes A$ is the tensor of A with X in the category of commutative ring spectra [**10**, Chapter VII, §2, §3], one can identify

$$\mathsf{THH}(\mathsf{THH}(A)) = S^1 \otimes (S^1 \otimes A)$$

with $(S^1 \times S^1) \otimes A$ and this is the *torus homology of* A. Similarly, any n-fold iteration of algebraic K-theory of A has an iterated trace map to $(S^1)^n \otimes A$. There are calculations of torus homology of $H\mathbb{F}_p$ for small n by Rognes, Veen [**25**] and Ausoni–Dundas, but a general result is missing. However, the homotopy type of $S^n \otimes H\mathbb{F}_p$ is known for every n and for small n, $(S^1)^n \otimes A$ splits as follows: We have that $\Sigma(S^1)^n \simeq \Sigma(\bigvee_{i=1}^n \bigvee_{\binom{n}{i}} S^1)$ and one obtains for small n

$$(S^1)^n \otimes H\mathbb{F}_p \simeq (\bigvee_{i=1}^{n} \bigvee_{\binom{n}{i}} S^i) \otimes H\mathbb{F}_p.$$

This gave rise to the question whether $\mathcal{L}_X^R(A;C)$ is a *stable invariant*, *i.e.*, whether the homotopy type of $\mathcal{L}_X^R(A;C)$ only depends on the homotopy type of ΣX. There are positive results: $\mathcal{L}_X^{Hk}(HA)$ is a stable invariant if k is a field and A is a commutative Hopf algebra over k [**2**, Theorem 1.3] or if k is an arbitrary commutative ring and A is a smooth k-algebra [**9**, Example 2.6]. But Dundas and Tenti also show [**9**, §3.8] that $\mathcal{L}_X^{H\mathbb{Q}}(H\mathbb{Q}[t]/t^2)$ is *not* a stable invariant. They show that $\mathcal{L}_{S^1 \vee S^1 \vee S^2}^{H\mathbb{Q}}(H\mathbb{Q}[t]/t^2)$ and $\mathcal{L}_{S^1 \times S^1}^{H\mathbb{Q}}(H\mathbb{Q}[t]/t^2)$ differ and that reducing the coefficients from $H\mathbb{Q}[t]/t^2$ to $H\mathbb{Q}$ doesn't eliminate this discrepancy. If we work over the sphere spectrum S, since $S_{\mathbb{Q}} \simeq H\mathbb{Q}$ this also implies that $\mathcal{L}_X^S(H\mathbb{Q}[t]/t^2)$ and $\mathcal{L}_X^S(H\mathbb{Q}[t]/t^2; H\mathbb{Q})$ are not stable invariants.

Our aim is to investigate the question of stability in a systematic manner. We start by defining several different notions of stability. Instead of asking for equivalent homotopy types of $\mathcal{L}_X^R(A;C)$ and $\mathcal{L}_Y^R(A;C)$ if $\Sigma X \simeq \Sigma Y$ we are asking when we actually get an equivalence $\mathcal{L}_X^R(A;C) \simeq \mathcal{L}_Y^R(A;C)$ of augmented commutative C-algebras. There are intermediate notions that ask for less structure to be preserved, for instance, that the equivalence $\mathcal{L}_X^R(A;C) \simeq \mathcal{L}_Y^R(A;C)$ is one of commutative R-algebras or of C- or R-modules.

We establish that stability is preserved by several constructions such as base-change and products but we also show which procedures do *not* preserve stability. For instance stability is not a transitive property: if $R \to A$ and $A \to B$ satisfy stability then this does not imply that $R \to B$ has this property.

A central purpose of this paper is to establish new cases where stability holds. For instance for any regular quotient $R \to R/(a_1, \ldots, a_n)$ of a commutative ring R we obtain stability for the induced map of commutative ring spectra $HR \to HR/(a_1, \ldots, a_n)$. Free commutative ring spectra generated by a module spectrum satisfy stability and we suggest a notion of *really smooth* maps of commutative ring spectra. These are maps $R \to A$ that can be factored as the canonical inclusion of R into a free commutative R-algebra spectrum followed by a map that satisfies étale descent, so these maps model the local behaviour of smooth maps in the context of algebra, compare [**16**, Proposition E.2 (d)]. We show that really smooth maps satisfy stability. Other examples where stability holds are Thom spectra as well as $S \to KU$ and other spectra of the form $S \to R_h = (\Sigma^\infty_+ W_h)[x^{-1}]$ considered in [**5**]. Using Galois descent we also obtain stability for $S \to KO$.

For calculations like that of torus homology, one often doesn't really need stability, but the property of the suspension to decompose products is the crucial feature that one wants to have on the level of $\mathcal{L}^R_{(-)}(A;C)$. Therefore we say that $R \to A \to C$ decomposes products if

$$\mathcal{L}^R_{X\times Y}(A;C) \simeq \mathcal{L}^R_{X\vee Y\vee X\wedge Y}(A;C)$$

for all pointed simplicial sets X and Y. We use Greenlees' spectral sequence [**12**, Lemma 3.1] in the case $C = Hk$ for k a field to show that this decomposition property is preserved under forming suitable retracts.

In Section 7 we close with some observations on stability in characteristic zero, using that rationally the suspension of pointed simply connected simplicial sets splits into a pointed sum of rational spheres and using [**2**, Proposition 4.2] where Berest, Ramadoss and Yeung describe the behaviour of representation homology and higher order Hochschild homology under rational equivalences.

Acknowledgments

We thank Bjørn Dundas for many helpful discussions and for spotting several dumb mistakes in earlier versions of this paper. The second named author thanks the Isaac Newton Institute for Mathematical Sciences for support and hospitality during the programme *K-theory, algebraic cycles and motivic homotopy theory* when work on this paper was undertaken. This work was supported by Simons Collaboration Grant 359565 and NSF Grant DMS 2004300 for the first author and EPSRC grant number EP/R014604/1 for the second.

1.1. Definition of $\mathcal{L}^R_X(A;C)$

We denote the category of simplicial sets by sSets and the one of pointed simplicial sets by sSets_*. Let X be a finite pointed simplicial set and let $R \to A \to C$ be a sequence of maps of commutative ring spectra. We assume that R is a cofibrant commutative S-algebra and that A and C are cofibrant commutative R-algebras. The cofibrancy assumptions on R, A and C will ensure that the homotopy type of $\mathcal{L}^R_X(A;C)$ is well-defined:

The *Loday construction with respect to X of A over R with coefficients in C* is the simplicial commutative augmented C-algebra spectrum $\mathcal{L}^R_X(A;C)$ whose p-simplices are

$$C \wedge \bigwedge_{x\in X_p\setminus *} A,$$

where the smash products are taken over R. Here, $*$ denotes the basepoint of X and we place a copy of C at the basepoint. As the smash product over R is the coproduct in the category of commutative R-algebra spectra, the simplicial structure is straightforward: Face maps d_i on X induce multiplication in A or the A-action on C if the basepoint is involved. The degeneracy maps s_i on X cause the insertion of the unit map $\eta_A\colon R \to A$ over all n-simplices which are not hit by $s_i\colon X_{n-1} \to X_n$. As defined, $\mathcal{L}^R_X(A; C)$ is a simplicial commutative augmented C-algebra spectrum. We use the same symbol $\mathcal{L}^R_X(A; C)$ for its geometric realization. For $C = A$ we abbreviate $\mathcal{L}^R_X(A; A)$ by $\mathcal{L}^R_X(A)$.

For $X = S^n$ we write $\mathsf{THH}^{[n],R}(A; C)$ for $\mathcal{L}^R_{S^n}(A; C)$ and if $R = S$, then we omit it from the notation, so $\mathsf{THH}^{[n]}(A; C) = \mathcal{L}^S_{S^n}(A; C)$. For $n = 1$ this gives the classical topological Hochschild homology of A with coefficients in C, $\mathsf{THH}(A; C)$. Note that $\mathcal{L}^R_X(A)$ is by definition [**10**, VII, §2, §3] equal to $X \otimes A$ where $X \otimes A$ is formed in the category of commutative R-algebras.

As we assume that R is a cofibrant commutative S-algebra and that A and C are cofibrant commutative R-algebras, the simplicial spectrum $\mathcal{L}^R_X(A; C)$ is proper in the sense of [**10**, Definition X.2.2], compare the argument for THH in [**10**, Proposition IX.2.7]. Levelwise weak equivalences of proper simplicial spectra induce weak equivalences on geometric realizations by [**10**, Theorem X.2.4]. We will make heavy use of this fact later.

If $X \in \mathsf{sSets}_*$ is an arbitrary object, then we can write it as the colimit of its finite pointed subcomplexes and the Loday construction with respect to X can then also be expressed as the colimit of the Loday construction for the finite pointed subcomplexes.

2. Notions of stability

The weakest notion of stability just asks for an abstract equivalence in the stable homotopy category:

Definition 2.1.

1. Let $R \to A$ be a cofibration of commutative S-algebras with R cofibrant. We say that $R \to A$ is *stable* if for every pair of pointed simplicial sets X and Y an equivalence $\Sigma X \simeq \Sigma Y$ implies that $\mathcal{L}^R_X(A) \simeq \mathcal{L}^R_Y(A)$.
2. Let $S \to R \to A \to B$ be a sequence of cofibrations of commutative S-algebras. Then we say that (R, A, B) is *stable*, if for every pair of pointed simplicial sets X and Y an equivalence $\Sigma X \simeq \Sigma Y$ in sSets_* implies that $\mathcal{L}^R_X(A; B) \simeq \mathcal{L}^R_Y(A; B)$.

Examples 2.2.

- Dundas and Tenti show that for any discrete smooth commutative k-algebra A we have that $Hk \to HA$ is stable [**9**, Example 2.6].
- They show, however, that $H\mathbb{Q} \to H\mathbb{Q}[t]/t^2$ and $(H\mathbb{Q}, H\mathbb{Q}[t]/t^2, H\mathbb{Q})$ are not stable.
- If A is a commutative Hopf algebra over a field k, then Berest, Ramadoss and Yeung prove [**2**, §5] that $Hk \to HA$ and (Hk, HA, Hk) are stable by comparing higher order Hochschild homology to representation homology. For a purely homotopy-theoretic proof see [**14**, Theorem 3.8].
- In [**3**] we show that for any sequence of cofibrations of commutative S-algebras $S \to A \to B \to A$ we get that
$$\mathcal{L}^B_X(A) \simeq \mathcal{L}^A_{\Sigma X}(B; A)$$
as augmented commutative A-algebras and hence $B \to A$ is stable if B is a cofibrant commutative augmented A-algebra.

In the above definition we just require an abstract weak equivalence, but one can also pose additional conditions on the equivalence $\mathcal{L}_X^R(A;B) \simeq \mathcal{L}_Y^R(A;B)$. A strong version of stability is the following:

Definition 2.3.

1. Let $R \to A$ be a cofibration of commutative S-algebras with R cofibrant. We say that $R \to A$ is *multiplicatively stable* if for every pair of pointed simplicial sets X and Y an equivalence $\Sigma X \simeq \Sigma Y$ in sSets_* implies that $\mathcal{L}_X^R(A) \simeq \mathcal{L}_Y^R(A)$ as commutative augmented A-algebra spectra.

2. Let $S \longrightarrow R \xrightarrow{\alpha} A \xrightarrow{\beta} B$ be a sequence of cofibrations of commutative S-algebras. Then we say that $R \to A \to B$ is *multiplicatively stable* if for every pair of pointed simplicial sets X and Y an equivalence $\Sigma X \simeq \Sigma Y$ in sSets_* implies that $\mathcal{L}_X^R(A;B) \simeq \mathcal{L}_Y^R(A;B)$ and $\mathcal{L}_X^R(B) \simeq \mathcal{L}_Y^R(B)$ as commutative augmented B-algebras such that the diagram

$$\begin{tikzcd} \mathcal{L}_X^R(A;B) \arrow[r, dash, "\simeq"] \arrow[d, "\mathcal{L}_X^R(\beta)"'] & \mathcal{L}_Y^R(A;B) \arrow[d, "\mathcal{L}_X^R(\beta)"] \\ \mathcal{L}_X^R(B) \arrow[r, dash, "\simeq"] & \mathcal{L}_Y^R(B) \end{tikzcd}$$

commutes.

Note that we are using different notations, (R, A, B) to denote stability with coefficients and $R \to A \to B$ to denote multiplicative stability with coefficients.

Of course, there is a whole hierarchy of notions of stability. Instead of asking that the equivalence $\mathcal{L}_X^R(A) \simeq \mathcal{L}_Y^R(A)$ is one of augmented commutative A-algebras, we could ask for one of augmented commutative R-algebras or A- or just R-modules.

Definition 2.4. Let $R \to A$ be a cofibration of commutative S-algebras with R cofibrant. We call $R \to A$ *A-linearly stable* if for every pair of pointed simplicial sets X and Y an equivalence $\Sigma X \simeq \Sigma Y$ in sSets_* implies that $\mathcal{L}_X^R(A) \simeq \mathcal{L}_Y^R(A)$ as A-modules. Similarly, we call $R \to A$ *R-linearly stable* if for every pair of pointed simplicial sets X and Y an equivalence $\Sigma X \simeq \Sigma Y$ in sSets_* gives rise to an equivalence of R-modules $\mathcal{L}_X^R(A) \simeq \mathcal{L}_Y^R(A)$.

Remark 2.5. If $R \to A$ is A-linearly stable, then (R, A, B) is stable because

$$\mathcal{L}_X^R(A;B) \simeq \mathcal{L}_X^R(A) \wedge_A B.$$

If $R \to A$ is multiplicatively stable, then so is $R \to A \to B$ for every cofibrant commutative A-algebra B.

A converse might not be true: Even if B is faithful as an A-module, we might not know that the equivalence $\mathcal{L}_X^R(A) \wedge_A B \simeq \mathcal{L}_Y^R(A) \wedge_A B$ is of the form $f \wedge_A B$, so we cannot deduce that $\mathcal{L}_X^R(A) \simeq \mathcal{L}_Y^R(A)$.

Let us start with several examples of multiplicative stability.

Proposition 2.6. *If B is an augmented commutative A-algebra, then $B \to A$ and $A \to \mathcal{L}_{\Sigma X}^A(B;A) \to A$ are multiplicatively stable for all $X \in \mathsf{sSets}_*$.*

Proof. In the augmented case $A \to B \to A$, as an equivalence $\Sigma X \simeq \Sigma Y$ in sSets_* implies that $\mathcal{L}^A_{\Sigma X}(B;A) \simeq \mathcal{L}^A_{\Sigma Y}(B;A)$ as augmented commutative A-algebras, we also get that $\mathcal{L}^B_X(A) \simeq \mathcal{L}^B_Y(A)$ as augmented commutative A-algebras by applying [**3**, Theorem 3.3] to the sequence of maps $A = A \to B \to A$, so $B \to A$ is multiplicatively stable.

For the second claim we observe that there is an equivalence of augmented commutative A-algebras

$$\mathcal{L}^A_Y(\mathcal{L}^A_{\Sigma X}(B;A);A) \simeq \mathcal{L}^A_{Y \wedge \Sigma X}(B;A) = \mathcal{L}^A_{\Sigma Y \wedge X}(B;A).$$

As we have that $\mathcal{L}^A_X(A) \simeq A$ for all X, the map $A \to \mathcal{L}^A_{\Sigma X}(B;A) \to A$ is multiplicatively stable. □

Loday constructions for suspensions are stable:

Theorem 2.7. *Let $R \to A$ be a cofibration of commutative S-algebras with R cofibrant. Then $A \to \mathcal{L}^R_{\Sigma X}(A)$ is multiplicatively stable for all $X \in \mathsf{sSets}_*$.*

Proof. We have to show that $\mathcal{L}^A_Y(\mathcal{L}^R_{\Sigma X}(A))$ only depends on the homotopy type of ΣY. We first identify $\mathcal{L}^A_Y(\mathcal{L}^R_{\Sigma X}(A))$ with the help of [**3**, Lemma 3.1] as an augmented commutative A-algebra as

$$\begin{aligned}
\mathcal{L}^A_Y(\mathcal{L}^R_{\Sigma X}(A)) &\simeq \mathcal{L}^R_Y(\mathcal{L}^R_{\Sigma X}(A)) \wedge^L_{\mathcal{L}^R_Y(A)} A \\
&\simeq \mathcal{L}^R_{Y \times \Sigma X}(A) \wedge^L_{\mathcal{L}^R_Y(A)} A \\
&\simeq \mathcal{L}^R_{(Y \times \Sigma X) \cup_Y *}(A) \\
&\simeq \mathcal{L}^R_{Y_+ \wedge \Sigma X}(A) \cong \mathcal{L}^R_{\Sigma(Y_+) \wedge X}(A).
\end{aligned}$$

As $\Sigma(Y_+) \simeq \Sigma Y \vee S^1$ for $Y \in \mathsf{sSets}_*$, this depends only on ΣY. □

Example 2.8. Applying Theorem 2.7 to $H\mathbb{F}_p$ and $\Sigma X = S^2$ gives that the map

$$H\mathbb{F}_p \to \mathsf{THH}^{[2]}(H\mathbb{F}_p) \simeq H\mathbb{F}_p \vee \Sigma^3 H\mathbb{F}_p$$

is multiplicatively stable for all primes p.

As we know from the algebraic setting that smooth algebras are stable, it is natural to consider free commutative A-algebra spectra. Let M be an A-module spectrum for some commutative S-algebra A. We consider the free commutative A-algebra on M,

$$\mathbb{P}_A(M) = \bigvee_{n \geqslant 0} M^{\wedge_A n}/\Sigma_n$$

with the usual convention that $M^{\wedge_A 0}/\Sigma_0 = A$.

In the following we use several categories, so let's fix some notation. Let $\mathcal{U}$ denote the category of unbased (compactly generated weak Hausdorff) spaces. For a commutative ring spectrum R, $\mathcal{M}_R$ denotes the category of R-module spectra and $\mathcal{C}_R$ denotes the category of commutative R-algebras.

Lemma 2.9. *For every simplicial set X and for every $M \in \mathcal{M}_A$ there is a weak equivalence of commutative A-algebras*

$$\mathcal{L}^A_X(\mathbb{P}_A(M)) \simeq \mathbb{P}_A(X_+ \wedge M).$$

Proof. For the proof we use the fact that the category of commutative A-algebras is tensored over unpointed topological spaces and simplicial sets in a compatible way [**10**, VII §2, §3]. Note that $\mathcal{L}_X^A(\mathbb{P}_A(M)) = X \otimes_A \mathbb{P}_A(M)$ in the notation of [**10**].

We have the following chain of bijections for an arbitrary commutative A-algebra B:

$$\begin{aligned}\mathcal{C}_A(X \otimes_A \mathbb{P}_A(M), B) &\cong \mathcal{U}(X, \mathcal{C}_A(\mathbb{P}_A(M), B))\\ &\cong \mathcal{U}(X, \mathcal{M}_A(M, B))\\ &\cong \mathcal{M}_A(X_+ \wedge M, B)\\ &\cong \mathcal{C}_A(\mathbb{P}_A(X_+ \wedge M), B),\end{aligned}$$

where $X_+ \wedge M$ is the tensor of X with M in the category of A-modules. Hence the Yoneda lemma implies the claim. □

Corollary 2.10. *In the setting above, if $\Sigma X \simeq \Sigma Y$, then $\mathcal{L}_X^A(\mathbb{P}_A(M)) \simeq \mathcal{L}_Y^A(\mathbb{P}_A(M))$ as commutative A-algebras.*

Proof. If $\Sigma X \simeq \Sigma Y$, then $\Sigma_+^\infty X \simeq \Sigma_+^\infty Y$ and as $X_+ \wedge M = \Sigma_+^\infty X \wedge M$ this implies that $\mathbb{P}_A(X_+ \wedge M) \simeq \mathbb{P}_A(Y_+ \wedge M)$ as commutative A-algebras. □

The following example was also considered in [**19**, Lemma 5.5]. A cofibration $A \to B$ of commutative S-algebras with A cofibrant is called THH-étale if the canonical map $B \to \mathsf{THH}^A(B)$ is a weak equivalence.

Proposition 2.11. *If $A \to B$ is THH-étale, then for all connected pointed X the canonical map $B \to \mathcal{L}_X^A(B)$ is an equivalence. Hence, as this map is a map of augmented commutative B-algebras, $\mathcal{L}_X^A(B) \simeq \mathcal{L}_Y^A(B)$ for any pair of connected simplicial sets X and Y.*

Proof. The proof is by induction on the top dimension of a non-degenerate simplex in a finite connected simplicial set, and then by taking colimits in the infinite case. A connected 0-dimensional simplicial set consists of a point, where there is nothing to prove. Any 1-dimensional connected finite simplicial set is homotopy equivalent to a wedge of circles, so if $X \simeq S^1 \vee S^1 \vee \ldots \vee S^1$ and $B \simeq \mathcal{L}_{S^1}^A B$,

$$\mathcal{L}_X^A B \simeq B \wedge_B B \wedge_B \cdots \wedge_B B \simeq B.$$

Once we know the result for simplicial sets of dimension $\leqslant n-1$, if we get a simplicial set X with a finite number of non-degenerate n-cells we proceed by induction on the number of those. As in the proof of Proposition 8.4 in [**3**], using the homotopy invariance of the construction and subdivision, if needed, we can assume that X can be constructed by adding a new non-degenerate simplex with an embedded boundary to a simplicial set homotopy equivalent to X with one non-degenerate n-cell deleted, for which the proposition holds by the induction on the number of non-degenerate n-cells. By the inductive hypothesis it also holds for the embedded boundary $\partial\Delta^n$, and since the new simplex being added is homotopy equivalent to a point, the proposition holds for it. By the connectivity and by homotopy invariance we can also assume that the basepoint of X is contained in the boundary of the new simplex being attached, so the identifications of all three Loday constructions with B are compatible. Then $\mathcal{L}_X^A(B) \simeq B \wedge_B B \simeq B$. □

Remark 2.12. Examples of THH-étale maps $A \to B$ are Galois extensions in the sense of [**21**] but also étale maps in the sense of Lurie [**17**, Definition 7.5.1.4]. For a careful discussion of these notions and for comparison results see [**18**].

3. Inheritance properties and descent

With the assumption of multiplicative stability we get a descent result:

Theorem 3.1. *If $R \to A \to B$ is multiplicatively stable, then $A \to B$ is multiplicatively stable.*

Proof. Let's assume that $\Sigma X \simeq \Sigma Y$ in sSets_*. Then by assumption we get that $\mathcal{L}_X^R(B) \simeq \mathcal{L}_Y^R(B)$ and $\mathcal{L}_X^R(A;B) \simeq \mathcal{L}_Y^R(A;B)$ as commutative augmented B-algebras, compatibly with the module structure of the former over the latter. The Juggling Lemma [**3**, Lemma 3.1] yields an equivalence of augmented commutative B-algebras

$$\mathcal{L}_X^A(B) \simeq B \wedge_{\mathcal{L}_X^R(A;B)} \mathcal{L}_X^R(B) \text{ and } \mathcal{L}_Y^A(B) \simeq B \wedge_{\mathcal{L}_Y^R(A;B)} \mathcal{L}_Y^R(B).$$

Our assumptions guarantee that therefore $\mathcal{L}_X^A(B) \simeq \mathcal{L}_Y^A(B)$ as commutative augmented B-algebras. □

One can upgrade this slightly and introduce coefficients:

Corollary 3.2. *If $S \to R \to A \to B \to C$ is a sequence of cofibrations of commutative A-algebras and both $R \to A \to C$ and $R \to B \to C$ are multiplicatively stable, then $A \to B \to C$ is multiplicatively stable as well.*

Lemma 3.3. *Let $A \leftarrow R \to B$ be cofibrations of commutative S-algebras with R cofibrant. Then $\mathcal{L}_X^A(A \wedge_R B) \cong A \wedge_R \mathcal{L}_X^R(B)$ as simplicial commutative augmented $A \wedge_R B$-algebras and hence on realizations as commutative augmented $A \wedge_R B$-algebras.*

Proof. There is a direct isomorphism sending $A \wedge_R (B \wedge_R \ldots \wedge_R B)$ to $(A \wedge_R B) \wedge_A \ldots \wedge_A (A \wedge_R B)$ and this isomorphism is compatible with the multiplication. □

This implies that stability is closed under base-change:

Proposition 3.4. *Let A and E be cofibrant commutative R-algebra spectra. If $R \to A$ is R-linearly stable, then so is $E \to E \wedge_R A$. If $R \to A$ is multiplicatively stable, then so is $E \to E \wedge_R A$.*

Proof. Assume that $\Sigma X \simeq \Sigma Y$ in sSets_*. Then by assumption $\mathcal{L}_X^R(A) \simeq \mathcal{L}_Y^R(A)$ as R-modules or as augmented commutative A-algebras. But then also $E \wedge_R \mathcal{L}_X^R(A) \simeq E \wedge_R \mathcal{L}_Y^R(A)$ and by Lemma 3.3 this implies

$$\mathcal{L}_X^E(E \wedge_R A) \simeq \mathcal{L}_Y^E(E \wedge_R A).$$ □

Remark 3.5. Note that the above implication cannot be upgraded to an equivalence: starting with the assumption that $\mathcal{L}_X^E(E \wedge_R A) \simeq \mathcal{L}_Y^E(E \wedge_R A)$, we get $E \wedge_R \mathcal{L}_X^R(A) \simeq E \wedge_R \mathcal{L}_Y^R(A)$. Even if $\mathcal{L}_X^R(A)$ and $\mathcal{L}_Y^R(A)$ are E-local in the category of R-modules, however, we don't know that the weak equivalence $E \wedge_R \mathcal{L}_X^R(A) \simeq E \wedge_R \mathcal{L}_Y^R(A)$ is of the form $E \wedge_R f$ (or a zigzag of such maps), but for the E-local Whitehead Theorem [**4**, Lemma 1.2] we have to have a map and not just an abstract isomorphism of E_*-homology groups.

Smashing with a fixed commutative R-algebra preserves stability:

Lemma 3.6. *Let A, B and C be cofibrant commutative R-algebras. Then there is an equivalence of commutative augmented $C \wedge_R B$-algebras*

$$\mathcal{L}_X^{C\wedge_R A}(C \wedge_R B) \simeq C \wedge_R \mathcal{L}_X^A(B).$$

Hence if $f\colon A \to B$ is multiplicatively stable, then so is $C \wedge_R f\colon C \wedge_R A \to C \wedge_R B$.

Proof. The equivalence

$$\mathcal{L}_X^{C\wedge_R A}(C \wedge_R B) \simeq C \wedge_R \mathcal{L}_X^A(B)$$

is based on the equivalence

$$(C \wedge_R B) \wedge_{(C\wedge_R A)} (C \wedge_R B) \simeq C \wedge_R (B \wedge_A B). \qquad \square$$

Proposition 3.7. *Let R be a commutative ring and let $a \in R$ be a regular element. Then $HR \to HR/a$ is multiplicatively stable.*

Proof. We consider the pushout $HR \wedge^L_{HR[t]} HR$ where the right algebra map $R[t] \to R$ sends t to zero and the left algebra map sends t to a. Note that with respect to both of these maps $HR[t]$ is an augmented commutative HR-algebra spectrum. The Künneth spectral sequence for $\pi_*(HR \wedge^L_{HR[t]} HR)$ has as its E^2-term $\mathrm{Tor}^{R[t]}_{*,*}(R, R)$ and we take the standard free $R[t]$ resolution

$$0 \longrightarrow R[t] \xrightarrow{t} R[t]$$

of R. Applying $(-) \otimes_{R[t]} R$ yields

$$\begin{tikzcd}
0 \arrow[r] & R[t] \otimes_{R[t]} R \arrow[rr, "t\otimes\mathrm{id}", "=\mathrm{id}\otimes a"'] \arrow[d, "\cong"'] & & R[t] \otimes_{R[t]} R \arrow[d, "\cong"] \\
0 \arrow[r] & R \arrow[rr, "a"] & & R.
\end{tikzcd}$$

Note, that the regularity of a is needed to ensure injectivity on the left hand side.

We apply Lemma 3.3 and choose a cofibrant model of HR as a commutative $HR[t]$-algebra and obtain

$$\mathcal{L}_X^{HR}(HR/a) \simeq \mathcal{L}_X^{HR}(HR \wedge_{HR[t]} HR) \simeq HR \wedge_{HR[t]} \mathcal{L}_X^{HR[t]}(HR),$$

where the right $HR[t]$-module structure of $\mathcal{L}_X^{HR[t]}(HR)$ factors through the augmentation map sending t to 0. Assume that $\Sigma X \simeq \Sigma Y$ in sSets_*. By Proposition 2.6 we have that $HR[t] \to HR$ is multiplicatively stable, so $\mathcal{L}_X^{HR[t]}(HR) \simeq \mathcal{L}_Y^{HR[t]}(HR)$ as commutative augmented HR-algebras. This yields an equivalence of commutative augmented $HR \wedge_{HR[t]} HR \simeq HR/a$-algebras between $\mathcal{L}_X^{HR}(HR/a)$ and $\mathcal{L}_Y^{HR}(HR/a)$. $\square$

Remark 3.8. The above result can be used for calculating torus homology for instance for $H\mathbb{Z} \to H\mathbb{Z}/p\mathbb{Z}$ for every prime p: We know the homotopy type of $\mathcal{L}_{S^k}^{H\mathbb{Z}}(H\mathbb{Z}/p\mathbb{Z})$ by [**3**, Proposition 5.3] for all k and therefore we get the homotopy type of $\mathcal{L}_{(S^1)^n}^{H\mathbb{Z}}(H\mathbb{Z}/p\mathbb{Z})$ as smash products over $H\mathbb{Z}/p\mathbb{Z}$ of $\binom{n}{k}$ copies of $\mathcal{L}_{S^k}^{H\mathbb{Z}}(H\mathbb{Z}/p\mathbb{Z})$ for $1 \leqslant k \leqslant n$.

Corollary 3.9. *For every commutative ring R and every regular element $a \in R$ the square-zero extension*

$$HR/a \to HR/a \vee \Sigma HR/a$$

is multiplicatively stable. In particular, for every commutative ring R, $HR \to HR \vee \Sigma HR$ is multiplicatively stable.

Proof. As $HR \to HR/a$ is multiplicatively stable we get by Lemma 3.3 that

$$\mathcal{L}_X^{HR/a}(HR/a \wedge_{HR} HR/a) \simeq HR/a \wedge_{HR} \mathcal{L}_X^{HR}(HR/a)$$

as augmented commutative $HR/a \wedge_{HR} HR/a$-algebras and so $HR/a \to HR/a \wedge_{HR} HR/a$ is multiplicatively stable. The Künneth spectral sequence gives $\pi_*(HR/a \wedge_{HR} HR/a) \cong \Lambda_{R/a}(x)$ with $|x| = 1$. By [**8**, Proposition 2.1] this implies that

$$HR/a \wedge_{HR} HR/a \simeq HR/a \vee \Sigma HR/a$$

as a commutative augmented HR/a-algebra.

Considering the regular element $t \in R[t]$ gives that $HR \to HR \vee \Sigma HR$ is multiplicatively stable. □

Stability is inherited by Loday constructions.

Proposition 3.10. *If $R \to A$ is multiplicatively stable, then so is $R \to \mathcal{L}_Z^R(A)$ for any Z.*

Proof. Assume that $\Sigma X \simeq \Sigma Y$. As $\Sigma(X \times Z) \simeq \Sigma X \vee \Sigma Z \vee \Sigma X \wedge Z$ we get that

$$\Sigma(X \times Z) \simeq \Sigma X \vee \Sigma Z \vee \Sigma X \wedge Z \simeq \Sigma Y \vee \Sigma Z \vee \Sigma Y \wedge Z \simeq \Sigma(Y \times Z)$$

and thus, as $R \to A$ is multiplicatively stable

$$\mathcal{L}_X^R(\mathcal{L}_Z^R(A)) \simeq \mathcal{L}_{X \times Z}^R(A) \simeq \mathcal{L}_{Y \times Z}^R(A) \simeq \mathcal{L}_Y^R(\mathcal{L}_Z^R(A)).$$ □

Remark 3.11. One can interpret Proposition 3.10 as the statement that Loday constructions preserve stability because for all Z there is an equivalence of augmented commutative R-algebras $R \simeq \mathcal{L}_Z^R(R)$.

The Loday construction behaves nicely with respect to pushouts:

Lemma 3.12. *If $C \leftarrow A \to B$ is a diagram of cofibrations of commutative R algebras and if A is cofibrant as a commutative R-algebra, then*

$$\mathcal{L}_X^R(C \wedge_A B) \simeq \mathcal{L}_X^R(C) \wedge_{\mathcal{L}_X^R(A)} \mathcal{L}_X^R(B).$$

Proof. This equivalence is proven using an exchange of priorities in a colimit diagram based on the equivalence

$$(C \wedge_A B) \wedge_R (C \wedge_A B) \simeq (C \wedge_R C) \wedge_{A \wedge_R A} (B \wedge_R B).$$ □

Remark 3.13. Beware that the above identification does *not* imply that multiplicative stability is closed under pushouts in the category of commutative R-algebras. Knowing that $\mathcal{L}_X^R(D) \simeq \mathcal{L}_Y^R(D)$ as commutative augmented D-algebras for $D = A, B$ and C does not imply that $\mathcal{L}_X^R(C \wedge_A B)$ is equivalent to $\mathcal{L}_Y^R(C \wedge_A B)$ because we cannot guarantee that the equivalences $\mathcal{L}^R_X(D) \simeq \mathcal{L}^R_Y(D)$ commute with the structure maps in the pushout diagram.

For example we know that $H\mathbb{Q} \to H\mathbb{Q}[t]$ and $H\mathbb{Q} \to H\mathbb{Q}[t,x]$ are multiplicatively stable, but $H\mathbb{Q} \to H\mathbb{Q}[t]/t^2$ is not stable by [**9**], despite the fact that we can express the latter as a pushout $H\mathbb{Q}[t] \wedge_{H\mathbb{Q}[t,x]} H\mathbb{Q}[t]$ where x maps to t^2 on the left hand side and to 0 on the right hand side.

In the case of $A = R$ we *do* get a stability result:

Corollary 3.14. *Assume that $R \to B$ and $R \to C$ are multiplicatively stable. Then so is $R \to B \wedge_R C$.*

Proof. If $\Sigma X \simeq \Sigma Y$ in sSets_*, then by assumption we have that $\mathcal{L}^R_X(B) \simeq \mathcal{L}^R_Y(B)$ and $\mathcal{L}^R_X(C) \simeq \mathcal{L}^R_Y(C)$ and these equivalences are of commutative augmented B- and C-algebras, so in particular of commutative augmented R-algebras. Note that $\mathcal{L}^R_X(R) \simeq R$ for all pointed X. Hence by Lemma 3.12 we obtain

$$\mathcal{L}^R_X(B \wedge_R C) \simeq \mathcal{L}^R_X(B) \wedge_R \mathcal{L}^R_X(C) \simeq \mathcal{L}^R_Y(B) \wedge_R \mathcal{L}^R_Y(C) \simeq \mathcal{L}^R_Y(B \wedge_R C)$$

and this is an equivalence of commutative augmented $B \wedge_R C$-algebras. □

Example 3.15. We know from Proposition 3.7 that $HR \to HR/a$ is multiplicatively stable for every commutative ring R and every regular element $a \in R$. Corollary 3.14 implies that $HR \to HR/a \wedge_{HR} HR/a$ is multiplicatively stable and as before we know that $HR/a \wedge_{HR} HR/a \simeq HR/a \vee \Sigma HR/a$, so $HR \to HR/a \vee \Sigma HR/a$ is multiplicatively stable. For instance $H\mathbb{Z} \to H\mathbb{Z}/p \vee \Sigma H\mathbb{Z}/p$ is multiplicatively stable for all primes p.

Example 3.16. Taking the coproduct (with a cofibrant model of $H\mathbb{Z}[t]$ as a commutative $H\mathbb{Z}$-algebra)

$$\begin{array}{ccc} H\mathbb{Z} & \longrightarrow & H\mathbb{Z}/p \\ \downarrow & & \downarrow \\ H\mathbb{Z}[t] & \longrightarrow & H\mathbb{Z}[t] \wedge^L_{H\mathbb{Z}} H\mathbb{Z}/p \simeq H\mathbb{Z}/p[t] \end{array}$$

shows that $H\mathbb{Z} \to H\mathbb{Z}/p[t]$ is multiplicatively stable.

Corollary 3.17. *Let R be a commutative ring and assume that $(a_1, \ldots, a_n)$ is a regular sequence in R. Then $HR \to HR/(a_1, \ldots, a_n)$ is multiplicatively stable.*

Proof. We use induction. We have shown in Proposition 3.7 that $HR \to HR/a_1$ is multiplicatively stable, so we can inductively assume that $HR \to HR/(a_1, \ldots, a_{n-1})$ is multiplicatively stable. We use the fact that the coproduct $HR/(a_1, \ldots, a_{n-1}) \wedge^L_{HR} HR/a_n$ of

$$\begin{array}{ccc} HR & \longrightarrow & HR/(a_1, \ldots, a_{n-1}) \\ \downarrow & & \\ HR/a_n & & \end{array}$$

is $HR/(a_1, \ldots, a_n)$, and then by Corollary 3.14 the claim follows.

This identification of the coproduct can be proven using the Künneth spectral sequence

$$Tor_*^R(R/(a_1,\dots,a_{n-1}), R/a_n) \Rightarrow \pi_*(HR/(a_1,\dots,a_{n-1}) \wedge^L_{HR} HR/a_n).$$

The Tor can be calculated by taking the standard free resolution $0 \longrightarrow R \xrightarrow{a_n} R$ of R/a_n and tensoring it with $R/(a_1,\dots,a_{n-1})$ to obtain

$$0 \longrightarrow R/(a_1,\dots,a_{n-1}) \otimes_R R \xrightarrow{\mathrm{id}\otimes a_n} R/(a_1,\dots,a_{n-1}) \otimes_R R.$$

Since multiplication by a_n is injective on $R/(a_1,\dots,a_{n-1})$, the E^2 term of the spectral sequence consists only of $R/(a_1,\dots,a_n)$ and we are done. □

Proposition 3.18. *Assume that $S \to A$ and $S \to B$ are cofibrations of commutative S-algebras such that $S \to A$ and $S \to B$ are multiplicatively stable. If X and Y are connected and $\Sigma X \simeq \Sigma Y$, then*

$$\mathcal{L}^S_X(A \times B) \simeq \mathcal{L}^S_Y(A \times B)$$

as commutative S-algebras.

Proof. This follows from [**3**, Proposition 8.4] because $\mathcal{L}^S_X(A \times B) \simeq \mathcal{L}^S_X(A) \times \mathcal{L}^S_X(B)$ as commutative S-algebras. □

The following notion is investigated in [**19, 18**].

Definition 3.19. Let $R \to A \to B$ be a sequence of cofibrations of commutative S-algebras with R cofibrant. Then this sequence *satisfies étale descent* if for all connected X the canonical map

$$\mathcal{L}^R_X(A) \wedge_A B \to \mathcal{L}^R_X(B)$$

is an equivalence.

If $R \to A \to B$ satisfies étale descent and if X is not connected, so for example $X = X_1 \sqcup X_2$ with X_i connected for $i = 1, 2$, then the formula becomes

$$\mathcal{L}^R_X(B) = \mathcal{L}^R_{X_1 \sqcup X_2}(B) \simeq \mathcal{L}^R_{X_1}(B) \wedge_R \mathcal{L}^R_{X_2}(B) \simeq \mathcal{L}^R_{X_1}(A) \wedge_A B \wedge_R \mathcal{L}^R_{X_2}(A) \wedge_A B.$$

The property of satisfying étale descent is closed under smashing with a fixed commutative S-algebra:

Lemma 3.20. *If $R \to A \to B$ satisfies étale descent and if C is a cofibrant commutative R-algebra, then $C \to C \wedge_R A \to C \wedge_R B$ satisfies étale descent.*

Proof. We know from Lemma 3.3 that $\mathcal{L}^C_X(C \wedge_R A) \simeq C \wedge_R \mathcal{L}^R_X(A)$. Therefore an exchange of pushouts yields

$$\begin{aligned}\mathcal{L}^C_X(C \wedge_R A) \wedge_{(C \wedge_R A)} (C \wedge_R B) &\simeq (C \wedge_R \mathcal{L}^R_X(A)) \wedge_{(C \wedge_R A)} (C \wedge_R B)\\ &\simeq (C \wedge_C C) \wedge_{R \wedge_R R} (\mathcal{L}^R_X(A) \wedge_A B)\\ &\simeq C \wedge_R \mathcal{L}^R_X(B) \simeq \mathcal{L}^C_X(C \wedge_R B).\end{aligned}$$ □

In the case of étale descent we can extend stable maps and get maps that are stable for connected X:

Proposition 3.21. *Let $R \to A \to B$ be a sequence of cofibrations of commutative S-algebras with R cofibrant. If $R \to A$ is multiplicatively stable and if $R \to A \to B$ satisfies étale descent, then if $\Sigma X \simeq \Sigma Y$ in sSets_* for connected X and Y we can conclude that there is a weak equivalence of augmented commutative B-algebras*

$$\mathcal{L}_X^R(B) \simeq \mathcal{L}_Y^R(B).$$

Proof. As X and Y are connected and as $R \to A$ is stable, the equivalence $\Sigma X \simeq \Sigma Y$ in sSets_* implies that $\mathcal{L}_X^R(A) \simeq \mathcal{L}_Y^R(A)$ and with étale descent we can upgrade this to

$$\mathcal{L}_X^R(B) \simeq \mathcal{L}_X^R(A) \wedge_A B \simeq \mathcal{L}_Y^R(A) \wedge_A B \simeq \mathcal{L}_Y^R(B). \qquad \square$$

Remark 3.22. We know that $H\mathbb{Q} \to H\mathbb{Q}[t]$ is stable and as $\mathbb{Q}[t]/t^2$ and $\mathbb{Q}[t]$ are commutative augmented $\mathbb{Q}$-algebras, we also know that $H\mathbb{Q}[t]/t^2 \to H\mathbb{Q}$ and $H\mathbb{Q}[t] \to H\mathbb{Q}$ are stable, but since $H\mathbb{Q} \to H\mathbb{Q}[t]/t^2$ and $H\mathbb{Q} \to H\mathbb{Q}[t]/t^2 \to H\mathbb{Q}$ are not stable, we won't have general descent results. For instance in the diagram

$$\begin{tikzcd} & \mathbb{Q}[t]/t^2 \arrow[d, "\varepsilon"] \\ \mathbb{Q} \arrow[ur, "\eta"] \arrow[r, equal] & \mathbb{Q} \end{tikzcd}$$

the maps $H(\varepsilon)$ and the identity on $H\mathbb{Q}$ are (even multiplicatively) stable, but $H\eta$ isn't.

For morphisms that are faithful Galois extensions and satisfy étale descent, we obtain a descent result for multiplicative stability:

Theorem 3.23. *Let $A \to B$ be a faithful Galois extension with finite Galois group G and assume that $A \to B$ satisfies étale descent. Assume that $\Sigma X \simeq \Sigma Y$ for connected X and Y implies that there is a G-equivariant equivalence $\mathcal{L}_X^S(B) \simeq \mathcal{L}_Y^S(B)$ as commutative B-algebras. Then also $\mathcal{L}_X^S(A) \simeq \mathcal{L}_Y^S(A)$ as commutative A-algebras.*

Proof. The base-change result for Galois extensions [**21**, Lemma 7.1.1] applied to the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ \mathcal{L}_X^S(A) \arrow[r] & B \wedge_A \mathcal{L}_X^S(A) \end{tikzcd}$$

yields that $\mathcal{L}_X^S(A) \to B \wedge_A \mathcal{L}_X^S(A)$ is a G-Galois extension and by étale descent there is an equivalence of augmented commutative B-algebras $B \wedge_A \mathcal{L}_X^S(A) \simeq \mathcal{L}_X^S(B)$ which is G-equivariant where on the left hand side the only non-trivial G-action is on the B-factor and on the right hand side G-acts on $\mathcal{L}_X^S(B)$ by naturality in B. Hence we get a chain of G-equivariant equivalences of commutative B-algebras

$$B \wedge_A \mathcal{L}_X^S(A) \simeq \mathcal{L}_X^S(B) \simeq \mathcal{L}_Y^S(B) \simeq B \wedge_A \mathcal{L}_Y^S(A).$$

Taking G-homotopy fixed points then gives an equivalence of augmented commutative A-algebras

$$\mathcal{L}_X^S(A) \simeq \mathcal{L}_X^S(B)^{hG} \simeq \mathcal{L}_Y^S(B)^{hG} \simeq \mathcal{L}_Y^S(A). \qquad \square$$

There exist several definitions of smoothness in the literature (see for instance [**21, 19**]) using THH-étaleness and TAQ-étaleness. Using the local behaviour of smooth commutative k-algebras [**16**, Appendix E, Proposition E.2 (d)] as a template we suggest the following variant.

Definition 3.24. We call a map of cofibrant S-algebras $\varphi\colon R \to A$ *really smooth* if it can be factored as $R \xrightarrow{i_R} \mathbb{P}_R(X) \xrightarrow{f} A$ where i_R is the canonical inclusion, X is an R-module, and $R \xrightarrow{i_R} \mathbb{P}_R(X) \xrightarrow{f} A$ satisfies étale descent.

Combining Proposition 3.21 and Corollary 2.10 we get:

Proposition 3.25. *If $R \to A$ is really smooth then $\Sigma X \simeq \Sigma Y$ for connected X and Y implies*

$$\mathcal{L}_X^R(A) \simeq \mathcal{L}_Y^R(A)$$

as commutative R-algebras.

The notion of being really smooth is transitive and closed under base change.

Lemma 3.26.

- *If $\varphi\colon R \to A$ and $\psi\colon A \to B$ are really smooth, then so is $\psi \circ \varphi\colon R \to B$.*
- *If $\varphi\colon R \to A$ is really smooth and if C is a cofibrant commutative R-algebra, then $C \to C \wedge_R A$ is really smooth.*

Proof. To prove the transitivity, take the factorizations $\varphi = R \xrightarrow{i_R} \mathbb{P}_R(X) \xrightarrow{f} A$ and $\psi = A \xrightarrow{i_A} \mathbb{P}_A(Y) \xrightarrow{g} B$ and combine them to give

$$R \xrightarrow{i_R} \mathbb{P}_R(X \vee Y) \simeq \mathbb{P}_R(X) \wedge_R \mathbb{P}_R(Y) \xrightarrow{f \wedge_R \mathrm{id}} A \wedge_R \mathbb{P}_R(Y) \simeq \mathbb{P}_A(Y) \xrightarrow{g} B.$$

So it is enough to show that for general maps $f\colon D \to A, g\colon A \to B, h\colon B \to C$ of commutative R-algebras:

1. If f satisfies étale descent, then so does $f \wedge_R \mathrm{id}_C$ for every commutative R-algebra C.
2. If g and h satisfy étale descent, then so does $h \circ g$.

For (1) let X be connected. As $\mathcal{L}_X^R(-)$ commutes with pushouts (see Lemma 3.12), we get that $\mathcal{L}_X^R(A \wedge_R C) \simeq \mathcal{L}_X^R(A) \wedge_R \mathcal{L}_X^R(C)$. As f satisfies étale descent,

$$\mathcal{L}_X^R(A) \wedge_R \mathcal{L}_X^R(C) \simeq A \wedge_D \mathcal{L}_X^R(D) \wedge_R \mathcal{L}_X^R(C) \simeq A \wedge_D \mathcal{L}_X^R(D \wedge_R C)$$

and this in turn is equivalent to $(A \wedge_R C) \wedge_{(D \wedge_R C)} \mathcal{L}_X^R(D \wedge_R C)$.

The proof of (2) is straightforward because

$$\begin{aligned} C \wedge_A \mathcal{L}_X^R(A) &\simeq C \wedge_B (B \wedge_A \mathcal{L}_X^R(A)) \\ &\simeq C \wedge_B \mathcal{L}_X^R(B) \\ &\simeq \mathcal{L}_X^R(C). \end{aligned}$$

For the claim about base change consider the diagram

$$\begin{tikzcd} R \arrow[r, "i_R"] \arrow[d, "\eta"'] & \mathbb{P}_R(X) \arrow[d] \\ C \arrow[r] & C \wedge_R \mathbb{P}_R(X). \end{tikzcd}$$

Adjunction gives that $C \wedge_R \mathbb{P}_R(X) \simeq \mathbb{P}_C(C \wedge_R X)$. Since $R \to \mathbb{P}_R(X) \to A$ satisfies étale descent we obtain with Lemma 3.20 that $C \to C \wedge_R \mathbb{P}_R(X) \to C \wedge_R A$ satisfies étale descent. □

4. Truncated polynomial algebras

Note that we know that the square zero extensions $H\mathbb{F}_p \to H\mathbb{F}_p \vee \Sigma^3 H\mathbb{F}_p$ (Example 2.8) and $H\mathbb{F}_p \to H\mathbb{F}_p \vee \Sigma H\mathbb{F}_p$ (Corollary 3.9) are multiplicatively stable. However, if we place the module $H\mathbb{F}_p$ in degree zero, then the following result shows that the square zero extension $H\mathbb{F}_p \to H\mathbb{F}_p \vee H\mathbb{F}_p \simeq H\mathbb{F}_p[t]/t^2$ is *not* multiplicatively stable for odd primes p. The proof is a direct adaptation of [**9**, §3.8].

Theorem 4.1. *Let p be an odd prime. Then $(H\mathbb{F}_p, H\mathbb{F}_p[t]/t^2, H\mathbb{F}_p)$ is not stable.*

Corollary 4.2. *For an odd prime p, neither is $H\mathbb{F}_p \to H\mathbb{F}_p[t]/t^2$ multiplicatively stable nor is it $H\mathbb{F}_p[t]/t^2$-linearly stable.*

Proof. If it were, then this would imply that $(H\mathbb{F}_p, H\mathbb{F}_p[t]/t^2, H\mathbb{F}_p)$ is stable. □

Remark 4.3. In [**14**, Theorem 4.18] we extend Theorem 4.1 to $\mathbb{F}_p[t]/t^n$ for $2 \leqslant n < p$.

Proof of Theorem 4.1. We know that

$$\begin{aligned}\pi_*\mathcal{L}^{H\mathbb{F}_p}_{S^1\vee S^1\vee S^2}&(H\mathbb{F}_p[t]/t^2; H\mathbb{F}_p)\\ &\cong \pi_*\mathcal{L}^{H\mathbb{F}_p}_{S^1}(H\mathbb{F}_p[t]/t^2; H\mathbb{F}_p)^{\otimes_{\mathbb{F}_p} 2} \otimes_{\mathbb{F}_p} \pi_*\mathcal{L}^{H\mathbb{F}_p}_{S^2}(H\mathbb{F}_p[t]/t^2; H\mathbb{F}_p)\end{aligned}$$

and by [**3**] we know what the tensor factors are:

$$\pi_*\mathcal{L}^{H\mathbb{F}_p}_{S^1}(H\mathbb{F}_p[t]/t^2; H\mathbb{F}_p) \cong \mathsf{HH}^{\mathbb{F}_p}_*(\mathbb{F}_p[t]/t^2; \mathbb{F}_p) \cong \Lambda_{\mathbb{F}_p}(\varepsilon t) \otimes_{\mathbb{F}_p} \Gamma_{\mathbb{F}_p}(\varphi^0 t)$$

and

$$\begin{aligned}\pi_*\mathcal{L}^{H\mathbb{F}_p}_{S^2}(H\mathbb{F}_p[t]/t^2; H\mathbb{F}_p) &\cong \mathsf{HH}^{[2],\mathbb{F}_p}_*(\mathbb{F}_p[t]/t^2; \mathbb{F}_p)\\ &\cong \Gamma_{\mathbb{F}_p}(\varrho^0\varepsilon t) \otimes \bigotimes_k (\Lambda_{\mathbb{F}_p}(\varepsilon\varphi^k t) \otimes \Gamma_{\mathbb{F}_p}(\varphi^0\varphi^k t)).\end{aligned}$$

Here, the degrees of the generators are $|\varepsilon w| = 1 + |w|$ for any w, $|\varrho^0\varepsilon t| = 2$, and $|\varphi^k w| = p^k(2 + p|w|)$ for any w.

Torus homology is the total complex of the bicomplex for $\mathcal{L}^{\mathbb{F}_p}_{S^1\times S^1}(\mathbb{F}_p[t]/t^2; \mathbb{F}_p)$ as in [**9**]. In the bicomplex in bidegree (n, m) we have the term

$$\mathcal{L}^{\mathbb{F}_p}_{[n]\times[m]}(\mathbb{F}_p[t]/t^2; \mathbb{F}_p) \cong \mathbb{F}_p \otimes_{\mathbb{F}_p} (\mathbb{F}_p[t]/t^2)^{(n+1)(m+1)-1} \cong (\mathbb{F}_p[t]/t^2)^{(n+1)(m+1)-1}.$$

In total degree one we have contributions from $(0, 1)$ and $(1, 0)$ that we call y^v_1 and y^h_1 as in [**9**, §3.8]. Everything is a cycle here and these elements correspond to $\begin{smallmatrix}1\\ \otimes\\ t\end{smallmatrix}$ and $1 \otimes t$.

From now on we suppress the tensor signs from the notation and we denote the generators by matrices. In total degree two there are three possibilities $(0, 2)$, $(1, 1)$

and $(2,0)$. There are the classes y_2^v in bidegree $(0,2)$, and y_2^h in bidegree $(2,0)$ corresponding to the standard Hochschild generators $\begin{pmatrix} 1 \\ t \\ t \end{pmatrix}$ and $\begin{pmatrix} 1 & t & t \end{pmatrix}$.

In bidegree $(1,1)$ there are the following possibilities for non-degenerate cycles:

$$\begin{pmatrix} 1 & t \\ t & t \end{pmatrix}, \begin{pmatrix} 1 & t \\ t & 1 \end{pmatrix}, \begin{pmatrix} 1 & t \\ 1 & t \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ t & t \end{pmatrix}, \text{ and } \begin{pmatrix} 1 & 1 \\ 1 & t \end{pmatrix}.$$

As we are working over $\mathbb{F}_p$ for an odd prime p, 2 is invertible. The boundary of $\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & t \\ 1 & t \end{pmatrix}$ is $\begin{pmatrix} 1 & t \\ 1 & t \end{pmatrix}$, the boundary of $\frac{1}{2}\begin{pmatrix} 1 & 1 & 1 \\ 1 & t & t \end{pmatrix}$ is $\begin{pmatrix} 1 & 1 \\ t & t \end{pmatrix}$. Finally, we identify $\begin{pmatrix} 1 & t \\ t & t \end{pmatrix}$ as the boundary of $\begin{pmatrix} 1 & 1 \\ 1 & t \\ t & t \end{pmatrix}$.

The element $\begin{pmatrix} 1 & 1 \\ t & 1 \\ 1 & t \end{pmatrix}$ ensures that $\begin{pmatrix} 1 & t \\ t & 1 \end{pmatrix}$ is homologous to $\begin{pmatrix} 1 & 1 \\ t & t \end{pmatrix}$, so we are left with the generator in bidegree $(1,1)$ given by $\begin{pmatrix} 1 & 1 \\ 1 & t \end{pmatrix}$.

So we get (at most) a 3-dimensional vector space in total degree 2.

In $\pi_2\mathcal{L}^{H\mathbb{F}_p}_{S^1\vee S^1\vee S^2}(H\mathbb{F}_p[t]/t^2; H\mathbb{F}_p)$, however, we get the generators $\varphi^0 t\otimes 1\otimes 1$, $1\otimes \varphi^0 t\otimes 1$, $\varepsilon t\otimes \varepsilon t\otimes 1$ and $1\otimes 1\otimes \varrho^0\varepsilon t$, so we have a 4-dimensional vector space. □

Remark 4.4. For odd primes 2 is invertible and this reduces the number of generators in total degree 2 to 3 in the torus homology of $\mathbb{F}_p[t]/t^2$ over $\mathbb{F}_p$ with $\mathbb{F}_p$-coefficients. For $p = 2$, there is an extra class coming from $\begin{pmatrix} 1 & t \\ t & 1 \end{pmatrix}$ which is homologous to $\begin{pmatrix} 1 & 1 \\ t & t \end{pmatrix}$ and to $\begin{pmatrix} 1 & t \\ 1 & t \end{pmatrix}$ so together with the class $\begin{pmatrix} 1 & 1 \\ 1 & t \end{pmatrix}$ this gives two generators in bidegree $(1,1)$ and the ones in $(2,0)$ and $(0,2)$ giving a total of dimension 4. As $\mathbb{F}_2[t]/t^2$ is a commutative Hopf algebra over $\mathbb{F}_2$, we know that $\mathbb{F}_2 \to \mathbb{F}_2[t]/t^2$ and $\mathbb{F}_2 \to \mathbb{F}_2[t]/t^2 \to \mathbb{F}_2$ are stable.

We can model $S[t]/t^n$ as $S \wedge \Pi_+$ for the commutative pointed monoid $\Pi_+ = \{+, 1, t, \ldots, t^{n-1}\}$. Using the generalization of Theorem 4.1 to all $2 \leqslant n < p$ in [**14**, Theorem 4.18], we obtain

Corollary 4.5. *For every $n \geqslant 2$ the map $S \to S[t]/t^n$ is not multiplicatively stable.*

Proof. If it were multiplicatively stable, then by Lemma 3.3 $H\mathbb{F}_p \to H\mathbb{F}_p[t]/t^n$ would be as well. For $n = 2$ this contradicts the result above. For higher n, there is a prime p with $p > n$, and then [**14**, Theorem 4.18] yields that $H\mathbb{F}_p \to H\mathbb{F}_p[t]/t^n$ isn't multiplicatively stable. □

Remark 4.6. Neither stability nor multiplicative stability are transitive: for every commutative ring k the map $k \to k[t]$ is smooth, hence (multiplicatively) stable and $k[t] \to k[t]/t^2$ is stable by Proposition 3.7, but for $k = \mathbb{Q}$ Dundas and Tenti show [**9**] that $\mathbb{Q} \to \mathbb{Q}[t]/t^2$ is not stable and for an odd prime p and $k = \mathbb{F}_p$ we know that $\mathbb{F}_p \to \mathbb{F}_p[t]/t^2$ is not multiplicatively stable.

Proposition 4.7. *Let k be a field and let Π_+ be a pointed commutative monoid. If $S \to Hk$ and $Hk \to Hk[\Pi_+]$ are multiplicatively stable, then $\Sigma X \simeq \Sigma Y$ in sSets_* implies that $\mathcal{L}^S_X(Hk[\Pi_+]) \simeq \mathcal{L}^S_Y(Hk[\Pi_+])$ as augmented commutative Hk-algebras.*

Proof. This follows from the splitting of $\mathcal{L}^S_X(Hk[\Pi_+])$ as a commutative augmented Hk-algebra [**15**, Theorem 7.1] as

$$\mathcal{L}^S_X(Hk[\Pi_+]) \simeq \mathcal{L}^S_X(Hk) \wedge_{Hk} \mathcal{L}^{Hk}_X(Hk[\Pi_+]). \tag{4.1}$$

□

It is important to know whether $S \to Hk$ is Hk-linearly stable, because if it is, then for all Hk-linearly stable $Hk \to HA$ that satisfy a splitting formula as in (4.1), such as polynomial algebras, we would get that $S \to HA$ is Hk-linearly stable.

Of course, $S \to H\mathbb{Q}$ is multiplicatively stable because $S_{\mathbb{Q}} \simeq H\mathbb{Q}$ and $H\mathbb{Q} \wedge_S H\mathbb{Q} \simeq H\mathbb{Q}$. We do not know whether $S \to H\mathbb{F}_p$ is stable. We will discuss Thom spectra and stability in the next section. We can express $H\mathbb{F}_p$ as a Thom spectrum, but this Thom spectrum structure comes from a double loop map, so it is not of the form needed for Corollary 5.1. So we leave this as an open question:

Is $H\mathbb{F}_p$ stable?

We close with a family of examples that show that several of the Juggling Formulas from [**3**] cannot be generalized to arbitrary pointed simplicial sets because that would contradict certain non-stability results.

Let k be a field. The case $k \to k[t] = R \to k[t]/t^m = R/t^m \to k = R/t$ for $m \geqslant 2$ is special in the sense that the quotient $k[t]/t^m$ is itself a commutative augmented k-algebra, so we can combine our splitting result for higher order Shukla homology [**3**, Proposition 7.5] with the Juggling Formula [**3**, Theorem 3.3]. We have [**3**, Theorem 7.6]:

$$\mathsf{THH}^{[n]}(k[t]/t^m; k) \simeq \mathsf{THH}^{[n]}(k[t]; k) \wedge_{Hk} \mathsf{THH}^{[n],Hk[t]}(k[t]/t^m; k) \tag{4.2}$$

for all $n \geqslant 1$ and for all $m \geqslant 2$. In this special case we can get the following Hk-version of this result:

Theorem 4.8. *Let k be a field and let m be greater or equal to 2. Then for all $n \geqslant 1$*

$$\mathsf{HH}^{[n],k}(k[t]/t^m; k) \simeq \mathsf{HH}^{[n],k}(k[t]; k) \wedge^L_{Hk} \mathsf{THH}^{[n],Hk[t]}(k[t]/t^m; k). \tag{4.3}$$

Proof. Consider the diagram

$$\begin{tikzcd}
Hk \arrow[r] \arrow[d] & \mathsf{THH}^{[n-1],Hk[t]}(k) \arrow[r] \arrow[d] & \mathsf{HH}^{[n],k}(k[t]; k) \arrow[d] \\
\mathsf{THH}^{[n],Hk[t]}(k[t]/t^m; k) \arrow[r] & \mathsf{THH}^{[n-1],Hk[t]/t^m}(k) \arrow[r] & \mathsf{HH}^{[n],k}(k[t]/t^m; k).
\end{tikzcd}$$

The left-hand square is a homotopy pushout square by [**3**, Proposition 7.5] and the juggling formula [**3**, Theorem 3.3] applied to $Hk \to Hk[t] \to Hk[t]/t^m \to Hk$ ensures that the right-hand square is also a homotopy pushout square because for all $n \geqslant 1$

$$\mathsf{HH}^{[n],k}(k[t]/t^m;k) \simeq \mathsf{HH}^{[n],k}(k[t];k) \wedge^L_{\mathsf{THH}^{[n-1],Hk[t]}(k)} \mathsf{THH}^{[n-1],Hk[t]/t^m}(k).$$

This yields that the outer rectangle is also a homotopy pushout square and this was the claim. □

Remark 4.9. Note that there cannot be a version of (4.2) and (4.3) for arbitrary connected X: We know that $Hk[t] \to Hk[t]/t^2 \to Hk$ is multiplicatively stable for all fields k and we know that $Hk \to Hk[t] \to Hk$ is stable. But for any odd prime p we know that $H\mathbb{F}_p \to H\mathbb{F}_p[t]/t^2 \to H\mathbb{F}_p$ is not stable and that there is an actual discrepancy between

$$\pi_2(\mathcal{L}^{H\mathbb{F}_p}_{S^1\times S^1}(H\mathbb{F}_p[t]/t^2;H\mathbb{F}_p)) \ncong \pi_2(\mathcal{L}^{H\mathbb{F}_p}_{S^1\vee S^1\vee S^2}(H\mathbb{F}_p[t]/t^2;H\mathbb{F}_p)),$$

so $\mathcal{L}^{H\mathbb{F}_p}_{S^1\times S^1}(H\mathbb{F}_p[t]/t^2;H\mathbb{F}_p)$ and $\mathcal{L}^{H\mathbb{F}_p}_{S^1\vee S^1\vee S^2}(H\mathbb{F}_p[t]/t^2;H\mathbb{F}_p)$ cannot be equivalent.

5. Thom spectra and topological K-theory

Christian Schlichtkrull gives a closed formula for the Loday construction on Thom spectra [**22**, Theorem 1.1]: Let $f\colon W \to BF_{hI}$ be an E_∞-map with W grouplike and let $T(f)$ denote the corresponding Thom spectrum. Here, BF_{hI} is a model of the classifying space for stable spherical fibrations. Then for any $T(f)$-module spectrum M one has

$$\mathcal{L}^S_X(T(f);M) \simeq M \wedge \Omega^\infty(E_W \wedge X)_+, \tag{5.1}$$

where E_W is the Omega spectrum associated to W (*i.e.*, $W \simeq \Omega^\infty E_W$). If M is a commutative $T(f)$-algebra spectrum, then the above equivalence is one of commutative $T(f)$-algebras. For $M = T(f)$ the equivalence also respects the augmentation.

An immediate consequence of Schlichtkrull's result is the following:

Corollary 5.1. *If $T(f)$ is a Thom spectrum as above, then $S \to T(f)$ is multiplicatively stable.*

Proof. If $\Sigma X \simeq \Sigma Y$ in sSets_*, then on the level of spectra we obtain

$$\Sigma(E_W \wedge X) \simeq E_W \wedge \Sigma X \simeq E_W \wedge \Sigma Y \simeq \Sigma(E_W \wedge Y),$$

but here suspension is invertible, hence $E_W \wedge X \simeq E_W \wedge Y$ and therefore

$$\mathcal{L}^S_X(T(f)) \simeq \mathcal{L}^S_Y(T(f)).$$

An equivalence of spectra induces an equivalence of infinite loop spaces and the $T(f)$-algebra structure on $T(f) \wedge \Omega^\infty(E_W \wedge X)_+$ just comes from the one on $T(f)$ and the infinite loop structure on $\Omega^\infty(E_W \wedge X)$. This gives the multiplicativity of the stability. □

The case of the suspension spectrum of an abelian topological group is a special case where we take $f\colon G \to BF_{hI}$ to be the trivial map. Then $T(f) \simeq \Sigma^\infty_+(G)$. Other examples are MU, MO, MSO, MSp or $MSpin$.

Remark 5.2. Nima Rasekh, Bruno Stonek, and Gabriel Valenzuela [**20**, Theorem 4.13] generalize Schlichtkrull's calculation to generalized Thom spectra, *i.e.*, Thom spectra that are formed with respect to a map of E_∞-groups $f\colon G \to \mathrm{Pic}(R)$ for some commutative ring spectrum R. They note (see [**20**, Remark 4.17]) that this implies stability for such Thom spectra.

Remark 5.3. Note that by Corollary 4.5 spherical abelian monoid rings are *not* stable in general, whereas spherical abelian group rings are.

Bruno Stonek calculates higher THH of periodic complex topological K-theory, KU, and he determines topological André–Quillen homology of KU [**24**]. He uses Snaith's description of KU as the Bott localization of $\Sigma^\infty_+\mathbb{C}P^\infty$. The latter is a Thom spectrum because $\mathbb{C}P^\infty = BU(1)$ can be realized as a topological abelian group.

Theorem 5.4. *If X and Y are connected and $\Sigma X \simeq \Sigma Y$ in sSets_*, then*

$$\mathcal{L}^S_X(KU) \simeq \mathcal{L}^S_Y(KU)$$

as commutative augmented KU-algebra spectra.

Proof. Let β denote the Bott element. Stonek uses Snaith's identification of KU as $\Sigma^\infty_+\mathbb{C}P^\infty[\beta^{-1}]$ to prove [**24**, Corollary 4.12] that there is a zigzag of equivalences

$$\begin{tikzcd} \mathsf{THH}(KU) \simeq \mathsf{THH}(\Sigma^\infty_+\mathbb{C}P^\infty[\beta^{-1}]) & \mathsf{THH}(\Sigma^\infty_+\mathbb{C}P^\infty) \wedge_{\Sigma^\infty_+\mathbb{C}P^\infty} \Sigma^\infty_+\mathbb{C}P^\infty[\beta^{-1}] \arrow[l, "\simeq"'] \arrow[d, "\simeq"] \\ & (\mathsf{THH}(\Sigma^\infty_+\mathbb{C}P^\infty))[\beta^{-1}]. \end{tikzcd}$$

The same argument yields that for any connected X the localization of $\mathcal{L}^S_X(\Sigma^\infty_+\mathbb{C}P^\infty)$ at β is equivalent to $\mathcal{L}^S_X(\Sigma^\infty_+\mathbb{C}P^\infty[\beta^{-1}]) = \mathcal{L}^S_X(KU)$.

The localization map $\Sigma^\infty_+\mathbb{C}P^\infty \to \Sigma^\infty_+\mathbb{C}P^\infty[\beta^{-1}]$ satisfies étale descent, and therefore the composite $S \to \Sigma^\infty_+\mathbb{C}P^\infty \to \Sigma^\infty_+\mathbb{C}P^\infty[\beta^{-1}]$ identifies KU as an étale extension of a Thom spectrum. By Proposition 3.21 we obtain multiplicative stability for connected simplicial sets. □

Corollary 5.5. *If X and Y are connected simplicial sets with $\Sigma X \simeq \Sigma Y$ then we get that $\mathcal{L}^S_X(KO) \simeq \mathcal{L}^S_Y(KO)$ as commutative KO-algebras.*

Proof. Rognes shows [**21**, §5.3] that the complexification map $KO \to KU$ is a faithful C_2-Galois extension of commutative ring spectra and Mathew [**18**, Example 4.6] deduces from [**6**, Example 5.9] that it satisfies étale descent. Schlichtkrull's equivalence from (5.1) is natural hence it preserves the C_2-action. Therefore the result follows from Theorem 3.23. □

In [**5**] Hood Chatham, Jeremy Hahn, and Allen Yuan construct interesting examples of E_∞-ring spectra. For a prime p they consider the infinite loop space

$$W_h = \Omega^\infty\Sigma^{2\nu(h)}BP\langle h\rangle,$$

where $\nu(h) = \frac{p^{h+1}-1}{p-1}$. This is the $2\nu(h)$th space of the Omega spectrum for the h-truncated Brown–Peterson spectrum $BP\langle h\rangle$; these spaces were extensively studied by Steve Wilson [**26**]. On the suspension spectrum of W_h they invert the generator x of the bottom non-trivial homotopy group $\pi_{2\nu(h)}(W_h) \cong \mathbb{Z}_{(p)}$ and obtain an E_∞-ring spectrum

$$R_h := (\Sigma_+^\infty W_h)[x^{-1}],$$

which has remarkable features [**5**, Theorem 1.13]: R_h has torsion-free homotopy groups that vanish in odd degrees, it is Landweber exact, and its Morava-$K(n)$ localization $L_{K(n)}R_h$ vanishes if and only if $n > h+1$, so R_h is of chromatic height $h+1$. As W_0 is $\mathbb{C}P^\infty$, this recovers Snaith's construction in this special case, but there are many more interesting examples. For all of these spectra, the above method of proof applies, so we obtain.

Theorem 5.6. *If X and Y are connected and $\Sigma X \simeq \Sigma Y$ in sSets_*, then*

$$\mathcal{L}_X^S(R_h) \simeq \mathcal{L}_Y^S(R_h)$$

as commutative augmented R_h-algebra spectra.

6. The Greenlees spectral sequence

Let k be a field and let $A \to B$ be a morphism of connective commutative S-algebras with an augmentation to Hk satisfying some mild finiteness assumption. Then by [**12**, Lemma 3.1] there is a spectral sequence

$$E_{s,t}^2 = \pi_s(B \wedge_A Hk) \otimes_k \pi_t(A) \Rightarrow \pi_{s+t}(B).$$

Let p be an odd prime. We consider the cofibration $S^1 \vee S^1 \hookrightarrow S^1 \times S^1 \to S^2$ and the associated pushout diagram

$$\begin{tikzcd}
\mathcal{L}^R_{S^1\vee S^1}(H\mathbb{F}_p[t]/t^2; H\mathbb{F}_p) \arrow[r] \arrow[d] & \mathcal{L}^R_{S^1\times S^1}(H\mathbb{F}_p[t]/t^2; H\mathbb{F}_p) \arrow[d] \\
H\mathbb{F}_p \arrow[r] & \mathcal{L}^R_{S^2}(H\mathbb{F}_p[t]/t^2; H\mathbb{F}_p).
\end{tikzcd}$$

Here, R can be S or $H\mathbb{F}_p$. For $R = S$ we obtain a Greenlees spectral sequence

$$\begin{aligned}\pi_s(\mathcal{L}^S_{S^2}(H\mathbb{F}_p[t]/t^2; H\mathbb{F}_p)) \otimes_{\mathbb{F}_p} \pi_t(\mathcal{L}^S_{S^1\vee S^1}(H\mathbb{F}_p[t]/t^2; H\mathbb{F}_p)) \\ \Rightarrow \pi_{s+t}(\mathcal{L}^S_{S^1\times S^1}(H\mathbb{F}_p[t]/t^2; H\mathbb{F}_p)),\end{aligned} \tag{6.1}$$

whereas for $R = H\mathbb{F}_p$ the spectral sequence is

$$\begin{aligned}\pi_s(\mathcal{L}^{H\mathbb{F}_p}_{S^2}(H\mathbb{F}_p[t]/t^2; H\mathbb{F}_p)) \otimes_{\mathbb{F}_p} \pi_t(\mathcal{L}^{H\mathbb{F}_p}_{S^1\vee S^1}(H\mathbb{F}_p[t]/t^2; H\mathbb{F}_p)) \\ \Rightarrow \pi_{s+t}(\mathcal{L}^{H\mathbb{F}_p}_{S^1\times S^1}(H\mathbb{F}_p[t]/t^2; H\mathbb{F}_p)).\end{aligned} \tag{6.2}$$

In (6.1) every term $\mathcal{L}^S_X(H\mathbb{F}_p[t]/t^2; H\mathbb{F}_p)$ splits as

$$\mathcal{L}^S_X(H\mathbb{F}_p) \wedge_{H\mathbb{F}_p} \mathcal{L}^{H\mathbb{F}_p}_X(H\mathbb{F}_p[t]/t^2; H\mathbb{F}_p)$$

naturally in X, and going from working over S to working over $H\mathbb{F}_p$ simply collapses the $\mathcal{L}^S_X(H\mathbb{F}_p)$ to $\mathcal{L}^{H\mathbb{F}_p}_X(H\mathbb{F}_p) \simeq H\mathbb{F}_p$. Therefore we get a surjection of the spectral sequence of (6.1) onto the one of (6.2), and if all the spectral sequence differentials vanish on the former, they have to vanish on the latter too. But we know that the rank of $\pi_2(\mathcal{L}^{H\mathbb{F}_p}_{S^1\times S^1}(H\mathbb{F}_p[t]/t^2; H\mathbb{F}_p))$ is less than the rank of the E^2-term in total degree 2, hence there has to be a non-trivial differential in (6.2) and hence also in (6.1). This implies the following result.

Theorem 6.1. *For every odd prime p, $(S, H\mathbb{F}_p[t]/t^2, H\mathbb{F}_p)$ is not stable.*

With the results of [**14**, §4] the above result can be generalized to $\mathbb{F}_p[t]/t^n$ for $2 \leqslant n < p$.

Instead of stability we can consider the following property of Loday constructions.

Definition 6.2. Let R be a cofibrant commutative ring spectrum and let $R \to A \to C$ be a sequence of cofibrations of commutative R-algebras. We say that $R \to A \to C$ *decomposes products* if for all pointed connected simplicial sets X and Y

$$\mathcal{L}^R_{X\times Y}(A;C) \simeq \mathcal{L}^R_{X\vee Y\vee X\wedge Y}(A;C).$$

Note that the right hand side is equivalent to $\mathcal{L}^R_X(A;C)\wedge_C \mathcal{L}^R_Y(A;C)\wedge_C \mathcal{L}^R_{X\wedge Y}(A;C)$.

Proposition 6.3. *Let $R \to A \to B \to A \to Hk$ be a sequence of maps of commutative S-algebras that turns B into an augmented commutative A-algebra. Assume that k is a field.*

If $R \to B \to Hk$ decomposess products then so does $R \to A \to Hk$.

Proof. The naturality of the Loday construction ensures that the vertical compositions in the diagram

$$\begin{array}{ccccc}
\mathcal{L}^R_{X\vee Y}(A;Hk) & \longrightarrow & \mathcal{L}^R_{X\times Y}(A;Hk) & \longrightarrow & \mathcal{L}^R_{X\wedge Y}(A;Hk) \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{L}^R_{X\vee Y}(B;Hk) & \longrightarrow & \mathcal{L}^R_{X\times Y}(B;Hk) & \longrightarrow & \mathcal{L}^R_{X\wedge Y}(B;Hk) \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{L}^R_{X\vee Y}(A;Hk) & \longrightarrow & \mathcal{L}^R_{X\times Y}(A;Hk) & \longrightarrow & \mathcal{L}^R_{X\wedge Y}(A;Hk)
\end{array}$$

are the identity. Therefore the spectral sequence

$$\pi_s(\mathcal{L}^R_{X\wedge Y}(A;Hk)) \otimes_k \pi_t(\mathcal{L}^R_{X\vee Y}(A;Hk)) \Rightarrow \pi_{s+t}(\mathcal{L}^R_{X\times Y}(A;Hk))$$

is a direct summand of the one for B. So if the spectral sequence for A has a non-trivial differential, then the one for B also has to have one, but as B decomposes products, this cannot happen. $\square$

Note that this gives an additive splitting, but we can't rule out multiplicative extensions.

If B does not decompose products, then this does *not* imply that A doesn't either. A concrete counterexample is $S \to H\mathbb{Q} \to H\mathbb{Q}[t]/t^2 \to H\mathbb{Q}$. Here, $S \to H\mathbb{Q}[t]/t^2$ does not decompose products, but $S \to H\mathbb{Q}$ is even multiplicatively stable.

7. Rational equivalence

The starting point for this section is the following result:

Proposition 7.1. *(Berest, Ramadoss, and Yeung [2]) If k is a field of characteristic zero and A is a commutative Hopf algebra over k, then any rational equivalence $f\colon X \to Y$ between simply connected spaces induces a weak equivalence*

$$f_*\colon \mathcal{L}_X^{Hk}(HA) \simeq \mathcal{L}_Y^{Hk}(HA).$$

If $f\colon X \to Y$ is a rational equivalence between simply connected pointed spaces then f induces a weak equivalence

$$f_*\colon \mathcal{L}_X^{Hk}(HA; Hk) \simeq \mathcal{L}_Y^{Hk}(HA; Hk).$$

Proof. This follows from [**2**, Theorem 1.3 (a)] which says that for such k and A and any unbased simplicial set X,

$$\pi_*\mathcal{L}_X^{Hk}(HA) \cong \mathsf{HR}_*(\Sigma(X_+), A),$$

where HR is representation homology, and from [**2**, Proposition 4.2], which says that rational equivalences between simply connected spaces induce isomorphisms on representation homology for such k and A. In the pointed setting, [**2**, Theorem 1.3 (b)] applies to give the equivalence

$$\pi_*\mathcal{L}_X^{Hk}(HA; Hk) \cong \mathsf{HR}_*(\Sigma X, A; k). \qquad \square$$

We can extend Proposition 7.1 to augmented commutative finitely generated k-algebras:

Proposition 7.2. *If k is a field of characteristic zero and A is a finitely generated augmented commutative k-algebra, then any rational equivalence $f\colon X \to Y$ of simply connected spaces induces a weak equivalence*

$$f_*\colon \mathcal{L}_X^{Hk}(HA; Hk) \simeq \mathcal{L}_Y^{Hk}(HA; Hk).$$

Proof. Let A be generated by $a_1, a_2, \ldots, a_\ell$ as a commutative k-algebra, let $\varepsilon\colon A \to k$ be its augmentation, and let $\eta\colon k \to A$ be the unit map. We denote by I the augmentation ideal, $I = \ker \varepsilon$. Then for all $1 \leqslant i \leqslant \ell$, $a_i - \eta(\varepsilon(a_i)) \in I$, so we can define a surjection of augmented commutative k-algebras

$$\varphi\colon k[x_1, \ldots, x_\ell] \to A, \quad \varphi(x_i) = a_i - \eta(\varepsilon(a_i)) \text{ for all } 1 \leqslant i \leqslant \ell.$$

Here we consider the augmentation of $k[x_1, \ldots, x_\ell]$ that sends every x_i to zero, so that its augmentation ideal is $(x_1, \ldots, x_\ell)$.

Since k is a field, $k[x_1, \ldots, x_\ell]$ is Noetherian so we can find finitely many polynomials $f_1, \ldots, f_m$ in the x_i to generate $\ker \varphi$. Since $\varepsilon \circ \varphi$ is the augmentation of $k[x_1, \ldots, x_\ell]$, we get that for all $1 \leqslant j \leqslant m$, f_j is an element in $(x_1, \ldots, x_\ell)$. Hence, we can define another map of augmented commutative k-algebras

$$\psi\colon k[u_1, \ldots, u_m] \to k[x_1, \ldots, x_\ell], \quad \psi(u_i) = f_j \text{ for all } 1 \leqslant j \leqslant m,$$

which maps $k[u_1, \ldots, u_m]$ onto $\ker \varphi$. The augmentation of $k[u_1, \ldots, u_m]$ is again the standard one. We express A as a pushout of commutative augmented k-algebras

$$\begin{tikzcd} k[u_1, \dots, u_m] \arrow[r, "\psi"] \arrow[d] & k[x_1, \dots, x_\ell] \arrow[d, "\varphi"] \\ k \arrow[r, "\eta"] & A, \end{tikzcd}$$

where all entries except A are known to be commutative Hopf algebras over k. So for them, f induces a weak equivalence $\mathcal{L}^{Hk}_X(H(-); Hk) \to \mathcal{L}^{Hk}_Y(H(-); Hk)$. Since both $\mathcal{L}^{Hk}_X(H(-); Hk)$ and $\mathcal{L}^{Hk}_Y(H(-); Hk)$ send pushouts of augmented commutative k-algebras to homotopy pushouts of augmented commutative Hk-algebras, f also induces a weak equivalence on the pushout. $\square$

Let X be a pointed simply connected simplicial set. Then rationally

$$\Sigma X_{\mathbb{Q}} \simeq \bigvee_{i \in I} S^{k_i}_{\mathbb{Q}}$$

for some indexing set I and some $k_i \geqslant 2$ (see for instance [**11**, Theorem 24.5]). In particular,

$$\bigvee_{i \in I} S^{k_i}_{\mathbb{Q}} \simeq \Sigma \left(\bigvee_{i \in I} S^{k_i - 1}_{\mathbb{Q}} \right).$$

So with the help of [**2**, Theorem 1.3] we obtain:

Theorem 7.3. *For every pointed simply connected X, every field of characteristic zero k and every commutative Hopf-algebra A over k, for a suitable indexing set I and integers $k_i \geqslant 2$ we get*

$$\pi_* \mathcal{L}^k_{X_{\mathbb{Q}}}(A; k) \cong \pi_* \mathcal{L}^k_{\bigvee_{i \in I} S^{k_i - 1}_{\mathbb{Q}}}(A; k).$$

For simply-connected spaces and k and A as above we know by [**2**, Proposition 4.2] that the homotopy type of the Loday construction only depends on the rational homotopy type of the suspension, so we can discard the rationalization in the above statement. This yields, for instance:

Example 7.4. Let $X = \mathbb{C}P^n$ for some $n \geqslant 1$. Then for every field of characteristic zero k and every commutative Hopf-algebra A over k, as $\Sigma \mathbb{C}P^n_{\mathbb{Q}} \simeq \Sigma \bigvee_{i=1}^n S^{2i}_{\mathbb{Q}}$, we obtain

$$\pi_* \mathcal{L}^k_{\mathbb{C}P^n}(A; k) \cong \pi_* \mathcal{L}^k_{\bigvee_{i=1}^n S^{2i}}(A; k).$$

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