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ON THE PICARD GROUP GRADED HOMOTOPY GROUPS OF A FINITE TYPE TWO $K(2)$-LOCAL SPECTRUM AT THE PRIME THREE

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Abstract

Consider Hopkins' Picard group of the stable homotopy category of $E(2)$-local spectra at the prime three, consisting of homotopy classes of invertible spectra. Then, it is isomorphic to the direct sum of an infinite cyclic group and two cyclic groups of order three. We consider the Smith–Toda spectrum $V(1)$ and the cofiber V_2 of the square α^2 of the Adams map, which is a ring spectrum. In this paper, we introduce imaginary elements which make computation clearer, and determine the module structures of the Picard group graded homotopy groups $\pi_\star(V(1))$ and $\pi_\star(V_2)$.

1. Introduction

We work in the stable homotopy category $\mathcal{S}_{(3)}$ of spectra localized at the prime three. Consider the Brown–Peterson spectrum BP with coefficient algebra $\mathbb{Z}_{(3)}[v_1, v_2, \dots]$ on the generators v_i of degree $2 \times 3^i - 2$ for $i \geqq 1$. Then, the second Johnson–Wilson spectrum $E(2) \in \mathcal{S}_{(3)}$ is the spectrum representing the Landweber exact functor $E(2)_*(X) = E(2)_* \otimes_{BP_*} BP_*(X)$ for $E(2)_* = \mathbb{Z}_{(3)}[v_1, v_2, v_2^{-1}]$ on $X \in \mathcal{S}_{(3)}$. Let $\mathcal{L}_2$ denote the full subcategory of $\mathcal{S}_{(3)}$ consisting of spectra localized with respect to $E(2)$ in the sense of Bousfield. Then, we have the Bousfield localization functor $L_2\colon \mathcal{S}_{(3)} \to \mathcal{L}_2$, which is a retraction. A spectrum $X \in \mathcal{L}_2$ is called invertible if there is a spectrum Y such that $X \wedge Y = L_2 S^0$ for the sphere spectrum S^0. Hopkins' Picard group $\mathrm{Pic}(\mathcal{L}_2)$ is defined to be a group consisting of the homotopy equivalence classes of invertible spectra with multiplication defined by the smash product. For an element $\lambda \in \mathrm{Pic}(\mathcal{L}_2)$, S^λ denotes an invertible spectrum that represents λ. Note that $E(2)_*(S^\lambda) = E(2)_*$ shown by Hovey and Sadofsky [**2**]. In [**1**], Goerss, Henn, Mahowald and Rezk showed that $\mathrm{Pic}(\mathcal{L}_2)$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/3$. The generator of the summand $\mathbb{Z}$ is represented by $S^1 = \Sigma L_2 S^0$. Let ω_i for $i = 1, 2$ denote a generator of the i-th summand of $\mathbb{Z}/3 \oplus \mathbb{Z}/3 \subset \mathrm{Pic}(\mathcal{L}_2)$. The Picard group graded

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homotopy groups $\pi_\star(X)$ of a spectrum X is

$$\pi_\star(X) = \bigoplus_{\lambda \in \mathrm{Pic}(\mathcal{L}_2)} [S^\lambda, L_2X].$$

Note that $S^{a+b\omega_1+c\omega_2}$ for $a \in \mathbb{Z}$ and $b, c \in \mathbb{Z}/3$ is represented by the invertible spectrum $\Sigma^a(S^{\omega_1})^{\wedge b} \wedge (S^{\omega_2})^{\wedge c}$.

Let M denote the mod 3 Moore spectrum fitting in the cofiber sequence

$$S^0 \xrightarrow{3} S^0 \xrightarrow{i} M \xrightarrow{j} S^1. \tag{1}$$

For an integer $e \in \{1, 2\}$, we have spectra V_e given by the cofiber sequence

$$\Sigma^{4e}M \xrightarrow{\alpha^e} M \xrightarrow{i_e} V_e \xrightarrow{j_e} \Sigma^{4e+1}M, \tag{2}$$

for the Adams map α satisfying $E(2)_*(\alpha) = v_1$. Then,

$$E(2)_*(V_e) = E(2)_*/(3, v_1^e). \tag{3}$$

Note that $E(2)_*(V_1) = K(2)_*$, the coefficient algebra of the second Morava K-theory. The spectrum V_1 is the first Smith–Toda spectrum $V(1)$. We note that Toda [**10**] showed that V_1 is not a ring spectrum, while Oka [**6**] showed that V_2 is a ring spectrum. We tried to determine homotopy groups of $L_2V_1 = L_2V(1)$, $V_1 \wedge S^{\omega_1}$ and $V_1 \wedge S^{\omega_2}$ [**8, 4, 3**]. Unfortunately, there are some missing relations on the differential d_9 in [**3**], and the result is not correct. In this paper, we correct the result (see Remark 2.7), and furthermore, determine the additive structure of the homotopy groups of L_2V_2, $V_2 \wedge S^{\pm\omega_1}$ and $V_2 \wedge S^{\pm\omega_2}$. Our main tool is the $E(2)$-based Adams spectral sequence

$$E_2^{s,t}(X) = \mathrm{Ext}^{s,t}_{E(2)_*(E(2))}(E(2)_*, E(2)_*(X)) \Longrightarrow \pi_{t-s}(L_2X)$$

for a spectrum X. The generators of the E_2-terms behave very complicated in the spectral sequences. To make the behavior clearer, we introduce some imaginary generators. In order to compute E_r-terms, we consider differential algebras C_e for $e \in \{1, 2\}$, whose cohomologies are easily determined, so that the E_∞-terms for V_e are obtained from the cohomologies.

In the next section, we state our main theorem, the homotopy groups $\pi_*(V_e \wedge S^{l\omega_2})$ for $l \in \mathbb{Z}/3$, after introducing the elements. We determine the E_2-terms $E_2^{*,*+l\omega_2}(V_e)$ in section three, and the Adams–Novikov differentials d_5 and d_9 for $\pi_{*+l\omega_2}(V_e)$ in section four. Sections five and six are devoted to compute the cohomologies of the differential algebras C_1g^l and C_2g^l for $l \in \mathbb{Z}/3$, respectively. Here, g denotes a generator of $E(2)_*(S^{\omega_2})$. In the last section, we deduce our main theorems Theorems 2.6 and 2.8 from the results of the cohomologies of C_1g^l and C_2g^l.

2. Statement of results

By the 3×3 lemma, the cofiber sequences in (2) give rise to another cofiber sequence

$$\Sigma^4V_1 \xrightarrow{\overline{\alpha}} V_2 \xrightarrow{\overline{i}} V_1 \xrightarrow{\overline{j}} \Sigma^5V_1. \tag{4}$$

On the generator $\omega_1 \in \mathrm{Pic}(\mathcal{L}_2)$, we have the following:

Theorem 2.1 ([**4**, Th. A])**.** *There is a homotopy equivalence* $v_2^3\colon \Sigma^{48}V_1 \simeq V_1 \wedge S^{\omega_1}$.

Since $\pi_{-5}(L_2V_1) = 0$ by [**8**, Th. 10.6] (see Theorem 4.5), this theorem implies that $\pi_{43}(V_1 \wedge S^{\omega_1}) = 0$. It follows that $(\bar{j} \wedge 1)v_2^3 i_1 i = 0$ for v_2^3 in Theorem 2.1, and so $v_2^3 i_1 i \in \pi_{48}(V_1 \wedge S^{\omega_1})$ is pulled back to $\pi_{48}(V_2 \wedge S^{\omega_1})$ under $(\bar{i} \wedge 1)_*$. Notice that V_2 is a ring spectrum, and we obtain the following:

Proposition 2.2. *There is a homotopy equivalence* $v_2^3\colon \Sigma^{48}V_2 \simeq V_2 \wedge S^{\omega_1}$.

Consider the $E(2)$-based Adams spectral sequence

$$E_2^{s,t}(X) = \mathrm{Ext}_{E(2)_*(E(2))}^{s,t}(E(2)_*, E(2)_*(X)) \Longrightarrow \pi_{t-s}(L_2X)$$

for a spectrum X. The E_2-term is given by the cohomology of the cobar complex $\Omega^* E(2)_*(X)$ of the $E(2)_*(E(2))$-comodules. Here,

$$E(2)_*(E(2)) = E(2)_*[t_1, t_2, \dots] \otimes_{BP_*} E(2)_*$$

with $|t_i| = 2(3^i - 1)$. Note that

$$E(2)_*(S^{\omega_i}) = E(2)_*\{g_i\}$$

for $i \in \{1, 2\}$ and generators $g_i \in E(2)_0(S^{\omega_i})$ (see [**2**, Th. 2.4]).

Proposition 2.3. *Let* $e \in \{1, 2\}$. *The Picard graded homotopy groups* $\pi_{s+l_1\omega_1+l_2\omega_2}(L_2V_e)$ *for* $s \in \mathbb{Z}$ *and* $l_1, l_2 \in \mathbb{Z}/3$ *is isomorphic to* $\pi_{s+48l_1+l_2\omega_2}(L_2V_e)$.

We concentrate the determination of the homotopy groups $\pi_{s+l\omega_2}(L_2V_e)$ for $s \in \mathbb{Z}$, $l \in \mathbb{Z}/3$ and $e \in \{1, 2\}$, and abbreviate ω_2 and g_2 to ω and g, respectively.

For the homotopy equivalences v_2^3 in Theorem 2.1 and Proposition 2.2, consider the composite map $B_e\colon \Sigma^{144}V_e \xrightarrow{v_2^3} \Sigma^{96}V_e \wedge S^{\omega_1} \xrightarrow{v_2^3\wedge 1} \Sigma^{48}V_e \wedge S^{\omega_1} \wedge S^{\omega_1} \xrightarrow{v_2^3\wedge 1} V_e \wedge S^{\omega_1} \wedge S^{\omega_1} \wedge S^{\omega_1} = V_e$, in which $S^{\omega_1} \wedge S^{\omega_1} \wedge S^{\omega_1} = L_2S^0$ since $3\omega_1 = 0$.

Proposition 2.4. *There exist self maps* $B_e\colon \Sigma^{144}V_e \to V_e$ *for* $e \in \{1, 2\}$ *such that* $E(2)_*(B_e) = v_2^9\colon E(2)_*(V_e) \to E(2)_*(V_e)$.

The maps B_e induce the isomorphisms $(B_e)_*\colon \pi_{*+l\omega}(L_2V_e) \to \pi_{*+l\omega}(L_2V_e)$ of the homotopy groups as well as the isomorphisms $v_2^9\colon E_r^{*,*+l\omega}(V_e) \to E_r^{*,*+l\omega}(V_e)$ of the Adams–Novikov E_r-terms, and so it suffices to determine $E_r^{*,*+l\omega}(V_e) \otimes_{K^{(2)}} \mathbb{Z}/3$ for $r \geqq 2$ for the homotopy groups $\pi_{*+l\omega}(L_2S^0)$. Here,

$$K^{(k)} = \mathbb{Z}/3[v_2^{3^k}, v_2^{-3^k}] \tag{5}$$

for $k \in \{0, 1, 2\}$. Note that $K^{(0)} = K(2)_*$. Moreover, $\mathbb{Z}/3$ is considered to be a $K^{(2)} = \mathbb{Z}/3[v_2^9, v_2^{-9}]$-module by sending v_2^9 to 1. Hereafter, we abuse notation, and a $K^{(2)}$-module M denotes

$$M \otimes_{K^{(2)}} \mathbb{Z}/3. \tag{6}$$

So degrees run over $\mathbb{Z}/144$, and $K^{(2)}$ is considered to be $\mathbb{Z}/3$. We also consider the algebra

$$P^{(k)} = K^{(k)}[b]$$

for a generator b corresponding to $b_0 \in E_2^{2,12}(V_e)$, which detects $i_e i\beta_1 \in \pi_{10}(V_e)$ for the well known generator $\beta_1 \in \pi_{10}(S^0)$.

Theorem 2.5 ([**8**, Th. 5.8]). *The E_2-term $E_2^{*,*}(V_1)$ is isomorphic to a free $P^{(0)}$-module*

$$K^{(0)} \otimes \left(F^b \oplus F^h \oplus F^{h\varphi} \oplus F^{b\varphi}\right) \otimes \Lambda(\zeta_2)$$

for

$$F^b = P^{(2)}\{1, \overline{b}_1\}, \quad F^h = P^{(2)}\{h_1, \overline{h}_0\},$$
$$F^{b\varphi} = P^{(2)}\{\overline{\psi}_0, \overline{\psi}_1\} \quad \text{and} \quad F^{h\varphi} = P^{(2)}\{\overline{\xi}, \overline{\xi}\overline{b}_1\}.$$

Here, $\zeta_2 \in E_2^{1,0}(V_1)$, $h_1 \in E_2^{1,12}(V_1)$ and

$$\overline{h}_0 = v_2^5 h_0 \in E_2^{1,84}(V_1), \quad \overline{b}_1 = v_2^3 b_1 \in E_2^{2,84}(V_1),$$
$$\overline{\xi} = -v_2^7 \xi \in E_2^{2,120}(V_1), \quad \overline{\psi}_0 = v_2^2 \psi_0 \in E_2^{3,48}(V_1) \quad \text{and} \quad \overline{\psi}_1 = -v_2^6 \psi_1 \in E_2^{3,120}(V_1)$$

*for the generators h_0, b_1, ξ, ψ_0 and ψ_1 in [**8**].*

By [**8**, Prop. 5.9], the generators satisfy the relations:

$$\begin{gathered} \overline{h}_0 h_1 = 0, \quad \overline{h}_0 \overline{\xi} = 0, \quad h_1 \overline{\xi} = 0, \\ \overline{h}_0 b_0 = h_1 \overline{b}_1, \quad h_1 b_0 = -\overline{h}_0 \overline{b}_1, \\ \overline{b}_1 \overline{\xi} = \overline{h}_0 \overline{\psi}_1 = -h_1 \overline{\psi}_0, \quad b_0 \overline{\xi} = \overline{h}_0 \overline{\psi}_0 = h_1 \overline{\psi}_1, \\ v_2^9 b_0^2 = -\overline{b}_1^2, \quad b_0 \overline{\psi}_1 = \overline{b}_1 \overline{\psi}_0 \quad \text{and} \quad b_0 \overline{\psi}_0 = -\overline{b}_1 \overline{\psi}_1, \end{gathered} \tag{7}$$

as well as

$$\overline{h}_0^2 = 0, \quad h_1^2 = 0, \quad \overline{\xi}^2 = 0, \quad \overline{\psi}_0^2 = 0, \quad \overline{\psi}_1^2 = 0 \quad \text{and} \quad \zeta^2 = 0. \tag{8}$$

We introduce imaginary generators u and φ such that

$$u^2 = -v_2^9 = -1, \quad \overline{\psi}_0 = b\varphi \quad \text{and} \quad \overline{\psi}_1 = ub\varphi, \tag{9}$$

and put $h = h_1$ and $\zeta = \zeta_2$. We further identify the elements as follows:

$$\overline{h}_0 = uh, \quad \overline{b}_1 = ub, \quad \overline{\xi} = uh\varphi.$$

Here, the bidegrees of the generators are

$$\begin{gathered} \|v_1\| = (0,4), \quad \|v_2\| = (0,16), \quad \|u\| = (0,72), \quad \|h\| = (1,12), \\ \|\varphi\| = (1,36), \quad \|\zeta\| = (1,0) \quad \text{and} \quad \|b\| = (2,12). \end{gathered} \tag{10}$$

3	$b\varphi, uhb$	$ub\zeta,$	$uh\varphi\zeta, ub\varphi, hb$	$b\zeta$
2		$uh\zeta, ub$	$uh\varphi$	$h\zeta, b$
1	ζ	uh		h
0	1			
$s\uparrow / t\rightarrow$	0	4	8	12

(mod $16 = |v_2|$)

In the table, we notice that

$$h\varphi \notin E_2^{2,48}(V_1) \quad \text{and} \quad hb\varphi \in E_2^{4,60}(V_1). \tag{11}$$

The modules in Theorem 2.5 are rewritten as

$$\begin{gathered} F^b = K^{(2)} \oplus bP_u^{(2)}, \quad F^h = hP_u^{(2)}, \quad F^{b\varphi} = b\varphi P_u^{(2)} \quad \text{and} \\ F^{h\varphi} = uh\varphi K^{(2)} \oplus h\varphi bP_u^{(2)} = uh\varphi F^b \end{gathered} \tag{12}$$

for

$$K_u^{(k)} = \mathbb{Z}/3[v_2^{3^k}, v_2^{-3^k}, u]/(u^2+1) \quad \text{and} \quad P_u^{(k)} = K_u^{(k)}[b], \tag{13}$$

where $k \in \{0,1,2\}$, and so

$$E_2^{*,*}(V(1)) \cong \Big(K^{(0)}\{1, uh, h, uh\varphi\} \oplus bP_u^{(0)} \otimes \Lambda(h,\varphi)\Big) \otimes \Lambda(\zeta). \tag{14}$$

We notice that the relations (7) follow from the two relations

$$u^2 = -1 \quad \text{and} \quad h^2 = 0.$$

Furthermore, we consider the element

$$\varsigma = u\varphi\zeta \quad (\in E_2^{2,108}(V_e)),$$

and modules

$$\underline{K} = \mathbb{Z}/3\{1, v_2, v_2^5\} \quad \text{and} \quad \underline{K}' = \mathbb{Z}/3\{1, v_2^5\}, \tag{15}$$

and

$$\begin{array}{rcl} P(k) &=& P^{(2)}/(b^k) \;=\; \mathbb{Z}/3[b]/(b^k), \\ P_u(k) &=& P_u^{(2)}/(b^k) \;=\; P(k) \oplus uP(k), \\ P(k,l) &=& P(k) \oplus v_2^3 P(l), \\ P(k, b^i l) &=& P(k) \oplus v_2^3 b^i P(l) \quad \text{and} \\ P(k,l,m) &=& P(k) \oplus v_2^3 P(l) \oplus v_2^6 P(m) \end{array} \tag{16}$$

for $i \in \{1,2\}$, $k,l,m \in \{-\} \cup \{n \in \mathbb{Z} \mid n \geqq 0\}$, where

$$P(-) = P^{(2)} \quad \text{and} \quad P(0) = 0.$$

We also note that

$$ub^t = (ub)b^{t-1} = \overline{b}_1 b^{t-1} \quad \text{for } t \geqq 1.$$

By use of these notation, we determine the homotopy groups:

Theorem 2.6. *The homotopy groups $\pi_{*+l\omega}(L_2V_1)$ for $l \in \mathbb{Z}/3$ are given by:*

$$\begin{aligned}\pi_*(L_2V_1) = \underline{K} \otimes \Lambda(\zeta) \otimes \Big[&(P(5) \oplus ubP(4) \oplus v_2h\,(P(2,2) \oplus uP(3,3)))\\ &\oplus \varphi\,(b\,(P(4) \oplus uP(5)) \oplus v_2h\,(bP(2,2) \oplus uP(3,3)))\Big] \quad \textit{and}\\ \pi_{*\pm\omega}(L_2V_1) = \Big[&b^2\,(P(3) \oplus uP(3)) \oplus v_2h\,\big(P(2,b1) \oplus uP(3,b^21)\big)\\ &\oplus \varsigma\big(b\,(P(3) \oplus uP(3)) \oplus v_2h\,(P(1,3) \oplus ubP(1,2))\big)\\ &\oplus \varphi\,(b\,(P(4) \oplus uP(5)) \oplus v_2h\,(bP(2,2) \oplus uP(3,3)))\\ &\oplus \zeta\big(\,(P(5) \oplus ubP(4)) \oplus v_2h\,(P(2,2) \oplus uP(3,3))\,\big)\Big] \otimes \underline{K}g^{\pm 1}.\end{aligned}$$

Remark 2.7. From the structure, we find missing differentials in the paper [**3**]:

$$\begin{aligned} d_9(v_2^{j-2}h_{11}g_q) &\equiv v_2^{j-4}\psi_0 b_{10}^3\zeta_2 g_q \qquad j \equiv 2,6,7 \ (9),\\ d_9(v_2^{j}h_{10}g_q) &\equiv v_2^{j+6}\psi_1 b_{10}^3\zeta_2 g_q \qquad j \equiv 0,1,5 \ (9),\\ d_9(v_2^{j}h_{10}b_{10}g_q) &\equiv v_2^{j+6}\psi_1 b_{10}^4\zeta_2 g_q \qquad j \equiv 0,1,5 \ (9)\end{aligned}$$

up to sign. Here, the notations are those used in [**3**].

Theorem 2.8. *The homotopy groups $\pi_{*+l\omega}(L_2V_2)$ for $l \in \mathbb{Z}/3$ are given by:*

$$\pi_*(L_2V_2) = (\mathcal{M} \oplus \varphi\mathcal{M}^\varphi) \otimes \Lambda(\zeta) \oplus S_2$$

for

$$\begin{aligned}\mathcal{M} &= v_1v_2^6\,(P(3,3) \oplus ubP(2,2)) \otimes \underline{K}' \oplus (P(5) \oplus ubP(4)) \otimes \Lambda(v_1v_2)\\ &\quad \oplus h\,(P(4) \oplus uP(5)) \otimes \underline{K}' \oplus v_2h\,(P(2,2) \oplus uP(3,3)) \otimes \Lambda(v_1v_2),\\ \mathcal{M}^\varphi &= v_1v_2^6 b\,(P(2,2) \oplus uP(3,3)) \otimes \underline{K}' \oplus b\,(P(4) \oplus uP(5)) \otimes \Lambda(v_1v_2)\\ &\quad \oplus h\,(bP(4) \oplus uP(5)) \otimes \underline{K}' \oplus v_2h\,(bP(2,2) \oplus uP(3,3)) \otimes \Lambda(v_1v_2), \quad \textit{and}\\ S_2 &= uv_1v_2hK^{(1)} \otimes \underline{K}' \otimes \Lambda(\varphi,\zeta); \quad \textit{and}\end{aligned}$$

$$\pi_{*\pm\omega}(L_2V_2) = \Big[(\underline{\mathcal{M}} \oplus \varsigma\overline{\mathcal{M}}^\varphi) \oplus \zeta\mathcal{M} \oplus \varphi\mathcal{M}^\varphi \oplus S_2\Big]\,g^{\pm 1}$$

for

$$\begin{aligned}\underline{\mathcal{M}} &= v_1v_2^6\,\big(P(3,b^21) \oplus ubP(2,b1)\big) \otimes \underline{K}' \oplus b^2P_u(3) \otimes \Lambda(v_1v_2)\\ &\quad \oplus hb^2\,(P(2) \oplus uP(3)) \otimes \underline{K}' \oplus v_2h\,\big(P(2) \oplus uP(3,b^21)\big) \otimes \Lambda(v_1v_2),\\ \overline{\mathcal{M}}^\varphi &= v_1v_2^6 b\,(P(1,3) \oplus uP(1,2)) \otimes \underline{K}' \oplus bP_u(3) \otimes \Lambda(v_1v_2)\\ &\quad \oplus h\,(P(3,1) \oplus ubP(3)) \otimes \underline{K}' \oplus v_2h\,(P(1,3,1) \oplus ubP(1,2)) \otimes \Lambda(v_1v_2).\end{aligned}$$

We notice that these are isomorphism of modules, and so the modules are not expressed uniquely. For example, in the summands of $\pi_{*+\omega}(L_2V_2)$,

$$\begin{aligned} g\big[\,&\big(hb^2P(2) \oplus h\varsigma P(3,1)\big) \otimes \underline{K}' \oplus (v_2hP(2) \oplus v_2h\varsigma P(1,3,1)) \otimes \Lambda(v_1v_2)\\ &= (hbP(3) \oplus h\varsigma P(3)) \otimes \underline{K}' \oplus (v_2hP(2,b1) \oplus v_2h\varsigma P(1,3)) \otimes \Lambda(v_1v_2)\big].\end{aligned}$$

Indeed, these are isomorphic to

$$\big(hb^2gP(2) \oplus h\varsigma gP(3) \oplus h\,\langle bg\rangle\,P(1)\big) \otimes \underline{K}' \\ \oplus \big(v_2hgP(2) \oplus v_2h\varsigma gP(1,3) \oplus v_2^4h\,\langle bg\rangle\,P(1)\big) \otimes \Lambda(v_1v_2)$$

for the element $\langle bg\rangle = bg + v_2^3\varsigma g$ in (37).

3. The Adams–Novikov E_2-terms for $\pi_*(V_e)$

By (14), we rewrite the E_2-term as follows:

$$E_2^{*,*}(V_1) = E^{(1)} \otimes \underline{K} \otimes \Lambda(\zeta) \tag{17}$$

for

$$E^{(1)} = K^{(1)} \otimes \left(F^b \oplus F^h \oplus F^{b\varphi} \oplus F^{h\varphi}\right).$$

Consider the exact sequence

$$E_2^{s,t-4}(V_1) \xrightarrow{v_1} E_2^{s,t}(V_2) \xrightarrow{\bar{i}_*} E_2^{s,t}(V_1) \xrightarrow{\delta} E_2^{s+1,t-4}(V_1) \tag{18}$$

associated to the cofiber sequence (4). Recall Landweber's formula $\eta_R(v_2) \equiv v_2 + v_1t_1^3 - v_1^3t_1 \bmod (3)$ in $BP_*(BP)$. Then, we see that

$$\delta(v_2^s) = sv_2^{s-1}h. \tag{19}$$

Indeed, $h = [t_1^3] \in E_2^{1,12}(V_1)$. Hereafter, $[c] \in E_2^{*,*}(V_e)$ for a cocycle $c \in \Omega^{*,*}E(2)_*(V_e)$ denotes the homology class of c. Under the exact sequence (18), (19) implies

$$v_1v_2^sh = 0 \in E_2^{1,*}(V_2) \quad \text{unless } s \equiv 2\ (3). \tag{20}$$

We also recall (3) that

$$E(2)_*(V_1) = K(2)_* \quad \text{and} \quad E(2)_*(V_2) = E(2)_*/(3, v_1^2).$$

For a cocycle $c \in \Omega^{s,4t}K(2)_*$, we have a cocycle $c^9 \in \Omega^{s,36t}E(2)_*/(3, v_1^2)$. Furthermore, we see that

$$\bar{i}_*([c^9]) = [v_2^{2t}c] \in E_2^{s,36t}(V_1),$$

since $t_k^9 = v_2^{3^k-1}t_k \in \Omega^{1,*}K(2)_*$.

Lemma 3.1. *The connecting homomorphism δ acts trivially on the submodule $E^{(1)}$ of $E_2^{*,*}(V_1)$.*

Proof. It suffices to show that, for each element $x \in E^{(1)}$, we have an element $(x)^\sim \in E_2^{*,*}(V_2)$ such that $\bar{i}_*((x)^\sim) = x$. For the generators of $E^{(1)}$, we may put

$$(b)^\sim = [b_{1,0}], \quad (ub)^\sim = [v_2^3b_{1,1}], \quad (h)^\sim = [t_1^3], \quad (uh)^\sim = [v_2^3t_1^9]$$
$$(uh\varphi)^\sim = [v_2^3X^9], \quad (b\varphi)^\sim = [v_2^3Y_0^9] \quad \text{and} \quad (ub\varphi)^\sim = [v_2^3Y_1^9].$$

Here, $b_{1,k} = (t_1 \otimes t_1^2 + t_1^2 \otimes t_1)^{3^k}$, and $X \in \Omega^{2,*}K(2)_*$, Y_0 and $Y_1 \in \Omega^{3,*}K(2)_*$ denote cocycles representing $\overline{\xi} = uh\varphi$, $\overline{\psi}_0 = b\varphi$ and $\overline{\psi}_1 = ub\varphi$, respectively. □

The exact sequence (18) together with an isomorphism (17) gives rise to the exact sequences

$$\begin{gathered}v_2^5E^{(1)} \xrightarrow{v_1} \widetilde{E}_0^{(1)} \xrightarrow{\bar{i}_*} E^{(1)} \xrightarrow{\delta} v_2^5E^{(1)},\ E^{(1)} \xrightarrow{v_1} \widetilde{E}_1^{(1)} \xrightarrow{\bar{i}_*} v_2E^{(1)} \xrightarrow{\delta} E^{(1)} \text{ and}\\ v_2E^{(1)} \xrightarrow{v_1} \widetilde{E}_5^{(1)} \xrightarrow{\bar{i}_*} v_2^5E^{(1)} \xrightarrow{\delta} v_2E^{(1)},\end{gathered} \tag{21}$$

and we obtain

$$E_2^{*,*}(V_2) = \left(\widetilde{E}_0^{(1)} \oplus \widetilde{E}_1^{(1)} \oplus \widetilde{E}_5^{(1)}\right) \otimes \Lambda(\zeta). \tag{22}$$

The homomorphism $\bar{i}_*$ induces an isomorphism

$$\mathbb{Z}/3\{(v_2^sh)^\sim\} = E_2^{1,16s+12}(V_2) \xrightarrow[\cong]{\bar{i}_*} E_2^{1,16s+12}(V_1) = \mathbb{Z}/3\{v_2^sh\}$$

for $v_2^s \in \underline{K}$ (see the chart below (10)). The representatives for $(v_2^sh)^\sim$ are given by

$$(v_2^sh)^\sim = [v_2^st_1^3 - sv_1v_2^{s-1}t_1^6].$$

It follows that:

Lemma 3.2. *In $E_2^{*,*}(V_2)$, the generators satisfy the relations:*

$$h(v_2h)^\sim = v_1v_2^{-3}ub, \quad h(v_2^2h)^\sim = -v_1v_2^{-2}ub \quad \text{and} \quad (v_2h)^\sim(v_2^2h)^\sim = v_1v_2^{-1}ub.$$

In other words, $(v_2^sh)^\sim(v_2^th)^\sim = (t-s)v_1v_2^{s+t-4}ub$.

Proof. This follows from the computation

$$\begin{aligned}h(v_2h)^\sim &= [t_1^3 \otimes v_2t_1^3 - v_1t_1 \otimes t_1^6] = [v_2t_1^3 \otimes t_1^3 + v_1t_1^6 \otimes t_1^3 - v_1t_1^3 \otimes t_1^6]\\ &= [d(v_2t_1^6) - v_1t_1^3 \otimes t_1^6 + v_1t_1^6 \otimes t_1^3 - v_1t_1 \otimes t_1^6] = v_1v_2^{-3}ub,\\ h(v_2^2h)^\sim &= [t_1^3 \otimes v_2^2t_1^3 + v_1v_2t_1 \otimes t_1^6] = [v_2^2t_1^3 \otimes t_1^3 - v_1v_2t_1^6 \otimes t_1^3 + v_1v_2t_1^3 \otimes t_1^6]\\ &= [d(v_2^2t_1^6) + v_1v_2t_1^3 \otimes t_1^6 - v_1v_2t_1^6 \otimes t_1^3 + v_1v_2t_1 \otimes t_1^6] = -v_1v_2^{-2}ub,\\ (v_2h)^\sim(v_2^2h)^\sim &= [v_2t_1^3 \otimes v_2^2t_1^3 + v_1v_2^2t_1 \otimes t_1^6 - v_1v_2^2t_1^6 \otimes t_1^3]\\ &= [v_2^3t_1^3 \otimes t_1^3 - v_1v_2^2t_1^6 \otimes t_1^3 + v_1v_2^2t_1 \otimes t_1^6 - v_1v_2^2t_1^6 \otimes t_1^3]\\ &= [d(v_2^3t_1^6) + v_1v_2^2t_1^3 \otimes t_1^6 + v_1v_2^2t_1 \otimes t_1^6] = v_1v_2^{-1}ub.\end{aligned}$$

□

We note that the multiplication by b (resp. ub) defines the monomorphism $b\colon E_2^{*,*}(V_e) \to E_2^{*+2,*+12}(V_e)$ (resp. $ub\colon E_2^{*,*}(V_e) \to E_2^{*+2,*+84}(V_e)$).

Lemma 3.3. *We have an element $(v_2^suh)^\sim \in E_2^{*,*}(V_2)$ satisfying*

$$(v_2^suh)^\sim b = (v_2^sh)^\sim ub \quad \textit{for } v_2^s \in \underline{K}.$$

Proof. Since $\delta(v_2^suh) = 0$, we have an element $(v_2^suh)' \in E_2^{*,*}(V_2)$ such that $\bar{i}_*((v_2^suh)') = v_2^suh$. Then, $\bar{i}_*((v_2^suh)'b) = v_2^suhb = \bar{i}_*((v_2^sh)^\sim ub)$. Thus, $(v_2^suh)'b - (v_2^sh)^\sim ub$ is an image of v_1. By degree reason, $(v_2^suh)'b - (v_2^sh)^\sim ub = kv_1v_2^{s-4}b\zeta$ for some $k \in \mathbb{Z}/3$. Thus the lemma follows by setting $(v_2^suh)^\sim = (v_2^suh)' - kv_1v_2^{s-4}\zeta$. □

We also have

$$(v_2^s uh\varphi)^\sim = [v_2^{3+s}X^9 - sv_1v_2^{s-4}Z^9] \in E_2^{*,*}(V_2)$$

for a cochain $Z \in \Omega^2 K(2)_*$ such that $d(Z) = t_1^3 \otimes X$. Since $v_2\psi_0 \in \langle h_1, h_1, \xi\rangle \subset E_2^{*,*}(V_1)$, we may put

$$(b\varphi)^\sim = \left[v_2^6 t_1^6 \otimes X^9 + t_1^3 \otimes Z^9\right] \in E_2^{*,*}(V_2).$$

We note that $v_2 Y_0 = t_1^6 \otimes X + t_1^3 \otimes Z$ for Y_0 in the proof of Lemma 3.1.

Lemma 3.4. *In $E_2^{*,*}(V_2)$, the generators satisfy the relations:*

$$(v_2^s h)^\sim (v_2^t uh\varphi)^\sim = (t-s)v_1v_2^{5+s+t}b\varphi \quad \text{and} \quad (v_2^s h)^\sim (b\varphi)^\sim = (v_2^s uh\varphi)^\sim ub$$

for $s, t \in \{1, 2\}$.

Proof. The first relation follows from

$$\begin{aligned}
(v_2^s h)^\sim (v_2^t uh\varphi)^\sim &= \left[(v_2^s t_1^3 - sv_1v_2^{s-1}t_1^6) \otimes (v_2^{3+t}X^9 - tv_1v_2^{t-4}Z^9)\right] \\
&= \left[\underline{v_2^{3+s+t}t_1^3 \otimes X^9}_{(1)} + tv_1v_2^{2+s+t}t_1^6 \otimes X^9 - \underline{\underline{tv_1v_2^{s+t-4}t_1^3 \otimes Z^9}}\right] \\
&\qquad - \left[sv_1v_2^{2+s+t}t_1^6 \otimes X^9\right] \;=\; (t-s)v_1v_2^{5+s+t}b\varphi \\
\Big(\because -d(v_2^{s+t-3}Z^9) &= -\underline{\underline{(s+t)v_1v_2^{s+t-4}t_1^3 \otimes Z^9}} - \underline{v_2^{s+t+3}t_1^3 \otimes X^9}_{(1)}\Big).
\end{aligned}$$

Here, the underlined terms with subscript (1) cancel each other out, and the coefficient of the sum of the doubly underlined terms is $t - s$.

Similarly, we verify the second relation by computing

$$\begin{aligned}
(v_2^s h)^\sim (b\varphi)^\sim &= \left[(v_2^s t_1^3 - sv_1v_2^{s-1}t_1^6) \otimes (v_2^6 t_1^6 \otimes X^9 + t_1^3 \otimes Z^9)\right] \\
&= \left[v_2^{s+6}t_1^3 \otimes t_1^6 \otimes X^9 + v_2^s t_1^3 \otimes t_1^3 \otimes Z^9\right] \\
&\qquad - sv_1v_2^{s-1}\left[t_1^6 \otimes (v_2^6 t_1^6 \otimes X^9 + t_1^3 \otimes Z^9)\right] \\
&= \left[v_2^{s+6}b_{1,1} \otimes X^9 - \underline{v_2^{s+6}t_1^6 \otimes t_1^3 \otimes X^9}_{(1)} + \underline{v_2^s t_1^3 \otimes t_1^3 \otimes Z^9}_{(2)}\right] \\
&\qquad - sv_1v_2^{s-1}\left[t_1^6 \otimes (v_2^6 t_1^6 \otimes X^9 + \underline{\underline{t_1^3 \otimes Z^9}})\right] \\
&= \left[v_2^{s+6}b_{1,1} \otimes X^9 - sv_1v_2^{s+5}t_1^6 \otimes t_1^6 \otimes X^9 - \underline{\underline{sv_1v_2^{s-1}b_{1,1} \otimes Z^9}}\right] \\
&= \left[b_{1,1} \otimes v_2^{s+6}X^9 - sv_1v_2^{s+5}\left(t_1^3 b_{1,1} + b_{1,1}t_1^3\right) \otimes X^9\right] \\
&\qquad - s\left[v_1v_2^{s+5}t_1^6 \otimes t_1^6 \otimes X^9 + v_1v_2^{s-1}b_{1,1} \otimes Z^9\right] \\
&= \left[b_{1,1} \otimes v_2^{s+6}X^9 - sv_1v_2^{s+5}\left(-\underline{t_1^6 \otimes t_1^6}_{(3)} + \underline{t_1^9 \otimes t_1^3 + t_1^3 \otimes t_1^9}_{(4)}\right) \otimes X^9\right] \\
&\qquad - s\left[\underline{v_1v_2^{s+5}t_1^6 \otimes t_1^6 \otimes X^9}_{(3)} + v_1v_2^{s-1}b_{1,1} \otimes Z^9\right] \;=\; (v_2^s uh\varphi)^\sim ub.
\end{aligned}$$

Indeed,

$$\begin{aligned}
-d(v_2^s t_1^6 \otimes Z^9) &= -\underline{\underline{sv_1v_2^{s-1}t_1^3 \otimes t_1^6 \otimes Z^9}} - \underline{v_2^s t_1^3 \otimes t_1^3 \otimes Z^9}_{(2)} + \underline{v_2^{s+6}t_1^6 \otimes t_1^3 \otimes X^9}_{(1)} \\
-sd(v_1v_2^{s+5}t_1^{12} \otimes X^9)& \\
&= sv_1v_2^{s+5}\left(\underline{t_1^3 \otimes t_1^9 + t_1^9 \otimes t_1^3}_{(4)}\right) \otimes X^9.
\end{aligned}$$

□

By (19) and (21), we see that

$$\begin{aligned}\mathrm{Im}\,(\delta\colon v_2^sE^{(1)}\to v_2^{s-1}E^{(1)}) &= v_2^{s-1}K^{(1)}\otimes(bF^h\oplus\overline{F}^{h\varphi})\oplus v_2^{s-1}hK^{(1)} \quad \text{and}\\ \mathrm{Ker}\,(\delta\colon v_2^sE^{(1)}\to v_2^{s-1}E^{(1)}) &= v_2^sK^{(1)}\otimes(F^h\oplus F^{h\varphi})\end{aligned}$$

for $s\in\{1,5\}$, where $\overline{F}^{h\varphi}=h\varphi bP_u^{(1)}$ such that $K^{(1)}\otimes F^{h\varphi}=uh\varphi K^{(1)}\oplus\overline{F}^{h\varphi}$. From this, we obtain the following:

Lemma 3.5. *The submodules $\widetilde{E}_s^{(1)}$ for $s\in\{0,1,5\}$ are:*

$$\begin{aligned}\widetilde{E}_0^{(1)} &= E^{(1)}\otimes\Lambda(v_1v_2^5) \quad \textit{and}\\ \widetilde{E}_s^{(1)} &= \left(\widetilde{F}_s^h\oplus\widetilde{F}_s^{h\varphi}\right)\oplus v_1v_2^{s-1}K^{(1)}\otimes\big(F^b\oplus F^{b\varphi}\oplus uhK^{(2)}\otimes\Lambda(\varphi)\big)\end{aligned}$$

for $s\in\{1,5\}$. Here,

$$\widetilde{F}_s^h=P^{(1)}\{(v_2^sh)^\sim,(uv_2^sh)^\sim\} \quad \textit{and} \quad \widetilde{F}_s^{h\varphi}=P^{(1)}\{(v_2^suh\varphi)^\sim,(v_2^suh\varphi)^\sim ub\}.$$

Hereafter, we abbreviate $(x)^\sim$ to x. Then, we may identify $\widetilde{F}_s^h=v_2^sK^{(1)}\otimes F^h$ and $\widetilde{F}_s^{h\varphi}=v_2^sK^{(1)}\otimes F^{h\varphi}$.

Corollary 3.6. *$E_2^{*,*}(V_2)$ is isomorphic to the tensor product of $K^{(1)}$, $\Lambda(\zeta)$ and the direct sum of*

$$\big(F^b\oplus F^{b\varphi}\oplus F^h\oplus F^{h\varphi}\big)\otimes\Lambda(v_1v_2^5)$$

and

$$v_2^5\underline{K}'\otimes\left(F^h\oplus F^{h\varphi}\oplus v_1v_2^5\left(F^b\oplus F^{b\varphi}\oplus uhK^{(2)}\otimes\Lambda(\varphi)\right)\right).$$

The generators satisfy $h^2=0$. Therefore, the relations in (7) also hold in $E_2^{*,*}(V_2)$.

We note that

$$\begin{aligned}E_2^{*,*}(V_2) &= K^{(1)}\otimes\Lambda(\zeta)\otimes\Big(\big(F^b\oplus F^{b\varphi}\oplus v_1(F^b\oplus F^{b\varphi})\otimes\underline{K}\big)\\ &\oplus\big(\big(F^h\oplus F^{h\varphi}\big)\otimes\underline{K}\oplus v_1v_2^5\big(F^h\oplus F^{h\varphi}\big)\big)\oplus v_1v_2uhK^{(2)}\otimes\Lambda(\varphi)\otimes\underline{K}'\Big).\end{aligned} \tag{23}$$

By Lemmas 3.2 and 3.4, we have

$$(v_2^sh)(v_2^th\varphi)=(t-s)v_1v_2^{s+t-4}ub\varphi=(v_2^sh)(v_2^th)\varphi.$$

4. The Adams–Novikov differentials on $E_r^{*,*+l\omega}(V_e)$ for $e\in\{1,2\}$ and $l\in\mathbb{Z}/3$

Let $\beta_1\in\pi_{10}(S^0)$ be the well known generator. Note that it is detected by $b=b_0\in E_2^{2,12}(S^0)$. Consider a spectrum W fitting in the cofiber sequence

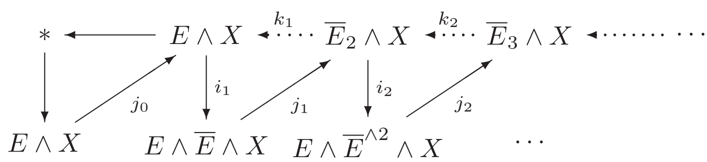
$$S^{10}\xrightarrow{\beta_1}S^0\xrightarrow{\iota}W\xrightarrow{\kappa}S^{11}. \tag{24}$$

Then, $E(2)_*(W)=E(2)_*\oplus E(2)_{*-11}\mathfrak{b}$ for a generator $\mathfrak{b}\in E(2)_{11}(W)$ such that $\kappa_*(\mathfrak{b})=1\in E(2)_0$.

Hereafter, we abbreviate the generators ω_2 of $\mathrm{Pic}(\mathcal{L}_2)$ and g_2 of $E(2)_0(S^{\omega_2})$ to ω and g, respectively. We set

$$V_e^{(l)} = V_e \wedge S^{l\omega} \quad \text{for } e \in \{1, 2\} \text{ and } l \in \mathbb{Z}/3.$$

Then, $E_2^{*,*-l\omega}(V_e) = E_2^{*,*}(V_e^{(l)})$ for $e \in \{1, 2\}$. Note that $E_2^{s,t}(V_e^{(l)}) = E_2^{s,t}(V_e)$ for $l \in \mathbb{Z}/3$, and β_1 induces a monomorphism $b\colon E_2^{s,t}(V_e^{(l)}) \to E_2^{s+2,t+12}(V_e^{(l)})$ by Theorem 2.5 and Corollary 3.6. For the next lemma, we recall an exact couple defining the Adams–Novikov spectral sequence:



for a spectrum X. Here, $E = E(2)$, and $S^0 \xrightarrow{i} E \xrightarrow{j} \overline{E}$ is a cofiber sequence.

Lemma 4.1. *The Adams–Novikov E_3-term $E_3^{s,*}(V_e^{(l)} \wedge W)$ is trivial for $e \in \{1, 2\}$, $l \in \mathbb{Z}/3$ and $s \geqq 6$.*

Proof. The cofiber sequence (24) induces a short exact sequence

$$0 \to E_2^{s,t}(V_e^{(l)}) \xrightarrow{\iota_*} E_2^{s,t}(V_e^{(l)} \wedge W) \xrightarrow{\kappa_*} E_2^{s,t-11}(V_e^{(l)}) \to 0. \tag{25}$$

Consider the generator $g^l \in E(2)_0(V_e^{(l)})$, and let $\mathfrak{i}^{(l)} \in \pi_2(\overline{E}_3 \wedge V_e^{(l)})$ be an element such that $k_1 k_2(\mathfrak{i}^{(l)}) = g^l$. Let $b' \in \pi_{12}(E \wedge \overline{E}^{\wedge 2} \wedge V_e^{(l)})$ be an element representing b. Since $(\overline{E}_3 \wedge \iota)_*(j_2)_*(b') = 0$, the element $\iota_*(b)$ in the E_2-term $E_2^{2,12}(V_e^{(l)} \wedge W)$ is in the image of a differential d_r of the spectral sequence. By degree reason, we have $d_2(\mathfrak{b}g^l) = b \in E_2^{2,12}(V_e^{(l)} \wedge W)$. Therefore, the induced connecting homomorphism from (25) of the d_2-differential modules is the multiplication by b and so we obtain an exact sequence of the Adams–Novikov-E_3-terms

$$E_3^{s,t}(V_e^{(l)}) \xrightarrow{b} E_3^{s+2,t+12}(V_e^{(l)}) \xrightarrow{\iota_*} E_3^{s+2,t+12}(V_e^{(l)} \wedge W) \xrightarrow{\kappa_*} E_3^{s+1,t}(V_e^{(l)}). \tag{26}$$

Here, note that $E_3^{s,t}(V_e^{(l)}) = E_2^{s,t}(V_e^{(l)})$ by degree reason.

Consider a commutative diagram

$$\begin{array}{ccccccccc}
E_2^{s-1,t} & \xrightarrow{\delta} & E_2^{s,t-4} & \xrightarrow{v_1} & E_2^{s,t}(V_2^{(l)}) & \xrightarrow{\bar{i}_*} & E_2^{s,t} & \xrightarrow{\delta} & E_2^{s+1,t-4} \\
\downarrow b & & \downarrow b & & \downarrow b & & \downarrow b & & \downarrow b \\
E_2^{s+1,t+12} & \xrightarrow{\delta} & E_2^{s+2,t+8} & \xrightarrow{v_1} & E_2^{s+2,t+12}(V_2^{(l)}) & \xrightarrow{\bar{i}_*} & E_2^{s+2,t+12} & \xrightarrow{\delta} & E_2^{s+3,t+8}
\end{array}$$

associated to the cofiber sequence (4), where $E_2^{s,t}$ denotes $E_2^{s,t}(V_1^{(l)})$. By Theorem 2.5, we see that $b\colon E_2^{s,t}(V_1^{(l)}) \to E_2^{s+2,t+12}(V_1^{(l)})$ is an isomorphism if $s \geqq 4$, and a monomorphism with $\mathrm{Coker}\, b = K^{(0)}\{hb\varphi\zeta\}$ if $s = 3$ (see (11)). The Five Lemma shows that $b\colon E_2^{s,t}(V_2^{(l)}) \to E_2^{s+2,t+12}(V_2^{(l)})$ is an isomorphism if $s \geqq 5$ and an epimorphism if $s = 4$. Therefore, the lemma follows from the exact sequence (26). $\square$

Lemma 4.2. *In $E_r^{*,*}(V_e^{(l)})$ for $e \in \{1,2\}$ and $l \in \mathbb{Z}/3$, if $d_r(xb) = yb$ for elements $x, y \in E_r^{*,*}(V_e^{(l)})$, then $d_r(x) = y$. Similarly, a relation $d_r(xub) = yub$ also implies $d_r(x) = y$.*

Proof. Since $E_3^{s,t}(V_e^{(l)}) = 0$ unless $4 \nmid t$, we see that $E_2^{*,*}(V_e^{(l)}) = E_5^{*,*}(V_e^{(l)})$. By Theorem 2.5 and (22), we see that b in (26) is a monomorphism on the E_2-terms. Therefore, the lemma holds for $r = 5$.

Suppose inductively that the lemma holds for s with $5 \leqq s < r$. Suppose also $d_r(xb) = yb \in E_r^{k,m}(V_e^{(l)})$ and put $d_r(x) = y'$. Then $by = by' \in E_r^{k,m}(V_e^{(l)})$, and so we have an integer $s < r$ and an element $z \in E_s^{k-s,m-s+1}(V_e^{(l)})$ such that $d_s(z) = b(y - y')$. Note that $r - s \geqq 4$. Since $k \geqq r + 2$, we see that $k - s \geqq r + 2 - s \geqq 6$. Therefore, $\iota_*(z) = 0$ in (26) by Lemma 4.1 and we have $\widetilde{z}$ such that $b\widetilde{z} = z$. It follows that $d_s(\widetilde{z}b) = d_s(z) = b(y - y')$, and by the inductive hypothesis we have $d_s(\widetilde{z}) = y - y'$ and $d_r(x) = y$ as desired.

Since ub is a permanent cycle (see Proposition 4.8), multiplying the relation $d_r(xub) = yub$ by ub implies $d_r(x(ub)^2) = y(ub)^2$. Therefore, $d_r(xb^2) = yb^2$, and we obtain $d_r(x) = y$. □

Corollary 4.3. *In $E_2^{*,*}(V_e^{(l)})$ for $e \in \{1,2\}$ and $l \in \mathbb{Z}/3$, if xb (resp. xub) is a permanent cycle, then so is x.*

By [**5**] and [**1**], the differential $d_5 \colon E_2^{*,*}(S^\omega) \to E_2^{*+5,*+4}(S^\omega)$ acts on g by

$$d_5(g) = \omega g \ (\equiv v_2 uhb\varphi\zeta g \in E_2^{*,*}(V_e \wedge S^\omega) \text{ for } e \in \{1,2\}). \tag{27}$$

By [**8**, Prop.s 8.4, 9.9, 9.10], we deduce that

$$d_5(v_2^{3t+s} g^l) \ = \ -tv_2^{3t+s-2} hb^2 g^l + l v_2^{3t+s} u(v_2 h) b\varphi\zeta g^l \in E_2^{*,*}(V_1 \wedge S^{l\omega}), \tag{28}$$

for $l \in \mathbb{Z}/3$ and $s \in \{0,1,5\}$, and

$$d_5(v_2^{3t+s} x g^l) = d_5(v_2^{3t+s} g^l) x \in E_2^{*,*}(V_1 \wedge S^{l\omega})$$

for $x \in \{b, h, uh, ub, uh\varphi, b\varphi, ub\varphi, hb\varphi, \zeta\} = \{b, \overline{h}_0, h_1, \overline{b}_1, \overline{\xi}, \overline{\psi}_0, \overline{\psi}_1, \overline{b}_1\overline{\xi}, \zeta_2\}$. In particular,

$$d_5(v_2^{3t+s} h g^l) = 0 \in E_2^{*,*}(V_1 \wedge S^{l\omega})$$

by (28) together with (8). We also have:

Proposition 4.4 ([**8**, Prop. 10.5])**.** *For $s \in \{0,1,5\}$, we have an integer $\sigma(s) \in \{1,2\}$ such that*

$$d_9(v_2^{7+s} h) = \sigma(s) v_2^s ub^5 \in E_9^{10,*}(V_1) \quad (ub^5 = \overline{b}_1 b^4).$$

The integer $\sigma(s)$ is not determined in [**8**]. We determine it to be two in Lemma 4.10.

Theorem 4.5 ([**8**, Th. 10.6])**.** *The E_{10}-term for V_1 is isomorphic to the tensor product of $\Lambda(\zeta)$, $\underline{K}$ and*

$$P^{(2)}/(b^4)\{ub, b\varphi\} \oplus P^{(2)}/(b^5)\{1, ub\varphi\}$$
$$\oplus \big(P^{(2)}/(b^2)\{v_2 h, v_2 hb\varphi\} \oplus P^{(2)}/(b^3)\{v_2 uh, v_2 uh\varphi\}\big) \otimes \mathbb{Z}/3\{1, v_2^3\}.$$

See (15) for $\underline{K}$.

In particular, we have:

Corollary 4.6. *Every element of* $\underline{K} \subset E_2^{0,*}(V_1)$ *and* $v_1\underline{K} \subset E_2^{0,*}(V_2)$ *is a permanent cycle in the spectral sequences.*

Corollary 4.7. *The elements* $v_2^s h \in E_2^{1,*}(V_1)$ *for* $s \in \{0,1,2,4,5,6\}$ *and* $v_1 v_2^s h \in E_2^{1,*}(V_1)$ *for* $s \in \{2,5\}$ *are permanent cycles in the spectral sequences. (See (20).)*

The following is well known (*cf.* [**7**]):

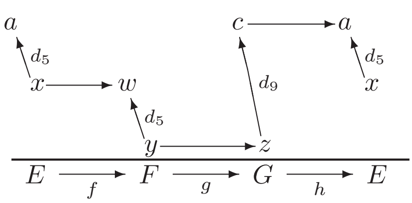
Proposition 4.8. *For* $e \in \{1,2\}$*, the elements* h *and* $v_2 h$ *in* $E_2^{1,*}(V_e)$ *and* b *and* ub *in* $E_2^{2,*}(V_e)$ *are permanent cycles detecting* $i_e\beta_1'$ *and* $i_e\beta_2'$ *in* $\pi_*(V_e)$ *and* $i_e i\beta_1$ *and* $i_e i\beta_{6/3}$ *in* $\pi_*(V_e)$*, respectively. Here,* i *and* i_e *are the maps in (1) and (2), the element* β_1 *is the one in (24),* $\beta_2 \in \pi_{26}(S^0)$ *is the generator, and* $\beta_s' \in \pi_{16s-5}(M)$ *for* $s \in \{1,2\}$ *denotes an element such that* $j\beta_s' = \beta_s$ *for the map* j *in (1).*

Among the Adams–Novikov differentials for $V_e^{(l)}$ for $e \in \{1,2\}$ and $l \in \mathbb{Z}/3$, the following relation is also well known (*cf.* [**9**]):

Proposition 4.9. *Consider the exact sequence of the* E_2*-terms*

$$\begin{aligned} E_2^{*,*}(V_1 \wedge S^{l\omega}) &\xrightarrow{\delta} E_2^{*,*}(V_1 \wedge S^{l\omega}) \xrightarrow{v_1} E_2^{*,*}(V_2 \wedge S^{l\omega}) \xrightarrow{\bar{i}_*} E_2^{*,*}(V_1 \wedge S^{l\omega}) \\ &\xrightarrow{\delta} E_2^{*,*}(V_1 \wedge S^{l\omega}), \end{aligned}$$

and let $E \xrightarrow{f} F \xrightarrow{g} G \xrightarrow{h} E$ *be a part of the exact sequence. Then, we have a relation described below:*



Lemma 4.10. *Let* $s \in \{0,1,5\}$ *and* $t \in \mathbb{Z}/3$*. Then, the integers* $\sigma(s)$ *for* $s \in \{0,1,5\}$ *in Proposition 4.4 are all two. Furthermore, in* $E_2^{*,*}(V_2)$*,*

$$\begin{aligned} d_5(v_2^{3t}) &= -tv_2^{3t-3}(v_2h)b^2, \\ d_5(v_2^{3t+s}h) &= t(1-s)v_1v_2^{3t+s-6}ub^3, \\ d_5(v_1v_2^{3t+s}) &= \begin{cases} -tv_1v_2^{3t-1}hb^2 & s=1 \\ 0 & s \in \{0,5\} \end{cases} \qquad \textit{and} \\ d_5(v_1v_2^{3t+2}h) &= 0. \end{aligned}$$

Proof. We read off $E_2^{5,48}(V_1) = \mathbb{Z}/3\{v_2^{-3}ub^2\zeta\}$ by Theorem 2.5, and may put $d_5(v_2^3) = -v_2hb^2 + kv_1v_2^{-3}ub^2\zeta \in E_2^{5,52}(V_2)$ for $k \in \mathbb{Z}/3$ by (28). Since the differential d_5 is a derivation, we have

$$\begin{aligned} d_5(v_2^{3t}) &= -tv_2^{3t-3}(v_2h)b^2 + tkv_1v_2^{3t+3}ub^2\zeta, \quad \text{and} \\ d_5(v_2^{3t+s}h) &= -tv_2^{3t-3}(v_2h)(v_2^sh)b^2 + tkv_1v_2^{3t+3+s}uhb^2\zeta + v_2^{3t}d_5(v_2^sh). \end{aligned} \tag{29}$$

It follows that $d_5(v_2^{3t+1}) = 0$ by Lemma 3.2, (20) and Proposition 4.8. Thus, we have $d_5(v_2^{3t+s}h)$ for $s = 1$ in the lemma.

Suppose that $s \in \{0, 5\}$. Put

$$\begin{aligned} &a = (s-1)\sigma(s-4)v_2^{s-5}uhb^5, && c = \sigma(s-4)v_2^{s-4}ub^5, && x = (s-1)\sigma(s-4)v_2^{s-3}ub^3, \\ &y = v_2^{3+s}h, && z = \bar{i}_*(y), && w = v_1x, \end{aligned}$$

and we have $d_9(z) = c$ by Proposition 4.4, $\delta(c) = a$ by (20) and $d_5(x) = a$ by (28). Therefore, we have $d_5(y) = w$ by Proposition 4.9, that is,

$$d_5(v_2^{3+s}h) = (s-1)\sigma(s-4)v_1v_2^{s-3}ub^3. \tag{30}$$

Similarly, put

$$\begin{aligned} &a = v_1c, && c = (1-s)\sigma(s)v_2^sub^5, && x = v_2^{6+s}hb^2, \\ &y = -v_2^{8+s}, && z = (1-s)v_2^{7+s}h, && w = \bar{i}_*(x), \end{aligned}$$

and we have $d_5(y) = w$ by (28), $\delta(y) = z$ by (19) and $d_9(z) = c$ by Proposition 4.4. Thus, we have $d_5(x) = a$. By Lemma 4.2,

$$d_5(v_2^{6+s}h) = (1-s)\sigma(s)v_1v_2^sub^3. \tag{31}$$

Since $(v_2h)(v_2^sh) = (s-1)v_1v_2^{s-3}ub$ by Lemma 3.2, the second relation of (29) is:

$$d_5(v_2^{3t+s}h) = \begin{cases} tv_1v_2^{3t-6}ub^3 & s = 0 \\ -tv_1v_2^{3t-1}ub^3 + tkv_1v_2^{3t+8}uhb^2\zeta + v_2^{3t}d_5(v_2^5h) & s = 5 \end{cases}$$

by (20) and Proposition 4.8. Comparing with (30) and (31), we obtain

$$\sigma(5) = -1 = \sigma(0); \quad v_2^3d_5(v_2^5h) = (1+\sigma(1))v_1v_2^2ub^3 - kv_1v_2^2uhb^2\zeta \quad \text{and}$$
$$v_2^6d_5(v_2^5h) = kv_1v_2^5uhb^2\zeta.$$

The last two relations show $\sigma(1) = -1$ and $k = 0$, and then $d_5(v_2^5) = 0$. Thus the top two relations of the lemma follow from (29).

The third relation of the lemma follows from the first one together with (20) and Corollary 4.6. Multiplying the permanent cycle v_1 in Corollary 4.6 to the second relation of the lemma implies the last one. □

Lemma 4.11. *The elements uh, $uh\varphi = \overline{\xi}$, $v_2^6uh\varphi = v_2^6\overline{\xi}$, $b\varphi = \overline{\psi}_0$, $v_2^6b\varphi = v_2^6\overline{\psi}_0$, $ub\varphi = \overline{\psi}_1$, $v_2^6ub\varphi = v_2^6\overline{\psi}_1$ and $\zeta = \zeta_2$ of $E_2^{*,*}(V_2)$ are permanent cycles.*

Proof. Let V_3 denote the cofiber of $\alpha^3\colon \Sigma^{12}M \to M$, and consider the cofiber sequence $\Sigma^4V_2 \xrightarrow{\overline{\alpha}'} V_3 \xrightarrow{\overline{i}'} V_1 \xrightarrow{\overline{j}'} \Sigma^5V_2$ obtained similarly to (4). Let $\delta_2\colon E_2^{*,*}(V_1) \to E_2^{*+1,*-8}(V_2)$ denote the associated connecting homomorphism. In the cobar complex $\Omega^*E(2)_*(V_3)$, we compute $d(v_2^5t_1^3 + v_1v_2^4t_1^6) = -v_1v_2^4t_1^3 \otimes t_1^3 + v_1^2v_2^3t_1^6 \otimes t_1^3 + v_1^2v_2^3t_1^3 \otimes$

$t_1^6 + v_1 v_2^4 t_1^3 \otimes t_1^3 = v_1^2 v_2^3 b_{1,1}$. It follows that $\delta_2(v_2^5 h\zeta) = ub\zeta$, and so $ub\zeta$ is a permanent cycle by the Geometric Boundary Theorem, since $v_2^5 h\zeta \in E_2^{*,*}(V_1)$ is a permanent cycle by Theorem 4.5. Therefore, ζ is a permanent cycle by Proposition 4.8 and Corollary 4.3. Since $(uh)b = h(ub)$ by Corollary 3.6 and (7), and h is a permanent cycle by Proposition 4.8, the element uh is a permanent cycle.

We also compute $\delta_2(v_2^{3t-4}\overline{\xi}) = v_2^{3t}\overline{\psi}_0$ by [**9**, Lemma 4.4], which is $\delta_2(v_2^{3t-4}uh\varphi) = v_2^{3t}b\varphi$ in our notation. Since $v_2^2 uh\varphi$ and $v_2^5 uh\varphi$ are permanent cycles of $E_r^{*,*}(V_1)$ by Theorem 4.5, their δ_2-images $v_2^6 b\varphi$ and $b\varphi$ are permanent cycles of $E_r^{*,*}(V_2)$ by the Geometric Boundary Theorem. By Proposition 4.8, Corollary 3.6 and (7), we have $uh(v_2^s b\varphi) = b(v_2^s uh\varphi)$ and $ub(v_2^s b\varphi) = b(v_2^s ub\varphi)$ in $E_2^{*,*}(V_2)$ for $s \in \{0, 6\}$. Noticing that uh and ub are permanent cycles, these show that $uh\varphi$, $v_2^6 uh\varphi$, $ub\varphi$ and $v_2^6 ub\varphi$ are all permanent cycles by Corollary 4.3. □

Here, consider an element

$$\mathfrak{g}^l = b^2 g^l + l v_2^3 ub\varphi\zeta g^l \in E_2^{4,24}(V_e \wedge S^{l\omega}) \quad \text{for } l \in \mathbb{Z}/3 \text{ and } e \in \{1, 2\}. \tag{32}$$

We notice that the element $v_2^3 ub\varphi\zeta g$ is not divisible by b in the E_2-term.

Lemma 4.12. *Let $s \in \{0, 1, 5\}$. In $E_9^{*,*}(V_1 \wedge S^\omega)$, we have*

$$d_9(v_2^{3t+s}(v_2 h)g) = \begin{cases} 0 & t = 0 \\ -v_2^s b^4 \varphi\zeta g & t = 1 \\ -v_2^s ub^3 \mathfrak{g} & t = 2. \end{cases}$$

In particular, $\mathfrak{g}(= \mathfrak{g}^1)$ is a permanent cycle.

Proof. We notice that

$$d_5(xg) = d_5(x)g + (-1)^{|x|} x(v_2 h) ub\varphi\zeta g \in E_2^{*,*+k\omega}(V_e)$$

for $e \in \{1, 2\}$ by (27). Suppose that $s \in \{0, 5\}$ and put

$$\begin{aligned} a &= v_1 c, & c &= (t-1)tv_2^{3t+s-6}ub^5 g - tv_2^{3t+s-3}b^4\varphi\zeta g, & y &= (s-1)v_2^{3t+s+2}g, \\ w &= \bar{i}_*(x), & x &= (1-s)\left((t-1)v_2^{3t+s}hb^2 g - v_2^{3t+s+3}uhb\varphi\zeta g\right), & z &= v_2^{3t+s+1}hg. \end{aligned}$$

Then, $d_5(x) = a \in E_2^{10,*}(V_2 \wedge S^\omega)$ by Lemmas 4.10, 4.11 and 3.2, $d_5(y) = w \in E_2^{5,*}(V_1 \wedge S^\omega)$ by (28), and $\delta(y) = z$ by (19). By Proposition 4.9, we have $d_9(z) = c$. For the case for $s = 1$, we set

$$\begin{aligned} a &= \delta(c), & c &= (t-1)tv_2^{3t-5}ub^5 - tv_2^{3t-2}b^4\varphi\zeta g, & y &= v_2^{3t+2}hg \\ w &= v_1 x, & x &= (1-t)v_2^{3t-4}ub^3 g - v_2^{3t-1}b^2\varphi\zeta g, & z &= \bar{i}_*(y). \end{aligned}$$

Then, $d_5(x) = a \in E_2^{10,*}(V_1 \wedge S^\omega)$ by (28) and Lemma 4.11, and $d_5(y) = w \in E_2^{5,*}(V_2 \wedge S^\omega)$ by Lemmas 4.10 and 3.2. By Proposition 4.9, we also have $d_9(z) = c$ in this case. □

Corollary 4.13. *In the spectral sequence $E_r^{*,*}(V_1 \wedge S^\omega)$, $v_2^s b\varphi g$ and $v_2^s ub\varphi g$ are permanent cycles for $s \in \{0, 1, 5\}$.*

Proof. Since we have a pairing $V_1 \wedge V_2 \to V_1$, we have $d_9(v_2^{7+s}u^\varepsilon hb\varphi g) = -v_2^s u^{1-\varepsilon} b^6 \varphi g$ in $E_9^{*,*}(V_1)g$ for $\varepsilon \in \{0, 1\}$ by Lemmas 4.11 and 4.12. This shows that $v_2^s u^{1-\varepsilon} b^6 \varphi g$ is a permanent cycle, and hence the corollary follows from Corollary 4.3. □

By Lemma 4.10, among the elements of $(v_1K^{(0)} \oplus K^{(1)}) \otimes F^b$ and $(v_1v_2^2K^{(1)} \oplus K^{(0)}) \otimes F^h$ in the E_2-term $E_2^{*,*}(V_2)$, the following elements survive to E_9-term

$$v_1v_2^{3t+s} \quad \text{for } s \in \{0,5\}, \quad v_1v_2,$$
$$v_1v_2^{3t+2}h, \quad h, \quad v_2^{3t+1}h \quad \text{and} \quad v_2^5h$$

for $t \in \mathbb{Z}/3$.

Lemma 4.14. *In $E_9^{*,*}(V_2)$, we have*

$$d_9(v_1v_2^3) = hb^4, \quad d_9(v_1v_2^8) = -v_2^5hb^4,$$
$$d_9(v_1v_2^8h) = -v_1v_2ub^5 \quad \textit{and} \quad d_9(v_2^7h) = -ub^5.$$

The following generators are permanent cycles:

$$v_1v_2^j \quad \textit{for } j \in \{0,1,2,5,6\}, \quad v_1v_2^jh \quad \textit{for } j \in \{2,5\}, \textit{ and}$$
$$v_2^jh \quad \textit{for } j \in \{0,1,4,5\}.$$

Proof. We begin with verifying the permanent cycles. The elements $v_1v_2^j$ for $j \in \{0,1,5\}$ and $v_1v_2^jh$ for $j \in \{2,5\}$ are permanent cycles by Corollaries 4.6 and 4.7. The second relation in Lemma 4.10 with $(t,s) = (1,0)$ and $= (1,5)$ shows that $v_1v_2^{-3}ub^3$ and $v_1v_2^2ub^3$ are permanent cycles. Corollary 4.3 implies that $v_1v_2^j$ for $j \in \{2,6\}$ are permanent. Similarly, the first relation in Lemma 4.10 with $t = 1$ and $= 2$ implies that v_2hb^2 and $v_2^4hb^2$ are permanent, and so v_2^jh for $j \in \{1,4\}$ is a permanent cycle by Corollary 4.3. By the same argument, the top two relations of this lemma imply that v_2^jh for $j \in \{0,5\}$ is permanent.

Turn to the top two relations. For $s \in \{0,5\}$, put

$$a = \bar{i}_*(c) \qquad c = (s-1)t(t+1)v_2^{3t+s-3}hb^4 \qquad w = -tv_2^{3t+s-2}hb^2$$
$$x = -(s-1)tv_2^{3t+s-1}b^2 \qquad y = v_2^{3t+s} \qquad z = v_1v_2^{3t+s}.$$

Then, these satisfy the relations in Proposition 4.9 other than $d_9(z) = c$ by (28) and (19). Hence, $d_9(z) = c$:

$$d_9(v_1v_2^{3t+s}) = (s-1)t(t+1)v_2^{3t+s-3}hb^4 \in E_9^{*,*}(V_2).$$

This with $t = 1$ shows the first two equalities.

Multiply by h to the second equality, and Lemma 3.2 implies

$$d_9(v_1v_2^8h) = -(v_2^5h)hb^4 = -v_1v_2ub^5,$$

which is the third one. Since $\bar{i}_*(v_2^7h) = v_2^7h \in E_9^{1,*}(V_1)$ and $d_9(v_2^7h) = -ub^5 \in E_9^{10,*}(V_1)$ by Proposition 4.4 and Lemma 4.10, we see that

$$d_9(v_2^7h) = -ub^5 + kv_1v_2^{-7}h\varphi b^2 = -ub^5 - d_5(kv_1v_2^4b^2\varphi)$$

for $k \in \mathbb{Z}/3$ by (28). Thus, the fourth d_9-differential follows. $\square$

Now, the next lemma follows from Lemma 4.10 (see also Lemma 3.2).

Lemma 4.15. *Let* $s \in \{0,1,5\}$ *and* $t, l \in \mathbb{Z}/3$. *Then, in* $E_2^{*,*}(V_2 \wedge S^{l\omega})$,

$$\begin{aligned} d_5(v_2^{3t} g^l) &= -tv_2^{3t-3}(v_2 h)b^2 g^l + lv_2^{3t}(v_2 h)ub\varphi\zeta g^l, \\ d_5(v_2^{3t+s} hg^l) &= t(1-s)v_1 v_2^{3t+s-6} ub^3 g^l + l(1-s)v_1 v_2^{3t+s-3} b^2\varphi\zeta g^l, \\ d_5(v_1 v_2^{3t+s} g^l) &= \begin{cases} -tv_1 v_2^{3t-1} hb^2 g^l + lv_1 v_2^{3t+2} uhb\varphi\zeta g^l & s=1 \\ 0 & s \in \{0,5\} \end{cases} \quad \textit{and} \\ d_5(v_1 v_2^{3t+2} hg^l) &= 0. \end{aligned}$$

By Lemma 4.15, among the elements of $\Big((v_1 K^{(0)} \oplus K^{(1)}) \otimes F^b \oplus (v_1 v_2^2 K^{(1)} \oplus K^{(0)}) \otimes F^h\Big)g$ in the E_2-term $E_2^{*,*}(V_2 \wedge S^\omega)$, the following elements survive to the E_9-term

$$v_1 v_2^{3t+s} g \quad \text{for } s \in \{0,5\}, \quad v_1 v_2^{3t+2} hg \quad \text{and} \quad v_2^{3t+1} hg$$

for $t \in \mathbb{Z}/3$.

The relation with $(t,s) = (2,0)$ in Lemma 4.12 is $d_9(v_2^7 hg) = -ub^3\mathfrak{g} \in E_9^{10,132}(V_1 \wedge S^\omega)$. We see that $v_1 E_2^{10,128}(V_1) = v_1 b^3 E_2^{4,92}(V_1) = \mathbb{Z}/3\{v_1 v_2^2 hb^4\varphi\} \subset E_2^{10,132}(V_2)$ by Theorem 2.5. The generator is zero in the E_9-term by $d_5(v_2^8 uhb\varphi g) = v_1 v_2^2 b^4 \varphi g$, which follows from the last relation in Lemma 4.15 multiplied by the permanent cycle $ub\varphi$ (Lemma 4.11). Thus, the relation in $E_9^{*,*}(V_1)$ is pulled back to the one in $E_9^{*,*}(V_2)$:

$$d_9(v_2^7 hg) = -ub^3\mathfrak{g} \in E_9^{10,132}(V_2 \wedge S^\omega).$$

It follows from Corollary 4.3 that

$$\mathfrak{g} = b^2 g + v_2^3 ub\varphi\zeta g \in E_9^{4,24}(V_2 \wedge S^\omega) \quad \text{is a permanent cycle}$$

for the element $\mathfrak{g} = \mathfrak{g}^1$ in (32).

Lemma 4.16. *In* $E_9^{*,*}(V_2 \wedge S^\omega)$, *we have*

$$\begin{aligned} d_9(v_1 v_2^{3t+s} g) &= \begin{cases} (s-1)v_2^s uhb^3\varphi\zeta g & t=0 \\ (1-s)v_2^s hb^2\mathfrak{g} & t=1 \\ 0 & t=2 \end{cases} \quad \textit{for } s \in \{0,5\}, \textit{ and} \\ d_9(v_2^{3t+1} hg) &= \begin{cases} 0 & t=0 \\ -b^4\varphi\zeta g & t=1 \\ -ub^3\mathfrak{g} & t=2. \end{cases} \end{aligned}$$

Proof. For a permanent cycle x of $E_2^{*,*}(V_2)$ with $d_5(xg) = 0$, we have $d_9(x\mathfrak{g}) = 0 \in E_9^{*,*}(V_2 \wedge S^\omega)$, and so

$$d_9(xb^2 g) = -d_9(xv_2^3 ub\varphi\zeta g) = -d_9(xv_2^3 g)ub\varphi\zeta \in E_9^{*,*}(V_2 \wedge S^\omega). \tag{33}$$

Put $x_t^{(0)} = v_1 v_2^{3t+s}$ and $x_t^{(1)} = v_2^{3t+4}h$. By Lemma 4.15, $d_5(x_t^{(\varepsilon)} g) = 0$ for $\varepsilon \in \{0,1\}$, and so $x_t^{(\varepsilon)} g \in E_9^{*,*}(V_2 \wedge S^\omega)$. Furthermore, Lemma 4.14 shows that $x_t^{(\varepsilon)}$ for $\varepsilon \in \{0,1\}$

is a permanent cycle unless $t = 1$. Therefore, by (33), we compute

$$d_9(x_0^{(\varepsilon)}b^2g) = -d_9(x_1^{(\varepsilon)}g)ub\varphi\zeta, \quad \text{and}$$
$$d_9(x_2^{(\varepsilon)}b^4g) = -d_9(x_0^{(\varepsilon)}b^2g)ub\varphi\zeta = d_9(x_1^{(\varepsilon)}g)(ub\varphi\zeta)^2 = 0.$$

Thus, the relations for $t = 0$ follow from those for $t = 1$, and the relations for $t = 2$ follow from Corollary 4.3.

Now we consider the differential d_9 on $x_1^{(\varepsilon)}g$. Lemma 4.15 together with Lemma 4.11 also shows that

$$v_2uhb\varphi\zeta g, \quad v_1v_2^6b^2\varphi\zeta g \quad \text{and} \quad v_1v_2^2b^2\varphi\zeta g \tag{34}$$

are zero in $E_9^{*,*}(V_2 \wedge S^\omega)$. Therefore,

$$d_9(x_1^{(0)}b^3g) \underset{(34)}{=} d_9(x_1^{(0)}(b^3g + v_2^3ub^2\varphi\zeta g)) = d_9(v_1v_2^{3+s}b\mathfrak{g}) \underset{4.14}{=} (1-s)v_2^shb^5\mathfrak{g}, \quad \text{and}$$
$$d_9(x_1^{(1)}b^2g) \underset{(34)}{=} d_9(x_1^{(1)}(b^2g + v_2^3ub\varphi\zeta g)) = d_9(v_2^7h\mathfrak{g}) \underset{4.14}{=} -ub^5\mathfrak{g}$$

for $s \in \underline{K}'$. By Corollary 4.3, we obtain the relations for $d_9(x_1^{(\varepsilon)})$. □

5. The cohomology of a differential algebra C_1

Consider algebras $K^{(k)}$, $K_u^{(k)}$, $P^{(k)}$ and $P_u^{(k)}$ in (5) and (13) and

$$A_1^{(k)} = P_u^{(k)} \otimes \Lambda(v_2h)$$

for $k \in \{0, 1, 2\}$. Recall that these algebras are considered to be the tensor products with $\mathbb{Z}/3$ over $K^{(2)}$ (see (6)). In this section, we consider the module

$$C_1g^l = \left(A_1^{(0)} \otimes \Lambda(\varphi, \zeta)\right) g^l$$

for $l \in \mathbb{Z}/3$, which contains $E_2^{*,*}(V_1)g^l = E_2^{*,*}(V_1 \wedge S^{l\omega})$. We use the relation

$$g^lg^m = g^{l+m} \quad \text{for } l, m \in \mathbb{Z}/3.$$

In order to consider a differential algebra, we consider the subalgebra

$$C_1^{(1)} = A(1)^{(1)} \otimes \Lambda(\varphi, \zeta) \subset C_1.$$

We begin with introducing a differential algebra structure on $C_1^{(1)}[g]/(g^3)$ so that the inclusion $E_2^{*,*}(V)[g]/(g^3) \to C_1[g]/(g^3)$ is the one of differential $C_1^{(1)}$-modules with differential ∂_5:

$$\begin{aligned} \partial_5(x) &= 0 \quad \text{for } x \in \{1, u, b, v_2h, \varphi, \zeta\}, \\ \partial_5(v_2^{3t}) &= -tv_2^{3t-3}(v_2h)b^2 \quad \text{for } t \in \mathbb{Z}/3, \text{ and} \\ \partial_5(g) &= \omega g \end{aligned} \tag{35}$$

on the generators, where

$$\omega = uv_2hb\varphi\zeta = v_2hb\varsigma \in \varsigma A(1)^{(2)} \quad (\varsigma = u\varphi\zeta). \tag{36}$$

We make $C_1 = C_1^{(1)} \otimes \underline{K}$ a differential module by setting

$$\partial_5(v_2^s) = 0 \quad \text{and} \quad \partial_9(v_2^s) = 0 \quad \text{for } v_2^s \in \underline{K},$$

and we obtain

$$H^*(C_1 g^l, \partial_5) = H^*(C_1^{(1)} g^l, \partial_5) \otimes \underline{K}.$$

In addition to (16), we consider $P_u^{(2)}$-algebras

$$P_u(b^{e_1}k) = b^{e_1}P_u(k), \quad P_u(b^{e_1}k, b^{e_2}l) = b^{e_1}P_u(k) \oplus v_2^3 b^{e_2} P_u(l) \quad \text{and}$$
$$P_u(b^{e_1}k, b^{e_2}l, b^{e_3}m) = b^{e_1}P_u(k) \oplus v_2^3 b^{e_2} P_u(l) \oplus v_2^6 b^{e_3} P_u(m)$$

for $k, l, m, e_i \in \{-\} \cup \{n \in \mathbb{Z} \mid n > 0\}$, and we set $b^- = 0$. We notice that

$$P_u^{(1)} = P_u(-,-,-).$$

Since ∂_5 acts as $P_u(-,-,-) \to v_2 h P_u(b^2-, b^2-) \subset v_2 h P_u(-,-,-)$, we immediately obtain the following lemma from the second equality of (35):

Lemma 5.1. *The cohomology* $H^*(A_1^{(1)}, \partial_5)$ *is isomorphic to*

$$\mathbb{A}_1^{(1)} = P_u^{(2)} \oplus v_2 h P_u(2,2,-)$$

as an algebra.

Put

$$B_1^{(1)} = A_1^{(1)} \otimes \Lambda(\varsigma) \qquad \text{for } \varsigma = u\varphi\zeta.$$

Consider an element

$$\langle bg^l \rangle = bg^l + l v_2^3 \varsigma g^l, \tag{37}$$

and we see that this is a ∂_5-cocycle. Note that the element $\mathfrak{g}$ in (32) equals $b\langle bg\rangle$, but that

$$\partial_5(v_2^3 g) = -v_2 h b^2 g + v_2^3 (v_2 h) b \varsigma g \neq -v_2 h b \langle bg \rangle$$

by (35).

Lemma 5.2. *The cohomology* $H^*(B_1^{(1)} g^{\pm 1}, \partial_5)$ *is isomorphic to*

$$\mathbb{B}_1^{(1)} g^{\pm 1} = \langle bg^{\pm 1} \rangle P_u^{(2)} \oplus \Big(v_2 h P_u(2,2,-) \oplus \varsigma \Big(P_u^{(2)} \oplus v_2 h P_u(1,2,-) \Big) \Big) g^{\pm 1}.$$

Then, Lemmas 5.1 and 5.2 imply the following:

Corollary 5.3. *The cohomology* $H^*(C_1^{(1)} g^l, \partial_5)$ *for* $l \in \mathbb{Z}/3$ *is isomorphic to*

$$\mathbb{C}_1^{(1)} g^l = \begin{cases} \mathbb{A}_1^{(1)} \otimes \Lambda(\varphi, \zeta) & l = 0 \\ \Big(\mathbb{B}_1^{(1)} \oplus \mathbb{A}_1^{(1)}\{\varphi, \zeta\} \Big) g^l & l = \pm 1, \end{cases}$$

and $H^*(C_1 g^l, \partial_5)$ *is isomorphic to* $\mathbb{C}_1 g^l = \mathbb{C}_1^{(1)} g^l \otimes \underline{K}$.

Now, we introduce $\mathbb{C}_1 g^l$ for $l \in \mathbb{Z}/3$ a differential module structure with differential ∂_9 given by

$$\partial_9(v_2^{3t+s+1} h g^l) = \begin{cases} 0 & t = 0 \\ -luv_2^s b^4 \varsigma g^l & t = 1 \\ -uv_2^s b^4 \langle bg^l \rangle & t = 2 \end{cases} \tag{38}$$

for $t \in \mathbb{Z}/3$ and $s \in \{0, 1, 5\}$. In particular, we assume that

$$\partial_9(\langle bg^l \rangle) \ = \ 0 \ = \ \partial_9(\varsigma g^l) \quad \text{for } l \in \mathbb{Z}/3. \tag{39}$$

By definition, we immediately obtain the following:

Lemma 5.4.

$$H^*(\mathbb{A}_1^{(1)}, \partial_9) \ = \ \mathcal{A}_1^{(1)} \quad \textit{and}$$
$$H^*(\mathbb{B}_1^{(1)} g^l, \partial_9) \ = \ \underline{\mathcal{A}}_1^{(1)} g^l \oplus \varsigma \overline{\mathcal{A}}_1^{(1)} g^l$$

for $l \in \{1, 2\}$. *Here,*

$$\mathcal{A}_1^{(1)} = P_u(5) \oplus v_2 h P_u(2, 2),$$
$$\underline{\mathcal{A}}_1^{(1)} = bP_u(4) \oplus v_2 h P_u(2, b1) \quad \textit{and} \quad \overline{\mathcal{A}}_1^{(1)} = P_u(4) \oplus v_2 h P_u(1, 2).$$

Since $H^*(\mathbb{C}_1 g^l, \partial_9) = H^*(\mathbb{C}_1^{(1)} g^l, \partial_9) \otimes \underline{K}$, we obtain

Corollary 5.5. *The cohomology* $H^*(\mathbb{C}_1 g^l, \partial_9)$ *for* $l \in \mathbb{Z}/3$ *is isomorphic to*

$$\mathcal{C}_1 g^l = \begin{cases} \mathcal{A}_1^{(1)} \otimes \underline{K} \otimes \Lambda(\varphi, \zeta) & l = 0 \\ \left[\left(\underline{\mathcal{A}}_1^{(1)} \oplus \varsigma \overline{\mathcal{A}}(1)^{(1)} \right) \oplus \mathcal{A}_1^{(1)} \otimes \mathbb{Z}/3\{\varphi, \zeta\} \right] \otimes \underline{K} g^{\pm 1} & l = \pm 1. \end{cases}$$

Corollary 5.6. *On* $H^{*,*}(C_1 g^l, \partial_5)$*, there is no more non-trivial differential* ∂_9 *other than those in (38). Furthermore, no more differential* ∂_r *for* $r \geqq 10$ *can be defined on the cohomologies on them.*

Proof. Since the submodule with the homology dimension of $\mathcal{C}_1^{(1)} g^l$ greater than ten is trivial, ∂_r is trivial for each $r \geqq 10$. For $r = 9$, ∂_9 originates $H^{s,*}(C_1 g, \partial_5)$ for $s \in \{0, 1\}$, on which the differential ∂_9 is defined. □

6. The cohomology of the differential algebra C

We consider an algebra $E = \mathbb{Z}/3[v_1, v_2, v_2^{-1}]/(v_1^2)$ and E-algebras

$$Q_u \ = \ v_1 P_u^{(0)} \oplus P_u^{(1)}, \quad \text{and} \quad Q_u^h \ = \ v_1 v_2^2 h P_u^{(1)} \oplus h P_u^{(0)}, \tag{40}$$

in which h is an element with bidegree $||h|| = (1, 12)$, and the E-action and the multiplication on Q_u^h satisfies

$$\begin{aligned} & v_1 v_2^s h = 0 \quad \text{ unless } s \equiv 2 \ (3), \\ & xy = 0 \quad \text{for } x \in v_1 v_2^2 h P_u^{(1)} \text{ and } y \in Q_u^h, \end{aligned} \tag{41}$$

and

$$(v_2^s h)(v_2^t h) = (t-s)v_1 v_2^{s+t-4} ub. \tag{42}$$

We notice that Q_u^h has a Q_u-module structure by (41). In this section, we consider the algebras

$$A = Q_u \oplus Q_u^h, \quad C = A \otimes \Lambda(\varphi, \zeta) \quad \text{and} \quad C_g = C[g]/(g^3-1) \tag{43}$$

for generators φ, ζ (*cf.* above (9)) and g with $g^3 = 1$. We introduce differentials $\partial_5 \colon C_g \to C_g$ and $\partial_9 \colon H^*(C_g, \partial_5) \to H^*(C_g, \partial_5)$ so that $H^*(H^*(C_g, \partial_5), \partial_9)$ is closely related to $E_{10}^{*,*}(V_2)$. We moreover assume that ∂_r is a derivation. For the generators u, φ, ζ, $v_1 v_2^s$, $v_2^s h$, b and g, we set

$$\begin{gathered} \partial_r(u) = 0, \quad \partial_r(\varphi) = 0, \quad \partial_r(\zeta) = 0, \quad \partial_r(v_1 v_2^s) = 0, \quad \partial_r(b) = 0, \\ \partial_r(v_2^s h) = 0 \quad \text{and} \quad \partial_5(g) = \omega g = v_2 h b \varsigma g \end{gathered} \tag{44}$$

for $r \in \{5, 9\}$, $s \in \{0, 1, 5\}$, and ω and ς of (36). We define the differential ∂_5 by

$$\partial_5(v_2^{3t}) \;=\; -t v_2^{3t-2} h b^2 \quad \text{for } v_2^{3t} \in K^{(1)}. \tag{45}$$

We notice that the relations in Lemma 4.15 hold after replacing d_5 with ∂_5 by (41) and (42). We define differential ∂_9 on the algebra $\mathbb{C}_g = H^*(C_g, \partial_5)$ by

$$\begin{aligned} \partial_9(v_1 v_2^{3t+s} g^l) \;&=\; \begin{cases} (s-1) l v_2^s h b^3 \varsigma g^l & t = 0 \\ (1-s) v_2^s h b^3 \langle b g^l \rangle & t = 1 \\ 0 & t = 2 \end{cases} \quad \text{for } s \in \{0, 5\}, \\ \partial_9(v_2^{3t+1} h g^l) \;&=\; \begin{cases} 0 & t = 0 \\ -l u b^4 \varsigma g^l & t = 1 \\ -u b^4 \langle b g^l \rangle & t = 2 \end{cases} \end{aligned} \tag{46}$$

for $l \in \mathbb{Z}/3$ and $\langle b g^l \rangle$ in (37). We also assume that the relations in (39) hold in C_g. We further notice that

$$\begin{aligned} Q_u \;&=\; v_1 v_2^3 P_u^{(1)} \otimes \underline{K}' \oplus P_u^{(1)} \otimes \Lambda(v_1 v_2) \quad \text{and} \\ Q_u^h \;&=\; h P_u^{(1)} \otimes \underline{K}' \oplus v_2 h P_u^{(1)} \otimes \Lambda(v_1 v_2). \end{aligned}$$

By (44), (45) and (46), we easily obtain the following:

Lemma 6.1. *The cohomology $H^*(A, \partial_5)$ is isomorphic to*

$$\begin{aligned} \mathbb{A} \;=\; &\Big(v_1 v_2^6 P_u(3,3,-) \otimes \underline{K}' \oplus P_u^{(2)} \otimes \Lambda(v_1 v_2)\Big) \\ &\oplus \Big(h P_u^{(2)} \otimes \underline{K}' \oplus v_2 h P_u(2,2,-) \otimes \Lambda(v_1 v_2)\Big). \end{aligned}$$

The cohomology $H^*(\mathbb{A}, \partial_9)$ is isomorphic to

$$\begin{aligned} \mathcal{A} \;=\; &\big(v_1 v_2^6 P_u(3,3) \otimes \underline{K}' \oplus P_u(5) \otimes \Lambda(v_1 v_2)\big) \\ &\oplus \big(h P_u(4) \otimes \underline{K}' \oplus v_2 h P_u(2,2) \otimes \Lambda(v_1 v_2)\big). \end{aligned}$$

Consider a differential subalgebra of C

$$B = A \otimes \Lambda(\varsigma).$$

Then, in the same manner as the proof of Lemma 6.1, we verify the following lemma easily by (41), (42), (44), (45) and (46) (*cf.* Lemma 4.15):

Lemma 6.2. *The cohomology* $H^*(Bg^{\pm 1}, \partial_5)$ *is isomorphic to*

$$\mathbb{B}g^{\pm 1} = \left(\underline{\mathbb{A}} \oplus \varsigma\overline{\mathbb{A}}\right) g^{\pm 1},$$

where

$$\begin{aligned} \underline{\mathbb{A}} &= \left(v_1 v_2^6 P_u(3,3,-) \otimes \underline{K}' \oplus bP_u^{(2)} \otimes \Lambda(v_1 v_2)\right) \\ &\quad \oplus \left(hbP_u^{(2)} \otimes \underline{K}' \oplus v_2 hP_u(2,2,-) \otimes \Lambda(v_1 v_2)\right) \quad \textit{and} \\ \overline{\mathbb{A}} &= \left(v_1 v_2^6 P_u(2,3,-) \otimes \underline{K}' \oplus P_u^{(2)} \otimes \Lambda(v_1 v_2)\right) \\ &\quad \oplus \left(hP_u^{(2)} \otimes \underline{K}' \oplus v_2 hP_u(1,2,-) \otimes \Lambda(v_1 v_2)\right). \end{aligned}$$

The cohomology $H^*(\mathbb{B}g^{\pm 1}, \partial_9)$ *is isomorphic to*

$$\mathcal{B}g^{\pm 1} = \left(\underline{\mathcal{A}} \oplus \varsigma\overline{\mathcal{A}}\right) g^{\pm 1},$$

where

$$\begin{aligned} \underline{\mathcal{A}} &= \left(v_1 v_2^6 P_u(3,b2) \otimes \underline{K}' \oplus bP_u(4) \otimes \Lambda(v_1 v_2)\right) \\ &\quad \oplus \left(hbP_u(3) \otimes \underline{K}' \oplus v_2 hP_u(2,b1) \otimes \Lambda(v_1 v_2)\right) \quad \textit{and} \\ \overline{\mathcal{A}} &= \left(v_1 v_2^6 P_u(2,3) \otimes \underline{K}' \oplus P_u(4) \otimes \Lambda(v_1 v_2)\right) \\ &\quad \oplus \left(hP_u(3) \otimes \underline{K}' \oplus v_2 hP_u(1,2) \otimes \Lambda(v_1 v_2)\right). \end{aligned}$$

Remark 6.3. In $\underline{\mathcal{A}}g^{\pm 1}$, the elements $v_1 b^k g^{\pm 1}$, $b^k g^{\pm 1}$, $hb^k g^{\pm 1}$ and $v_2^4 hbg^{\pm 1}$ are the classes of $v_1 b^k g^{\pm 1} + v_1 v_2^3 b^{k-1} \varsigma g^{\pm 1} = v_1 b^{k-1} \langle bg^{\pm 1} \rangle$, $b^k g^{\pm 1} \pm v_2^3 b^{k-1} \varsigma g^{\pm 1} = b^{k-1} \langle bg^{\pm 1} \rangle$, $hb^k g^{\pm 1} + v_2^3 hb^{k-1} \varsigma g^{\pm 1} = hb^{k-1} \langle bg^{\pm 1} \rangle$ and $v_2^4 hb - v_2^7 h\varsigma g^{\pm 1} = v_2^4 h \langle bg^{\pm 1} \rangle$, respectively.

Corollary 6.4. *The cohomology* $H^*(H^*(Cg^l, \partial_5), \partial_9)$ *for* $l \in \mathbb{Z}/3$ *is isomorphic to*

$$\mathcal{C}g^l = \begin{cases} \mathcal{A} \otimes \Lambda(\varphi, \zeta) & l = 0 \\ (\mathcal{B} \oplus \mathcal{A}\{\varphi, \zeta\})\, g^l & l = \pm 1. \end{cases}$$

Corollary 6.5. *The other differentials* $\partial_r \colon \mathbb{C}_g^s \to \mathbb{C}_g^{s+r}$ *for* $r \geqq 9$ *are all trivial.*

Proof. By Corollary 6.4, the submodules of $\mathcal{C}g^l$ for $l \in \mathbb{Z}/3$ with the homology dimension greater than nine are:

$$\begin{aligned} \mathcal{C}^{10,*} &= v_1 v_2 b^4 \varsigma K_u^{(2)} \oplus b^4 \varsigma K_u^{(2)} \quad \text{and} \\ \mathcal{C}^{s,*} g^l &= 0 \quad \text{for } s = 10 \text{ and } l = \pm 1 \text{ or } s \geqq 11. \end{aligned}$$

Therefore, $\partial_r = 0$ for $r \geqq 10$. The differential ∂_9 is defined on each element of $\mathcal{C}^{\varepsilon,*} g^l$ for $\varepsilon \in \{0, 1\}$ and $l \in \mathbb{Z}/3$, and no more differential can be defined. □

7. The E_r-terms from the cohomologies of C_1 and C

In this section, we show a lemma by which the E_∞-terms $E_\infty^{*,*}(V_e)g^l$ for $l \in \mathbb{Z}/3$ are deduced from $\mathcal{C}_e g^l$ for $e \in \{1,2\}$. Hereafter, $C_2 = C$, $\mathbb{C}_2 = \mathbb{C}$ and $\mathcal{C}_2 = \mathcal{C}$. Let R_e and S_e denote modules fitting in the diagram

$$\begin{array}{ccccc} & & S_e g^l & & \\ & & \downarrow \mathfrak{i} & & \\ bC_e g^l & \xrightarrow{\subset} & E_2^{*,*}(V_e)g^l & \xrightarrow{\mathfrak{p}} & R_e g^l \\ & \searrow_{\bar{\mathfrak{j}}} & \downarrow \mathfrak{j} & & \\ & & C_e g^l & & \end{array} \tag{47}$$

in which the row and the column are exact. Then, $\bar{\mathfrak{j}}$ and $\mathfrak{p}\mathfrak{i}$ are monomorphisms. Indeed, if $\mathfrak{p}\mathfrak{i}(x) = 0$, then we have an element $bc \in bC_e g^l$ such that $bc = \mathfrak{i}(x)$. $bc = \bar{\mathfrak{j}}(bc) = \mathfrak{j}\mathfrak{i}(x) = 0$ and so $\mathfrak{i}(x) = 0$. Since $\mathfrak{i}$ is a monomorphism, $x = 0$ as desired. Here, $S_1 g^l = 0$, $S_2 g^l = \overline{S}_2 \otimes \Lambda(\zeta)g^l$ and $R_e g^l = \overline{R}_e \otimes \Lambda(\zeta)g^l$ for

$$\begin{aligned} \overline{S}_2 &= uv_1v_2hK^{(1)} \otimes \Lambda(\varphi) \otimes \underline{K}', \\ \overline{R}_1 &= K^{(0)}\{1, h, uh, uh\varphi\} \\ &= (P(1,1,1) \oplus v_2hP_u(1,1,1) \oplus uv_2h\varphi P(1,1,1)) \otimes \underline{K} \quad \text{and} \\ \overline{R}_2 &= K^{(0)}\{v_1, h\} \oplus K^{(1)} \otimes \Lambda(v_1v_2^2h) \\ &\quad \oplus uh\left(K^{(0)} \oplus v_1v_2^2K^{(1)}\right) \otimes \Lambda(\varphi) \oplus \overline{S}_2 \\ &= \left(v_1v_2^6P(1,1,1) \otimes \underline{K}' \oplus P(1,1,1) \otimes \Lambda(v_1v_2)\right) \\ &\quad \oplus \left(hP_u(1,1,1) \otimes \underline{K}' \oplus v_2hP_u(1,1,1) \otimes \Lambda(v_1v_2)\right) \\ &\quad\quad \oplus u\varphi\left(hP(1,1,1) \otimes \underline{K}' \oplus v_2hP(1,1,1) \otimes \Lambda(v_1v_2)\right) \oplus \overline{S}_2. \end{aligned} \tag{48}$$

Indeed, we deduce $\overline{S}_2$ and $\overline{R}_2$ from (23), (43) and isomorphisms

$$\begin{aligned} bQ_u \oplus K^{(1)} \oplus v_1K^{(0)} &= (K^{(1)} \oplus v_1K^{(0)}) \otimes F^b, \\ bQ_u\varphi &= (K^{(1)} \oplus v_1K^{(0)}) \otimes F^{b\varphi}, \\ bQ_u^h \oplus h(K_u^{(0)} \oplus v_1v_2^2K_u^{(1)}) &= (K^{(0)} \oplus v_1v_2^2K^{(1)}) \otimes F^h \quad \text{and} \\ bQ_u^{h\varphi} \oplus uh\varphi(K^{(0)} \oplus v_1v_2^2K^{(1)}) &= (K^{(0)} \oplus v_1v_2^2K^{(1)}) \otimes F^{h\varphi} \end{aligned}$$

obtained by (12) and (40).

We see that

$$\mathfrak{p}\left(\bigoplus\nolimits_{s \geqq 4} E_2^{s,*}(V_e)g^l\right) = 0. \tag{49}$$

Lemma 7.1. *Every element of $S_2g^l \subset E_2^{*,*}(V_2)g^l$ is a permanent cycle.*

Proof. Since $v_1v_2^suhbg^l = 0 \in E_2^{*,*}(V_2)g^l$ (by (19)) unless $s \equiv 2 \bmod 3$, we see that $bS_2g^l = 0$, and so the lemma follows from Corollary 4.3. □

Put

$$b_*\mathbb{C}_e^l = H^*(bC_eg^l, \partial_5) \quad \text{and} \quad b_*\mathcal{C}_e^l = H^*(b_*\mathbb{C}_e^l, \partial_9).$$

We notice that the generator b induces isomorphisms $\mathbb{C}_eg^l \to b_*\mathbb{C}_eg^l$ and $\mathcal{C}_eg^l \to b_*\mathcal{C}_eg^l$. Since $\mathfrak{p}$ in (47) is an epimorphism, for each $x \in R_e$, we have an element $\widetilde{x} \in E_2^{*,*}(V_e)$ such that $\mathfrak{p}(\widetilde{x}) = x$.

Lemma 7.2. *There is an isomorphism*

$$E_{10}^{*,*+l\omega}(V_e) \cong b_*\mathcal{C}_e^l/D_e^l \oplus Z_e^l \quad \textit{for } l \in \mathbb{Z}/3$$

of modules. Here,

$$\begin{aligned} D_e^l &= \{[[xg^l]] \in b_*\mathcal{C}_e^l \mid xg^l = d_5(\widetilde{w}g^l) \textit{ or } [xg^l] = d_9([\widetilde{w}g^l]) \textit{ for } w \in R_e\} \quad \textit{and} \\ Z_e^l &= \{xg^l \in R_eg^l \mid d_5(\widetilde{x}g^l) = 0 \textit{ and } d_9([\widetilde{x}g^l]) = 0\}. \end{aligned}$$

Proof. Note that the differentials d_5 and d_9 act on R_eg^l trivially by (48) (and (49)). Indeed, it has no element of cohomology dimension greater than two. The short exact sequence in (47) induces the long exact sequence

$$R_eg^l \xrightarrow{\delta_5} b_*\mathbb{C}_e^l \xrightarrow{inc_\star} E_6^{*,*}(V_e)g^l \xrightarrow{\mathfrak{p}_\star} R_eg^l$$

of d_5-cohomologies. Hereafter, $inc_\star$ denotes an homomorphism induced from the inclusion. This gives rise to the short exact sequence

$$0 \to b_*\mathbb{C}_e^l/(\mathrm{Im}\ \delta_5) \xrightarrow{inc_\star} E_6^{*,*}(V_e)g^l \xrightarrow{\mathfrak{p}_\star} \mathrm{Ker}\ \delta_5 \to 0.$$

Here, $\delta_5(x) = d_5(\widetilde{x}) \in E_2^{*,*}(V_e)$, and so $\mathrm{Im}\ \delta_5 = \{[x] \mid x = d_5(w),\ w \in R_e\}$. For d_9-cohomologies, we obtain a long exact sequence

$$\mathrm{Ker}\ \delta_5 \xrightarrow{\delta_9} H^*(b\mathbb{C}_e^l/(\mathrm{Im}\ \delta_5), \partial_9) \xrightarrow{inc_\star} E_{10}^{*,*}(V_e)g^l \xrightarrow{\mathfrak{p}_\star} \mathrm{Ker}\ \delta_5 \xrightarrow{\delta_9} \cdots,$$

which splits into a short exact sequence

$$0 \to H^*(b\mathbb{C}_e^l/(\mathrm{Im}\ \delta_5), \partial_9)/(\mathrm{Im}\ \delta_9) \xrightarrow{inc_\star} E_{10}^{*,*}(V_e)g^l \xrightarrow{\mathfrak{p}_\star} \mathrm{Ker}\ \delta_9 \to 0.$$

Now we deduce the lemma by verifying that $H^*(b\mathbb{C}_e^l/(\mathrm{Im}\ \delta_5), \partial_9)/(\mathrm{Im}\ \delta_9) = b_*\mathcal{C}_e^l/D_e^l$ and $\mathrm{Ker}\ \delta_9 = Z_e^l$. □

Since V_e is an M-module spectrum, the homotopy groups $\pi_*(L_2V_e)$ are $\mathbb{Z}/3$-modules, and hence $\pi_{t-s}(L_2V_e) \cong \bigoplus E_{10}^{s,t}(V_e)$. So it suffices to determine the structures of E_{10}-terms.

Proof of Theorem 2.6. The structure of $E_{10}^{*,*}(V_1)$ follows from Theorem 4.5.

For $E_{10}^{*,*\pm\omega}(V_1)$, we obtain

$$\begin{aligned} Z_1^{\pm 1} &= \big[v_2hP_u(1) \oplus uv_2h\varphi P(1,1) \\ &\qquad \oplus\zeta\,(P(1) \oplus v_2hP_u(1,1)) \oplus v_2h\varsigma P(1,1,1)\big] \otimes \underline{K}g^{\pm 1} \quad \text{and} \\ D_1^{\pm 1} &= \big[v_2hb\varsigma P(1) \oplus v_2hb^2P(1,1) \oplus b^5P_u(1) \oplus b^4\varsigma P_u(1) \oplus b^5\varphi P(1) \\ &\qquad \oplus\zeta\,\big(v_2hb^2P(1,1) \oplus b^5P_u(1)\big)\,\big] \otimes \underline{K}g^{\pm 1} \end{aligned}$$

from $\overline{R}_1$ in (48) by (28) and Lemma 4.12 (*cf.* (35) and (38)). We notice that the last summand of $Z_1^{\pm 1}$ is given by the permanent cycles of (38) by setting $\widetilde{v_2^7 h\varsigma g^{\pm 1}} = (v_2^7 h\varsigma \pm v_2^4 hb)g^{\pm 1}$. Therefore, by Corollary 5.5, the module $b_*\mathcal{C}_1^{\pm 1}/D_1^{\pm 1}$ is isomorphic to the tensor product of $\underline{K}g^{\pm 1}$ and

$$b^2 P_u(3) \oplus v_2 hbP(1) \oplus uv_2 hbP(2, b1) \oplus \varsigma \left(bP_u(3) \oplus v_2^4 hbP(2) \oplus uv_2 hbP(1, 2)\right)$$
$$\oplus \varphi\left(bP(4) \oplus ubP(5) \oplus v_2 hbP_u(2, 2)\right) \oplus \zeta\left(bP_u(4) \oplus v_2 hbP(1, 1) \oplus uv_2 hbP(2, 2)\right),$$

and the structure of the E_{10}-terms follow from Lemma 7.2. We add the summand $v_2^4 hbP(1) \otimes \underline{K}g^{\pm 1}$ to the E_{10}-term instead of the last summand $v_2^7 h\varsigma P(1) \otimes \underline{K}g^{\pm 1}$ of $Z_1^{\pm 1}$, since both of the generators of the modules represent the generator $v_2^4 h\left\langle bg^{\pm 1}\right\rangle$. □

Proof of Theorem 2.8. By Proposition 4.8, and Lemmas 4.10, 4.11, 4.14, and 7.1, we read off from (48):

$$Z_2^0 = (\overline{Z}_2 \oplus u\varphi \overline{Z}_2^{\varphi} \oplus \overline{S}_2) \otimes \Lambda(\zeta) \quad \text{and} \quad D_2^0 = (\overline{D}_2 \oplus \varphi \overline{D}_2^{\varphi}) \otimes \Lambda(\zeta),$$

for

$$\begin{aligned}
\overline{Z}_2 &= v_1 v_2^6 P(1, 1) \otimes \underline{K}' \oplus P(1) \otimes \Lambda(v_1 v_2) \oplus hP_u(1) \otimes \underline{K}' \oplus v_2 hP_u(1, 1) \otimes \Lambda(v_1 v_2),\\
\overline{Z}_2^{\varphi} &= hP(1) \otimes \underline{K}' \oplus v_2 hP(1, 1) \otimes \Lambda(v_1 v_2),\\
\overline{D}_2 &= hb^4 P(1) \otimes \underline{K}' \oplus v_2 hb^2 P(1, 1) \otimes \Lambda(v_1 v_2)\\
&\qquad \oplus v_1 v_2^6 b^3 P_u(1, 1) \otimes \underline{K}' \oplus b^5 P_u(1) \otimes \Lambda(v_1 v_2) \quad \text{and}\\
\overline{D}_2^{\varphi} &= v_1 v_2^6 b^3 P(1, 1) \otimes \underline{K}' \oplus b^5 P(1) \otimes \Lambda(v_1 v_2).
\end{aligned}$$

By Lemmas 4.15, 4.16 and 7.1,

$$\begin{aligned}
Z_2^{\pm 1} &= v_1 v_2^6 P(1) \otimes \underline{K}' \oplus v_2 hP_u(1) \otimes \Lambda(v_1 v_2) \oplus \zeta \overline{Z}_2 \oplus u\varphi \overline{Z}_2^{\varphi}\\
&\qquad \oplus \varsigma\left(hP(1, 1) \otimes \underline{K}' \oplus v_2 hP(1, 1, 1) \otimes \Lambda(v_1 v_2)\right) \oplus S_2 \quad \text{and}\\
D_2^{\pm 1} &= \left(hb^3 \varsigma P(1) \oplus hb^4 P(1)\right) \otimes \underline{K}' \oplus \left(v_2 hb^2 P(1, 1) \oplus v_2 hb\varsigma P(1)\right) \otimes \Lambda(v_1 v_2)\\
&\quad \oplus \left(v_1 v_2^6 b^3 P_u(1, 1) \oplus v_1 v_2^6 \varsigma b^2 P_u(1)\right) \otimes \underline{K}' \oplus \left(b^5 P_u(1) \oplus b^4 \varsigma P_u(1)\right) \otimes \Lambda(v_1 v_2)\\
&\qquad \oplus \zeta \overline{D}_2 \oplus \varphi \overline{D}_2^{\varphi} \oplus u\varsigma\left(v_1 b^3 P(1) \otimes \underline{K}'\right).
\end{aligned}$$

Here, every element of $\varsigma \overline{Z}^{\varsigma} g^{\pm 1}$ for

$$\overline{Z}^{\varsigma} = v_2^3 hP(1) \otimes \underline{K}' \oplus v_2^7 hP(1) \otimes \Lambda(v_1 v_2)$$

is a permanent cycle. Indeed, $v_2^{7+s} h\varsigma g^{\pm 1}$ for $s \in \{0, 1, 5\}$ denotes a permanent cycle $(v_2^{7+s} h\varsigma \mp v_2^{4+s} hb)g^{\pm 1}$. Furthermore, for

$$\overline{Z}^{g} = v_1 v_2^6 P(1) \otimes \underline{K}' \oplus v_2 hP_u(1) \otimes \Lambda(v_1 v_2),$$

we have

$$Z_2^{\pm 1} = \overline{Z}^{g} \oplus \zeta \overline{Z}_2 \oplus u\varphi \overline{Z}_2^{\varphi} \oplus \varsigma\left(\overline{Z}_2^{\varphi} \oplus \overline{Z}^{\varsigma}\right) \oplus S_2.$$

Put

$$\overline{D}_2^{\pm1,\varphi} = \big(v_1v_2^6b^2P_u(1) \oplus uv_1b^3P(1)\big) \otimes \underline{K}' \oplus b^4P_u(1) \otimes \Lambda(v_1v_2)$$
$$\oplus hb^3P(1) \otimes \underline{K}' \oplus v_2hbP(1) \otimes \Lambda(v_1v_2),$$

and we see that

$$D_2^{\pm1} = \overline{D}_2 \oplus \varsigma\overline{D}_2^{\pm1,\varphi} \oplus \zeta\overline{D}_2 \oplus \varphi\overline{D}_2^{\varphi}.$$

Then, we notice that

$$b_*\mathcal{C}^0/D_2^0 = \left(b_*\mathcal{A}/\overline{D}_2 \oplus \varphi\left(b_*\mathcal{A}/\overline{D}_2^{\varphi}\right)\right) \otimes \Lambda(\zeta), \quad \text{and}$$
$$b_*\mathcal{C}^{\pm1}/D_2^{\pm1} = \left(b_*\underline{\mathcal{A}}/\overline{D}_2 \oplus \varsigma\left(b_*\overline{\mathcal{A}}/\overline{D}_2^{\pm1,\varphi}\right)\right) \oplus \left(\zeta\left(b_*\mathcal{A}/\overline{D}_2\right) \oplus \varphi\left(b_*\mathcal{A}/\overline{D}_2^{\varphi}\right)\right)$$

by Corollary 6.4. Furthermore, we read off the summands:

$$b_*\mathcal{A}/\overline{D}_2 = \big(v_1v_2^6bP_u(2,2) \otimes \underline{K}' \oplus bP_u(4) \otimes \Lambda(v_1v_2)\big)$$
$$\oplus \big(hb\,(P(3) \oplus uP(4)) \otimes \underline{K}' \oplus v_2hb\,(P(1,1) \oplus uP(2,2)) \otimes \Lambda(v_1v_2)\big),$$
$$b_*\mathcal{A}/\overline{D}_2^{\varphi} = \big(v_1v_2^6b\,(P(2,2) \oplus uP(3,3)) \otimes \underline{K}' \oplus b\,(P(4) \oplus uP(5)) \otimes \Lambda(v_1v_2)\big)$$
$$\oplus \big(hbP_u(4) \otimes \underline{K}' \oplus v_2hbP_u(2,2) \otimes \Lambda(v_1v_2)\big),$$
$$b_*\underline{\mathcal{A}}/\overline{D}_2 = \big(v_1v_2^6bP_u(2,b1) \otimes \underline{K}' \oplus b^2P_u(3) \otimes \Lambda(v_1v_2)\big)$$
$$\oplus \big(hb^2\,(P(2) \oplus uP(3)) \otimes \underline{K}' \oplus v_2hb\,(P(1) \oplus uP(2,b1)) \otimes \Lambda(v_1v_2)\big) \text{ and}$$
$$b_*\overline{\mathcal{A}}/\overline{D}_2^{\pm1,\varphi} = v_1v_2^6b\,(P(1,3) \oplus uP(1,2)) \otimes \underline{K}' \oplus bP_u(3) \otimes \Lambda(v_1v_2)$$
$$\oplus \big(hb\,(P(2) \oplus uP(3)) \otimes \underline{K}' \oplus v_2hb\,\big(v_2^3P(2) \oplus uP(1,2)\big) \otimes \Lambda(v_1v_2)\big).$$

Put that $\mathcal{M} = b_*\mathcal{A}/\overline{D}_2 \oplus \overline{Z}_2$, $\mathcal{M}^{\varphi} = b_*\mathcal{A}/\overline{D}_2^{\varphi} \oplus u\overline{Z}_2^{\varphi}$, $\underline{\mathcal{M}} = b_*\underline{\mathcal{A}}/\overline{D}_2 \oplus \overline{Z}^g$ and $\overline{\mathcal{M}}^{\varphi} = b_*\overline{\mathcal{A}}/\overline{D}_2^{\pm1,\varphi} \oplus \left(\overline{Z}_2^{\varphi} \oplus \overline{Z}^{\varsigma}\right)$, and we obtain the E_{10}-terms from Lemma 7.2, and the homotopy groups of the M-module spectrum V_2 are isomorphic to the corresponding E_{10}-terms. □

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