AN UPPER BOUND ON THE TOPOLOGICAL COMPLEXITY OF DISCRIMINANTAL VARIETIES

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Abstract

We give an upper bound on the topological complexity of varieties $\mathcal{V}$ obtained as complements in $\mathbb{C}^m$ of the zero locus of a polynomial. As an application, we determine the topological complexity of unordered configuration spaces of the plane.

1. Introduction

Let X be a path-connected topological space. The topological complexity $\mathbf{TC}(X)$ is a numerical invariant introduced by Farber [6] to measure the complexity of a motion planner on X. Denote by $X^{[0,1]}$ the path-space of X, whose points are all continuous maps $\gamma\colon [0,1] \to X$, and by $p_X\colon X^{[0,1]} \to X \times X$ the fibration which assigns to each path $\gamma\colon [0,1] \to X$ (i.e. each element $\gamma \in X^{[0,1]}$) the couple $(\gamma(0), \gamma(1)) \in X \times X$.

Definition 1.1 (Topological complexity)**.** A *motion planner* on a path-connected space X is the datum of:

- a covering of $X \times X$ with open sets $U_0, \dots, U_k$;
- for each $U_i \subset X \times X$, a section $s_i\colon U_i \to X \times X$ of the fibration p_X.

The *topological complexity* of X, denoted $\mathbf{TC}(X)$, is the minimum k for which there exists a motion planner on X with $k+1$ open sets $U_0, \dots, U_k$ as above.

The connection of the previous definition to robotics is immediate: if X is the space of configurations that a robot (or a system of robots) can assume, a motion planner on X will give, for each couple x_1, x_2 of points in X, a path to pass from the configuration x_1 to the configuration x_2. The number k measures the "robustness" of a motion planner: the smaller is k, the less the motion planner is sensitive to small perturbations of x_1 and x_2 and thus the more it is reliable.

A variation of the notion of topological complexity is the *equivariant topological complexity*, introduced by Colman and Grant [5]. Let X be a path-connected space with a continuous action of a topological group G; then G acts diagonally on $X \times X$ and on $X^{[0,1]}$: in the second case the "diagonal" is the natural map $G \mapsto G^{[0,1]}$ sending $g \in G$ to the constant function $[0,1] \to G$ with value g; hence $(g \cdot \gamma)(t) := g \cdot (\gamma(t))$. Note that the fibration $p_X \colon X^{[0,1]} \to X \times X$ is a G-equivariant map.

Definition 1.2 (Equivariant topological complexity)**.** A G-*motion planner* on X is given by:

- a covering of $X \times X$ by G-invariant open sets $U_0, \dots, U_k$;
- for each $U_i \subset X \times X$, a G-equivariant section $s_i \colon U_i \to X \times X$ of the fibration p_X.

The G-*equivariant topological complexity* of X, denoted $\mathbf{TC}_G(X)$, is the minimum k for which there exists a G-motion planner on X with $k+1$ open sets $U_0, \dots, U_k$ as above.

Clearly a G-motion planner is also a motion planner, hence $\mathbf{TC}(X) \leqslant \mathbf{TC}_G(X)$ for all path-connected G-spaces X.

In this paper we address the problem of bounding from above the topological complexity of algebraic varieties $\mathcal{V}$ obtained as complements in $\mathbb{C}^k$ of the zero locus of a polynomial.

Definition 1.3. A *discriminantal variety* is a complex algebraic variety $\mathcal{V}$ obtained as the complement in $\mathbb{C}^m$ of the zero locus of a non-zero polynomial $\Delta(z_1, \dots, z_m)$, called the *discriminant*.

As we will see, discriminantal varieties often carry a natural, nice action of an s-dimensional torus $\mathbb{T}^s = (\mathbb{S}^1)^s$, so that we can compare $\mathbf{TC}(\mathcal{V})$ and $\mathbf{TC}_{\mathbb{T}^s}(\mathcal{V})$.

The motivating example for considering discriminantal varieties is given by *configuration spaces of points in the plane*, which come usually in two different flavours.

Definition 1.4. The *ordered configuration space* of n points in $\mathbb{C}$ is the space

$$F_n = \{(w_1, \dots, w_n) \in \mathbb{C}^n \mid w_i \neq w_j \ \forall i \neq j\}.$$

There is a natural action of the symmetric group $\mathfrak{S}_n$ on F_n, by permuting the labels of a configuration. The quotient space is the *unordered configuration space* of n points in $\mathbb{C}$, and is denoted

$$C_n := F_n/\mathfrak{S}_n.$$

Roughly speaking, the topological complexities $\mathbf{TC}(F_n)$ and $\mathbf{TC}(C_n)$ measure the complexity of a motion planner which coordinates n robots in the plane, avoiding collisions between them. The difference between the two cases is subtle: in the first case each robot is labeled by a number $1, \dots, n$, and the motion planner takes as input, for each robot, its initial and its final position; in the second case the robots are indistinguishable, and the motion planner's task is to move all robots together, starting from the set of n initial positions, so that at the end the robots occupy (in some order) the n final positions.

One can consider the action of $\mathbb{T} = \mathbb{S}^1 = SO(2)$ by rotations on both F_n and C_n: then the topological complexities $\mathbf{TC}_{\mathbb{T}}(F_n)$ and $\mathbf{TC}_{\mathbb{T}}(C_n)$ measure the complexity of a motion planner coordinating n ordered or unordered robots in the plane, with the additional hypothesis that the motion planner should be equivariant with respect to a "change of coordinate system" by rotation. A motion planner with this property might be particularly useful, for instance, because it prescribes to the robots the same behaviour independently of the spatial point of view of the observer who measures their positions while coordinating them. In this paper we restrict for simplicity to the compact group $SO(2)$, but the reader will see that essentially the same arguments work when studying the equivariant topological complexity of F_n and C_n with respect to the entire group $\mathrm{Iso}^+(\mathbb{R}^2)$ of orientation-preserving isometries of $\mathbb{R}^2$, in which also translations are taken into account. However there is an intrinsic issue to extending the discussion to the group $\mathrm{Iso}(\mathbb{R}^2)$ of *all* isometries of $\mathbb{R}^2$: for $n \geqslant 2$, neither F_n nor C_n is $\mathrm{Iso}(\mathbb{R}^2)$-connected (see Definition 2.3), hence the $\mathrm{Iso}(\mathbb{R}^2)$-equivariant topological complexity of both spaces is not defined (or is equal to infinity).

The topological complexity of F_n was computed by Farber and Yuzvinsky [7] and is equal to $2n - 3$. For the space C_n, Recio-Mitter and the author [3] proved the bounds $2n - 2\lfloor \sqrt{n/2} \rfloor - 3 \leqslant \mathbf{TC}(C_n) \leqslant 2n - 2$, conjectured that $\mathbf{TC}(C_n) = 2n - 3$, and showed that the lower bound $2n - 3 \leqslant \mathbf{TC}(C_n)$ would follow from a proof that the cohomological dimension $\mathrm{cd}([P_n, P_n])$ is equal to $n - 2$, where $P_n = \pi_1(F_n)$ is the *pure braid group on n strands* and $[P_n, P_n]$ is its commutator subgroup. The equality $\mathrm{cd}([P_n, P_n]) = n - 2$ was then proved by the author [2]. In this paper we prove the upper bound for $\mathbf{TC}(C_n)$, and compute also the $\mathbb{T}$-equivariant topological complexities of F_n and C_n.

Theorem 1.5. *Consider the action of $\mathbb{T} = \mathbb{S}^1 = SO(2)$ on F_n and C_n by rotation of configurations of points in the plane. For all $n \geqslant 2$ we have $\mathbf{TC}_{\mathbb{T}}(F_n) \leqslant 2n - 3$ and $\mathbf{TC}_{\mathbb{T}}(C_n) \leqslant 2n - 3$.*

Note that, together with the already known lower bound estimate, we obtain the chain of equalities $\mathbf{TC}(F_n) = \mathbf{TC}_{\mathbb{T}}(F_n) = \mathbf{TC}(C_n) = \mathbf{TC}_{\mathbb{T}}(C_n) = 2n - 3$ for all $n \geqslant 2$. We will obtain Theorem 1.5 as an application of the following, more general statement.

Theorem 1.6. *Let $\mathcal{V} \subset \mathbb{C}^m$ be a discriminantal variety. Assume that, for some $s \geqslant 0$, the torus $\mathbb{T}^s = (\mathbb{S}^1)^s$ acts by scalar multiplication on $\mathcal{V}$ (see Definition 5.2), and assume that all stabilisers of the action have dimension $\leqslant t$. Then we have*

$$\mathbf{TC}(\mathcal{V}) \leqslant \mathbf{TC}_{\mathbb{T}^s}(\mathcal{V}) \leqslant 2m - s + t.$$

The paper is organised as follows. In Sections 2, 3 and 4 we recall some standard material about actions of groups on spaces, equivariant Morse theory and CW complexes in the equivariant setting. The exposition is in a few points non-standard, and it is adapted to the content of the rest of the paper. In Section 5 we prove Theorem 1.6 up to finding a $\mathbb{T}^s$-invariant Morse function on $\mathcal{V} \times \mathcal{V}$ which satisfies some additional properties. The existence of such a function is shown in Section 6. Finally, in Section 7 we apply Theorem 1.6 to prove Theorem 1.5, and in Section 8 we briefly discuss the problem of finding an explicit, $\mathbb{T}$-equivariant motion planner on C_n (the case of F_n being quite analogous).

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2.   Actions of groups on spaces

Before starting we introduce some classical notions and some notation related to spaces with an action of a group. We fix a compact, connected Lie group G throughout the next three sections.

Definition 2.1. Two G-spaces X and Y are G-equivariantly homotopy equivalent if there are two G-equivariant maps $\alpha\colon X \to Y$ and $\beta\colon Y \to X$ such that $\beta \circ \alpha\colon X \to X$ is homotopic to Id_X through G-equivariant maps $X \to X$, and $\alpha \circ \beta$ is homotopic to Id_Y through G-equivariant maps $Y \to Y$.

Definition 2.2. Let $H \leqslant G$ be a closed subgroup and let X be a G-space. We denote by $X^H \subseteq X$ the subspace of points $x \in X$ which are fixed by the action of H, i.e. satisfy $H \leqslant \mathrm{Stab}_x \leqslant G$.

Note that if X and Y are G-homotopy equivalent G-spaces, then X^H and Y^H are homotopy equivalent spaces for all closed $H \leqslant G$. The following definition differs slightly from the one usually given in the literature, e.g. in [5].

Definition 2.3. A G-space X is G-connected if X^H is path-connected *or empty* for all closed subgroups $H \leqslant G$.

The G-equivariant Lyusternik-Schnirelmann category was first introduced in [9]. We give a slightly non-standard definition of it.

Definition 2.4. Let X be a G-space; an open set $U \subset X$ is G-categorical if U is G-invariant and there is a G-equivariant homotopy $U \times [0,1] \to X$ starting from the inclusion $U \hookrightarrow X$ and ending with a map with values in a *finite union of G-orbits* of X. The G-equivariant Lyusternik-Schnirelmann category of X, denoted $\mathbf{cat}_G(X)$, is the minimum k such that X can be covered by $k + 1$ G-categorical open sets $U_0, \dots, U_k$.

The previous definition differs from the usual one because we allow a G-categorical set $U \subset X$ to deform G-equivariantly into the union of more than a single G-orbit of X (clearly, this is in principle possible only if U is disconnected).

Similarly to the classical Lyusternik-Schnirelmann category, $\mathbf{cat}_G$ is a G-homotopy invariant of G-spaces: if two G-spaces X and Y are G-equivariantly homotopy equivalent, then $\mathbf{cat}_G(X) = \mathbf{cat}_G(Y)$. The argument of the proof is the usual one also with our slightly different definition, and we repeat it for completeness. Let $\alpha\colon X \to Y$ be

a G-equivariant homotopy equivalence with G-homotopy inverse $\beta\colon Y \to X$, and suppose that Y is covered by G-categorical sets $U_0, \ldots, U_k$ (in the sense of Definition 2.4); then we claim that $V_0 = \alpha^{-1}(U_0), \ldots, V_k = \alpha^{-1}(U_k)$ are G-categorical sets covering X. To see that V_i is G-categorical, note that the G-homotopy $\beta \circ \alpha \simeq_G \mathrm{Id}_X$ restricts to a G-homotopy from the inclusion $V_i \hookrightarrow X$ to the composition $\beta \circ \alpha\colon V_i \to X$; moreover $\alpha\colon V_i \to Y$ has image in U_i, so we can G-equivariantly deform $\alpha\colon V_i \to Y$ to a map $V_i \to Y$ with image contained in finitely many orbits. Composing this latter homotopy with β yields a G-homotopy from $\beta \circ \alpha\colon V_i \to X$ to a map $V_i \to X$ with image in finitely many orbits.

The following is [**5**, Proposition 5.6].

Proposition 2.5. *Let X be a G-connected G-space; then $\mathbf{TC}_G(X) \leqslant \mathbf{cat}_G(X \times X)$, where $X \times X$ is endowed with the diagonal action of G.*

The proof given in [**5**] works also using our slightly different Definitions 2.3 and 2.4. The main difference is that, in our setting, we have to remark that for each closed subgroup $H \leqslant G$, either $(X \times X)^H = X^H \times X^H$ is empty, or it is path-connected and intersects the diagonal of $X \times X$ in some point.

3. Equivariant Morse theory

Recall that if M is a smooth manifold and $f\colon M \to \mathbb{R}$ is a proper and bounded below Morse function, then M is homotopy equivalent to a CW complex X whose k-cells are in one-to-one correspondence with critical points of f of index k, for all $k \geqslant 0$.

In this section we establish the analogous result for manifolds M endowed with a smooth left action of a compact connected Lie group G: if M is endowed with a G-invariant Morse function, then M is G-equivariantly homotopy equivalent to a G-cell complex X whose k-cells are in one-to-one correspondence with critical manifolds $N \subset M$ of index k.

We will consider equivariant Morse theory and equivariant cell complexes in the general case, although later we will specialise to manifolds M with an action of a torus $\mathbb{T}^s$. The material of this section is an adaptation of the results of [**12**, §4].

Let M be a smooth real manifold, let G be a compact Lie group, and assume G acts smoothly on the left on M.

Let $f\colon M \to \mathbb{R}$ be a smooth function, and assume that f is *proper* (i.e. for all closed intervals $[a, b] \subset \mathbb{R}$ the preimage $f^{-1}([a, b]) \subset M$ is compact) and f is G-invariant.

Definition 3.1. A submanifold $N \subset M$ is called *critical* for f if the following conditions hold:

- N is an orbit of the action of G on M;
- every point $p \in N$ is critical for f, i.e. $df(p) = 0$.

If N is critical for f and $p \in N$, then the Hessian $H(f)_p$ is a well-defined bilinear form on $T_p(M)$, with $T_p(N) \subset \ker H(f)_p$, i.e. $H(f)\langle v, w \rangle = 0$ for all $v \in T_p(N)$ and $w \in T_p(M)$. It follows that there is an induced bilinear form $\bar{H}(f)_p$ on the vector space $T_p(M)/T_p(N)$.

Definition 3.2. A critical manifold $N \subset M$ for f is *non-degenerate* if $\bar{H}(f)_p$ is a non-degenerate bilinear form on $T_p(M)/T_p(N)$. The *index* of N, denoted $\mathrm{ind}(N)$, is the negativity index ι_- of the bilinear form $\bar{H}(f)_p$ on $T_p(M)/T_p(N)$.

To see that the index of N does not depend on the choice of $p \in N$, note that the signature of $\bar{H}(f)_p$ always has the form $(\iota_0(p), \iota_-(p), \iota_+(p))$ with $\iota_0(p) = 0$, for all $p \in N$; since the positivity and negativity indices ι_+ and ι_- are lower semicontinuous functions $N \to \mathbb{N}$ and the sum $\iota_- + \iota_+$ is constantly equal to $\dim M - \dim N$ on N, it follows that ι_+ and ι_- are continuous, hence constant functions on N. Thus Definition 3.2 is well-posed.

Definition 3.3. A G-invariant function $f \colon M \to \mathbb{R}$ is a *G-invariant Morse function* if it is bounded below, is proper and the critical locus of f is the union of finitely many non-degenerate critical manifolds $N_1, \ldots, N_r \subset M$.

Lemma 4.8 in [**12**] is stated only for closed manifolds, but it can be generalised to the following lemma with the same proof.

Lemma 3.4. *Let $\tilde{f} \colon M \to \mathbb{R}$ be a smooth, proper, G-invariant function, and assume that $\tilde{f}$ has no critical points outside a compact set $K \subset M$. Let U be a G-invariant, relatively compact open neighbourhood of K in M. Then for all $l \geqslant 1$ there is a G-invariant Morse function $f \colon M \to \mathbb{R}$ such that $\tilde{f} - f$ vanishes outside U, and $\tilde{f}$ is arbitrarily close to f in the C^l norm.*

We will need the case $l = 2$. To state the main result about G-invariant Morse functions we need the following definition.

Definition 3.5. Let $H \leqslant G$ be a closed subgroup, let R be an orthogonal representation of H of real dimension k, and denote by $D(R) \subset R$ the closed unit ball, and by $S(R) = \partial D(R) \subset D(R)$ the unit sphere. We say that the G-space Y is obtained from the G-space X by a *cell attachment of type (H, R)* if there is a pushout diagram of G-spaces

$$
\begin{array}{ccc}
G \times_H S(R) & \xrightarrow{\ \varphi\ } & X \\
\big\downarrow{\scriptstyle \subset} & & \big\downarrow \\
G \times_H D(R) & \xrightarrow{\hspace{1cm}} & Y
\end{array}
$$

for some G-equivariant map $\varphi \colon G \times_H S(R) \to X$. We denote by $\psi \colon G \times_H D(R) \to Y$ the induced map, which is called a *characteristic map* of a cell, whereas φ is referred to as an *attaching map*. We represent a cell by its characteristic map ψ; the *index* of ψ is defined as the real dimension of R.

A *finite G-cell complex X* is a space obtained starting from $\emptyset$ and applying finitely many cell attachments as above. The *index* of a G-cell complex X, denoted by $\mathrm{ind}(X)$, is the maximal index of its cells.

Clearly the first cell that we attach to $\emptyset$ while constructing a G-cell complex X must have index 0. Note however that we do not require in Definition 3.5 that the cells are attached in an order for which the index is weakly increasing; in particular, if X is a finite G-cell complex and H, R and φ are as in Definition 3.5, it may

happen that φ hits points belonging to cells with index greater or equal than the one we are currently attaching. It is not true in general that we can homotope φ in a G-equivariant way to a new attaching map φ' hitting only points of X belonging to cells of index lower than $\dim R$: in other words, there is no G-equivariant cellular approximation theorem for G-cell complexes as in Definition 3.5.

The index $\mathrm{ind}(X)$ coincides with the geometric dimension of X (seen as a space homeomorphic to a plain CW complex) only when all G-cells of X are defined using subgroups $H \leqslant G$ of finite index, i.e. codimension 0. Our case of interest will be quite the opposite, namely the one in which every G-cell is defined using a subgroup $H \leqslant G$ of small dimension.

Corollary 4.11 in [**12**] is stated for compact manifolds, but it can be generalised, with the same proof, to the following theorem.

Theorem 3.6. *Let M be a smooth manifold with a smooth action of G, and let $f\colon M \to \mathbb{R}$ be a G-invariant Morse function. Then M is G-equivariantly homotopy equivalent to a finite G-cell complex X, whose cells $\psi_1, \ldots, \psi_r$ are in one-to-one correspondence with the critical manifolds $N_1, \ldots, N_r$ of f, such that $\mathrm{ind}(\psi_i) = \mathrm{ind}(N_i)$ for all $1 \leqslant i \leqslant r$.*

4. Equivariant CW complexes

The results of this section are probably standard, but I could not find a source stating them explicitly; nevertheless they are straightforward consequences of results that can be found in [**11**, II] and [**8**], which are the main references for this section.

Our aim is to replace, up to G-equivariant homotopy equivalence, a G-cell complex by a G-CW complex, which is defined as follows.

Definition 4.1. A *finite G-pre-CW complex* X is a finite G-cell complex X obtained by attaching cells of type (H, ε^k), where ε^k is the trivial k-dimensional representation of some closed subgroup $H \leqslant G$, for some $k \geqslant 0$.

A *finite G-CW complex* X is a particular type of finite G-pre-CW-complex: it is a G-space X filtered by G-invariant subspaces $\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X_\nu = X$, for some $\nu \geqslant 0$, such that, for all $0 \leqslant k \leqslant \nu$, X_k is obtained from X_{k-1} by *simultaneously* attaching finitely many G-cells of type ε^k. More precisely, for all $0 \leqslant k \leqslant \nu$ there are:

- a finite set I_k;
- a closed subgroup $H_j < G$ for all $j \in I_k$;
- a G-equivariant map $\varphi_j \colon G \times_{H_j} S(\varepsilon^k) \to X_{k-1}$ for all $j \in I_k$,

such that X_k is obtained as pushout of the following diagram:

$$
\begin{array}{ccc}
\coprod_{j \in I_k} G \times_{H_j} S(\varepsilon^k) & \xrightarrow{\,\coprod \varphi_j\,} & X_{k-1} \\
\downarrow{\scriptstyle \subset} & & \downarrow \\
\coprod_{j \in I_k} G \times_{H_j} D(\varepsilon^k) & \longrightarrow & X_k.
\end{array}
$$

The following proposition is a direct consequence of [**11**, Theorem II.2.1].

Proposition 4.2. *Every G-pre-CW complex X is G-equivariantly homotopy equiva-lent to a G-CW complex Y with the same number of cells in every index.*

Thus our aim reduces to replacing a G-cell complex by a G-pre-CW complex up to G-equivariant homotopy equivalence. Before proceeding, we state the following theorem, which is a weak form of [**8**, Theorem 7.1].

Theorem 4.3. *Let W be a smooth, compact manifold of dimension k with a smooth action of G; then W is homeomorphic to a G-CW complex with G-cells of index $\leqslant k$.*

Suppose now that we are given a G-pre-CW complex X, a closed subgroup $H \leqslant G$, a possibly non-trivial representation R of H of some dimension k and a G-equivariant map $\varphi \colon G \times_H S(R) \to X$. Then $G \times_H S(R)$ is a compact manifold (it is the total space of a bundle over G/H with fibre $S(R)$), so by Theorem 4.3 there is a decom-position of $G \times_H S(R)$ as a G-CW complex.

We can extend this to a structure of G-CW-complex on $G \times_H D(R)$, by using a cell of index 0 to represent the zero section of the disc bundle $G \times_H D(R) \to G/H$, and then by coning off the G-CW structure on $G \times_H S(R)$ inside $G \times_H D(R)$: note that the cells filling the cone have index precisely 1 more than the corresponding ones in $G \times_H S(R)$. We thus obtain a structure of G-pre-CW complex on the pushout of $G \times_H D(R)$ and X along φ.

Repeating this argument for all cells in a G-cell complex, we obtain a decomposition of any G-cell complex X as a G-pre-CW complex: the total number of cells will increase, but the index of X is the same when X is considered as a G-cell complex (before the decomposition) or as a G-pre-CW complex (after the decomposition).

Putting together this argument, Theorem 3.6 and Proposition 4.2 we obtain the following theorem, for which I could not find a direct reference in the literature.

Theorem 4.4. *Let M be a smooth manifold with a smooth action of G, and let $f \colon M \to \mathbb{R}$ be a G-invariant Morse function whose critical manifolds have index $\leqslant k$. Then M is G-equivariantly homotopy equivalent to a finite G-CW complex X with $\mathrm{ind}(X) \leqslant k$.*

The index of a G-CW complex X gives an upper bound for the G-equivariant Lyusternik-Schnirelmann category $\mathbf{cat}_G(X)$, as we see in the following proposition.

Proposition 4.5. *Let X be a finite G-CW complex of index ν; then $\mathbf{cat}_G(X) \leqslant \nu$.*

Proof. Let $\emptyset = X_{-1} \subset X_0 \subset \cdots \subset X_\nu$ be the filtration of X as in Definition 4.1, and for $0 \leqslant k \leqslant \nu$ let $U_k \subset X$ be a G-invariant neighbourhood of $X_k \setminus X_{k-1}$ inside $X \setminus X_{k-1}$, such that U_k deformation retracts G-equivariantly onto $X_k \setminus X_{k-1}$. Each difference $X_k \setminus X_{k-1}$ takes the form $\coprod_{i \in I_k} G/H_i \times \mathring{D}(\varepsilon^k)$, where I_k is the finite set indexing cells of index k in X, H_i is a closed subgroup of G for all $i \in I_k$, and $\mathring{D}$ denotes the open unit disc. There is a G-equivariant retraction of $G/H_i \times \mathring{D}(\varepsilon^k)$ onto its core $G/H_i \times \{0\}$; carrying out these retractions simultaneously shows that U_k is G-categorical. $\square$

Note that in the previous proof we crucially used our non-standard definition of the G-equivariant Lyusternik-Schnirelmann category.

5.   Strategy of proof of Theorem 1.6

We fix $m \geqslant 1$ and consider the complex vector space $\mathbb{C}^m$ with complex coordinates $z_1, \ldots, z_m$; the real coordinates of $\mathbb{C}^m$ are $x_i = \Re(z_i)$ and $y_i = \Im(z_i)$ for all $1 \leqslant i \leqslant m$. Let $\Delta(z_1, \ldots, z_m)$ be a non-zero polynomial with complex coefficients and denote by $\mathcal{V} \subset \mathbb{C}^m$ the subspace of points of $\mathbb{C}^m$ where Δ does not vanish. We write explicitly

$$\Delta(z_1, \ldots, z_m) = \sum_{\underline{i} \in I} \alpha_{\underline{i}} \cdot z_1^{i_1} \ldots z_m^{i_m},$$

where $I \subset \mathbb{Z}_{\geqslant 0}^m$ is a finite set of multi-indices $\underline{i} = (i_1, \ldots, i_m)$, and $\alpha_{\underline{i}} \in \mathbb{C} \setminus \{0\}$ for all $\underline{i} \in I$.

Definition 5.1. A *homogenisation* of Δ is the assignment of a degree $d_i \in \mathbb{Z}$ to each of the variables z_i, such that Δ becomes a homogeneous polynomial of some degree $N = N(d_1, \ldots, d_m)$. We represent a homogenisation by a vector of integers $(d_1, \ldots, d_m) \in \mathbb{Z}^m$. The set of all homogenisations of Δ is denoted by $\mathrm{Homog}(\Delta) \subset \mathbb{Z}^m$.

In formulas, the condition for $(d_1, \ldots, d_m)$ to lie in $\mathrm{Homog}(\Delta)$ is that the integer $N = i_1 d_1 + \cdots + i_m d_m$ be the same for all $\underline{i} \in I$.

For example, if $m = 3$ and $\Delta = z_1^2 - z_2 z_3$, then $(1, 1, 1)$ and $(2, 4, 0)$ are homogenisations of Δ: in the two cases Δ becomes homogeneous, of degree 2 and 4 respectively. Note that $(0, \ldots, 0)$ is always a homogenisation of Δ, making it homogeneous of degree 0; in some cases this is the only possible homogenisation, e.g. for $m = 2$ and $\Delta = z_1 + z_1^2 + z_2^2 + z_2^3$. Note that $\mathrm{Homog}(\Delta)$ is a sub-$\mathbb{Z}$-module of $\mathbb{Z}^m$.

Definition 5.2. For $s \geqslant 0$ let $\mathbb{T}^s = (\mathbb{S}^1)^s$ be the s-dimensional torus, considered as a Lie group (it is the trivial group for $s = 0$). We denote by $\underline{\theta} = (\theta_1, \ldots, \theta_s)$ a generic element of $\mathbb{T}^s$, where $\theta_i \in \mathbb{S}^1 \subset \mathbb{C}$ is a unit complex number. Assume that we are given a matrix $\Xi = (\xi_{i,j})_{1 \leqslant i \leqslant s, 1 \leqslant j \leqslant m}$ with coefficients in $\mathbb{Z}$, whose rows (of length m) lie in $\mathrm{Homog}(\Delta)$. Then Ξ induces an action of $\mathbb{T}^s$ on $\mathcal{V}$ *by scalar multiplication* by the formula

$$\underline{\theta} \cdot (z_1, \ldots, z_m) = (z_1', \ldots, z_m'),$$

where for all $1 \leqslant j \leqslant m$ we set $z_j' = \theta_1^{\xi_{1,j}} \cdot \theta_2^{\xi_{2,j}} \ldots \theta_s^{\xi_{s,j}} \cdot z_j$.

The condition that all rows of Ξ lie in $\mathrm{Homog}(\Delta)$ ensures the equality

$$\Delta(z_1', \ldots, z_m') = \theta_1^{N(\xi_{1,1}, \ldots, \xi_{1,m})} \cdot \theta_2^{N(\xi_{2,1}, \ldots, \xi_{2,m})} \ldots \theta_s^{N(\xi_{s,1}, \ldots, \xi_{s,m})} \cdot \Delta(z_1, \ldots, z_m).$$

In particular $(z_1, \ldots, z_m)$ lies in $\mathcal{V}$ (i.e. $\Delta(z_1, \ldots, z_m) \neq 0$) if and only if $(z_1', \ldots, z_m')$ lies in $\mathcal{V}$ (i.e. $\Delta(z_1', \ldots, z_m') \neq 0$), and thus the formula gives a well-defined action of $\mathbb{T}^s$ on $\mathcal{V}$. The action is not assumed to be free or faithful in any way: for instance we allow Ξ to be the zero matrix, yielding a trivial action of $\mathbb{T}^s$ on $\mathcal{V}$. In the rest of the section and in the next section we fix a matrix Ξ as in Definition 5.2. We will prove the following proposition.

Proposition 5.3. *Assume that all stabilisers of the action of $\mathbb{T}^s$ on $\mathcal{V}$ have dimension $\leqslant t$, for some $0 \leqslant t \leqslant s$. Then $\mathcal{V} \times \mathcal{V}$ is G-equivariantly homotopy equivalent to a G-CW complex X with $\mathrm{ind}(X) \leqslant 2m - s + t$.*

Using Theorem 4.4, in order to prove Proposition 5.3 it suffices to exhibit a $\mathbb{T}^s$-invariant Morse function on the manifold $M = \mathcal{V} \times \mathcal{V}$ with indices $\leqslant 2m - s + t$: we will do it in the next section. In the remainder of this section we will see how Theorem 1.6 follows from Proposition 5.3.

Lemma 5.4. *The $\mathbb{T}^s$-space $\mathcal{V}$ is G-connected.*

Proof. First, note that $\mathcal{V}$ is path-connected, since $\mathcal{V}$ is the complement in $\mathbb{C}^m$ of the zero-locus of a (non-zero) polynomial (incidentally, this shows that our attempt to estimate $\mathbf{TC}(\mathcal{V})$ is not completely pointless!). Let $H \leqslant \mathbb{T}^s$ be a non-trivial closed subgroup. For a point $(z_1, \ldots, z_m) \in \mathcal{V}$, the property that $(z_1, \ldots, z_m)$ belongs to $\mathcal{V}^H$ can be characterised as follows: for all $\underline{\theta} \in H$ and for all $1 \leqslant j \leqslant m$, either $z_j = 0$ or $\theta_1^{\xi_{1,j}} \cdot \ldots \cdot \theta_s^{\xi_{s,j}} = 1 \in \mathbb{S}^1$. In particular every $\underline{\theta} \in H$ imposes a finite, possibly empty collection of restrictions for points of $\mathcal{V}^H$, taking the form $z_j = 0$ for some $1 \leqslant j \leqslant m$.

Reasoning the other way around, let $\{j_1, \ldots, j_l\} \subset \{1, \ldots, m\}$ be the subset of indices j for which there exists $\underline{\theta} \in H$ with $\theta_1^{\xi_{1,j}} \cdot \ldots \cdot \theta_s^{\xi_{s,j}} \neq 1$: then $\mathcal{V}^H$ is the intersection of $\mathcal{V}$ with the sub-vector space of $\mathbb{C}^m$ determined by the equations $z_{j_i} = 0$ for all $1 \leqslant i \leqslant l$. This intersection is again the complement of the zero locus of a polynomial in a smaller complex vector space, so it is either empty (if the polynomial restricts to the zero polynomial on the subspace) or path-connected. $\square$

By Lemma 5.4 and Proposition 2.5 we have $\mathbf{TC}_G(\mathcal{V}) \leqslant \mathbf{cat}_G(M \times M)$, and since the G-equivariant Lyusternik-Schnirelmann category is a G-homotopy invariant, we can write $\mathbf{TC}_G(M) \leqslant \mathbf{cat}_G(X)$, where X is given by Proposition 5.3. Since X is a G-CW complex of index $\leqslant 2m - s + t$, using Proposition 4.5 we obtain $\mathbf{cat}_G(X) \leqslant 2m - s + t$.

6.   A good Morse function on $\mathcal{V} \times \mathcal{V}$

We need to find a $\mathbb{T}^s$-equivariant Morse function $f \colon \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ with indices $\leqslant 2m - s + t$, using the notation from Proposition 5.3. We will use essentially the same method used by Andreotti and Frankel [1] in their proof of the Lefschetz hyperplane theorem.

We first define a function $g \colon \mathcal{V} \to \mathbb{R}$ by the equation

$$g(z_1, \ldots, z_m) = |z_1|^2 + \cdots + |z_m|^2 + \left| \frac{1}{\Delta(z_1, \ldots, z_m)} \right|^2 .$$

Note that g is invariant with respect to the action of $\mathbb{T}^s$ by scalar multiplication. Note also that g takes values $\geqslant 0$ and is proper: for $\lambda > 0$ let $\{g \leqslant \lambda\} \subset \mathcal{V}$ denote the subspace of points $(z_1, \ldots, z_m) \in \mathcal{V}$ for which $g(z_1, \ldots, z_m) \leqslant \lambda$; then $\{g \leqslant \lambda\}$ is contained in the closed subspace $\left\{ |\Delta| \geqslant \frac{1}{\sqrt{\lambda}} \right\} \subset \mathbb{C}^m$, so $\{g \leqslant \lambda\}$ is also closed in $\mathbb{C}^m$; moreover $\{g \leqslant \lambda\}$ is contained in the closed ball of radius $\sqrt{\lambda}$ centred at 0, hence $\{g \leqslant \lambda\}$ is compact.

The critical locus $\mathrm{crit}(g) \subseteq \mathcal{V}$, i.e. the subspace of points of $\mathcal{V}$ where dg vanishes, can be characterised by algebraic equations and inequalities in the real coordinates $x_i = \Re(z_i)$ and $y_i = \Im(z_i)$, hence $\mathrm{crit}(g)$ is a real semi-algebraic subset of $\mathbb{C}^m = \mathbb{R}^{2m}$.

As such, $\mathrm{crit}(g)$ has finitely many connected components (see for instance [**4**, Theorem 2.4.4]).

Note also that the vanishing of dg on a connected component of $\mathrm{crit}(g)$ implies that g is constant on this connected component. Since g is proper on $\mathcal{V}$ and since $\mathrm{crit}(g)$ is closed in $\mathcal{V}$, every component of $\mathrm{crit}(g)$ is compact; since $\mathrm{crit}(g)$ has finitely many connected components, also $\mathrm{crit}(g)$ is compact.

Recall that if $p \in \mathbb{C}^m$ is a point, U is a neighbourhood of p and $h \colon U \to \mathbb{R}$ is a smooth function, we can define a Euclidean Hessian $\mathfrak{H}(h)_p$ as the $2m \times 2m$ matrix of second partial derivatives, computed with respect to the Euclidean coordinates x_i and y_i of $\mathbb{C}^m$. The Euclidean Hessian $\mathfrak{H}(h)_p$ can be considered as a bilinear form on $T_p(\mathbb{C}^m)$, but its definition strongly depends on the choice of the coordinates we use to parametrise a neighbourhood of p, in our case the Euclidean coordinates. On the other hand, if p happens to be a critical point for h, i.e. dh vanishes at p, then the Hessian $H(h)_p$ is a bilinear form on $T_p\mathbb{C}^m$ which only depends on p and g but not on the coordinates used to parametrise a neighbourhood of p; moreover $\mathfrak{H}(h)_p$ and $H(h)_p$ agree if $p \in \mathrm{crit}(h)$.

Lemma 6.1. *Let $p \in \mathbb{C}^m$ be a point and let U be an open neighbourhood of p; let $\eta \colon U \to \mathbb{C} \setminus \{0\}$ be a holomorphic function, and define $h \colon U \to \mathbb{C}^m$ by $h(p) = |\eta(p)|^2$. Consider the Euclidean Hessian $\mathfrak{H}(h)_p$: then the negativity index of $\mathfrak{H}(h)_p$ is at most m.*

Proof. Up to a change of coordinates of U by translation in $\mathbb{C}^m$, we can assume $p = 0$. We consider the Taylor expansion of η around 0:

$$\eta(z_1, \ldots, z_m) = c + \sum_{i=1}^m \lambda_i z_i + \sum_{i,j=1}^m \mu_{i,j} z_i z_j + \text{h.o.t.},$$

where $c = \eta(0)$, $\lambda_i = \frac{\partial \eta}{\partial z_i}(0)$, $\mu_{i,j} = \frac{\partial^2 \eta}{\partial z_i \partial z_j}$ and the higher order terms are those of degree $\geqslant 3$. Up to multiplying η by $1/c$ (and consequently h by the positive number $1/|c|^2$), we can assume $c = 1$. Up to a $\mathbb{C}$-linear change of coordinates of U, we can further assume that the $m \times m$ matrix $(\mu_{i,j})$ is diagonal, with entries $\mu_{1,1}, \ldots, \mu_{r,r}$ equal to 1 and entries $\mu_{r+1,r+1}, \ldots, \mu_{m,m}$ equal to 0, for some $0 \leqslant r \leqslant m$. We can therefore assume that the Taylor expansion of η has the form:

$$\eta(z_1, \ldots, z_m) = 1 + \sum_{i=1}^m \lambda_i z_i + z_1^2 + z_2^2 + \cdots + z_r^2 + \text{h.o.t.}$$

Then the Taylor expansion of h is the following, using the variables $z_1, \ldots, z_m$ and their conjugates $\bar{z}_1, \ldots, \bar{z}_m$, and recalling the identity $h = \eta \cdot \bar{\eta}$:

$$h(z_1, \ldots, z_m) = 1 + \left(\sum_{i=1}^m \lambda_i z_i + \bar{\lambda}_i \bar{z}_i \right) + \left(\sum_{i,j=1}^m \lambda_i \bar{\lambda}_j z_i \bar{z}_j \right) + \left(\sum_{i=1}^r z_i^2 + \bar{z}_i^2 \right) + \text{l.o.t.}$$

The Euclidean Hessian $\mathfrak{H}(h)_0$ can be regarded as the sum of two $\mathbb{R}$-bilinear forms $\mathfrak{H}^1$ and $\mathfrak{H}^2$, the first corresponding to the quadratic form $\left(\sum_{i,j=1}^m \lambda_i \bar{\lambda}_j z_i \bar{z}_j \right)$, the second corresponding to the quadratic form $\left(\sum_{i=1}^r z_i^2 + \bar{z}_i^2 \right)$.

The bilinear form $\mathfrak{H}^2$ can be written, with respect to the coordinates x_i and y_i, as a $2m \times 2m$ real diagonal matrix, with r occurrences of 2, r occurrences of -2

and $2m - 2r$ occurrences of 0 on the main diagonal: this follows from the equality $z_i^2 + \bar{z}_i^2 = 2x_i^2 - 2y_i^2$ for all $1 \leqslant i \leqslant r$. In particular $\mathfrak{H}^2$ has signature $(2m - 2r, r, r)$, so it has negativity index $r \leqslant m$.

The bilinear form $\mathfrak{H}^1$ comes from the sesquilinear form on $\mathbb{C}^m$ represented by the $m \times m$ complex matrix $\Lambda \cdot \bar{\Lambda}^T$, where Λ is the $m \times 1$ matrix with entries $\lambda_1, \ldots, \lambda_m$. Every sesquilinear form of type $\Lambda \cdot \bar{\Lambda}^T$ is positive semidefinite on $\mathbb{C}^m$, and hence positive semidefinite when considered as a $\mathbb{R}$-bilinear form on $\mathbb{R}^{2m}$; hence $\mathfrak{H}^1$ is positive semidefinite.

It follows that the negativity index of $\mathfrak{H}(h)_0 = \mathfrak{H}^1 + \mathfrak{H}^2$ is at most $r \leqslant m$. $\qquad \square$

We apply Lemma 6.1 to the function $\eta(z_1, \ldots, z_m) = 1/\Delta(z_1, \ldots, z_m)$, obtaining that for all $p \in \mathcal{V}$ the matrix $\mathfrak{H}(|\eta|^2)$ has negativity index $\leqslant m$; in other words, the sum of its nullity and positivity indices is at least m.

On the other hand $\mathfrak{H}(|z_1|^2 + \cdots + |z_m|^2)_p$ is (strictly) positive definite: we have indeed that $\mathfrak{H}(|z_1|^2 + \cdots + |z_m|^2)_p$ is the $2m \times 2m$ identity matrix, independently of $p \in \mathcal{V}$. We obtain therefore that for all $p \in \mathcal{V}$ the matrix $\mathfrak{H}(g)_p$ has positivity index $\geqslant m$.

Now we consider the function $\tilde{f} \colon \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ defined by $\tilde{f}(p, p') = g(p) + g(p')$. Note that $\tilde{f}$ is invariant with respect to the diagonal action of $\mathbb{T}^s$, takes values $\geqslant 0$ and is proper.

The differential of $\tilde{f}$ is the "direct" sum of the pullbacks of the differentials of g at p and p': this implies that a critical point (p, p') for $\tilde{f}$ is precisely a couple of critical points p, p' for g; as a consequence $\mathrm{crit}\left(\tilde{f}\right) = \mathrm{crit}(g) \times \mathrm{crit}(g)$ is compact.

The Euclidean Hessian $\mathfrak{H}(\tilde{f})_{(p,p')}$ computed at any point $(p, p') \in \mathcal{V} \times \mathcal{V}$ is the block sum of the Euclidean Hessians $\mathfrak{H}(g)_p$ and $\mathfrak{H}(g)_{p'}$: in particular the positivity index of $\mathfrak{H}(\tilde{f})_{(p,p')}$ is at least $2m$ for all $(p, p') \in \mathcal{V} \times \mathcal{V}$.

By Lemma 3.4 we can find a $\mathbb{T}^s$-invariant Morse approximation $f \colon \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ of $\tilde{f}$; we can ensure that $\tilde{f}$ and f agree outside a $\mathbb{T}^s$-invariant relatively compact open subspace $U \subset \mathcal{V} \times \mathcal{V}$ containing $\mathrm{crit}\left(\tilde{f}\right)$ in its interior; and we can ask that $\tilde{f}$ and f be close in the C^2 norm, so that in particular also f, as $\tilde{f}$, has the property that $\mathfrak{H}(f)_{(p,p')}$ has positivity index at least $2m$ for all $(p, p') \in U$.

We claim that f has critical points of index $\leqslant 2m + s - t$. Let (p, p') be a critical point for f: then $H(f)_{(p,p')}$ can be represented in coordinates as $\mathfrak{H}(f)_{(p,p')}$, and thus it has positivity index at least $2m$. Denote by $N \subset \mathcal{V} \times \mathcal{V}$ the critical manifold containing (p, p'); then $N \cong \mathbb{T}^s/H$, for some closed subgroup $H \leqslant \mathbb{T}^s$. We have that H is the intersection of the stabilisers of p and p' by the action of $\mathbb{T}^s$ on $\mathcal{V}$, therefore our hypothesis on t implies that $\dim H \leqslant t$, and hence $\dim N \geqslant s - t$. Since the bilinear form $H(f)_{(p,p')}$ vanishes on $T_{(p,p')}N$, we have that the nullity index of $H(f)_{(p,p')}$ is at least $s - t$. It follows that the negativity index of $H(f)_{(p,p')}$ is at most $4m - (2m + s - t) = 2m - s + t$.

7. Applications

In this section we apply Theorem 1.6 to configuration spaces. Fix $n \geqslant 2$, and note that F_n deformation retracts onto its subspace F_n^0 of ordered configurations

$(w_1, \ldots, w_n)$ whose barycentre $(w_1 + \cdots + w_n)/n$ is equal to 0. The explicit deformation retraction $\rho \colon F_n \times [0, 1] \to F_n$ is given by the following formula:

$$\rho(w_1, \ldots, w_n; t) = \left(w_1 - t\frac{w_1 + \cdots + w_n}{n}, \ldots, w_n - t\frac{w_1 + \cdots + w_n}{n} \right).$$

Note that F_n^0 is a $\mathbb{T}$-invariant subspace of F_n and that ρ is a $\mathbb{T}$-equivariant deformation retraction, so that F_n^0 and F_n are $\mathbb{T}$-equivariantly homotopy equivalent: we can study the space F_n^0 instead of F_n. Note also that ρ is equivariant with respect to the action of the symmetric group $\mathfrak{S}_n$ on F_n by permutation of labels: thus ρ induces a $\mathbb{T}$-equivariant deformation retraction of C_n onto its $\mathbb{T}$-invariant subspace C_n^0 of unordered configurations $\{w_1, \ldots, w_n\}$ with barycentre $(w_1 + \cdots + w_n)/n$ equal to 0.

We first focus on the unordered case, so we want to apply Theorem 1.6 to the space C_n^0. For each $\{w_1, \ldots, w_n\} \in C_n^0$ we can form the polynomial

$$P(w) = w^n + \sum_{i=2}^{n} (-1)^i a_i w^{n-i} := (w - w_1) \cdot (w - w_2) \cdots \cdots (w - w_n),$$

where the coefficient a_i is the i-th elementary symmetric function in $w_1, \ldots, w_n$; note that $a_1 = \sum_{i=1}^{n} w_i$ vanishes by hypothesis, therefore we have omitted the corresponding term from the sum above. The discriminant

$$\Delta(P) = \prod_{1 \leqslant i < j \leqslant n} (w_i - w_j)^2$$

is a symmetric polynomial in the variables $w_1, \ldots, w_n$, hence it can be expressed as a polynomial $\Delta^C(a_2, \ldots, a_n)$ in the elementary symmetric functions $a_2, \ldots, a_n$: again we omit a_1 because this elementary symmetric function is constantly zero on C_n^0.

Note also that for $2 \leqslant i \leqslant n$ the elementary symmetric function a_i has total degree i in the variables w_i, whereas the polynomial Δ^C has total degree $n(n-1)$: in particular, for all $\theta \in \mathbb{T} = \mathbb{S}^1$, if we rotate the configuration $\{w_1, \ldots, w_n\} \in C_n^0$ by θ, i.e. we replace it with the configuration $\{\theta w_1, \ldots, \theta w_n\}$, then the symmetric function a_i is multiplied by θ^i and the value of Δ^C is multiplied by $\theta^{n(n-1)}$.

Consider now $\mathbb{C}^{n-1}$ with coordinates $z_2, \ldots, z_n$, and let $\mathcal{V}^C$ be the complement of the zero locus of the polynomial $\Delta^C(z_2, \ldots, z_n)$. The fundamental theorem of algebra establishes a homeomorphism between C_n^0 and $\mathcal{V}^C$; moreover the previous discussion shows that $(2, 3, \ldots, n)$ is a homogenisation of $\Delta^C(z_2, \ldots, z_n)$, so we can consider the corresponding action of $\mathbb{T}$ on $\mathcal{V}^C$ by scalar multiplication: this action corresponds under the homeomorphism $C_n^0 \cong \mathcal{V}^C$ to the action of $\mathbb{T}$ on C_n^0 by rotations.

It is left to check that the stabilisers of the action of $\mathbb{T}$ on $\mathcal{V}^C$ have dimension at most 0, i.e. they are finite groups. Since the only closed subgroups of $\mathbb{T}$ are either finite cyclic groups or the entire $\mathbb{T}$, it suffices to check that no point of $\mathcal{V}^C$ is fixed by the entire $\mathbb{T}$, or equivalently no point of C_n^0 is fixed by the entire $\mathbb{T}$: the second statement is a straightforward consequence of the assumption $n \geqslant 2$.

Applying Theorem 1.6 we obtain that $\mathbf{TC}_{\mathbb{T}}(C_n^0) = \mathbf{TC}_{\mathbb{T}}(\mathcal{V}^C) \leqslant 2(n-1) - 1 + 0 = 2n - 3$.

We pass now to the ordered case. Similarly as above, we can parametrise F_n^0 using the coordinates $w_1, \ldots, w_{n-1}$, because on F_n^0 we can express the last coordinate as $w_n = -(w_1 + \cdots + w_{n-1})$. The defining inequality for $F_n \subset \mathbb{C}^m$, which is

$\prod_{1 \leqslant i < j \leqslant n}(w_i - w_j) \neq 0$, restricts to the following defining inequality for F_n^0, expressed only in terms of the variables $w_1, \ldots, w_{n-1}$:

$$\Delta^F(w_1, \ldots, w_{n-1}) := \left(\prod_{1 \leqslant i < j \leqslant n-1} (w_i - w_j) \right) \left(\prod_{i=1}^{n-1} (w_i + (w_1 + \cdots + w_{n-1})) \right) \neq 0.$$

In particular F_n^0 is homeomorphic to the complement $\mathcal{V}^F \subset \mathbb{C}^{n-1}$ of the zero locus of the polynomial $\Delta^F(z_1, \ldots, z_{n-1})$. Since the polynomial Δ^F is homogeneous in the variables $z_1, \ldots, z_{n-1}$ we have that $(1, \ldots, 1)$ is a homogenisation for Δ^F; the corresponding action of $\mathbb{T}$ on $\mathcal{V}$ by scalar multiplication corresponds, under the homeomorphism $F_n^0 \cong \mathcal{V}^F$, to the action of $\mathbb{T}$ on F_n^0 by rotation.

It is left to check that the stabilisers of the action of $\mathbb{T}$ on $\mathcal{V}^F$, or equivalently on F_n^0, are finite groups, and this follows again, as in the unordered case, from the inequality $n \geqslant 2$. Theorem 1.6 applies again to give $\mathbf{TC}_\mathbb{T}(F_n^0) = \mathbf{TC}_\mathbb{T}(\mathcal{V}^F) \leqslant 2(n-1) - 1 + 0 = 2n - 3$.

8. Outlook: finding explicit motion planners

In principle one can use the strategy of the proof of Theorem 1.5 to construct an explicit $\mathbb{T}$-equivariant motion planner on C^n (or on F^n). One fixes a $\mathbb{T}$-invariant Morse function $f \colon C_n^0 \times C_n^0 \to \mathbb{R}$, and makes a list of its critical manifolds $N_1, \ldots, N_r \subset C_n^0 \times C_n^0$. For each $N_i \cong \mathbb{T}/H_i$, where H_i is some finite cyclic group, one chooses a point $(q_i, q_i') \in N_i$ and fixes a path from q_i to q_i' inside the subspace $(C_n^0)^{H_i}$, which we know is path-connected (see Lemma 5.4); we extend this path, in the unique possible way, to a $\mathbb{T}$-equivariant motion rule on each N_i.

Now, given two configurations $p, p' \in C_n$, the motion planner will connect p to p' as follows:

1. we connect both p and p', respectively, with the configurations $\rho(p, 1)$ and $\rho(p', 1)$ in C_n^0, using the homotopy ρ;
2. we use the gradient flow of f on $C_n^0 \times C_n^0$ to move the couple $(\rho(p, 1), \rho(p', 1))$ to a new couple (q, q') belonging to some critical manifold N_i, thus connecting p to q and p' to q';
3. we connect q to q' using the fixed motion rule on N_i.

We can define, for all $0 \leqslant i \leqslant 2n - 4$, the subspace $K_i \subset C_n \times C_n$ as the (not necessarily open) subspace containing pairs (p, p') that in step (2) are mapped to a pair (q, q') belonging to a critical manifold N_j of index precisely i. Then the hope is that the above motion rule is continuous on each K_i: if this is the case, one can argue that each K_i admits a $\mathbb{T}$-invariant open neighbourhood U_i that deformation retracts onto K_i, and thus one can define a continuous motion rule on U_i. Since K_i is a subspace of a manifold, one can expect that the deformation of U_i onto K_i can be given by a very explicit formula, for example by selecting, for all $(p, p') \in U_i$, the shortest path (with respect to the Euclidean metric on $\mathcal{V}^C \subset \mathbb{C}^{m-1}$) joining (p, p') with a pair in K_i; so one can expect that passing from K_i to U_i is not a big issue. As far as I can see, there are three other, major issues in implementing this strategy:

- Lemma 3.4 does not give an explicit way to perturb the explicit function $\tilde{f}$ given in Section 6 to a $\mathbb{T}$-invariant Morse function f.

- The motion rule described above may not be continuous on the subspaces K_i: this problem is reflected in the fact that Theorem 3.6 only produces a $\mathbb{T}$-cell complex, and we need to further subdivide cells (and to change the attaching maps by a $\mathbb{T}$-equivariant homotopy) in order to obtain a $\mathbb{T}$-CW complex.

- We need to list all critical manifolds N_i for f.

For the first issue, one can use directly the gradient flow of $\tilde{f}$ to perturb a pair in $C_n^0 \times C_n^0$ to one in $\mathrm{crit}\left(\tilde{f}\right) = \mathrm{crit}(g) \times \mathrm{crit}(g)$, using the function $g \colon C_n^0 \to \mathbb{R}$. Even assuming that g is Morse (something we did not speculate on, and for a good reason, as we will see soon), the connected components of $\mathrm{crit}\left(\tilde{f}\right)$ have the form $\mathbb{T}/H_1 \times \mathbb{T}/H_2$ and are thus 2-dimensional, so $\tilde{f}$ is *surely* not a $\mathbb{T}$-invariant Morse function. Nevertheless one can fairly easily decompose each component $\mathbb{T}/H_1 \times \mathbb{T}/H_2$ of $\mathrm{crit}\left(\tilde{f}\right)$ into two $\mathbb{T}$-invariant pieces, each of which has a $\mathbb{T}$-equivariant deformation retraction onto an orbit of the action of $\mathbb{T}$ on $C_n^0 \times C_n^0$. So up to assuming that g is Morse, the first issue can be solved or bypassed.

For the second issue, we need to understand the representation theory (over $\mathbb{R}$) of $\mathbb{T}$ and of its finite subgroups, in order to decompose the $\mathbb{T}$-cell complex from Theorem 3.6 into a $\mathbb{T}$-pre-CW complex: this is due to the actual construction underlying Theorem 4.3. The representation theory of $\mathbb{T}$ and of finite cyclic groups is well understood, so we can assume to be able to obtain an explicit $\mathbb{T}$-pre-CW complex X onto which $C_n^0 \times C_n^0$ deformation retracts $\mathbb{T}$-equivariantly. We can also hope that X is already a $\mathbb{T}$-CW complex, and redefine K_i as the set of pairs (p, p') that are mapped by the first two steps of the motion planner to a pair (q, q') in $X_i \setminus X_{i-1}$. Let us assume for a moment that the second issue can also be solved.

The third issue is, in my opinion, the biggest obstacle to implementing an explicit motion planner. Understanding $\mathrm{crit}\left(\tilde{f}\right) = \mathrm{crit}(g) \times \mathrm{crit}(g)$ boils down to understanding $\mathrm{crit}(g)$, and even assuming that g is a $\mathbb{T}$-invariant Morse function we would need to make a list of the components of $\mathrm{crit}(g)$. And anyway, we would still have to prove that g is Morse!

We have given an explicit formula for g, and all properties that we used are essentially the following: g is a proper $\mathbb{T}$-invariant function on C_n^0 obtained as a sum of absolute values of (locally defined) algebraic holomorphic functions.

Let us try to replace $g \colon C_n^0 \to \mathbb{R}$ with another, similar function $g' \colon C_n^0 \to \mathbb{R}$, given by the formula

$$g'(w_1, \ldots, w_n) = \sum_{i=1}^{n} |w_i|^2 + \sum_{1 \leqslant i < j \leqslant n} \frac{1}{|w_i - w_j|},$$

for all $\{w_1, \ldots, w_n\} \in C_n^0$.

The configurations in $\mathrm{crit}(g')$ are known in celestial mechanics as the *central configurations for the planar n-body problem*; for $n \leqslant 7$ it is known that g' is a $\mathbb{T}$-invariant Morse function, and there is a complete list of components of $\mathrm{crit}(g')$ [10], but for higher n there is no such a list, and it is even unknown whether g is a $\mathbb{T}$-equivariant Morse function or not: this latter question is also known as the 6th problem in Smale's eighteen problems for the 21st century. If the third issue is such a hard problem for g',

I do not expect that it can be easily solved for g (or for any other function with similar properties).

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