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ERRATUM TO "PROPERNESS AND SIMPLICIAL RESOLUTIONS FOR THE MODEL CATEGORY **dgCat**"

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(*communicated by J. Daniel Christensen*)

Abstract

There were two errors in the proof of the main theorem in our paper "Properness and simplicial resolutions for the model category **dgCat**" [*Homology Homotopy Appl.*, 16(2):263–273, 2014]. We explain the problems, close one gap and provide a reference dealing with the other.

Recall that in [2, Definition 3.2] we constructed for any dg category $\mathscr{C}$ a simplicial resolution $\mathscr{C}_\bullet$, i.e., a Reedy fibrant simplicial dg category such that each $\mathscr{C}_n$ is weakly equivalent ot $\mathscr{C}$. As pointed out to me by Daria Poliakova there are mistakes in the proof that $\mathscr{C}_\bullet$ is indeed a simplicial resolution of $\mathscr{C}$.

To discuss this, we need to distinguish between the Dwyer–Kan and Morita model structures. We fix a ground ring k and denote the category of dg categories over k by $\mathbf{dgCat}_k$. Let us first consider the Dwyer–Kan model structure on $\mathbf{dgCat}_k$ and fix a dg category $\mathscr{C}$.

The argument in [2] has two errors: Firstly, we assume $\mathscr{C}$ has cones in the proof of Proposition 3.10. This follows if $\mathscr{C}$ is Morita fibrant, but we are working in the Dwyer–Kan model structure and may not assume this. One can work in the pre-triangulated hull of $\mathscr{C}$ to overcome this. More seriously, in the proof of the same proposition we need to show we can lift a homotopy invertible map h. To do this we consider the cone of a lift H with $\pi(H) = h$. This approach falls short as H, as constructed, is not necessarily closed.

More careful, explicit proofs are given by Arhkipov and Poliakova, see Theorem 3.10 in [1], which shows that our construction in Definition 3.2 gives simplicial resolutions in the Dwyer–Kan model structure.

Next we consider the case of the Morita model structure. The issue in this case is that it is *a priori* harder to check that the maps $\mathscr{C}_n \to M_n\mathscr{C}$ are Morita fibrations.

We note that the Morita and Dwyer–Kan model structures on $\mathbf{dgCat}_k$ induce two different Reedy model structures on simplicial dg categories. The construction in Definition 3.2 applied to a Morita fibrant (i.e., pre-triangulated) dg category provides a level-wise Morita fibrant dg category $\mathscr{C}_\bullet$ that is weakly equivalent to $c\mathscr{C}$ in either Reedy model structure. It is moreover Dwyer–Kan Reedy fibrant using our main

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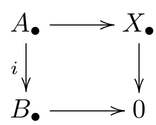
results with corrections from [1]. It remains to show that $\mathscr{C}_\bullet$ is also Morita Reedy fibrant. This is entirely formal.

In fact, the following argument applies to any Bousfield localization of model categories.

Lemma 1. *Let $X_\bullet$ be a simplicial dg category that is Dwyer–Kan Reedy fibrant and level-wise Morita fibrant. Then $X_\bullet$ is Morita Reedy fibrant.*

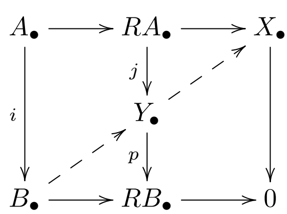
Proof. We take a Morita Reedy acyclic cofibration $A_\bullet \to B_\bullet$ and consider a diagram



Taking level-wise pre-triangulated envelopes gives a level-wise Morita fibrant replacement $RA_\bullet \to RB_\bullet$. As all objects are now Morita fibrant this is a level-wise weak equivalence in either model structure.

As $X_\bullet$ is level-wise Morita fibrant our diagram factors through $RA_\bullet \to RB_\bullet$ as the pre-triangulated envelope is left adjoint to inclusion.

Now we may factor $RA_\bullet \to RB_\bullet$ as an acyclic cofibration followed by an acyclic fibration in the Morita Reedy model category, write this $RA_\bullet \to Y_\bullet \to RB_\bullet$. Note $Y_\bullet$ is level-wise Morita fibrant, so $j : RA_\bullet \to Y_\bullet$ is a level-wise weak equivalence in the Dwyer–Kan model structure as well, and thus j is an acyclic cofibration in the Dwyer–Kan Reedy model structure.



To show $X_\bullet$ is Reedy Morita fibrant we need to find a lift $B_\bullet \to X_\bullet$ in our original diagram. We first find the lift $B_\bullet \to Y_\bullet$ as i is an acyclic cofibration and p is a fibration in the Morita Reedy model category.

Then we find the lift $Y_\bullet \to X_\bullet$ as j is an acyclic cofibration and $X_\bullet \to 0$ a fibration in the Dwyer–Kan Reedy model structure. □

References

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