

# THE TAMARKIN–TSYGAN CALCULUS OF AN ASSOCIATIVE ALGEBRA À LA STASHEFF

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## *Abstract*

We show how to compute the Tamarkin–Tsygan calculus of an associative algebra by providing, for a given cofibrant replacement of it, a ‘small’  $\text{Calc}_\infty$ -model of its calculus, which we make somewhat explicit at the level of  $\text{Calc}$ -algebras. To do this, we prove that the operad  $\text{Calc}$  is inhomogeneous Koszul; to our best knowledge, this result is new. We illustrate our technique by carrying out some computations for two monomial associative algebras using the cofibrant replacement obtained by the author in [39].

## 1. Introduction

To every associative algebra we may associate its Hochschild homology and cohomology groups. These are a priori graded spaces, but in fact are acted upon by several operads. In the simplest level, Hochschild cohomology is a graded commutative algebra under the cup product, and in fact a Gerstenhaber algebra, and Hochschild homology is a module for both of these algebra structures. It is the case that the Hochschild complex admits higher brace operations [32], refining the Gerstenhaber bracket, and the dg operad of such braces along with the cup product is quasi-isomorphic to the dg operad of singular chains on the little disks operad. In this way McClure and Smith give, in [32], a solution to Deligne’s conjecture. Another approach to the solution of the conjecture, preceding that of McClure and Smith, was proposed by Tamarkin in [38]. In [18], Hinich provided further details to the approach of Tamarkin.

As originally observed in [6], there is another operad that acts on Hochschild cohomology and homology, the 2-colored operad of Tamarkin–Tsygan calculi [36, §3.6], which yields for every associative algebra  $A$  a pair

$$\text{Calc}_A = (\text{HH}^*(A), \text{HH}_*(A)),$$

which we call the *Tamarkin–Tsygan calculus of  $A$* . In this paper we focus our attention on giving formulas for the action of it in terms of cofibrant resolutions in  $\mathbf{Alg}$ , the category of dga algebras with the projective model structure or, what is the

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same, a *homotopy invariant* description of the action on the chain level. Initially, we focused our attention on the Gerstenhaber bracket on Hochschild cohomology, originally defined in [12], since computing the resulting Gerstenhaber algebra structure on Hochschild cohomology has been of interest [15, 28, 29, 34, 41], and is agreed to be a non-trivial task; the reason for this is nicely explained in [34].

Quillen [35, Part II, §3] identified the Hochschild cochain complex of an algebra with the space of coderivations of its bar construction. Later, in the same spirit, Stasheff [33] gave a definition of the Gerstenhaber bracket of an associative algebra as the Lie bracket of coderivations of its bar construction, which deserves to be thought of as intrinsic to the category of associative algebras. Interest for a description of the Tamarkin–Tsygan calculus of an associative algebra à la Stasheff appears in [42, Remark 7]. Our main result here is giving such an intrinsic description of the action of Calc that also lends itself to computations.

To do this, consider a quasi-free associative algebra  $B = (TV, d)$ , and the following complexes of nc poly vector fields and of nc differential forms on  $B$ , respectively,

$$\mathcal{X}^*(B) = \text{cone}(\text{Ad}: B \longrightarrow \text{Der}(B)), \quad \Theta_*(B) = \text{cone}(C: V \otimes B \longrightarrow B),$$

where  $C$  is the commutator map. The space  $\Theta_*(B)$  is spanned by nc-forms  $\omega = b + b'dv$  where  $b, b' \in B$  and  $v \in V$  and  $\mathcal{X}^*(B)$  is spanned by nc-fields  $X = \lambda + f$  where  $\lambda \in B$  and  $f$  is a derivation.

**Theorem 1.1.** *The Tamarkin–Tsygan calculus  $\text{Calc}_A$  of an associative algebra  $A$  can be computed using the datum*

$$(\mathcal{X}^*(B), \Theta_*(B))$$

*of nc poly vector fields and nc differential forms obtained from a model  $B \longrightarrow A$ . In terms these two complexes, if we write a poly vector field by  $X = \lambda + f$  and a differential form by  $\omega = b + b'dv$ , then*

1. *The cup product can be computed through a brace operation  $\{X, Y; d\}$ .*
2. *The Lie bracket is the usual bracket of derivations.*
3. *The boundary of  $\omega$  is computed as*

$$d\omega = \sum_{i=1}^n (-1)^{\varepsilon} v_{i+1} \cdots v_n v_1 \cdots v_{i-1} dv_i.$$

To achieve this, we prove the seemingly new result that the colored operad Calc is inhomogeneous Koszul. The theory developed in [11] to obtain resolutions of inhomogeneous Koszul operads then allows us to produce a cofibrant model  $\text{Calc}_\infty$  for Calc and endow the pair

$$(\mathcal{X}^*(B), \Theta_*(B))$$

with a  $\text{Calc}_\infty$ -algebra structure.

**Theorem 1.2.** *The operad Calc is inhomogeneous Koszul. It admits a cofibrant model  $\text{Calc}_\infty$  of Calc with underlying symmetric sequence isomorphic to  $T(\delta) \circ \text{PreCalc}^1$ .*

Here  $\text{PreCalc}^1$  is the Koszul dual cooperad of the Koszul operad  $\text{PreCalc}$  and  $T(\delta)$  is a polynomial algebra with generator  $\delta$  of degree 2. Having this model for homotopy coherent calculi, we gather the following results from [7, 24]:

**Theorem 1.3.** *There is a dg colored operad  $\text{KS}$ , the Kontsevich–Soibelman operad, and a topological colored operad  $\text{Cyl}$  such that*

1. *The operad  $C_*(\text{Cyl})$  is formal and its homology is  $\text{Calc}$ .*
2. *There is a quasi-isomorphism  $\text{KS} \rightarrow C_*(\text{Cyl})$  of dg-operads.*
3.  *$\text{KS}$  acts on the pair  $(C^*(-), C_*(-))$  in such a way that:*
4. *On homology we obtain the usual  $\text{Calc}$ -algebra structure on  $(\text{HH}^*(-), \text{HH}_*(-))$ .*

*In particular, for every cofibrant replacement  $Q$  of  $\text{Calc}$ , the pair  $(C^*(A), C_*(A))$  is a  $Q$ -algebra which on homology gives  $\text{Calc}_A$ . □*

Specializing this to  $Q = \text{Calc}_\infty$ , we obtain on the pair  $(C^*(A), C_*(A))$  a  $\text{Calc}_\infty$ -algebra which on homology gives  $\text{Calc}_A$  and which we write  $\text{Calc}_{\infty,A}$ . With this at hand, the homotopy theory developed in [11] for inhomogeneous Koszul operads implies the following theorem.

**Theorem 1.4.** *For any quasi-free model  $B$  of  $A$ , the pair  $(\mathcal{X}^*(B), \Theta_*(B))$  admits a  $\text{Calc}_\infty$ -algebra structure which is  $\infty$ -quasi-isomorphic to the  $\text{Calc}_\infty$ -algebra  $\text{Calc}_{\infty,A}$  so that the action of degree 0 elements in  $\text{Calc}_\infty$  is given by the formulas above. In particular, this structure recovers  $\text{Calc}_A$  by taking homology.*

We remark that computing cofibrant resolutions in  $\text{Alg}$  is remarkably complicated. The author has solved the problem of computing models of *monomial* algebras in [39] and, as explained there, using this and some ideas of deformation theory, it is possible to attempt to compute models of certain algebras with a chosen Gröbner basis, although the general description of the minimal model of such algebras is, at the moment, missing.

In Section 2 we recall the elements of Hochschild (co)homology and necessary background on models of associative algebras. In Section 3 where we recall the basics of (colored) operads, their algebras, and the relevant Koszul duality theory for them, which we use to give a inhomogeneous Koszul model of  $\text{Calc}$ . In Section 5, we introduce the main algebraic structure of our paper, Tamarkin–Tsygan calculi. We then give two complexes to compute Hochschild (co)homology in terms of models, and give the promised formulas for the Tamarkin–Tsygan calculus of an algebra in terms of these. Finally, in Section 6, following the notion of Koszul  $A_\infty$ -algebras of Berglund–Börjeson [5], we show that the Tamarkin–Tsygan calculus of an  $A_\infty$ -Koszul algebra is dual to that of its Koszul dual algebra, extending a result of Herscovich [16]. In Section 6 we give some examples of computations to illustrate our methods.

In what follows  $\mathbb{k}$  is a field of characteristic zero, and all unadorned  $\otimes$  and  $\text{hom}$  are with respect to this base field. All algebras are non-unital and non-negatively homologically graded unless stated otherwise, and are defined over the base field. Implicit signs follow the Koszul sign rule, and we make them as explicit as we can. We will adhere to the convention that  $A$  will always denote a non-dg associative algebra concentrated in homological degree 0, while  $B$  will always denote a dg associative algebra concentrated in non-negative degrees. We write  $V^\#$  for the graded dual of a graded vector space.

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## 2. Preliminaries

In this section we quickly recall the classical definitions of Hochschild homology and cohomology. For details, the reader is referred to [25] and [27]. We also recall the basics on models of associative algebras; these are the objects we use to compute their calculus. The reader familiar with the theory may wish to skip this section. Throughout, we fix an associative algebra  $A$ , which we always assume is concentrated in homological degree zero.

### 2.1. Hochschild (co)homology

Let us begin by recalling what the Hochschild (co)chain complex of a dga algebra  $(B, \partial_B)$  looks like. For details, we refer the reader to the book [27, Section 7.1] and also to [1, Sections 1 and 2]. To this end form the first quadrant double complex  $C_*(B)_*$  so that for each  $p, q \in \mathbb{N}_0$  we have

$$C_p(B)_q = (B \otimes (s\overline{B})^{\otimes p})_{p+q},$$

the space of elements of  $B \otimes (s\overline{B})^{\otimes p}$  of total homological degree  $p + q$ . Observe that since we have shifted the degree of  $\overline{B}$ , a generic element of bidegree  $(p, q)$  is of the form  $b_0[b_1 | \cdots | b_p]$  where  $\sum_{i=0}^p |b_i| = q$ .

The horizontal differential is the Hochschild differential that incorporates the Koszul signs of  $B$ , so that for a generic basis element  $b_0[b_1 | \cdots | b_p]$  of our double complex we have that

$$\begin{aligned} d_h(b_0[b_1 | \cdots | b_p]) &= -\overline{b_0}b_1[b_2 | \cdots | b_p] + \sum_{i=1}^{p-1} \overline{b_0}[\overline{b_1} | \cdots | \overline{b_i}b_{i+1} | \cdots | b_p] \\ &\quad + (-1)^{a|b_p|+|b_0|} b_p b_0[\overline{b_1} | \cdots | \overline{b_{p-1}}], \end{aligned}$$

where we use the usual notation  $\overline{b} = (-1)^{|b|+1}b$ , and  $a = |b_0| + |b_1| + \cdots + |b_{p-1}| + p - 1$ . The vertical differential is simply the induced differential on the tensor product of complexes  $B \otimes (s\overline{B})^{\otimes p}$ , which incorporates Koszul signs and suspension signs.

Concretely, for a generic basis element  $b_0[b_1|\cdots|b_p]$  we have that

$$d_v(b_0[b_1|\cdots|b_p]) = db_0[b_1|\cdots|b_p] - \sum_{i=1}^{p-1} \overline{b_0}[\overline{b_1}|\cdots|\overline{b_i}|db_{i+1}|\cdots|b_p].$$

Dually, we can form the double complex  $C^*(B)^*$  so that for each  $p, q \in \mathbb{N}_0$  we have

$$C^p(B)^q = \text{hom}((s\overline{B})^{\otimes p}, B)^{p+q},$$

where the outer index indicates the cohomological degree of a map. In this way, elements in  $C^p(B)^q$  correspond to maps  $\overline{B}^{\otimes p} \rightarrow B$  that lower the homological degree by  $q$ . Observe this complex lives in the first two quadrants, since  $q$  can be negative. For a homogeneous map  $f: (s\overline{B})^{\otimes p} \rightarrow B$  and a basis element  $[b_1|\cdots|b_p]$  of  $(s\overline{B})^{\otimes p}$ , we have

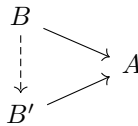
$$\begin{aligned} (d_h f)[b_1|\cdots|b_p] &= (-1)^{|b_1||f|} b_1 f[b_1|\cdots|b_p] + (-1)^{|f|} \sum_{i=1}^{p-1} f[\overline{b_1}|\cdots|\overline{b_i}b_{i+1}|\cdots|b_p] \\ &\quad + (-1)^{|f|} f[\overline{b_1}|\cdots|\overline{b_{p-2}}]b_{p-1}. \end{aligned}$$

### 2.2. Models of associative algebras

Let us write  $\mathbf{Alg}$  for the category of dga algebras. A surjective quasi-isomorphism  $B \rightarrow A$  from a quasi-free dg algebra  $B = (TV, \partial_B)$  is a *model* of  $A$ . One usually calls  $B$  a model of  $A$ , without explicit mention to the map  $B \rightarrow A$  which is understood from context. We say a model is *minimal* if  $B$  is

1. *triangulated*: there is a gradation  $V = \bigoplus_{j \geq 1} V(j)$  such that for each  $j \in \mathbb{N}$ , we have that  $\partial_B(V^{(j+1)}) \subseteq T(V^{(\leq j)})$ , and
2. its differential is *decomposable*, that is,  $\partial_B(V) \subseteq (TV)^{\geq 2}$ .

There is a model structure on  $\mathbf{Alg}$  whose weak equivalences are quasi-isomorphisms, the fibrations are the degree-wise surjections, and the cofibrant algebras are the retracts of triangulated quasi-free algebras; see [40]. In particular, minimal algebras are cofibrant, and may be used to cofibrantly resolve objects in  $\mathbf{Alg}$ . It is important to recall that not every dga algebra admits a minimal model; see the unpublished notes [19] for details.



Minimal models of algebras, when they exist, are unique up to unique isomorphism, meaning that any solid diagram where the diagonal arrows are minimal models, can be uniquely completed to a homotopy commutative triangle where the vertical dashed map is an isomorphism in  $\mathbf{Alg}$ . For brevity, we will use the term *model* to speak about triangulated quasi-free algebras with homology concentrated in degree zero.

In [39], we obtain a description of the minimal model of any monomial quiver algebra. Concretely, the model is free over the quiver with arrows the chains, also known as overlappings or ambiguities of the algebra, and the differential is given by

deconcatenation. If  $\gamma$  is an Anick chain of length  $r$ , a decomposition of it is a sequence  $(\gamma_1, \dots, \gamma_n)$  of Anick chains of lengths  $r_1, \dots, r_n$  so that their concatenation, in this order, is  $\gamma$ , and  $r - 1 = r_1 + \dots + r_n$ . What follows is the main result of that paper.

**Theorem 2.1.** *For each monomial algebra  $A$  there is a minimal model  $B \rightarrow A$  where  $B$  is the  $\infty$ -cobar construction on  $\text{Tor}_A(\mathbb{k}, \mathbb{k})$ . The differential  $d$  is such that for a chain  $\gamma \in \text{Tor}_A(\mathbb{k}, \mathbb{k})$ ,*

$$d\gamma = - \sum_{n \geq 2} (-1)^{\binom{n+1}{2} + |\gamma_1|} \gamma_1 \gamma_2 \cdots \gamma_n,$$

where the sum ranges through all possible decompositions of  $\gamma$ . □

Observe that the differential is manifestly decomposable, and that the gradation on  $\text{Tor}_A$  provides us with a triangulation of  $B$ , so that indeed  $B$  is minimal. We give examples of computation of this model in Section 6.3 which we then use to compute the Tamarkin–Tsygan calculus of  $A$  explicitly.

### 3. Elements of operad theory

We now briefly recall here the elements from the theory of (colored) operads, their algebras, and Koszul duality theory which allows us to give explicit models of algebras ‘up to homotopy’ of certain well behaved operads. With this at hand, we then explain in Section 5.2 how to produce a homotopy coherent calculus on nc forms and nc fields obtained from any cofibrant replacement of an algebra. We also prove that the operad  $\text{Calc}$  controlling Tamarkin–Tsygan calculi is inhomogeneous Koszul; this result is probably known to experts, but we could not find a proof in the literature.

#### 3.1. Symmetric operads

Let us write  ${}_{\Sigma}\text{Mod}$  for the category of  $\Sigma$ -modules, that is, collections of graded  $\mathbb{k}$ -modules  $X = \{X(n)\}_{n \geq 0}$  where for each  $n \in \mathbb{N}_0$  the space  $X(n)$  is a graded  $S_n$ -module. There is a monoidal product on  ${}_{\Sigma}\text{Mod}$  that corresponds to the composition of formal series, so that for two  $\Sigma$ -modules  $X$  and  $Y$  with  $Y(\emptyset) = 0$  we have, for each finite set  $I$

$$(X \circ Y)(n) = \bigoplus_{k \geq 0} X(k) \otimes_{S_k} Y[k, n],$$

where for each  $n \in \mathbb{N}$  and each  $k \leq n$ ,

$$Y[k, n] := \bigoplus_{\lambda_1 + \dots + \lambda_k = n} S_{\lambda}^n \uparrow Y(\lambda_1) \otimes \cdots \otimes Y(\lambda_k)$$

is the sum of induced representations from the subgroup

$$S_{\lambda} = S_{\lambda_1} \times \cdots \times S_{\lambda_k} \subseteq S_n.$$

Note it is both an  $S_n$ -module and an  $S_k$ -module (by permutation of the  $k$  factors) in a compatible way.

We like to think of an element in  $(X \circ Y)(n)$  as an  $n$ -corolla —a rooted tree with a single vertex of degree  $n + 1$ — with its root decorated with an element of  $X(n)$  and its leaves decorated with elements of  $Y$ . This makes  ${}_{\Sigma}\text{Mod}$  into a monoidal category

with unit the symmetric sequence  $(0, \mathbb{k}, 0, \dots)$ . Considering the particular case when  $Y = Y(1)$ , we obtain an endofunctor  $X: \mathbf{Vect}_{\mathbb{k}} \rightarrow \mathbf{Vect}_{\mathbb{k}}$  such that

$$X(V) = \bigoplus_{k \geq 0} X(k) \otimes_{S_k} V^{\otimes k}.$$

A symmetric operad is by definition an associative algebra in the monoidal category  $(\Sigma\mathbf{Mod}, \circ, 1)$ . It is customary to write  $\gamma: P \circ P \rightarrow P$  for the product of  $P$  which is instead called the composition map of  $P$ . In this way, to every element  $(\mu; \nu_1, \dots, \nu_t) \in P \circ P$  we assign an output  $\gamma(\mu; \nu_1, \dots, \nu_t)$  which we think of as grafting  $\nu_1, \dots, \nu_t$  at the leaves of  $\mu$  to obtain a new operation in  $P$ . In particular, one can consider the case when all but one of the operations grafted at the leaves are the unit  $1 \in P(1)$ . This gives us for each partial composition operations

$$\circ_i: P(n) \otimes P(m) \rightarrow P(m + n - 1)$$

which are sufficient to describe  $\gamma$ : any element  $\gamma(a; b_1, \dots, b_t)$  can be obtained by iterated partial compositions. In this language, associativity of  $\gamma$  is equivalent to the parallel and sequential composition axioms: for operations  $a, b$  and  $c \in P$  with of respective arities  $l, m$  and  $n$ , we have that

$$\begin{aligned} (a \circ_i b) \circ_{i+j-1} c &= a \circ_i (b \circ_j c) && \text{for } i \in [l] \text{ and } j \in [m], \\ (a \circ_i b) \circ_{k+l-1} c &= (-1)^{|c||b|} (a \circ_k c) \circ_i b && \text{for } i < k \in [l]. \end{aligned}$$

We will often make use of the partial composition description of an operad to define the operads of interest to us.

To every operad  $P$  we can assign its category of algebras. These are vector spaces  $V$  endowed with a linear map  $\gamma_V: P(V) \rightarrow V$  that is compatible with the composition law of  $P$  in the sense that for  $v_1 \otimes \dots \otimes v_t \in V^{\otimes n_1} \otimes \dots \otimes V^{\otimes n_t}$

$$\gamma_V(\gamma(a, b_1, \dots, b_t), v_1, \dots, v_t) = \gamma_V(a, \gamma_V(b_1, v_1), \dots, \gamma_V(b_t, v_t)),$$

and such that for each  $\sigma \in S_n$  we have that

$$\gamma_V(a \cdot \sigma, v_1, \dots, v_n) = \gamma_V(a, v_{\sigma 1}, \dots, v_{\sigma n}).$$

We like to think of  $\gamma_V(a; v_1, \dots, v_n)$  as the result of applying operation  $a$  to the string  $v_1 \otimes \dots \otimes v_n \in V^{\otimes n}$ . We also require that the unit of  $P$  (if there is any) acts as the identity endomorphism of the space  $V$ .

Alternatively, one can consider the endomorphism operad  $\text{End}_V$  of a vector space  $V$  so that for each  $n \in \mathbb{N}$  we have

$$\text{End}_V(n) = \text{End}(V^{\otimes n}, V)$$

and the operadic composition is given by

$$\gamma(f; g_1, \dots, g_n) = f \circ (g_1 \otimes \dots \otimes g_t)$$

while the symmetric group acts by permuting the variables of  $V^{\otimes n}$ . With this at hand, a  $P$ -algebra structure on  $V$  is given by a map of operads  $P \rightarrow \text{End}_V$ .

A useful observation, that allows us to define operads through generators and relation, is that operads can be defined as algebras over a monad

$$\mathcal{T}: \Sigma\mathbf{Mod} \rightarrow \Sigma\mathbf{Mod},$$

which we call the monad of trees. Concretely, for each  $\Sigma$ -module  $X$ , we define a

new  $\Sigma$ -module  $\mathcal{T}(X)$  so that for each  $n \in \mathbb{N}$  we have that  $\mathcal{T}(X)(n)$  is spanned by isomorphism classes of trees with  $n$  leaves with each vertex  $v$  decorated by an element in  $X(k)$ , where  $k$  is the number of inputs of  $v$ . The multiplication map

$$\mu: \mathcal{T} \circ \mathcal{T} \longrightarrow \mathcal{T}$$

is obtained by erasing the boundaries of vertices in a ‘tree of trees’.

In this way, an operad is just an associative algebra over  $\mathcal{T}$  and, in particular,  $\mathcal{T}(X)$  is the free operad on  $X$ :  $\mathcal{T}$  is the left adjoint to the forgetful functor from the category of operads to the category of  $\Sigma\text{Mod}$ . In this way, it makes sense to say that an operad  $P$  is generated by  $X$ : this means there is a surjective map of operads

$$\mathcal{T}(X) \longrightarrow P.$$

One then defines the notion of a double sided ideal, as usual, and hence it makes sense to say that an operad is generated by  $X$  subject to some  $\Sigma$ -module of relations  $R$ , as in the case of associative algebras.

Since it is useful for our purposes, let us consider the example where  $X = X(2)$  consists of the trivial representation of the symmetric group  $S_2$ , with a generator  $m$ , so that  $\mathcal{T}(X)(2)$  consists of a single binary tree which we identify with a commutative operation  $x_1x_2$ . The relation  $m \circ_1 m = m \circ_2 m$  which we can write more familiarly as the following equation:

$$x_1(x_2x_3) = (x_1x_2)x_3.$$

It recovers the usual associativity of commutative algebras. We call  $T(X)/(R)$  the operad of commutative algebras and write it  $\text{Com}$ . Note that for each  $n \in \mathbb{N}$  the space  $\text{Com}(n)$  is the one dimensional trivial representation of  $S_n$ , and it is spanned by an operation which we can unambiguously write  $x_1 \dots x_n$ . Similarly, we can consider the case  $X(2)$  is the regular representation of  $S_2$ , in which case following the same prescription we would obtain the operad  $\text{Ass}$  governing associative algebras. In this case, for each  $n \in \mathbb{N}$  the space  $\text{Ass}(n)$  is  $n!$ -dimensional and it is spanned by the operation  $x_1 \dots x_n$  along with all possible permutations of its variables.

We can similarly define the operad  $\text{Lie}$  governing Lie algebras: it is generated by a single binary operation  $b$  spanning the sign representation of  $S_2$ , subject to the Jacobi relation  $(1 + \tau + \tau^2)(b \circ_1 b) = 0$  where  $\tau = (123) \in S_3$ . We can again write this more familiarly as

$$[[x_1, x_2], x_3] + [[x_3, x_1], x_2] + [[x_2, x_3], x_1] = 0.$$

For each  $n \in \mathbb{N}$ , the space  $\text{Lie}(n)$  has a basis consisting of  $(n - 1)!$  Lie words

$$[x_{\sigma 1}, x_{\sigma 2}, \dots, x_{\sigma n}],$$

where  $\sigma \in S_n$  fixes 1, and where we are using the convention that we bracket a Lie word to the left. Thus, for example,  $[x_1, x_2, x_3] = [[x_1, x_2], x_3]$ .

We can use these two operads to define the operad  $\text{Ger}$  of Gerstenhaber algebras. It is generated by an associative commutative product  $m = x_1x_2$  in degree 0 and a Jacobi bracket  $b = [x_1, x_2]$  in degree 1 subject to the Leibniz rule  $b \circ_1 m - m \circ_2 b - \sigma(m \circ_2 b) = 0$ , where  $\sigma = (12) \in S_3$ , which we can again write in the familiar form

$$[x_1x_2, x_3] = x_1[x_2, x_3] + x_2[x_1, x_3].$$

We remark that both in the Jacobi relation and in the Leibniz rule are written



without signs since they are relations in the operad *Lie* and *Ger* respectively and, since  $x_1$  and  $x_2$  are merely placeholders, they carry no degree. When these are evaluated at elements of a fixed algebra  $V$ , the Koszul sign rule will create the appropriate signs. For example, in our last equation,  $x_1$  and  $x_2$  are permuted in the last summand, so that when evaluating this on some  $v_1 \otimes v_2 \otimes v_3$  in  $V^{\otimes 3}$ , we will create a sign in the step where we permute  $v_1$  and  $v_2$  and then apply  $m \circ_2 b$ :

$$v_1 \otimes v_2 \otimes v_3 \mapsto (-1)^{|v_1||v_2|} v_2 \otimes v_1 \otimes v_3.$$

Given an algebra  $V$  over an operad  $P$ , and a vector space  $M$ , let us consider the subspace  $P(V, M) \subseteq P(V \oplus M)$  of elements of the form

$$(\mu; v_1, \dots, v_{i-1}, m, v_{i+1}, \dots, v_t), \quad \text{where } \mu \in P, v_j \in V \text{ and } m \in M.$$

It is useful to think of this as a corolla with its root decorated by an element of  $P$ , and its leaves decorated by elements of  $V$  except one, which is decorated by an element of  $M$ , which results on in an operation

$$\mu: V^{\otimes(i-1)} \otimes M \otimes V^{\otimes(t-i)} \longrightarrow M.$$

Note that  $(P \circ P)(V, M)$  is naturally isomorphic to  $P(P(V), P(V, M))$ . A left  $V$ -module is a vector space  $M$  along with a structure map

$$\gamma_M: P(V, M) \longrightarrow M$$

compatible with the algebra structure of  $V$ , in the sense that, using the previous identification  $\gamma_M \circ P(\gamma_V, \gamma_M) = \gamma_M \circ \gamma$ .

To get acquainted with this notion of operadic module, it is useful to note that for  $V$  an associative algebra, a  $V$ -module is the same as a  $V$ -bimodule in the usual sense, the left and right actions given by the operations

$$mx = \gamma_M(\mu; m, x), \quad xm = \gamma_M(\mu; x, m).$$

In case  $V$  is a commutative algebra, a  $V$ -module is the same as a symmetric  $V$ -bimodule since in this case the symmetry of  $\mu$  gives  $xm = mx$ , and for a Lie algebra  $V$ , we recover the usual definition of a representation of a Lie algebra.

### 4. Colored operads and Koszul duality

Operads allow us to describe algebras of certain kind. We can embellish them to describe pairs of algebras of a certain kind along with a module over this algebra, as follows.

Let  $P$  be an operad, which we think of as having color  $\circ$  (white). We define a 2-colored operad  $P^*$  with colors  $\circ$  and  $\bullet$  (black) as follows, where we use the notation  $P^*(\varepsilon; a, b)$  for the spaces of operations with  $a$  inputs of color  $\circ$ ,  $b$  inputs of color  $\bullet$ , and the single output of color  $\varepsilon \in \{\bullet, \circ\}$ .

1.  $P^*(\circ; a, b) = \begin{cases} 0 & \text{if } a \geq 1, \\ P(b) & \text{if } a = 0. \end{cases}$
2.  $P^*(\bullet; a, b) = \begin{cases} 0 & \text{if } a = 0 \text{ or } a \geq 2, \\ (\partial P)(b + 1) & \text{if } a = 1. \end{cases}$

Here  $\partial$  is the pointing operator on symmetric sequences:  $\partial X(n)$  is a copy of  $X(n+1)$  where  $n+1$  is ‘marked’ and  $S_n$  acts on the remaining ones. Alternatively, we are taking the restriction of  $X(n+1)$  for the inclusion  $S_n \rightarrow S_{n+1}$  as the stabilizer of  $n+1 \in [n+1]$ . We then color the marked point black, as well as the output. In terms of finite sets, we have that  $(\partial X)(I) = X(I \sqcup \{I\})$  with the obvious action of the symmetric group  $S_I$ .

The composition law in  $P^*$  is dictated by the simple rule we may only graft a tree with root colored  $\varepsilon$  into a leaf of the same color. Hence, we only have compositions coming from those of  $P$ , and two new compositions that either graft an operation of  $P$  into a white leaf of a marked operation of  $\partial P$ , or a new composition that grafts a black root into a black leaf of  $\partial P$ , both resulting in an operation in  $P^*(\bullet; 1, a)$ .

If  $V = (V_\circ, V_\bullet)$  is a pair of vector spaces, which we think of put in color  $(\circ, \bullet)$ , then we can consider the colored operad  $\text{End}_V$ , where for each  $s, t \in \mathbb{N}$  and  $\varepsilon \in \{\circ, \bullet\}$ ,

$$\text{End}_V(\varepsilon; s, t) = \text{hom}(V_\circ^{\otimes s} \otimes V_\bullet^{\otimes t}, V_\varepsilon).$$

In this way, a  $P^*$ -algebra structure on  $V$  is equivalent to a morphism of colored operads

$$P^* \rightarrow \text{End}_V.$$

With this at hand, we have the following elementary result. We point the reader to [23] for details.

**Proposition 4.1.** *A  $P^*$ -algebra structure on  $V = (V_\circ, V_\bullet)$  is equivalent to the datum of a  $P$ -algebra structure on  $V_\circ$  and a  $V_\circ$ -module structure on  $V_\bullet$ .  $\square$*

Let us record the following result, which we will use later.

**Proposition 4.2.** *The operad  $P^*$  is Koszul if  $P$  is Koszul.*

*Proof.* This follows from the fact that the homology of the free  $P^*$ -algebra on  $(V_\circ, V_\bullet)$  is nothing but the Koszul homology of the free  $P$ -algebra over  $V_\circ$  in the first color and the Koszul homology of the free  $P$ -algebra over  $V_\circ$  with coefficients in the free  $V_\circ$ -module on  $V_\bullet$ , and both these complexes are acyclic since  $P$  is Koszul. Since an operad is Koszul if and only if its free algebras have trivial Koszul (co)homology in non-negative degrees, this proves the theorem.  $\square$

Observe that this implies both operads  $\text{Com}^*$  and  $\mathcal{S}^{-1}\text{Lie}^*$  are Koszul. From the operad  $\text{Ger}$  we form the 2-colored operad  $\text{PreCalc} = (\text{Ger}, \text{M})$  by adding two operations  $[v, x]$  and  $v \cdot x$  in  $\text{PreCalc}(\bullet; 1, 1)$ , of degrees 0 and  $-1$  respectively, and relations as follows:

1. the operations  $x_1x_2$  and  $v \cdot x_1$  satisfy the same relations defining  $\text{Com}^*$ ,
2. the operations  $[x_1, x_2]$  and  $[v, x_1]$  satisfy the same relations defining  $\mathcal{S}^{-1}\text{Lie}^*$ ,
3. the Lie action and the commutative action satisfy the following mixed Leibniz relations:

$$v \cdot [x_1, x_2] = [v, x_1] \cdot x_2 - [v \cdot x_2, x_1] \tag{L1}$$

$$[v, x_1x_2] = [v \cdot x_1, x_2] + [v, x_1] \cdot x_2. \tag{L2}$$

**Theorem 4.3.** *The operad  $\text{PreCalc}$  is Koszul.*

*Proof.* We use the distributive law criterion, adapted to 2-colored operads. Namely, the result that if  $P$  and  $P'$  are Koszul operads and if  $P''$  is a third operad obtained from a distributive rule  $P' \circ P \longrightarrow P \circ P'$ , then  $P''$  is itself Koszul [26, Section 8.6]. We will apply this criterion in the case  $P = \text{Com}^*$  and  $P' = \mathcal{S}^{-1}\text{Lie}^*$ .

To do this orient the Leibniz rule (L1) to display it in the form

$$[v \cdot x_2, x_1] = [v, x_1] \cdot x_2 - v \cdot [x_1, x_2]$$

and rewrite the second Leibniz rule (L2) incorporating (L1) as follows:

$$[v, x_1x_2] = [v, x_2] \cdot x_1 - v \cdot [x_1, x_2] + [v, x_1] \cdot x_2.$$

In this way, along with the usual Leibniz rule, we have defined a map

$$V \circ_{(1)} V' \longrightarrow V' \circ_{(1)} V$$

on the generators  $V$  of  $\text{Com}^*$  and  $V'$  of  $\mathcal{S}^{-1}\text{Lie}^*$  which, in turn, induces a surjective map

$$\text{Com}^* \circ \mathcal{S}^{-1}\text{Lie}^* \longrightarrow \text{PreCalc}.$$

It is well known that, on the sub-operad where all outputs and inputs are of the same (white) color, this is indeed an isomorphism, so it suffices we check this on the components of mixed color.

It is straightforward to check that  $(\text{Com}^* \circ \mathcal{S}^{-1}\text{Lie}^*)(n)$  is of dimension  $2 \cdot n!$ , so it suffices to check that  $\text{PreCalc}(n)$  is of the same dimension. To do this, we recall that in [7] the authors provide us with a geometrical model of  $\text{PreCalc}$ . Indeed, their results show there exists a topological operad  $\text{PreCyl}$  along with homotopy equivalences

$$\text{PreCyl}(\circ; n, 0) \simeq \text{Conf}_n(\mathbb{R}^2) \quad \text{PreCyl}(\bullet; n - 1, 1) \simeq \text{Conf}_{n-1}(\mathbb{R}^2 \setminus 0),$$

and a surjection  $\text{PreCalc} \longrightarrow H_*(\text{PreCyl})$ , and the Poincaré series of these spaces are known [4]. This gives us the requisite lower bound on the dimensions of  $\text{PreCalc}$ , and shows that  $\text{PreCalc}$  is obtained from a distributive law from two Koszul operads. This shows it is Koszul, as we claimed.  $\square$

*Remark 4.4.* Observe that this dimension count along with the methods developed in [22] and the ‘quantum order’ of [9], allows us to exhibit a quadratic Gröbner basis of  $\text{PreCalc}$  where the leading terms of the mixed Leibniz rules (L1) and (L2) are  $[v \cdot x_1, x_2]$  and  $[v, x_1x_2]$ , as it was done for the Poisson operad in [8], for the pre-Poisson operad in [9], and for the Lie–Rinehart operad in [22].

Let us now add a square zero operator  $\delta$  of degree 1 to  $\text{PreCalc}$  along with the non-quadratic relation encoding the magic formula of Cartan

$$\delta(v \cdot x) - \delta v \cdot x = [v, x].$$

We write this operad  $\text{Calc}$ ; it is the operad governing Tamarkin–Tsygan calculi. If we consider the increasing filtration on  $\text{Calc}$  by the number of uses of  $\delta$ , we obtain an associated quadratic operad  $q\text{Calc}$  governing  $\text{PreCalc}$ -algebras with a degree  $-1$  square zero operation  $\delta$  such that

$$[\delta v, x] = \delta[v, x], \quad \delta(v \cdot x) = \delta v \cdot x.$$

Let us consider the (colored) operad of dual numbers  $\text{Diff} = T(\delta)/(\delta^2)$  put in color  $\bullet$ . Then we can present  $q\text{Calc}$  through a distributive rule  $\lambda$  between  $\text{PreCalc}$  and  $\text{Diff}$ ,

and we claim that this is a distributive law. To do this, we need to check that

$$\rho: \text{Diff} \circ \text{PreCalc} \longrightarrow \text{PreCalc} \vee_{\lambda} \text{Diff} = q\text{Calc}$$

is an isomorphism.

**Theorem 4.5.** *The operad  $q\text{Calc}$  is Koszul and  $\text{Calc}$  is inhomogeneous Koszul. Moreover, the map  $q\text{Calc} \rightarrow \text{gr Calc}$  is an isomorphism of symmetric sequences, so that the underlying collections of  $\text{Calc}$  and  $q\text{Calc}$  are isomorphic to  $\text{Diff} \circ \text{PreCalc}$ .*

*Proof.* Since  $\text{Diff}$  and  $\text{PreCalc}$  are both Koszul, we deduce that  $q\text{Calc}$  is also Koszul by another use of the distributive law argument of [26, Section 8.6]. To apply it, we need to check that  $\rho$  is an isomorphism, which is immediate, since it is surjective and by [7] the dimensions of the domain and the codomain match. This implies that  $\text{Calc}$  is inhomogeneous Koszul, and hence the Poincaré–Birkhoff–Witt theorem for Koszul operads shows that  $q\text{Calc} \rightarrow \text{gr Calc}$  is an isomorphism.  $\square$

From this, we obtain a model  $\text{Calc}_{\infty}$  of  $\text{Calc}$  with generators

$$\text{Calc}^i = \text{Diff}^i \circ \text{PreCalc}^i$$

through the methods of [26, Chapter 7]. Note that  $\text{Diff}^i = T[u]$  is a polynomial algebra generated by  $u = \delta^{\vee}$ , and when we write  $\text{Calc}^i$  we are doing it in the sense of inhomogeneous Koszul duality theory.

*Remark 4.6.* In [36] the authors state that  $\text{Calc}$  is Koszul and give a description of what they refer to as  $\text{Calc}_{\infty}$ -algebras. However, it is not clear that  $\text{Calc}$  admits a quadratic presentation, although it does admit a quadratic-cubic presentation (where the quadratic-cubic equations only appear in the operations of mixed color).

## 5. Main results

Throughout, we fix an associative algebra  $A$  concentrated in homological degree 0 and a model  $B$  of  $A$  (see Section 2), and show how to compute the Tamarkin–Tsygan calculus of  $A$  using only the dga algebra  $B$  and two small complexes  $\mathcal{X}^*(B)$  and  $\Theta_*(B)$  built out of  $B$ , which we like to think of as the complexes of non-commutative (nc) polyvector fields and non-commutative differential forms on  $B$ . We begin by recalling the essential details of this algebraic structure we want to compute.

### 5.1. Tamarkin–Tsygan calculi

We have already introduced the operad  $\text{Calc}$ . It turns out the pair

$$(\text{HH}^*(A), \text{HH}_*(A))$$

admits a structure of a  $\text{Calc}$ -algebra consisting of the cup product, the cap product, Gerstenhaber’s bracket and Connes’ operator. The Lie action on  $\text{HH}_*(A)$  is defined so that for each  $\omega \in \text{HH}_*(A)$  and each  $X \in \text{HH}^*(A)$ ,

$$L_X(\omega) = di_X(\omega) - (-1)^{|X|}i_X(d\omega),$$

where we are using the notation  $i_X$  for the cup product action of  $X$  on  $\text{HH}_*(A)$ ; this will be convenient later.

That these operations define a **Calc**-algebra structure was originally proved in [6]. See also [37]. We call this algebra–module pair  $(\mathrm{HH}^*(A), \mathrm{HH}_*(A))$  along with the operator  $d$  the *Tamarkin–Tsygan calculus* of  $A$ . We will write  $\mathrm{Calc}_A$  for this **Calc**-algebra, hoping it does not give rise to any confusion, in that we are not evaluating  $A$  on the endofunctor **Calc**, for example.

**Proposition 5.1.** *Let  $A$  be an associative algebra. Then  $(\mathrm{HH}^*(A), \smile, [-, -])$  is a Gerstenhaber algebra, and  $(\mathrm{HH}_*(A), i)$  is a module over  $(\mathrm{HH}^*(A), \smile)$  and  $(\mathrm{HH}_*(A), L)$  is a module over  $(\mathrm{HH}^*(A), [-, -])$ . Moreover, if for each  $X \in \mathrm{HH}^*(A)$  we write  $i_X$  for the action of  $X$ , for  $Y \in \mathrm{HH}^*(A)$ ,*

$$[i_X, L_Y] = i_{[X, Y]} \quad \text{and} \quad L_{X \cdot Y} = L_X i_Y + (-1)^{|X|} i_X L_Y.$$

We can describe all the operators of Proposition 5.1 on the cochain level using the pair of classical complexes  $(C^*(A), C_*(A))$  as follows. For cochains  $f, g \in C^*(A)$  homogeneous of degrees  $p$  and  $q$  and a chain

$$z = a[a_1 | \cdots | a_{p+q}] \in C_*(A)$$

of degree  $n = p + q$ , the cup product, the cap product, the Lie bracket and Connes’ differential are defined by the following formulas, where  $\circ$  is Gerstenhaber’s pre-Lie bracket

$$f \circ g = \sum_{i=1}^n f \circ_i g$$

on the endomorphism operad  $C^*(A) = \mathrm{End}_A$  (see Section 3 for details):

- Cup product:  $f \smile g = \mu(f \otimes g) \Delta$ ,
- Cap product:  $i_f(z) = af[a_1 | \cdots | a_p][a_{p+1} | \cdots | a_{p+q}]$ ,
- Lie bracket:  $[f, g] = f \circ g - (-1)^{(p-1)(q-1)} g \circ f$ ,
- Connes’ differential:  $dz = \sum_{j=0}^n (-1)^{jn} [a_{j+1} | \cdots | a_n | a | a_1 | \cdots | a_j]$ ,
- Lie action:  $L_f(\omega) = [d, i_f](\omega)$ .

In other words, the homology of the pair of complexes

$$(C^*(A), C_*(A))$$

along with the operators  $(\smile, [-, -], i, L, d)$  recovers the Tamarkin–Tsygan calculus of the associative algebra  $A$ .

The work of Keller [20, 21], and later of Keller and Armenta [2, 3] shows that the Tamarkin–Tsygan calculus of an algebra is derived invariant. Although our main result does imply it is homotopy invariant, in the sense it induces a well-defined functor (as in Theorem 8.5.3 in [17])

$$\mathrm{Ho}(\mathbf{Alg})^{\mathrm{iso}} \longrightarrow \mathrm{Ho}(\mathbf{Calc})^{\mathrm{iso}},$$

invariance is not a novel result, since homotopy equivalent algebras are derived equivalent. Rather, our main result is more concerned with the explicit computation of this calculus using a choice of resolution in **Alg**.

To be precise, the result of Armenta–Keller implies that one may *attempt* to compute the calculus of an associative algebra  $A$  by choosing some model  $B$  of it and, in some way or another, obtain a description of the spaces  $(\mathrm{HH}^*(A), \mathrm{HH}_*(A))$  and

formulas for the Calc-algebra structure here. Knowing this, our goal here is, having chosen  $B$ , to carry out the last step as explicitly as possible, and to give a datum depending on  $B$  that makes such computation feasible.

**5.2. Fields and forms**

Let us consider the  $B$ -bimodule resolution of  $B$  given by

$$S_*(B): 0 \longrightarrow B \otimes V \otimes B \longrightarrow B \otimes B \longrightarrow 0,$$

where the only non-trivial map is given by

$$b \otimes v \otimes b' \longmapsto bv \otimes b' - b \otimes vb'.$$

In case  $B$  has no differential, this is the usual small resolution of the free algebra  $TV$ . In our case, there is a caveat: this is a semi-free resolution, in the sense that

$$B \otimes V \otimes B$$

can be filtered by sub-bimodules (using the triangulation of  $V$ ) in such a way that the successive quotients are free  $B$ -bimodules on a basis of cycles. Indeed, once we have considered the filtration on  $B \otimes V \otimes B$  using a triangulation of  $V$ , the differential that is nontrivial on  $V$  will vanish. Since this is precisely the kind of resolutions needed to compute Hochschild (co)homology in the dg setting, we may use this resolution to proceed with our computations.

To understand where the differential of  $B \otimes V \otimes B$  comes from, it is useful to note it identifies with the bimodule of *Kahler differentials*  $\Omega_B^1$  of  $B$ . This is the free  $B$ -bimodule on  $B$ , say spanned by basis elements  $b \otimes b' \otimes b''$  subject to the relations

$$b' \otimes b_1 b_2 \otimes b'' = b' b_1 \otimes b_2 \otimes b'' + b' \otimes b_1 \otimes b_2 b''.$$

Note that since  $B$  is generated by  $V$ , any class in  $\Omega_B^1$ , which we write  $bdb'b''$ , can be written in the form  $b_1 d v b_2$ , and the identification is such that

$$g: b \otimes v \otimes b' \longmapsto b d v b'.$$

The following lemma makes this precise.

**Lemma 5.2.** *The map  $g: B \otimes V \otimes B \longrightarrow \Omega_B^1$  is an isomorphism of  $B$ -bimodules.*

*Proof.* The  $B$ -bimodule  $\Omega_B^1$  satisfies the universal property that the restriction map induced from  $B \longrightarrow \Omega_B^1$  sending  $b \longmapsto db$  induces a bijection

$$\text{hom}_{B^e}(\Omega_B^1, -) \longrightarrow \text{Der}(B, -).$$

Since  $B$  is free on  $V$ , this last functor is naturally isomorphic to the functor  $\text{hom}(V, -)$ . It is immediate that the inclusion of  $V$  into  $B \otimes V \otimes B$  satisfies the analogous universal property, which gives the result. □

Since  $S_*(B)$  gives us a resolution of  $B$  in  $B$ -bimodules the complexes

$$\mathcal{X}^*(B) = \text{hom}_{B^e}(S_*(B), B), \quad \Theta_*(B) = B \otimes_{B^e} S_*(B)$$

compute Hochschild (co)homology of  $B$  and hence, by invariance, that of  $A$ . We call  $\mathcal{X}^*(B)$  the *space of nc poly vector fields on  $B$* , and  $\Theta_*(B)$  the *space of nc differential forms on  $B$* . Let us now observe that from Lemma 5.2 it follows these complexes take a very simple form, which also explains the origin of their names.

**Corollary 5.3.** *We have natural identifications of complexes*

$$\mathcal{X}^*(B) = \text{cone}(\text{Ad}: B \longrightarrow \text{Der}(B)), \quad \Theta_*(B) = \text{cone}(\text{Co}: B \otimes V \longrightarrow B).$$

Here, the map Ad is the adjoint map of  $B$  and the map Co is the commutator map of  $B$ .

It is worthwhile to observe that since  $H_*(B) = H_0(B) = A$ , there is a four term exact sequence coming from the long exact sequence of the cone

$$0 \longrightarrow \text{HH}_1(A) \longrightarrow H_0(B \otimes V) \longrightarrow H_0(B) \longrightarrow \text{HH}_0(A) \longrightarrow 0$$

and it is straightforward to check that the image of the middle arrow is

$$[A, A] \subseteq A = H_0(B),$$

which recovers the usual description of  $\text{HH}_0(A)$ . Moreover, we see that  $\text{HH}_1(A)$  is the kernel of this map, and that for  $n \in \mathbb{N}_{\geq 2}$ , we an identification  $\text{HH}_n(A) = H_{n-1}(B \otimes V)$ . Dually, for  $n \in \mathbb{N}_{\geq 2}$  we have an identification

$$\text{HH}^n(A) = H^{n+1}(\text{Der}(B)),$$

and a four term exact sequence

$$0 \longrightarrow \text{HH}^0(A) \longrightarrow H^0(B) \xrightarrow{j^*} H^0(\text{Der}(B)) \longrightarrow \text{HH}^1(A) \longrightarrow 0,$$

that shows that  $\text{HH}^1(A) = H^0(\text{Der}(B))/\text{im}(j^*)$ . A far-reaching generalization of this, explaining the relation of operadic cohomology and Hochschild cohomology of operads under reasonable homotopical hypotheses is present in [31, Theorem 1.3.8].

We now show that the resolution  $S_*(B)$  is retraction of the double sided bar resolution of  $B$ , a fact known since the inception of Hochschild cohomology. Once we have developed the homotopy theory of calculi, this will give us a way of transporting the homotopy coherence calculus of chains on  $B$  onto the pair  $(\mathcal{X}^*(B), \Theta_*(B))$ . For convenience, we recall that the data of a homotopy deformation retract is a triple of maps of complexes  $(i, \pi, h)$

$$i: C' \longrightarrow C, \quad \pi: C \longrightarrow C'$$

such that  $\pi i = 1$  and  $i\pi - 1 = dh + hd$ .

**Lemma 5.4.** *There is a homotopy deformation retract  $(i, \pi, h)$  between  $S_*(B)$  and the double sided bar resolution  $\text{Bar}_*(B)$  of  $B$ , where we may take*

$$\pi: \text{Bar}_*(B) \longrightarrow S_*(B)$$

to be either zero or the identity except on  $B \otimes B \otimes B$  where for  $b = v_1 \cdots v_n$  we have

$$\pi(1 \otimes b \otimes 1) = \sum_{i=1}^n v_1 \cdots v_{i-1} \otimes v_i \otimes v_{i+1} \cdots v_n.$$

*In particular, the chain complex  $C_*(B)$  is homotopy equivalent to  $\Theta_*(B)$  and the cochain complex  $C^*(B)$  is homotopy equivalent to  $\mathcal{X}^*(B)$ .*

*Proof.* Let us begin by noting that the non-trivial component of  $\pi$  identifies with the quotient map  $q: B \otimes B \otimes B \longrightarrow \Omega_B^1$  under the isomorphism of Lemma 5.2, so it is

immediate it is a map of complexes. To construct  $i$ , we observe that if  $B$  has zero differential, we may choose it to be the inclusion of  $B \otimes V \otimes B$  into  $B \otimes B \otimes B$ .

In case  $B$  has a differential, we can filter the total complexes above using its columns, and then the extra differential coming from  $B$  lowers this filtration degree. In this way, the homological perturbation lemma guarantees that we can obtain again a homotopy retract.  $\square$

It would be desirable to have an explicit formula of the maps  $i$  and  $h$ .

### 5.3. The homotopy calculus

Having already set up things to obtain a retract from Hochschild (co)chains of  $B$  onto the pair  $(\mathcal{X}^*(B), \Theta_*(B))$ , we recall that Hochschild (co)chains on  $B$  carry a homotopy coherent calculus structure (which we can understand at the ‘strict’ level) and use this to obtain a homotopy coherent calculus structure on  $(\mathcal{X}^*(B), \Theta_*(B))$  whose homotopy type recovers  $\text{Calc}_A$ .

Let us recall from Section 3 that there is a 2-colored operad  $\text{PreCalc}$  so that a  $\text{PreCalc}$ -algebra is the same as a pair  $(V, M)$  where  $V$  is a Gerstenhaber algebra and  $M$  is both a module for the commutative algebra  $V$  and a Lie module for the shifted Lie algebra  $V$ , in a compatible way. As explained there, extending the operad  $\text{PreCalc}$  by a square zero unary operation  $d$  of the second color and imposing the Cartan formula gives us the operad  $\text{Calc}$ .

We proved that the operad  $\text{PreCalc}$  is quadratic Koszul, while the operad  $\text{Calc}$  is inhomogeneous Koszul. In particular,  $\text{Calc}$  admits a model where the only non-quadratic differential comes from the Cartan formula and, in particular, there is  $\text{Ger}_\infty$  placed in the first color. We will write  $\text{Calc}_\infty$  for this model, although it is not the same model that the authors consider in [7]. With this at hand, we recall from [7, Section 4.3] the following technical results:

**Theorem 5.5.** *There is a dg colored operad  $\text{KS}$ , the Kontsevich–Soibelman operad, and a topological colored operad  $\text{Cyl}$  such that*

1. *the operad  $C_*(\text{Cyl})$  is formal and its homology is  $\text{Calc}$ ,*
2. *there is a quasi-isomorphism  $\text{KS} \rightarrow C_*(\text{Cyl})$  of dg-operads,*
3.  *$\text{KS}$  acts on the pair  $(C^*(-), C_*(-))$  in such a way that*
4. *on homology we obtain the usual  $\text{Calc}$ -algebra structure on  $(\text{HH}^*(-), \text{HH}_*(-))$ .*

*In particular, for every cofibrant replacement  $Q$  of  $\text{Calc}$ , the pair  $(C^*(A), C_*(A))$  is a  $Q$ -algebra which on homology gives  $\text{Calc}_A$ .*

*Proof.* Claims (1)–(4) are in the cited references, and the last claim is standard, but let us explain it, since we will apply it for our choice of cofibrant replacement of  $\text{Calc}$ . That  $Q$  is a cofibrant replacement of  $\text{Calc}$  means that there is a quasi-isomorphism  $Q \rightarrow \text{Calc}$  and  $Q$  is a cofibrant operad in the model category of dg operads. Since we have a surjective quasi-isomorphism  $\text{KS} \rightarrow \text{Calc}$  or, what is the same, a trivial fibration. But  $Q$  is cofibrant, so we can obtain a lift  $Q \rightarrow \text{KS}$ . Finally, since  $\text{KS}$  acts on the pair  $(C^*(A), C_*(A))$ , so does  $Q$ , and on homology we obtain  $\text{Calc}_A$ .  $\square$

In particular, from the theorem it follows that  $(C^*(B), C_*(B))$  admits a  $\text{Calc}_\infty$ -algebra structure which on homology gives the  $\text{Calc}$ -algebra  $\text{Calc}_B = \text{Calc}_A$ , and we



will write  $\text{Calc}_{\infty, B}$  for this structure. We point out that the explicit construction of the operads  $\text{KS}$  and  $\text{Cyl}$  are not necessary for us here.

We remind the reader that  $\text{Calc}_{\infty}$  is our choice of cofibrant replacement obtained from the inhomogeneous Koszul duality theory. Our main result is that the pair  $(\mathcal{X}^*(B), \Theta_*(B))$  admits  $\text{Calc}_{\infty}$ -algebra structure which on homology gives the usual  $\text{Calc}$ -algebra structure; for some formulas see Theorem 5.7.

**Theorem 5.6.** *For any quasi-free model  $B$  of  $A$ , the pair  $(\mathcal{X}^*(B), \Theta_*(B))$  admits a  $\text{Calc}_{\infty}$ -algebra structure which is  $\infty$ -quasi-isomorphic to the  $\text{Calc}_{\infty}$ -algebra  $\text{Calc}_{\infty, A}$ . In particular, this structure recovers  $\text{Calc}_A$  by taking homology.*

*Proof.* By results of Section 2, the operad  $\text{Calc}$  is inhomogeneous Koszul. The methods developed in [11] imply that  $\text{Calc}$ -algebras then have the same rich homotopy theory that algebras over Koszul operads do. In particular by [11, Theorem 33], any  $\text{Calc}_{\infty}$ -algebra structure may be transported through a homotopy equivalence of chain complexes. By Lemma 5.4, there is a homotopy equivalence

$$(C^*(B), C_*(B)) \longrightarrow (\mathcal{X}^*(B), \Theta_*(B)),$$

which implies the  $\text{Calc}_{\infty}$ -algebra structure on  $(C^*(B), C_*(B))$  may be transported to one on  $(\mathcal{X}^*(B), \Theta_*(B))$ . The fact this structure is  $\infty$ -quasi-isomorphic to that one on  $\text{Calc}_{\infty, A}$  follows by another use of the transfer theorem, this time using  $\text{Calc}_{\infty, A}$  and  $\text{Calc}_{\infty, B}$ .  $\square$

The space  $\mathcal{X}^*(B)$  of nc poly vector fields on  $B$  is a dg Lie algebra under the usual bracket between derivations, with differential  $[\partial_B, -]$ . We can now state the next theorem, which in particular tells us that to compute Gerstenhaber brackets in  $A$  we may do so by choosing any model  $B$  of  $A$  and computing the usual Lie bracket in  $\mathcal{X}^*(B)$ . This last statement is a special case of [17, Theorem 8.5.3].

**Theorem 5.7.** *Let  $B = (TV, d)$  be a quasi-free model of  $A$ . Then*

1. *the cup product can be computed through the symmetrization of dual cobracket operation  $x_1x_2 = \{x_1, x_2; d\}$  of Theorem 6.1,*
2. *the Gerstenhaber bracket is the Lie bracket on the semi-direct product of a shifted copy of  $B$  with its Lie algebra of derivations,*
3. *the Connes boundary can be computed by*

$$d\omega = \sum_{i=1}^n (-1)^{\varepsilon} v_{i+1} \cdots v_n v_1 \cdots v_{i-1} dv_i$$

*for  $\omega = b + b'dv$  be a differential form in  $\Theta_*(B)$  and  $b = v_1 \cdots v_n$ .*

*Proof.* In Theorem 6.1 we will see that the cup product above is part of a brace algebra structure on  $\mathcal{X}(B)$  which has the same quasi-isomorphism type as the brace algebra  $C^*(B)$ , which implies that the symmetrization of the operation in that formula gives the desired cup product in cohomology. Similarly, the Lie bracket corresponds to the antisymmetrization of the brace operation  $\{x_1; x_2\}$  which is just the composition. Finally, the claim for the Connes boundary can actually be checked directly, using, for example, the definition of the boundary map of the usual LES of Hochschild and cyclic cohomology.  $\square$

## 6. Other results and computations

### 6.1. Brace operations

Let  $V$  be a graded vector space. A  $B_\infty$ -algebra structure on  $V$  is the datum of a structure of dg bialgebra on  $T(sV)$  where the comultiplication is given by deconcatenation [14]. It follows that the data required to define such structure amounts to a differential on  $T(sV)$ , which gives  $V$  the structure of an  $A_\infty$ -algebra, along with a multiplication on  $T(sV)$ . The fact this is a map of coalgebras means it is completely determined by a map  $T(sV) \otimes T(sV) \rightarrow sV$ . This defines, for each  $(p, q) \in \mathbb{N} \times \mathbb{N}$ , a map of degree  $1 - p - q$

$$\mu_{p,q}: V^{\otimes p} \otimes V^{\otimes q} \rightarrow V.$$

Let  $B$  be a quasi-free dga algebra. We proceed to show that its space of nc-vector fields  $\mathcal{X}^*(B)$  admits a  $B_\infty$ -algebra structure. We have already noted it admits an  $A_\infty$ -structure, so it suffices we define the family of maps corresponding to the multiplication. As it happens for Hochschild cochains, we can arrange it so that for each  $(p, q) \in \mathbb{N} \times \mathbb{N}$  we have  $\mu_{p,q} = 0$  whenever  $q > 1$ , in which case what we have is an algebra over the operad of braces  $\text{Br}$ .

For linear maps  $f_1, \dots, f_n, g \in \text{hom}(V, B)$  corresponding to a derivation in  $\mathcal{X}^*(B)$ , we define  $\{f_1, \dots, f_n; g\}$  as follows. Let  $\text{sh}(f_1, \dots, f_n)$  be the unique derivation on  $B$  that acts by zero on monomials of length less than  $n$ , and acts, for  $k \in \mathbb{N}$  on monomials of length  $n + k$  by the sum

$$\sum_{\sigma} \sigma(f_1 \otimes \dots \otimes f_n \otimes 1^{\otimes k})$$

as  $\sigma$  runs through  $(n, k)$ -shuffles in  $S_{n+k}$ . In other words, each  $\sigma$  keeps  $f_1, \dots, f_n$  in order and puts the identities in the middle slots. If  $G$  is the derivation that corresponds to  $g$ , we set  $\{f_1, \dots, f_n; g\}$  to be the linear map corresponding to the derivation  $\text{sh}(f_1, \dots, f_n) \circ G$ . We call  $\{f_1, \dots, f_n; g\}$  a *dual brace operation*.

Recall from [13] that if  $A$  is an associative algebra, there are *brace operations* defined on  $C^*(A)$  that make it, along with its usual structure of a dga algebra, into a brace algebra. Concretely, for each  $n \in \mathbb{N}$  and for  $f, g_1, \dots, g_n \in C^*(A)$  we have that

$$\{f; g_1, \dots, g_n\} = f \circ \text{sh}(g_1, \dots, g_n).$$

In other words, we are inserting  $g_1, \dots, g_n$  into  $f$  in all possible ways preserving their order. For example,  $\{f; g\}$  is the circle product of Gerstenhaber whose antisymmetrization gives the Gerstenhaber bracket. Note our notation is not ambiguous here:  $\{f; g\}$  is both a brace and a dual brace operation, in both cases it is induced by composition of (co)derivations.

**Theorem 6.1.** *Let  $B = (TV, d)$  be a quasi-free dga algebra. The dual braces give the space of nc fields  $\mathcal{X}^*(B)$  the structure of a  $\text{Br}$ -algebra. In particular, for every quasi-free algebra  $B$ , the space of nc fields on  $B$  is a  $\text{Ger}_\infty$ -algebra. Moreover,  $\mathcal{X}^*(B)$  is  $\text{Br}$ -quasi-isomorphic to the brace algebra  $C^*(A)$ .*

*Proof.* The first part of the first claim is proved, mutatis mutandis, in the same way one shows brace operations on  $C^*(A)$  satisfy the defining equations of a brace algebra: the definitions are dual to each other. The second claim follows from a deep

result of Tamarkin [18], which shows that there is a quasi-isomorphism of operads  $\text{Ger}_\infty \rightarrow \text{Br}$ . In this way, one can obtain from the brace algebra structure on  $\mathcal{X}^*(B)$  a  $\text{Ger}_\infty$ -structure.

For the last claim, we note that  $C^*(B)$  is  $\text{Br}$ -quasi-isomorphic to  $C^*(A)$ . Our theorem implies that  $\mathcal{X}^*(B)$  is, in particular,  $\text{Ger}_\infty$ -quasi-isomorphic to  $C^*(B)$  and, by Tamarkin’s result, we deduce that these two are also  $\text{Br}$ -quasi-isomorphic.  $\square$

**A word of caution**

The cup product we have defined using the differential  $d$  of  $B$  is not commutative on the nose, and in fact the cup product that one obtains using the result of Tamarkin is given by the symmetrization of this product. Since the cup product is commutative when passing to homology, there is no harm in using the non-symmetrized version for computations.

**6.2. Duality**

In [16], Herscovich fixes an augmented weight graded connected dg algebra  $A$  and proceeds to show that, writing  $E_A$  for the dual of the dg coalgebra  $BA$ , the Tamarkin–Tsygan calculi of  $A$  and  $E_A$  are dual to each other.

In the same context, let us pick a quasi-free model  $(TV, d) = B \rightarrow A$  of  $A$ , where  $V$  is a weight graded  $A_\infty$ -coalgebra. It then makes sense to consider the Tamarkin–Tsygan calculus of the  $A_\infty$ -algebra  $E_A = (sV)^\#$ . From our main theorem, one obtains the, now tautological, extension of Herscovich’s result.

We also point out that, following Berglund and Börjeson [5], the algebra  $A$  is  $A_\infty$ -Koszul with Koszul dual  $E_A$ . Their result also implies that  $\mathcal{X}^*(B)$  is  $A_\infty$ -quasi-isomorphic to the Hochschild dg-algebra  $C^*(A)$  with its canonical product and differential.

**Theorem 6.2.** *The Tamarkin–Tsygan calculi of  $A$  and  $E_A$  are dual to each other in the sense that there is an isomorphism of Gerstenhaber algebras and an isomorphism of Gerstenhaber modules,*

$$f: \text{HH}^*(E_A) \rightarrow \text{HH}^*(A) \quad g: \text{HH}_*(A)^\# \rightarrow \text{HH}_*(E_A)$$

respectively, where  $\text{HH}_*(A)$  is a module over  $\text{HH}^*(E_A)$  through  $f$ . Moreover, we have that  $\delta_E g = -g \delta_A^\#$  where  $\delta_E$  is the Connes boundary of  $\text{HH}_*(E_A)$  and  $\delta_A$  is that of  $\text{HH}_*(A)$ .

*Proof.* We already know that to compute the Tamarkin–Tsygan calculus of  $A$  we may use nc-fields and nc-forms obtained from  $B$ . To compute the Tamarkin–Tsygan calculus of  $E_A$ , we may use a quasi-free dg model of the dg coalgebra  $BE_A$ —the  $\infty$ -bar construction on  $E_A$ —and nc-cofields, that is, coderivations, and nc-coforms on it: this is just the classical definition. These two constructions are dual to each other: the space  $\text{hom}(BE_A, E_A)$  is equal to  $\text{hom}(V, TV)$  while  $BE_A \otimes E_A$  is dual to  $V \otimes TV$ , and as explained in [16], these isomorphisms are compatible with the cup and cap products, the Lie bracket, and Connes’ boundary.  $\square$

**6.3. Examples of computation**

We now give two examples where, in the spirit of [10], we compute Hochschild cohomology of an algebra  $A$  using a (minimal) model of it. We will also compute, in

some cases, Hochschild homology and cyclic homology, and the action of Hochschild cohomology on Hochschild homology using the results of the previous sections. We remark that the computations carried out here are many and for convenience we have omitted them. However, they are all simple and straightforward computations with no intricacies whatsoever, so nothing should be lost to this omission.

**A crown quiver algebra**

Let us consider the quiver as in Figure 1 with the single relation  $\alpha_1\alpha_2 \cdots \alpha_r\alpha_1$ , and its associated algebra  $A$ . In [30], the authors compute its Hochschild cohomology, including the Gerstenhaber bracket and the cup product. We will recover their results using the minimal model of  $A$ .

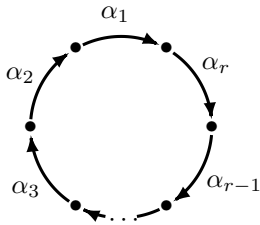


Figure 1: The wheel

We begin by noting that for  $n \in \mathbb{N}$  we have that  $\text{Tor}_A^{n+1}(\mathbb{k}, \mathbb{k})$  is one dimensional spanned by the class of the chain

$$\varepsilon_n = [\alpha_1 | \alpha_2 \cdots \alpha_r \alpha_1 | \cdots | \alpha_2 \cdots \alpha_r \alpha_1]$$

while, as usual,  $\text{Tor}_A^1$  is spanned by the arrows. One way to see this is to argue that the Anick resolution of the trivial  $A$ -module  $\mathbb{k}$ , which has the overlappings

$$\tau_n = \alpha_1(\alpha_2 \cdots \alpha_r \alpha_1)^n$$

corresponding to  $\varepsilon_n$  as  $A$ -module generators, is minimal, and hence these generate the corresponding Tor-groups. The minimal model  $B$  has then  $r$  generators in degree 0, and we write  $\varepsilon_0$  the one corresponding to  $[\alpha_1]$ , and for each  $n \in \mathbb{N}$  a generator  $\varepsilon_n$  in degree  $n$  whose differential is, by the main result of [39], as follows:

$$\partial(\varepsilon_n) = \sum_{s+t=n-1} (-1)^s \varepsilon_s \alpha_2 \cdots \alpha_r \varepsilon_t.$$

We are intentionally suppressing the sign given by the binomial coefficient since in this case there is exactly one non-vanishing higher coproduct,  $\Delta_{r+1}$ .

To find  $\text{HH}^*(A)$ , observe that for each natural number  $n \in \mathbb{N}_0$  there is an obvious cycle  $f_n$  of degree  $-n$  in  $\text{Der}(B, A)$  such that  $f_n(\varepsilon_n) = \varepsilon_0$ , and one can check, as it is done in [30], that it provides a generator for  $\text{HH}^{n+1}(A)$ , which is therefore one dimensional. We will now find a derivation of  $B$  that covers  $f_n$  under  $\alpha: B \rightarrow A$ , and then compute the Gerstenhaber bracket with these cycles: since the arrow  $\alpha_*$  is a quasi-isomorphism, we deduce that these cycles represent generators for the cohomology groups of  $\text{Der}(B)$ .

To do this, let us fix  $n \in \mathbb{N}_0$  and let  $F$  be a derivation of degree  $-n$  such that

$F(\varepsilon_n) = \varepsilon_0$ . Recursively solving for the values of  $F$  on generators using the equation

$$\partial F = (-1)^n F \partial,$$

shows that the following choice of lift works:

$$F_{2n}(\varepsilon_t) = (t - 2n + 1)\varepsilon_{t-2n}, \quad F_{2n+1}(\varepsilon_t) = \begin{cases} \varepsilon_{t-2n-1}, & \text{for } t \text{ odd,} \\ 0, & \text{for } t \text{ even.} \end{cases}$$

With this choice of generators of  $\text{HH}^*(A)$ , we compute that

$$[F_m, F_n] = \begin{cases} (m - n)F_{m+n}, & \text{for } m, n \text{ even,} \\ 0, & \text{for } m, n \text{ odd,} \\ mF_{m+n}, & \text{for } n \text{ even, } m \text{ odd.} \end{cases}$$

This coincides with the formulas obtained in [30]. However, observe that since we are using the natural grading in  $\text{Der}(B)$ , the derivations in odd degree represent elements of even degree in  $\text{HH}^*(A)$ , and those of even degree represent elements of odd degree in  $\text{HH}^*(A)$ , which explains the shift in our formulas.

Note that since  $B$  has no quadratic part in its differential, the cup product structure in  $\text{HH}^*(A)$  is trivial. This means, in particular, that the Gerstenhaber algebra structure on  $\text{HH}^*(A)$  is independent of the parameter  $r$ , but one can check that the higher products can be used to distinguish them: the  $A_\infty$ -structures on Hochschild cohomology are not quasi-isomorphic for distinct parameters, which shows in particular that these algebras are not derived equivalent; that is, their derived categories are not equivalent.

We now observe that  $\text{HC}_*(A)$  is easily computable: it is nothing but the homology of the abelianization of  $B$ , and this has a simple description. Indeed, since  $B$  is quasi-free, the space  $B/[B, B]$  is spanned by equivalence classes of cyclic words in  $B$  with respect to cyclic shifts. Moreover, the differential of  $B$  in 6.3 lands in  $[B, B]$ : this follows from the fact that the non-cyclic arrow  $\varepsilon_n$  lies in the commutator subspace, and the differential preserves it, hence we deduce that  $\text{HC}_*(A) = B/[B, B]$ .

**A non-3-Koszul algebra**

Let us consider the following quiver  $Q$  with relations  $R = \{xy^2, y^2z\}$ . We will compute its minimal model and with it its Hochschild cohomology, including the bracket. We will also compute the cup product; since the coproduct on  $\text{Tor}_A(\mathbb{k}, \mathbb{k})$  is non-vanishing only on the generator which we call  $\Gamma$ , this computation is straightforward.

By the main result in [39], the algebra  $A = \mathbb{k}Q/(xy^2, y^2z)$  has minimal model  $B$  given by the free algebra over the semisimple  $\mathbb{k}$ -algebra of vertices  $\underline{\mathbb{k}}$  with set of homogeneous generators  $\{x, y, z, \alpha, \beta, \Gamma, \Lambda\}$  such that

$$\partial x = \partial y = \partial z = 0, \quad \partial \alpha = xy^2, \quad \partial \beta = y^2z, \quad \partial \Gamma = \alpha z - x\beta, \quad \partial \Lambda = xy\beta - \alpha yz.$$

Here  $\Gamma$  corresponds to the overlap  $xy^2z$  while  $\Lambda$  corresponds to the overlap  $xy^3z$ , and  $\alpha$  and  $\beta$  correspond to the relations they cover under the differential  $\partial$  of the model, so that  $x, y$  and  $z$  are in degree 0,  $\alpha$  and  $\beta$  in degree 1, and  $\Gamma$  and  $\Lambda$  in degree 2. Thus  $B$  is the path algebra of the dg quiver in Figure 2.

The elements in  $B$  are the following, where  $r, t$  are elements of  $\{0, 1\}$  and  $s \in \mathbb{N}_0$ :

- *degree zero:*  $x^r y^s z^t$ ,

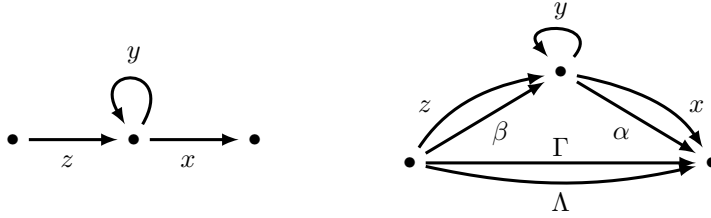


Figure 2: A quiver and its dg resolution with respect to the relations  $R$

- degree one:  $\alpha y^s z^t, x^r y^t \beta,$
- degree two:  $\Gamma, \Lambda, \alpha y^s \beta.$

Since we know that  $\text{Tor}_A^{\geq 4}(\mathbb{k}, \mathbb{k})$  is zero, it follows that  $\text{HH}^{\geq 4}(A)$  is zero. Moreover, it is straightforward to see that  $\text{HH}^0(A) = Z(A)$  has basis

$$\{xyz, xz, y^2, y^3, \dots\},$$

so we may focus our attention on derivations of degree 0,  $-1$  and  $-2$  to obtain bases for the remaining groups  $\text{HH}^1(A), \text{HH}^2(A)$  and  $\text{HH}^3(A)$ .

The following is a basis of derivations for the  $\mathbb{k}$ -bilinear 0-cycles, where  $s \in \mathbb{N}_0$ , and we adopt the convention that  $\alpha y^{-1} \beta = \Lambda$  and  $\alpha y^{-2} \beta = \Gamma$ :

$$\begin{aligned} E_s(x) &= 0, & E_s(y) &= y^{s+1}, & E_s(z) &= 0, & E_s(\alpha) &= 2\alpha y^s, \\ E_s(\beta) &= 2y^s \beta, & E_s(\Lambda) &= 3\alpha y^{s-1} \beta, & E_s(\Gamma) &= -2\alpha y^{s-2} \beta, \\ F_s(x) &= xy^s, & F_s(y) &= 0, & F_s(z) &= 0, & F_s(\alpha) &= \alpha y^s, \\ F_s(\beta) &= 0, & F_s(\Lambda) &= \alpha y^{s-1} \beta, & F_s(\Gamma) &= -\alpha y^{s-2} \beta, \\ G_s(x) &= 0, & G_s(y) &= 0, & G_s(z) &= y^s z, & G_s(\alpha) &= 0, \\ G_s(\beta) &= y^s \beta, & G_s(\Lambda) &= \alpha y^{s-1} \beta, & G_s(\Gamma) &= -\alpha y^{s-2} \beta. \end{aligned}$$

We now compute the  $\mathbb{k}$ -bilinear 0-boundaries. A basis for them is given by the following family of derivations, where  $s \in \mathbb{N}_0$ :

$$\begin{aligned} T_s(x) &= xy^{s+2}, & T_s(y) &= 0, & T_s(z) &= 0, & T_s(\alpha) &= \alpha y^{s+2}, \\ T_s(\beta) &= 0, & T_s(\Lambda) &= \alpha y^{s+1} \beta, & T_s(\Gamma) &= -\alpha y^s \beta, \\ R_s(x) &= 0, & R_s(y) &= 0, & R_s(z) &= y^{s+2} z, & R_s(\alpha) &= 0, \\ R_s(\beta) &= y^{s+2} \beta, & R_s(\Lambda) &= -\alpha y^{s+1} \beta, & R_s(\Gamma) &= \alpha y^s \beta. \end{aligned}$$

By direct inspection, we have that  $F_{s+2} = T_s, G_{s+2} = R_s$  for  $s \in \mathbb{N}_0$ , and no other relations, so that  $H^0(\text{Der}(B))$  has infinite dimension and is spanned the classes of the elements in the set  $\{F_0, F_1, G_0, G_1, E_s : s \in \mathbb{N}_0\}$ . Moreover, a basis for  $H^0(B)$  is of course given by the monomials belonging to  $A$ . We may only worry about the adjoint action on  $B$  of  $y$  and  $e_2$ , since they are the only ones that are  $\mathbb{k}$ -bilinear. We have that

$$\begin{aligned} \text{Ad}_y(x) &= -xy, & \text{Ad}_y(y) &= 0, & \text{Ad}_y(z) &= yz, \\ \text{Ad}_y(\alpha) &= -\alpha y, & \text{Ad}_y(\beta) &= y\beta, & \text{Ad}_y(\Lambda) &= 0, & \text{Ad}_y(\Gamma) &= 0, \end{aligned}$$

so that  $F_1 + \text{Ad}_y = G_1$ . Similarly,  $F_0 + \text{Ad}_{e_2} = G_0$ , so that  $\text{HH}^1(A)$  is infinite dimensional with basis the classes in  $\{F_0, G_0, E_s : s \in \mathbb{N}_0\}$ . Moreover, for each  $s, t \in \mathbb{N}$ ,

$$[E_s, E_t] = (s - t)E_{s+t},$$

and that  $[F_0, -]$  and  $[G_0, -]$  are identically zero. This determines  $\text{HH}^1(A)$  as a Lie algebra: it consists of abelian algebra  $\mathbb{k}^2$  acting trivially on the Witt algebra.

The following derivations form a basis of the 1-cycles in  $\text{Der}(B)$ , where unspecified values are zero,  $s \in \mathbb{N}_0$ , and we agree that  $y^{-1} = y^{-2} = 0$ :

$$\begin{array}{llll} \Phi_s(\alpha) = xy^s, & \Phi_s(\beta) = y^s z, & \Phi_s(\Lambda) = \alpha y^{s-1} z, & \Phi_s(\Gamma) = -\alpha y^{s-2} z, \\ \Phi'_s(\alpha) = 0, & \Phi'_s(\beta) = y^{s+2} z, & \Phi'_s(\Lambda) = -\alpha y^{s+1} z, & \Phi'_s(\Gamma) = \alpha y^s z, \\ \Pi_s(\alpha) = 0, & \Pi_s(\beta) = y^{s+2}, & \Pi_s(\Lambda) = \alpha y^{s+1}, & \Pi_s(\Gamma) = \alpha y^s, \\ \Pi'_s(\alpha) = xy^s z, & \Pi'_s(\beta) = 0, & \Pi'_s(\Lambda) = 0, & \Pi'_s(\Gamma) = 0, \\ \Psi_s(\alpha) = 0, & \Psi_s(\beta) = y^{s+2} z, & \Psi_s(\Lambda) = -\alpha y^{s+1} z, & \Psi_s(\Gamma) = xy^s \beta, \\ \Theta_s(\alpha) = 0, & \Theta_s(\beta) = 0, & \Theta_s(\Lambda) = \alpha y^s z - xy^s \beta, & \Theta_s(\Gamma) = 0, \\ \Xi_s(\alpha) = 0, & \Xi_s(\beta) = 0, & \Xi_s(\Lambda) = 0, & \Xi_s(\Gamma) = \Theta_s(\Lambda). \end{array}$$

Let us now compute the 1-boundaries of  $\text{Der}(B)$ . We observe that for every  $s \in \mathbb{N}_0$ , the elements  $\Xi_s, \Phi'_s, \Psi_s, \Theta_s$  have zero projection onto  $A$  under  $\alpha: B \rightarrow A$ , so these are boundaries. It is easy to check that the following set of derivations completes the list of 1-boundaries, where  $s \in \mathbb{N}_0$ :

$$\begin{array}{llll} X_s(\alpha) = xy^{s+2}, & X_s(\beta) = 0, & X_s(\Lambda) = xy^{s+1} \beta, & X_s(\Gamma) = -xy^s \beta, \\ Y_s(\alpha) = 2xy^{s+1}, & Y_s(\beta) = 2y^{s+1} z, & Y_s(\Lambda) = xy^s \beta - \alpha y^s z, & Y_s(\Gamma) = 0. \end{array}$$

Moreover, we have that  $\alpha(\Phi_{s+2} + \Phi'_s) = 0$ , so  $\Phi_{s+2}$  is a boundary for  $s \in \mathbb{N}_0$ . It follows that a basis of  $H^1(\text{Der}(B))$  is given by the classes of the derivations  $\Phi_0, \Phi_1$  so that  $\text{HH}^2(A)$  is two dimensional.

Every derivation of degree  $-2$  is a cycle and vanishes on every generator except, possibly,  $\Lambda$  and  $\Gamma$ , so that a basis for the 2-cycles is given by the following family of derivations, where  $s \in \mathbb{N}_0$  and  $t \in \{0, 1\}$ :

$$\Upsilon_s^t(\Lambda) = 0, \quad \Upsilon_s^t(\Gamma) = xy^s z^t, \quad \Upsilon_s^t(\Lambda) = xy^s z^t, \quad \Upsilon_s^t(\Gamma) = 0.$$

It is straightforward to check that all of these are boundaries except for  $\Upsilon_0^1$  and  $\Upsilon_0^0$ . More precisely, the following is a complete list of the 2-boundaries, where  $s \in \mathbb{N}_0$  and  $t \in \{0, 1\}$ :

$$\Upsilon_s^t(\Lambda) = 0, \quad \Upsilon_s^t(\Gamma) = xy^s z^t, \quad \Upsilon_{s+1}^t(\Lambda) = xy^{s+1} z^t, \quad \Upsilon_{s+1}^t(\Gamma) = 0.$$

From this it follows that  $\text{HH}^3(A)$  is also two dimensional. Finally, we compute the Gerstenhaber algebra structure. We already determined the bracket in  $\text{HH}^1(A)$ , while the bracket in  $\text{HH}^2(A)$  is trivial, since both generators vanish on  $\Lambda$  and  $\Gamma$ . The action

of  $\mathrm{HH}^1(A)$  on  $\mathrm{HH}^2(A)$  and  $\mathrm{HH}^3(A)$  is as follows, where  $s, t \in \mathbb{N}_0$  and  $r \in \{0, 1\}$ .

$$\begin{aligned} [E_{s+2}, \Phi_t] &= 3\Xi_{s+t+1} - 2\Theta_{s+t}, & [F_{s+2}, \Phi_t] &= \Theta_{t+s+1} - \Xi_{t+s}, \\ [G_{s+2}, \Phi_t] &= \Theta_{t+s+2} - \Xi_{t+s+2}, & [F_1, \Phi_t] &= [G_1, \Phi_t] = \Theta_t, \\ [E_s, \Upsilon_t^r] &= (t - 3\delta_{s,0})\Upsilon_{s+t}^r, & [E_s, \Omega_t^r] &= (t + 2\delta_{s,0})\Omega_{s+t}^r, \\ [F_0, -] &= [G_0, -] = 2, \text{ on } \langle \Omega_s^t, \Omega_s'^t : s \in \mathbb{N}_0 \rangle, \\ [F_0, -] &= [G_0, -] = [E_0, -] = 0, \text{ on } \langle \Upsilon_s^t, \Upsilon_s'^t, \Phi_s : s \in \mathbb{N}_0 \rangle, \\ [E_1, \Phi_t] &= 3\Xi_t. \end{aligned}$$

To compute cyclic and Hochschild homology of  $A$ , we begin by noting that for  $i \in \mathbb{N}$ , we have that  $[B, B]_i = B_i$ , and that  $(B/[B, B])_0$  is generated by the classes of  $y^j$  for  $j \in \mathbb{N}_0$ . This means that  $\mathrm{HC}_*(A)$  is concentrated in degree zero where it coincides with  $A/[A, A] = \mathbb{k}[\bar{y}]$ . The long exact sequence shows that  $\mathrm{HH}_*(A)$  is trivial in degrees larger than 1, and that  $d: \mathrm{HH}_0(A) \rightarrow \mathrm{HH}_1(A)$  is an isomorphism.

### The case of dg algebras that are not ‘aspherical’

In the above, we focused our attention on cofibrant algebras  $B$  for which  $H_*(B)$  is concentrated in degree zero. It is important to point out the work done here works equally well if  $B$  is only assumed to be cofibrant and non-negatively graded. Indeed, the only crucial property we used is that  $B$  is free as an associative algebra, so that the complexes of fields and forms compute what they are supposed to.

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