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TWO-DIMENSIONAL GOLOD COMPLEXES

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Abstract

We characterize two-dimensional Golod complexes combinatorially by vertex-breakability and topologically by the fat-wedge filtration of a polyhedral product. Applying the characterization, we consider a difference between Golodness over fields and rings, which enables us to give a two-dimensional simple Golod complex over any field such that the corresponding moment-angle complex is not a suspension.

1. Introduction

Throughout this paper, let K denote a simplicial complex over the vertex set $[m] = \{1, 2, \ldots, m\}$, where ghost vertices are not allowed. Recall that the Stanley–Reisner ring of K over a commutative ring R is defined by

$$R[K] = R[v_1, \ldots, v_m]/(v_{i_1} \cdots v_{i_k} \mid \{i_1, \ldots, i_k\} \notin K),$$

where we assume that each v_i is of degree 2. It is of particular interest to give a characterization of K which is equivalent to a given algebraic property of $R[K]$. For instance, Cohen–Macaulayness of $R[K]$ is completely characterized by a homological property of K. In this paper, we consider a property of $R[K]$, called Golodness, which was first introduced for a noetherian local ring [**3**].

Definition 1.1. A simplicial complex K is called *Golod* over R if all products and (higher) Massey products in $\mathrm{Tor}_+^{R[v_1,\ldots,v_m]}(R[K], R)$ vanish, where products and (higher) Massey products are given by the Koszul complex of $R[K]$.

We simply say that K is Golod if it is Golod over any ring.

Baskakov, Buchstaber and Panov [**2**] showed that there is a space Z_K, called the moment-angle complex for K, such that

$$H^*(Z_K; R) \cong \mathrm{Tor}_*^{R[v_1,\ldots,v_m]}(R[K], R),$$

where the isomorphism respects products and (higher) Massey products. This adds a topological viewpoint to the study of Stanley–Reisner rings, which is particularly useful in studying Golodness because K is Golod if Z_K is a suspension. The authors [**9**] developed a nice technology for the study of the homotopy type of Z_K, or more generally a polyhedral product, which is called the fat-wedge filtration. In particular,

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the following is proved in [**9**], where $\mathbb{R}Z_K$ denotes the real moment-angle complex for K.

Theorem 1.2. *If the fat-wedge filtration of* $\mathbb{R}Z_K$ *is trivial, then* Z_K *is a suspension, implying* K *is Golod.*

For several important Golod complexes such as the Alexander dual of sequentially Cohen–Macaulay complexes, the fat-wedge filtration of $\mathbb{R}Z_K$ has been proved to be trivial [**7, 8, 9**]. In particular, by describing a condition for the fat-wedge filtration of $\mathbb{R}Z_K$ combinatorially, combinatorial characterizations for Golodness of 1-dimensional complexes and triangulations of closed surfaces have been obtained in [**7, 9**]. Here we recall the result on surface triangulations. We say that K is k-neighborly if every $k+1$ vertices of K form a simplex of K.

Theorem 1.3. *If* K *is a triangulation of a connected closed surface, then the following conditions are equivalent:*

1. *K is Golod;*
2. *K is 1-neighborly;*
3. *the fat-wedge filtration of* $\mathbb{R}Z_K$ *is trivial.*

In this paper, we extend this result to all 2-dimensional simplicial complexes. Clearly, general 2-dimensional simplicial complexes are much more complicated than surface triangulations, and so neighborliness may not be enough to characterize Golodness of 2-dimensional simplicial complexes. Indeed, we have the following example.

Example 1.4. Let K be a wedge of two copies of the boundary of a 3-simplex. By definition, K is not 1-neighborly. On the other hand, it follows from [**9**, Corollary 7.5] that Z_K is of the homotopy type of a wedge of spheres. Then K is Golod.

Thus we need to consider a new notion to characterize Golodness of 2-dimensional simplicial complexes. Recall that the full subcomplex of K over a non-empty subset $I \subset [m]$ is defined by $K_I = \{\sigma \in K \mid \sigma \subset I\}$. For a vertex $v \in K$, we write $K_{[m]-v}$ by $\mathrm{dl}_K(v)$. Now we introduce a new notion.

Definition 1.5. We say that K has *vertex-breakable* n-th homology over an abelian group A if the map

$$\bigoplus_{v\in[m]}(i_v)_*\colon \bigoplus_{v\in[m]} H_n(\mathrm{dl}_K(v);A) \to H_n(K;A)$$

is not surjective, where $i_v\colon \mathrm{dl}_K(v) \to K$ denotes the inclusion. We simply say that K has vertex-breakable n-th homology if K has vertex-breakable n-th homology over *some* finitely generated abelian group.

Example 1.6. If K is a triangulation of a connected closed n-manifold, then K has vertex-breakable n-th homology.

Recall that a graph is called chordal if its minimal cycles are of length three. Now we state our results.

Theorem 1.7. *If* K *is a two-dimensional simplicial complex, then the following conditions are equivalent:*

1. *K is Golod;*

2. *the 1-skeleton of K is chordal, and every full subcomplex of K having vertex-breakable second homology is 1-neighborly;*
3. *the fat-wedge filtration of $\mathbb{R}Z_K$ is trivial.*

Golodness was originally defined for local rings by a certain equality involving the Poincaré series of their cohomology. Later, Golod [**3**] proved that the equality is equivalent to vanishing of products and (higher) Massey products as in Definition 1.1, which enables us to generalize the notion of Golodness over rings. Then it is natural to ask whether or not Golodness over fields and rings are different. Applying Theorem 1.7, we will prove that there is certainly a difference between them.

Theorem 1.8. *There is a two-dimensional simplicial complex which is Golod over any field but is not Golod over some ring.*

As in Theorems 1.3 and 1.7 as well as [**4, 5, 7, 8, 9**], Golodness over any field of several important classes of simplicial complexes has been proved to be a consequence of the corresponding moment-angle complexes being suspensions. Then it is natural to ask whether or not there is a simplicial complex K such that K is Golod over any field and Z_K is not a suspension. Yano and the first author [**10**] proved that there is such a simplicial complex by a direct calculation. We show that a simplicial complex of Theorem 1.8 is such a simplicial complex too, which is drastically simpler than the one of Yano and the first author.

Corollary 1.9. *There is a two-dimensional simplicial complex K such that K is Golod over any field and Z_K is not a suspension.*

Proof. Let K be a simplicial complex of Theorem 1.8. The first statement follows from Theorem 1.8. If Z_K is a suspension, then by Theorem 1.2, K must be Golod over any ring. Thus the second statement also follows from Theorem 1.8. □

Section 2 recalls properties of the fat-wedge filtration of a polyhedral product that we are going to use, and Section 3 considers a relation between Golodness and vertex-breakability. Section 4 proves Theorem 1.7, and Section 5 constructs a triangulation M of the Moore space $S^1 \cup_4 e^2$ which proves Theorem 1.8.

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2. Fat-wedge filtration

In this section, we recall from [**9**] properties of the fat-wedge filtration of a polyhedral product that we are going to use. First, we define a polyhedral product. Let $(\underline{X}, \underline{A}) = \{(X_i, A_i)\}_{i=1}^m$ be a collection of pairs of spaces. For a subset $\sigma \subset [m]$, let

$$(\underline{X}, \underline{A})^\sigma = Y_1 \times \cdots \times Y_m, \quad \text{where} \quad Y_i = \begin{cases} X_i & i \in \sigma, \\ A_i & i \notin \sigma. \end{cases}$$

The polyhedral product of $(\underline{X}, \underline{A})$ over K is defined by

$$Z_K(\underline{X}, \underline{A}) = \bigcup_{\sigma \in K} (\underline{X}, \underline{A})^\sigma.$$

Clearly, $Z_K(\underline{X},\underline{A})$ is natural with respect to $(\underline{X},\underline{A})$ and inclusions of subcomplexes of K. In particular, for $\emptyset \neq I \subset [m]$, $Z_{K_I}(\underline{X}_I,\underline{A}_I)$ is assumed to be a subspace of $Z_K(\underline{X},\underline{A})$, where $(\underline{X}_I,\underline{A}_I) = \{(X_i,A_i)\}_{i\in I}$. For a collection of pointed spaces $\underline{X} = \{X_i\}_{i=1}^m$, let $(C\underline{X},\underline{X}) = \{(CX_i,X_i)\}_{i=1}^m$. The polyhedral product $Z_K(C\underline{X},\underline{X})$ is particularly important. Indeed, the moment-angle complex and the real moment-angle complex for K are defined by

$$Z_K = Z_K(D^2,S^1) \quad \text{and} \quad \mathbb{R}Z_K = Z_K(D^1,S^0),$$

which play the fundamental role in toric topology. We refer to the comprehensive survey [**1**] for basic properties of polyhedral products.

Next we define the fat-wedge filtration of $Z_K(\underline{X},\underline{A})$. Let

$$Z_K^i(\underline{X},\underline{A}) = \{(x_1,\ldots,x_m) \in Z_K(\underline{X},\underline{A}) \mid \text{at least } m-i \text{ of } x_k \text{ are basepoints}\}$$

for $0 \leqslant i \leqslant m$. Clearly,

$$Z_K^i(\underline{X},\underline{A}) = \bigcup_{I\subset[m],\,|I|=i} Z_{K_I}(\underline{X}_I,\underline{A}_I).$$

The following is proved in [**9**, Theorem 3.1].

Theorem 2.1. *For each $\emptyset \neq I \subset [m]$, there is a map $\varphi_{K_I}\colon |K_I| \to \mathbb{R}Z_{K_I}^{|I|-1}$ such that*

$$\mathbb{R}Z_K^i = \mathbb{R}Z_K^{i-1} \bigcup_{I\subset[m],\,|I|=i} C|K_I|,$$

where the attaching maps are φ_{K_I}.

By the construction [**9**, Section 5] of φ_K, we have the following naturality.

Lemma 2.2. *Let L be a subcomplex of K such that the vertex set of L is the same as K. Then there is a commutative diagram*

$$\begin{array}{ccc} |L| & \xrightarrow{\varphi_L} & \mathbb{R}Z_L^{m-1} \\ \downarrow & & \downarrow \\ |K| & \xrightarrow{\varphi_K} & \mathbb{R}Z_K^{m-1}. \end{array}$$

We say that the fat-wedge filtration of $\mathbb{R}Z_K$ is trivial if φ_{K_I} is null-homotopic for each $\emptyset \neq I \subset [m]$. The main property of the fat-wedge filtration that we use is the following [**9**, Theorem 1.2], and Theorem 1.2 is its immediate corollary.

Theorem 2.3. *If the fat-wedge filtration of $\mathbb{R}Z_K$ is trivial, then for any $\underline{X} = \{X_i\}_{i=1}^m$, there is a homotopy equivalence*

$$Z_K(C\underline{X},\underline{X}) \simeq \bigvee_{\emptyset\neq I\subset[m]} |\Sigma K_I| \wedge \widehat{X}^I,$$

where $\widehat{X}^I = \bigwedge_{i\in I} X_i$.

To show that the map φ_K is null-homotopic, the following property is useful, which is proved in the proof of [**9**, Theorem 7.2]. Recall that a subset $\sigma \subset [m]$ is called a minimal non-face of K if $\sigma \notin K$ and $\sigma - i \in K$ for all $i \in \sigma$.

Lemma 2.4. *Let $\widehat{K}$ be a simplicial complex obtained by adding all minimal non-faces to K. Then the map φ_K factors through the inclusion $|K| \to |\widehat{K}|$.*

We will use the following criterion for triviality of the fat-wedge filtration of $\mathbb{R}Z_K$ [**9**, Theorem 1.6].

Theorem 2.5. *If K is $\lceil \frac{\dim K}{2} \rceil$-neighborly, then the fat-wedge filtration of $\mathbb{R}Z_K$ is trivial.*

3. Vertex-breakability

In this section, we prove a relation between vertex-breakability and Golodness. To this end, we recall a combinatorial description of the product in $\mathrm{Tor}_*^{R[v_1,\dots,v_m]}(R[K],R)$. By the classical theorem of Hochster, there is an isomorphism of R-modules

$$\mathrm{Tor}_i^{R[v_1,\dots,v_m]}(R[K],R) \cong \bigoplus_{\emptyset\neq I\subset[m]} \widetilde{H}^{i-|I|-1}(K_I;R).$$

It is remarkable that Baskakov, Buchstaber and Panov [**2**] proved that the product in $\mathrm{Tor}_*^{R[v_1,\dots,v_m]}(R[K],R)$ is nicely described through this isomorphism as follows. For disjoint simplicial complexes K, L, let $K * L$ denote the join of K and L, that is,

$$K * L = \{\sigma \sqcup \tau \mid \sigma \in K,\ \tau \in L\}.$$

Then $|K * L| = |K| * |L| \simeq \Sigma|K| \wedge |L|$. For $\emptyset \neq I, J \subset [m]$, we define a map $m_{I,J}\colon K_{I\cup J} \to K_I * K_J$ by

$$m_{I,J}(\sigma) = \sigma_I \sqcup \sigma_J$$

for $I \cap J = \emptyset$ and $m_{I,J} = *$ for $I \cap J \neq \emptyset$.

Theorem 3.1. *The product in $\mathrm{Tor}_*^{R[v_1,\dots,v_m]}(R[K],R)$ is identified with*

$$m_{I,J}^*\colon \widetilde{H}^{i-|I|-1}(K_I;R) \otimes \widetilde{H}^{j-|J|-1}(K_J;R) \to \widetilde{H}^{i+j-|I|-|J|-1}(K_{I\cup J};R).$$

We consider a map which is trivial in cohomology.

Lemma 3.2. *Let $f\colon X \to Y$ be a map between CW-complexes of finite type. If a map $f^*\colon H^n(Y;R) \to H^n(X;R)$ is trivial for any ring R, then the map $f_*\colon H_n(X;A) \to H_n(Y;A)$ is trivial for any finitely generated abelian group A.*

Proof. It is sufficient to prove the statement when A is a cyclic group. First, we consider the case $A = \mathbb{Z}/p^r$. As in [**12**, p. 239 - 240], $\mathbb{Z}/p^r$ is injective in the category of $\mathbb{Z}/p^r$-modules. Then

$$H^*(Z;\mathbb{Z}/p^r) \cong \mathrm{Hom}(H_*(Z;\mathbb{Z}/p^r),\mathbb{Z}/p^r)$$

for any space Z. This readily implies that f is trivial in homology over $\mathbb{Z}/p^r$.

Next, we consider the case $A = \mathbb{Z}$. By the universal coefficient theorem, for any abelian group B, there is a commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ext}(H_{n-1}(Y;\mathbb{Z}),B) & \longrightarrow & H^n(Y;B) & \longrightarrow & \mathrm{Hom}(H_n(Y;\mathbb{Z}),B) & \longrightarrow & 0 \\
 & & \big\downarrow (f_*)^* & & \big\downarrow f^* & & \big\downarrow (f_*)^* & & \\
0 & \longrightarrow & \mathrm{Ext}(H_{n-1}(X;\mathbb{Z}),B) & \longrightarrow & H^n(X;B) & \longrightarrow & \mathrm{Hom}(H_n(X;\mathbb{Z}),B) & \longrightarrow & 0.
\end{array}$$

Since the middle f^* is trivial for $B = \mathbb{Z}$ by assumption, the right $(f_*)^*$ is trivial for

$B = \mathbb{Z}$. For any finitely generated abelian group C, there is a natural isomorphism

$$\mathrm{Hom}(C, \mathbb{Z}) \cong \mathrm{Hom}(C/\mathrm{Tor}(C), \mathbb{Z})$$

where $\mathrm{Tor}(C)$ denotes the torsion part of C. Then since $C/\mathrm{Tor}(C)$ is a free abelian group, the map $f_*\colon H_n(X;\mathbb{Z})/\mathrm{Tor}(H_n(X;\mathbb{Z})) \to H_n(Y;\mathbb{Z})/\mathrm{Tor}(H_n(Y;\mathbb{Z}))$ is trivial, implying $f_*(H_n(X;\mathbb{Z})) \subset \mathrm{Tor}(H_n(Y;\mathbb{Z}))$. On the other hand, the right $(f_*)^*$ is trivial for $B = \mathbb{Z}/p^r$, r is arbitrary. Since $H_n(Y;\mathbb{Z})$ is finitely generated, if we take r large enough, then we can see that the p-torsion part $f_*(H_n(X;\mathbb{Z}))$ is trivial. Thus we obtain that $f_*\colon H_n(X;\mathbb{Z}) \to H_n(Y;\mathbb{Z})$ is trivial. □

The following lemma is a key to understand a relation between Golodness and 1-neighborliness, which is a refinement of [**11**, Lemma 4.3].

Lemma 3.3. *Given a non-trivial finitely generated abelian group A, suppose that for two vertices v, w of K, the map*

$$(i_v)_* \oplus (i_w)_*\colon H_n(\mathrm{dl}_K(v); A) \oplus H_n(\mathrm{dl}_K(w); A) \to H_n(K; A) \tag{1}$$

is not surjective. Then $\{v, w\}$ is an edge of K if and only if the map

$$(m_{I,J})_*\colon H_n(K; A) \to H_n(K_I * K_J; A)$$

is trivial for $I = \{v, w\}$ and $J = [m] - \{v, w\}$.

Proof. We set notation. The link of a vertex u of K is defined by

$$\mathrm{lk}_K(u) = \{\sigma \in K \mid u \notin \sigma \text{ and } \sigma \cup \{u\} \in K\}.$$

For an n-chain $c = \sum_i a_i[j_{i,0}, \ldots, j_{i,n}]$ of $\mathrm{lk}_K(u)$ for $a_i \in A$ and $[j_{i,0}, \ldots, j_{i,n}] \in \mathrm{lk}_K(u)$, we abbreviate the $(n+1)$-chain $\sum_i a_i[u, j_{i,0}, \ldots, j_{i,n}]$ of K by $u * c$.

Assume $\{v, w\}$ is not an edge of K. Let c be an n-cycle of K representing a homology class which is not in the image of the map (1). Then by the assumption above, there are $(n-1)$-chain c_v of $\mathrm{lk}_K(v)$, $(n-1)$-chain c_w of $\mathrm{lk}_K(w)$ and an n-chain d of K_J such that

$$c = v * c_v + w * c_w + d.$$

Since c_v is a chain of $\mathrm{lk}_K(v)$, c_v is a chain of K_J by the assumption above. One also gets c_w is a chain of K_J. Since $\partial c = 0$, one has $c_v - v * \partial c_v + c_w - w * \partial c_w + \partial d = 0$, implying

$$\partial c_v = \partial c_w = c_v + c_w + \partial d = 0.$$

Then it follows that

$$(m_{I,J})_*([c]) = [v * c_v - w * c_v + \partial(w * d)] = [v * c_v - w * c_v] \in H_n(K_I * K_J; A).$$

Since $K_I * K_J = \Sigma K_J$ by assumption, the map

$$H_{n-1}(K_J; A) \to H_n(K_I * K_J; A), \quad x \mapsto v * x - w * x$$

is an isomorphism. Thus since c_v is a cycle of K_J, if $(m_{I,J})_*([c]) = 0$, then there is an n-chain e of K_J such that $c_v = \partial e$, implying

$$\partial(w * c_w + d + e) = c_w + \partial d + \partial e = c_w + \partial d + c_v = 0, \quad \partial(v * c_v - e) = c_v - \partial e = 0.$$

Therefore $w * c_w + d + e$ and $v * c_v - e$ are cycles of $\mathrm{dl}_K(v)$ and $\mathrm{dl}_K(w)$, respectively, such that $[c] = (i_v)_*([w * c_w + d + e]) + (i_w)_*([v * c_v - e])$. This contradicts the definition of c, so the only if part is proved. The if part is obvious because $H_n(K_I * K_J; A) = 0$. ☐

Proposition 3.4. *If* $\dim K \leqslant 2$ *and* K *is Golod, then the following statements hold:*

1. *the 1-skeleton of* K *is chordal;*
2. *every full subcomplex of* K *having vertex-breakable second homology is 1-neighborly.*

Proof. The first statement is proved in the proof of [**9**, Proposition 8.17]. Suppose that K_I has vertex-breakable second homology. Clearly, the second statement holds for $|I| \leqslant 2$, and so we assume $|I| \geqslant 3$. Take any two vertices $v, w \in I$, and let $J = I - \{v, w\}$. Assume that $\{v, w\}$ is not an edge of K. Since $H^*(K_{\{v,w\}}; R)$ is a free R-module, the strong form of the Künneth formula holds as

$$\widetilde{H}^n(K_{\{v,w\}} * K_J; R) \cong \bigoplus_{i+j=n-1} \widetilde{H}^i(K_{\{v,w\}}; R) \otimes \widetilde{H}^j(K_J; R).$$

Then it follows from Lemma 3.2 that the map $(m_{\{v,w\},J})_* \colon H_2(K_I; A) \to H_2(K_{\{v,w\}} * K_J; A)$ is trivial for any finitely generated abelian group A. Thus by Lemma 3.3, $\{v, w\}$ is an edge of K_I. This is a contradiction, and so $\{v, w\}$ must be an edge of K. ☐

4. Proof of Theorem 1.7

We will use the following simple lemmas.

Lemma 4.1. *In a commutative diagram of abelian groups*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \\ 0 & \longrightarrow & B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & 0, \end{array}$$

with exact rows, suppose f_3 *is injective. Then* f_1 *is injective if and only if* f_2 *is.*

Proof. By the snake lemma, there is an exact sequence

$$0 \to \operatorname{Ker} f_1 \to \operatorname{Ker} f_2 \to \operatorname{Ker} f_3.$$

Since f_3 is injective, f_1 is injective if and only if f_2 is, completing the proof. ☐

Lemma 4.2. *For an exact sequence* $0 \to A_1 \to A_2 \to A_3 \to 0$ *of abelian groups, the following statements are equivalent:*

1. *the map* $\operatorname{Ext}(A_3, A) \to \operatorname{Ext}(A_2, A)$ *is injective for any abelian group* A*;*
2. *the exact sequence* $0 \to A_1 \to A_2 \to A_3 \to 0$ *splits.*

Proof. Consider the exact sequence

$$\begin{aligned} 0 \to \operatorname{Hom}(A_3, A) \to \operatorname{Hom}(A_2, A) \to \operatorname{Hom}(A_1, A)& \\ \to \operatorname{Ext}(A_3, A) \to \operatorname{Ext}(A_2, A) \to \operatorname{Ext}(A_1, A) \to 0.& \end{aligned}$$

Then the first statement is equivalent to the map $\operatorname{Hom}(A_2, A) \to \operatorname{Hom}(A_1, A)$ being surjective for any abelian group A. By setting $A = A_1$, this turns out to be equivalent to the second statement. ☐

Lemma 4.3. *Let $f\colon A \to B$ be a surjection between finitely generated abelian groups such that $f_*\colon \mathrm{Tor}(A, \mathbb{Z}/p^r) \to \mathrm{Tor}(B, \mathbb{Z}/p^r)$ is surjective for any prime p and any positive integer r. Then f admits a section.*

Proof. Since B is finitely generated, there is a decomposition

$$B \cong \mathrm{Free}(B) \oplus \bigoplus_p \mathrm{Tor}_p(B),$$

where $\mathrm{Free}(B)$ is a free abelian group, $\mathrm{Tor}_p(B)$ is the p-torsion part of B and p ranges over all primes. Since f is surjective, there is a map $s\colon \mathrm{Free}(B) \to A$ such that $f \circ s = 1_{\mathrm{Free}(B)}$. Let g_i be a generator of $\mathbb{Z}/p^{r_i}$ in $\mathrm{Tor}_p(B) \cong \mathbb{Z}/p^{r_1} \oplus \cdots \oplus \mathbb{Z}/p^{r_n}$. Then since

$$\mathrm{Tor}(B, \mathbb{Z}/p^r) = \{x \in B \mid p^r x = 0\} \tag{2}$$

and $f_*\colon \mathrm{Tor}(A, \mathbb{Z}/p^{r_i}) \to \mathrm{Tor}(B, \mathbb{Z}/p^{r_i})$ is surjective, there is an element h_i of A such that h_i is of order r_i and $f(h_i) = g_i$. Thus there is a map $s_p\colon \mathrm{Tor}_p(B) \to A$ such that $f \circ s_p = 1_{\mathrm{Tor}_p(B)}$. Therefore we obtain a section $s \oplus \bigoplus_p s_p\colon B \to A$ of f. □

Lemma 4.4. *Let $f\colon X \to Y$ be a map between finite complexes. If $f_*\colon H_*(X; A) \to H_*(Y; A)$ is surjective for any finitely generated abelian group A and $* = n-1, n$. Then $f^*\colon H^n(Y; A) \to H^n(X; A)$ is injective for any finitely generated abelian group A.*

Proof. By the universal coefficient theorem, there is a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ext}(H_{n-1}(Y;\mathbb{Z}), A) & \longrightarrow & H^n(Y;A) & \longrightarrow & \mathrm{Hom}(H_n(Y;\mathbb{Z}), A) & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle (f_*)^*} & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle (f_*)^*} & & \\
0 & \longrightarrow & \mathrm{Ext}(H_{n-1}(X;\mathbb{Z}), A) & \longrightarrow & H^n(X;A) & \longrightarrow & \mathrm{Hom}(H_n(X;\mathbb{Z}), A) & \longrightarrow & 0,
\end{array}$$

where rows are exact. By Lemma 4.1, it is sufficient to show that both left and right $(f_*)^*$ are injective. Since $f_*\colon H_n(X;\mathbb{Z}) \to H_n(Y;\mathbb{Z})$ is surjective, the right $(f_*)^*$ is injective. By Lemma 4.2, the left $(f_*)^*$ is injective if and only if the map $f_*\colon H_{n-1}(X;\mathbb{Z}) \to H_{n-1}(Y;\mathbb{Z})$ has a section. By the universal coefficient theorem, there is also a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_n(X;\mathbb{Z}) \otimes A & \longrightarrow & H_n(X;A) & \longrightarrow & \mathrm{Tor}(H_{n-1}(X;\mathbb{Z}), A) & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle f_*\otimes 1} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \\
0 & \longrightarrow & H_n(X;\mathbb{Z}) \otimes A & \longrightarrow & H_n(Y;A) & \longrightarrow & \mathrm{Tor}(H_{n-1}(Y;\mathbb{Z}), A) & \longrightarrow & 0,
\end{array}$$

where rows are exact. Since the middle f_* is surjective, so is the right f_*. Note that (2) holds for any abelian group B. Then since $H_{n-1}(X;\mathbb{Z})$ and $H_{n-1}(Y;\mathbb{Z})$ are finitely generated abelian groups, it follows from Lemma 4.3 that the surjectivity of the right f_* implies the existence of a section of the map $f_*\colon H_{n-1}(X;\mathbb{Z}) \to H_{n-1}(Y;\mathbb{Z})$. □

We apply Lemma 4.4 to our case.

Proposition 4.5. *If the 1-skeleton of K is chordal and the map*

$$\bigoplus_{v\in[m]} (i_v)_*\colon \bigoplus_{v\in[m]} H_2(\mathrm{dl}_K(v); A) \to H_2(K; A) \tag{3}$$

is surjective for any finitely generated abelian group A, then the map

$$\bigoplus_{v\in[m]} i_v^*\colon H^2(K;A)\to\bigoplus_{v\in[m]} H^2(\mathrm{dl}_K(v);A) \tag{4}$$

is injective.

Proof. If K is the boundary of a 2-simplex, then $H_2(K;A)=0$ and $H^2(K;A)=0$ for any abelian group A. Thus the statement holds obviously. Assume that K is not the boundary of a 2-simplex. By Lemma 4.4, it is sufficient to show that the map

$$\bigoplus_{v\in[m]} (i_v)_*\colon \bigoplus_{v\in[m]} H_1(\mathrm{dl}_K(v);A)\to H_1(K;A)$$

is surjective for any finitely generated abelian group A. For $m\leqslant 3$, $H_1(K;A)=0$ since K is not the boundary of a 2-simplex, where m is the number of vertices of K. Then the claim holds. For $m\geqslant 4$, every minimal cycle of K is in $\mathrm{dl}_K(v)$ for some v since the 1-skeleton of K is chordal. Then the claim also holds. □

We consider triviality of the fat-wedge filtration of $\mathbb{R}Z_K$.

Proposition 4.6. *Suppose* $\dim K\leqslant 2$ *and* φ_{K_I} *is null-homotopic for all* $I\subset[m]$ *with* $I\neq\emptyset,[m]$. *If the 1-skeleton of K is chordal and the map* (3) *is surjective for any finitely generated abelian group A, then φ_K is null homotopic.*

Proof. By definition, $\mathbb{R}Z_K^{m-1}$ is path-connected, and then it has the universal cover U_K. Since K is chordal, $|\widehat{K}|$ is simply-connected as in [**9**, Proposition 8.17]. Then by Lemma 2.4, $(\varphi_K)_*\colon\pi_1(|K|)\to\pi_1(\mathbb{R}Z_K^{m-1})$ is trivial for any basepoint of $|K|$. In particular, we get a lift $\widetilde{\varphi}_K\colon|K|\to U_K$. Choosing any vertex v of K, let $L=\mathrm{dl}_K(v)\sqcup v$. Then by arguing verbatim as above, we get a lift $\widetilde{\varphi}_L\colon|L|\to U_L$ of $\varphi_L\colon|L|\to\mathbb{R}Z_L^{m-1}$. Since there is a commutative diagram

$$\begin{array}{ccc} |L| & \xrightarrow{\varphi_L} & \mathbb{R}Z_L^{m-1}\\ \downarrow & & \downarrow\\ |K| & \xrightarrow{\varphi_K} & \mathbb{R}Z_K^{m-1}, \end{array}$$

it follows from the uniqueness of lifts of φ_L and φ_K that the square diagram

$$\begin{array}{ccc} |L| & \xrightarrow{\widetilde{\varphi}_L} & U_L\\ \downarrow & & \downarrow\\ |K| & \xrightarrow{\widetilde{\varphi}_K} & U_K \end{array}$$

commutes. Let $A=\pi_2(\mathbb{R}Z_K^{m-1})$ and $B=\pi_2(\mathbb{R}Z_L^{m-1})$. Then we get a homotopy commutative diagram

$$\begin{array}{ccccc} |L| & \xrightarrow{\widetilde{\varphi}_L} & U_L & \xrightarrow{u_L} & K(B,2)\\ \downarrow & & \downarrow & & \downarrow\\ |K| & \xrightarrow{\widetilde{\varphi}_K} & U_K & \xrightarrow{u_K} & K(A,2), \end{array}$$

where u_K and u_L are isomorphisms in π_2. Note that $\varphi_L=\varphi_{\mathrm{dl}_K(v)}\sqcup *$. By assumption, $\varphi_{\mathrm{dl}_K(v)}\simeq *$. Then since $\mathbb{R}Z_L^{m-1}$ is path-connected, φ_L is null-homotopic, implying $\widetilde{\varphi}_L$

is null-homotopic too. Thus the cohomology class $u_K \circ \widetilde{\varphi}_K$ belongs to the kernel of the map $H^2(K;A) \to H^2(L;A)$. Since this map is identified with $i_v^*\colon H^2(K;A) \to H^2(\mathrm{dl}_K(v);A)$, the cohomology class $u_K \circ \widetilde{\varphi}_K$ belongs to the kernel of the map (4).

By Theorem 2.1, $\mathbb{R}Z_K^{m-1}$ is a suspension. Then $A \cong H_2(\mathbb{R}Z_K;\mathbb{Z}) \otimes \mathbb{Z}\pi_1(\mathbb{R}Z_K^{m-1})$, and in particular, A is a sum of copies of $H_2(\mathbb{R}Z_K;\mathbb{Z})$ which is a finitely generated abelian group. Thus by Proposition 4.5, the map $i_v^*\colon H^2(K;A) \to \bigoplus_{v\in[m]} H^2(\mathrm{dl}_K(v);A)$ is injective, implying $u_K \circ \widetilde{\varphi}_K$ is null-homotopic. Since $\dim K \leqslant 2$, we obtain φ_K is null-homotopic. □

Now we prove Theorem 1.7.

Proof of Theorem 1.7. The implication (1) ⇒ (2) is proved by Proposition 3.4, and the implication (2) ⇒ (3) is proved by Theorem 2.5 and Proposition 4.6. The implication (3) ⇒ (1) is proved by Theorems 1.2 and 2.3. □

5. Golodness over fields and rings

As mentioned in Section 1, it is natural to ask whether or not there is a difference between Golodness over fields and over rings. This section gives an answer by giving a simplicial complex which is Golod over any field but is not Golod over some ring.

First, we prove that the converse of Proposition 3.4 holds over a field.

Proposition 5.1. *Let $\Bbbk$ be a field. A two-dimensional simplicial complex K is Golod over $\Bbbk$ if and only if the following conditions hold:*

1. *the 1-skeleton of K is chordal;*
2. *every full subcomplex of K having vertex-breakable second homology over $\Bbbk$ is 1-neighborly.*

Proof. The proof of Proposition 3.4 implies that if K is Golod over $\Bbbk$, then the two conditions hold.

Suppose conversely that the two conditions hold. Since $\Bbbk$ is a field, the Künneth formula in the strong form holds as $H^*(X \times Y;\Bbbk) \cong H^*(X;\Bbbk) \otimes H^*(Y;\Bbbk)$. It is proved by Kätthan [**11**, Theorem 6.3] that for a simplicial complex L of dimension at most three, the condition for (higher) Massey products in Definition 1.1 is redundant. Then to see that K is Golod over $\Bbbk$, it is sufficient to show that the map $H^*(K_{I_1} * K_{I_2};\Bbbk) \to H^*(K_I;\Bbbk)$ is trivial for $* = 1,2$, where I_1, I_2 are non-empty subsets of $[m]$ such that $I_1 \cap I_2 = \emptyset$ and $I_1 \cup I_2 = I$.

Let K^1 denote the 1-skeleton of K. It is proved in [**9**, Proposition 3.2] that a graph is Golod if and only if it is chordal. Then, in particular, K_I^1 is Golod (over $\Bbbk$) for each $\emptyset \neq I \subset [m]$. Consider a commutative diagram

$$\begin{array}{ccc} H^1(K_{I_1} * K_{I_2};\Bbbk) & \longrightarrow & H^1(K_I;\Bbbk) \\ \big\downarrow & & \big\downarrow \\ H^1(K^1_{I_1} * K^1_{I_2};\Bbbk) & \longrightarrow & H^1(K^1_I;\Bbbk), \end{array}$$

where I_1, I_2 are non-empty subsets of $[m]$ such that $I_1 \cap I_2 = \emptyset$ and $I_1 \cup I_2 = I$. Since K_I^1 is Golod over $\Bbbk$, the lower horizontal arrow is trivial. Then since the vertical arrows are injective, the upper horizontal arrow is trivial too.

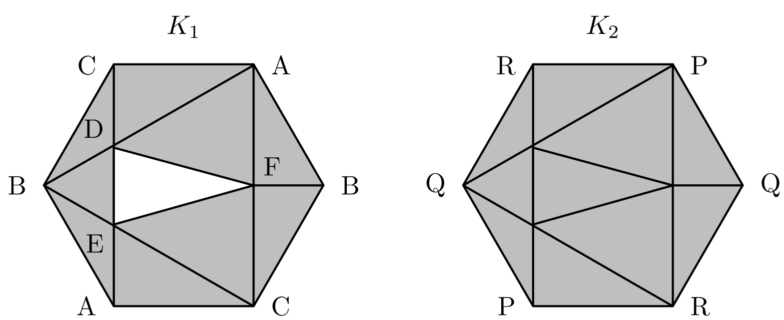
We show that the map $H^2(K_{I_1} * K_{I_2}; \Bbbk) \to H^2(K_I; \Bbbk)$ is trivial by induction on $|I|$. For $|I| = 2$, the map is trivial because $H^2(K_I; \Bbbk) = 0$. Suppose that K_J is Golod for $|J| < |I|$. If K_I has vertex-breakable second homology over $\Bbbk$, then it is 1-neighborly by assumption. Thus by Theorems 1.2, 2.3 and 2.5, K_I is Golod over $\Bbbk$. Consider a commutative diagram

$$\begin{array}{ccc} H^2(K_{I_1} * K_{I_2}; \Bbbk) & \longrightarrow & \bigoplus_{v \in I} H^2(L_1 * L_2; \Bbbk) \\ \downarrow & & \downarrow \\ H^2(K_I; \Bbbk) & \longrightarrow & \bigoplus_{v \in I} H^2(\mathrm{dl}_{K_I}(v); \Bbbk), \end{array}$$

where $L_i = K_{I_i}$ for $v \notin I_i$ and $L_i = \mathrm{dl}_{K_{I_i}}(v)$ for $v \in I_i$. If K_I does not have vertex-breakable second homology over $\Bbbk$, then the lower horizontal arrow is injective. By assumption, the right horizontal arrow is trivial, implying the left horizontal arrow is trivial too. Therefore the proof is complete. □

Remark 5.2. The proof of Proposition 5.1 does not work over a ring in general because the Künneth formula in the strong form does not hold and $\mathrm{Hom}(-, R)$ is not right exact.

We consider the following simplicial complexes K_1 and K_2, where vertices and edges having the same labels are identified.



Note that

$$|K_1| \simeq S^1 \quad \text{and} \quad |K_2| \simeq \mathbb{R}P^2.$$

We define a simplicial complex M by gluing K_1 and K_2 along the triangles DEF and PQR. Since the inclusion of the triangle DEF into K_1 is equivalent to the degree two self-map of S^1, M is a triangulation of the Moore space $S^1 \cup_4 e^2$.

Proof of Theorem 1.8. Let $\Bbbk$ be a field of characteristic 2. By definition, $H_2(M_I; \Bbbk) = 0$ unless M_I includes K_2, and M_I does not have vertex-breakable second homology over $\Bbbk$ whenever K_2 is a proper subcomplex of M_I. Thus M_I is Golod over $\Bbbk$ for $M_I \neq K_2$. If $M_I = K_2$, then M_I is 1-neighborly, implying M_I is Golod over $\Bbbk$ by Theorems 1.2, 2.3 and 2.5.

Let $\Bbbk$ be a field of characteristic $\neq 2$. Since $H_2(M_I; \Bbbk) = 0$ for each $\emptyset \neq I \subset [m]$, it follows from Proposition 5.1 that M is Golod over $\Bbbk$. Thus we obtain that M is Golod over any field.

Since $H_2(\mathrm{dl}_M(v);\mathbb{Z}/4)$ is either 0 or $\mathbb{Z}/2$ for each vertex v and $H_2(M;\mathbb{Z}/4) \cong \mathbb{Z}/4$, M has vertex-breakable second homology over $\mathbb{Z}/4$. Then since M is not 1-neighborly, M is not Golod over some ring by Theorem 1.7, where one actually sees from Lemma 3.3 that M is not Golod over $\mathbb{Z}/4$. Thus the proof is complete. □

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