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INVARIANTS FOR TAME PARAMETRISED CHAIN COMPLEXES

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(*communicated by Peter Bubenik*)

Abstract

We set the foundations for a new approach to Topological Data Analysis (TDA) based on homotopical methods at the chain complex level. We present the category of tame parametrised chain complexes as a comprehensive environment that includes several cases that usually TDA handles separately, such as persistence modules, zigzag modules, and commutative ladders. We extract new invariants in this category using a model structure and various minimal cofibrant approximations. Such approximations and their invariants retain some of the topological, and not just homological, aspects of the objects they approximate.

Introduction

Data analysis is often about simplifying, ignoring most of the information available and extracting what might be meaningful for the task at hand. The same strategy of extracting summaries is also at the core of topology. In recent years, these two branches have merged, giving rise to Topological Data Analysis (TDA) [**7**].

TDA can benefit from a broad spectrum of existing homotopical tools for extracting such summaries. Currently, the most popular is persistent homology. The first step in persistence theory is to transform data into spatial information via, for example, the Vietoris–Rips construction. The second step is typically the extraction of the homology of the obtained spaces, resulting in a so-called persistence module, effectively studied by enumerating its indecomposables [**20**].

Despite its success, TDA based on persistent homology has some limitations. Firstly, one is limited to objects for which it is possible to list their indecomposable summands [**20**], and whose decompositions can be computed algorithmically. For example, the class of *commutative ladders* cannot be analysed using its indecomposables because it is of wild representation type [**6**, **12**]. On the other hand, the indecomposables of the class of *zigzag modules* are fully described, but so far there is no efficient software to analyse them [**8**, **9**]. Secondly, applying homology might be too drastic, disregarding a large amount of geometric information.

The main goal of this paper is to show how to use homotopy theory not only to overcome the previous issues but also to open the way towards new invariants. The

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same strategy of disregarding some information and focusing on aspects that might be relevant is also at the core of homotopy theory, with *colocalization* being an example of such a process. In colocalization, the simplification is achieved by approximating arbitrary objects by other objects that are simpler and more manageable, such as the class of *cofibrant objects* in a model category. Our work is based on the realisation that the category $\text{tame}\big([0,\infty),\text{ch}\big)$ of tame $[0,\infty)$-parametrised chain complexes over a field admits a model structure for which there is a surprisingly simple decomposition theorem describing all indecomposable cofibrant objects (see Theorem 4.2). This is so even though the entire category $\text{tame}\big([0,\infty),\text{ch}\big)$ is of wild representation type. The structure Theorem 4.2 identifies cofibrant objects in $\text{tame}\big([0,\infty),\text{ch}\big)$ with sequences of persistence diagrams augmented with points on the diagonal (see Section 5) which we call Betti diagrams.

The cofibrant objects can be then used to approximate arbitrary objects in $\text{tame}\big([0,\infty),\text{ch}\big)$. Proving that such minimal approximations exist (see Theorem 2.4) has been essential in this work.

Model categories are convenient for ensuring the existence of certain morphisms or approximations. A common difficulty in working with model categories, however, is the lack of algorithmic constructions producing such morphisms and approximations. Approximations in model categories are often constructed using universal properties and require performing large limits. Extracting calculable invariants from such approximations, which is essential in TDA, is often not feasible. In this article, we make a great effort to describe all the constructions, factorisations, and approximations explicitly. All the steps we perform for the tame $[0,\infty)$-parametrised chain complexes in perspective can be implemented.

Considering tame $[0,\infty)$-parametrised chain complexes instead of vector spaces has another advantage. Persistence modules, zigzag modules, and commutative ladders are special objects in $\text{tame}\big([0,\infty),\text{ch}\big)$. Thus, this category allows for a comprehensive theory in which different objects that are handled separately by standard persistence theory can be studied and compared together. Furthermore, for persistence modules, cofibrant minimal approximations provide complete invariants (see Section 5.1).

In conclusion, we propose a refined approach to the persistence pipeline: first, convert the input into a parametrised simplicial complex. Second, extract a parametrised chain complex. Third, form a minimal cofibrant approximation of the extracted parametrised chain complex. Finally, represent the minimal cofibrant approximation by its Betti diagrams.

Related works

The model structure described in Section 2 is a special case of a projective model structures on a tame subcategory of functor categories [**15**, **16**].

The structure Theorem 4.2 describing cofibrant objects in $\text{tame}\big([0,\infty),\text{ch}\big)$ appears also in, for example, [**4**, **19**, **23**], although the language of model categories is not used there. An interpretation from the point of view of Morse theory was given in [**4**]. In [**19**], Meehan, Pavlichenko and Segert show that the category of filtered chain complexes is a Krull–Schmidt category. In [**23**], Usher and Zhang generalise the theory of barcodes to filtered Floer-type complexes, considering chain complexes of infinite dimension whose parametrisation is not tame. They prove a singular value decomposition theorem for such complexes and identify two types of barcodes of them:

the *verbose* and the *concise*. In the finite case, such barcodes correspond respectively to the Betti diagrams and the minimal Betti diagrams of cofibrant objects in our setting (see Section 5).

The point of view of homotopy theory is entering the TDA subject also for purposes different from ours. For example, [**5**,**13**,**14**,**17**] are about lifting the stability theorem of persistence to homotopy stability theorems, to make it applicable to a wider class of datasets.

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1. Minimality

Let $\mathcal{M}$ be a model category [**10**,**21**]. This means that three classes of morphisms in $\mathcal{M}$ are chosen: **weak equivalences** ($\xrightarrow{\sim}$), **fibrations** ($\twoheadrightarrow$), and **cofibrations** ($\hookrightarrow$). These classes and $\mathcal{M}$ are required to satisfy the following axioms:

MC1. Finite limits and colimits exist in $\mathcal{M}$.

MC2. If f and g are morphisms in $\mathcal{M}$ such that gf is defined and if two of the three morphisms f, g, gf are weak equivalences, then so is the third.

MC3. If f is a retract of g and g is a fibration, a cofibration, or a weak equivalence, then so is f.

MC4. Consider a commutative square in $\mathcal{M}$ consisting of the solid morphisms:

$$\begin{array}{ccc} X & \longrightarrow & E \\ {\scriptstyle\alpha}\big\downarrow & \nearrow & \big\downarrow{\scriptstyle\beta} \\ Y & \longrightarrow & B \end{array}$$

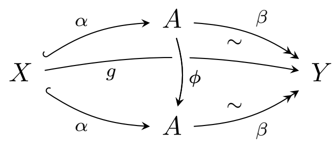
Then a morphism, depicted by the dotted arrow and making this diagram commutative, exists under either of the following two assumptions: (i) α is a cofibration and a weak equivalence and β is a fibration, or (ii) α is a cofibration and β is a fibration and a weak equivalence.

MC5. Any morphism g can be factored in two ways: (i) $g = \beta\alpha$, where α is a cofibration and β is both a fibration and a weak equivalence, and (ii) $g = \beta\alpha$, where α is both a cofibration and a weak equivalence and β is a fibration.

In particular, MC1 guarantees the existence of the initial object, denoted by $\emptyset$, and of the terminal object, denoted by $*$. An object X in a model category $\mathcal{M}$ is called **cofibrant** if the morphism $\emptyset \to X$ is a cofibration. If the morphism $X \to *$ is a fibration, then X is called **fibrant**.

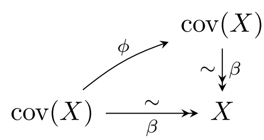
Axiom MC5 above guarantees existence of certain factorisations of morphisms. It does not specify any uniqueness. Typically, a morphism in a model category admits many such factorisations. There are however model categories in which among all these factorisations there is a canonical one called minimal [**3**, **22**]:

Definition 1.1. Let $g\colon X \to Y$ be a morphism in $\mathcal{M}$. A factorisation $g = \beta\alpha$, where α is cofibration and β is a fibration and a weak equivalence, is called **minimal** if every morphism ϕ which makes the following diagram commutative is an isomorphism:



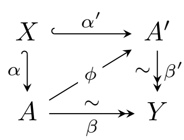
A minimal factorisation of $\emptyset \to X$ is called a **minimal cover** of X.

According to the above definition, we can think about a minimal cover of X as a morphism $\beta\colon \mathrm{cov}(X) \to X$ such that: (i) $\mathrm{cov}(X)$ is cofibrant, (ii) β is both a fibration and a weak equivalence, and (iii) any morphism ϕ which makes the following diagram commutative is an isomorphism:

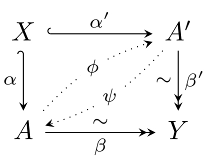


Minimal factorisations are unique:

Proposition 1.2. *Let $g\colon X \to Y$ be a morphism in $\mathcal{M}$. Assume $\beta\alpha = g = \beta'\alpha'$ are minimal factorisations. Then there is an isomorphism ϕ making the following diagram commutative:*



Proof. Let ϕ and ψ be any morphisms making the following diagram commute, which exist by the lifting axiom MC4:



Then by the definition of minimal factorisations, the compositions $\phi\psi$ and $\psi\phi$ are isomorphisms. Consequently, so are ϕ and ψ. □

Two objects X and Y in $\mathcal{M}$ are called **weakly equivalent** if there is a sequence of weak equivalences of the form:

$$X \xleftarrow{\sim} A_0 \xrightarrow{\sim} A_1 \xleftarrow{\sim} \cdots \xleftarrow{\sim} A_k \xrightarrow{\sim} Y$$

Similarly to factorisations of morphisms in a model category, the collection of objects weakly equivalent to a given object is large. There are model categories, however, where this collection contains a canonical object called a minimal representative:

Definition 1.3. An object X in $\mathcal{M}$ is called **minimal** if it is cofibrant, fibrant, and any weak equivalence $\phi\colon X \to X$ is an isomorphism. A **minimal representative** of an object X in $\mathcal{M}$ is a minimal object in $\mathcal{M}$ which is weakly equivalent to X.

Minimal representatives are unique up to isomorphisms:

Proposition 1.4. *Let X' and Y' be minimal representatives of respectively X and Y. Then X and Y are weakly equivalent if and only if X' and Y' are isomorphic.*

Proof. If X' and Y' are isomorphic, then X and Y are weakly equivalent. Assume X and Y are weakly equivalent. Then X' and Y' are also weakly equivalent. Since they are both cofibrant and fibrant there are weak equivalences $\phi\colon X' \xrightarrow{\sim} Y'$ and $\psi\colon Y' \xrightarrow{\sim} X'$. By the definition of the minimality, the compositions $\phi\psi$ and $\psi\phi$ are isomorphisms. Consequently, so are ϕ and ψ and hence X' and Y' are isomorphic. □

Proposition 1.2 and Proposition 1.4 ensure the uniqueness of minimal factorisations, minimal covers and minimal representatives. These propositions however do not imply their existence, which has to be proven separately and it does depend on the considered model category.

Definition 1.5. A model category satisfies the **minimal factorisation axiom** if all minimal factorisations exist in this category. It satisfies the **minimal representative axiom** if all minimal representatives exist in this category.

Many model categories, particularly of combinatorial flavour, satisfy the minimal factorisation axiom. However the standard model structure on topological spaces [10] does not.

2. Tame $[0, \infty)$-parametrised objects

Let $\mathcal{M}$ be a category. The symbol $[0, \infty)$ denotes the poset of non-negative real numbers. Functors of the form $X\colon [0, \infty) \to \mathcal{M}$ are also referred to as $[0, \infty)$-parametrised objects. The value of X at t in $[0, \infty)$ is denoted by X^t and $X^{s\leqslant t}\colon X^s \to X^t$ denotes the morphism in $\mathcal{M}$ that X assigns to $s \leqslant t$. The morphism $X^{s\leqslant t}$ is also referred to as the transition morphism in X from s to t.

Definition 2.1. A sequence $\tau_0 < \cdots < \tau_k$ in $[0, \infty)$ **discretises** $X\colon [0, \infty) \to \mathcal{M}$ if $X^{s\leqslant t}\colon X^s \to X^t$ may fail to be an isomorphism only when there is $a \in \{0, \ldots, k\}$ such that $s < \tau_a \leqslant t$. A functor $X\colon [0, \infty) \to \mathcal{M}$ is called **tame** if there is a sequence that discretises it. The symbol $\operatorname{tame}\big([0, \infty), \mathcal{M}\big)$ denotes the category whose objects are tame functors $X\colon [0, \infty) \to \mathcal{M}$ and whose morphisms are all of the natural transformations.

If $\tau_0 < \cdots < \tau_k$ discretises $X\colon [0,\infty) \to \mathcal{M}$, then the transitions of the restrictions of X to the intervals $[0,\tau_0), \ldots, [\tau_{k-1},\tau_k)$, and $[\tau_k,\infty)$ are isomorphisms. Note that if $\tau_0 < \cdots < \tau_k$ discretises $X\colon [0,\infty) \to \mathcal{M}$, then so does any of its refinements (a sequence $\mu_0 < \cdots < \mu_n$ is a **refinement** of $\tau_0 < \cdots < \tau_k$ if $\{\tau_0 \ldots, \tau_k\}$ is a subset of $\{\mu_0 \ldots, \mu_n\}$).

2.1. Kan extensions

Consider a sequence of k composable morphisms in $\mathcal{M}$:

$$X^0 \xrightarrow{X^{0<1}} \cdots \xrightarrow{X^{k-1<k}} X^k$$

Such a sequence encodes a functor $X\colon [k] \to \mathcal{M}$, where $[k]$ is the standard poset on the set $\{0,\ldots,k\}$. Consider also a sequence $\tau_0 < \cdots < \tau_k$ of elements in $[0,\infty)$, which encodes an inclusion of categories $\tau\colon [k] \subset [0,\infty)$. The **(left) Kan extension** of X along τ [**18**, Sect. X.3] is a functor $LX\colon [0,\infty) \to \mathcal{M}$ whose values are given by:

$$LX^t = \begin{cases} \emptyset & \text{if } t < \tau_0, \\ X^{\max\{a \mid \tau_a \leqslant t\}} & \text{if } t \geqslant \tau_0. \end{cases}$$

For morphisms, $LX^{s<t}$ is the identity if $\max\{a \mid \tau_a \leqslant s\} = \max\{a \mid \tau_a \leqslant t\}$, and otherwise it is the composition of:

$$X^{\max\{a \mid \tau_a \leqslant s\} < \max\{a \mid \tau_a \leqslant s\}+1}, \ldots, X^{\max\{a \mid \tau_a \leqslant t\}-1 < \max\{a \mid \tau_a \leqslant t\}}$$

The functor $LX\colon [0,\infty) \to \mathcal{M}$ is tame with $\tau_0 < \cdots < \tau_k$ as a discretising sequence.

To describe a natural transformation $g\colon LX \to Y$ to any other functor $Y\colon [0,\infty) \to \mathcal{M}$, it is enough to specify a sequence of morphisms $\{g^{\tau_a}\colon X^{\tau_a} \to Y^{\tau_a}\}_{a=0,\ldots,k}$ for which the following diagram commutes for every $a = 1,\ldots,k$:

$$\begin{array}{ccc} X^{\tau_{a-1}} & \xrightarrow{X^{\tau_{a-1}<\tau_a}} & X^{\tau_a} \\ {\scriptstyle g^{\tau_{a-1}}}\downarrow & & \downarrow{\scriptstyle g^{\tau_a}} \\ Y^{\tau_{a-1}} & \xrightarrow{Y^{\tau_{a-1}<\tau_a}} & Y^{\tau_a} \end{array}$$

If $k=0$, for an object X in $\mathcal{M}$ (representing a functor $X\colon [0] \to \mathcal{M}$) and an element τ_0 in $[0,\infty)$ (representing an inclusion $\tau_0\colon [0] \subset [0,\infty)$), the induced Kan extension is a functor $LX\colon [0,\infty) \to \mathcal{M}$ such that $LX^t = \emptyset$ if $t < \tau_0$ and $LX^t = X$ if $\tau_0 \leqslant t$. In this case, the set of natural transformations $LX \to Y$ is in bijection with the set of morphisms in $\mathcal{M}$ from X to Y^{τ_0}. If $k = 1$, for a morphism $X^{0<1}\colon X^0 \to X^1$ (representing a functor $X\colon [1] \to \mathcal{M}$) and two elements $\tau_0 < \tau_1$ in $[0,\infty)$ (representing an inclusion $[1] \subset [0,\infty)$), the induced Kan extension is a functor $LX\colon [0,\infty) \to \mathcal{M}$ such that $LX^t = \emptyset$ if $t < \tau_0$, $LX^t = X^0$ if $\tau_0 \leqslant t < \tau_1$, and $LX^t = X^1$ if $\tau_1 \leqslant t$. In this case, the set of natural transformations $LX \to Y$ is in bijection with commutative squares of the form:

$$\begin{array}{ccc} X^0 & \xrightarrow{X^{0<1}} & X^1 \\ \downarrow & & \downarrow \\ Y^{\tau_0} & \xrightarrow{Y^{\tau_0<\tau_1}} & Y^{\tau_1} \end{array}$$

Let $X\colon [0,\infty) \to \mathcal{M}$ be tame with $0 = \tau_0 < \cdots < \tau_k$ as a discretising sequence. Then X is isomorphic to the Kan extension along $0 = \tau_0 < \cdots < \tau_k$ of the following sequence of morphisms:

$$X^0 \xrightarrow{X^{0\leqslant\tau_1}} \cdots \xrightarrow{X^{\tau_{k-1}<\tau_k}} X^{\tau_k}$$

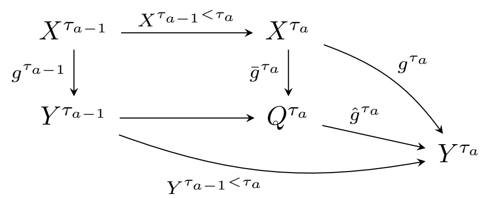
2.2. Factorisation

Let $\mathcal{M}$ admit all finite colimits. Let $g\colon X\to Y$ be a morphism in $\text{tame}\big([0,\infty),\mathcal{M}\big)$ and $0=\tau_0<\cdots<\tau_k$ a sequence discretising both X and Y. By induction on a in $[k]$, define morphisms $\bar{g}^{\tau_a}: X^{\tau_a}\to Q^{\tau_a}$ and $\hat{g}^{\tau_a}: Q^{\tau_a}\to Y^{\tau_a}$ in $\mathcal{M}$ as follows:

For $a=0$: $\quad (\bar{g}^0\colon X^0\to Q^0) \coloneqq (1\colon X^0\to X^0) \quad (\hat{g}^0\colon Q^0\to Y^0)\coloneqq(g^0\colon X^0\to Y^0)$

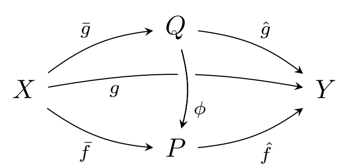
For $a=1\dots,k$: $\quad Q^{\tau_a}\coloneqq \operatorname{colim}(Y^{\tau_{a-1}} \xleftarrow{g^{\tau_{a-1}}} X^{\tau_{a-1}} \xrightarrow{X^{\tau_{a-1}<\tau_a}} X^{\tau_a})$

$\bar{g}^{\tau_a}: X^{\tau_a}\to Q^{\tau_a}$ and $\hat{g}^{\tau_a}: Q^{\tau_a}\to X^{\tau_a}$ are the unique morphisms making the following diagram commutative, where the inside square is a pushout:



For $a=1,\dots,k$, define $Q^{\tau_{a-1}<\tau_a}: Q^{\tau_{a-1}}\to Q^{\tau_a}$ to be the composition of the morphism $Y^{\tau_{a-1}}\to Q^{\tau_a}$ represented by the bottom horizontal arrow in the above diagram and $\hat{g}^{\tau_{a-1}}: Q^{\tau_{a-1}}\to Y^{\tau_{a-1}}$. Let $Q\colon[0,\infty)\to\mathcal{M}$ be the tame functor given by the Kan extension of the sequence of morphisms $\{Q^{\tau_{a-1}<\tau_a}\}$ along $0=\tau_0<\cdots<\tau_k$ (see Section 2.1). Finally, denote by $\bar{g}\colon X\to Q$ and $\hat{g}\colon Q\to Y$ the natural transformation given by $\{\bar{g}^{\tau_a}\}_{a=0,\dots,k}$ and $\{\hat{g}^{\tau_a}\}_{a=0,\dots,k}$ (see Section 2.1). Note that $g=\hat{g}\bar{g}$.

The isomorphism type of the functor Q and the factorisation $g=\hat{g}\bar{g}$ do not depend on the choice of the sequence that discretises X and Y. If $\bar{f}\colon X\to P$ and $\hat{f}\colon P\to Y$ are natural transformations constructed with respect to another such a sequence, then there is a unique isomorphism $\phi\colon Q\to P$ for which the following diagram commutes:



Theorem 2.2. *Let $\mathcal{M}$ be a model category. The following choices of weak equivalences, fibrations and cofibrations form a model structure on* $\text{tame}\big([0,\infty),\mathcal{M}\big)$. *A morphism $g\colon X\to Y$ in* $\text{tame}\big([0,\infty),\mathcal{M}\big)$ *is a*

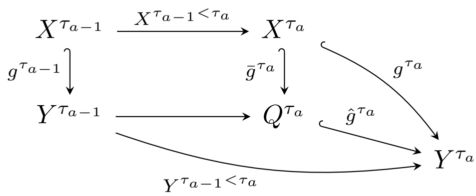
- *weak equivalence if $g^t\colon X^t\to Y^t$ is a weak equivalence for all t,*
- *fibration if $g^t\colon X^t\to Y^t$ is a fibration for all t,*
- *cofibration if $\hat{g}^t\colon Q^t\to Y^t$ (see Section 2.2) is a cofibration for all t.*

Due to tameness, to prove $g\colon X\to Y$ in $\text{tame}\big([0,\infty),\mathcal{M}\big)$ is a weak equivalence, or a fibration, or a cofibration, only finitely many verifications need to be performed. If $0=\tau_0<\cdots<\tau_k$ discretises both X and Y, then g is a weak equivalence (respectively, a fibration) if and only if $g^{\tau_a}: X^{\tau_a}\to Y^{\tau_a}$ is a weak equivalence (respectively, a fibration) in $\mathcal{M}$ for any $a = 0, \dots, k$. Similarly, g is a cofibration if and only if $\hat{g}^{\tau_a} : Q^{\tau_a} \to Y^{\tau_a}$ is a cofibration in $\mathcal{M}$ for any $a = 0, \dots, k$. It is important to realise however that for g to be a cofibration is it not enough for g^t to be a cofibration for all t.

Proposition 2.3. *Let $\mathcal{M}$ be a model category.*

1. *If $g\colon X \to Y$ is a cofibration in* $\operatorname{tame}\big([0,\infty),\mathcal{M}\big)$*, then $g^t\colon X^t \to Y^t$ is a cofibration in $\mathcal{M}$ for any t in $[0,\infty)$.*
2. *An object X in* $\operatorname{tame}\big([0,\infty),\mathcal{M}\big)$ *is cofibrant if and only if X^0 is cofibrant and, for any $s < t$ in $[0,\infty)$, the transition morphism $X^{s<t}\colon X^s \to X^t$ is a cofibration in $\mathcal{M}$.*

Proof. 1. Assume $g\colon X \hookrightarrow Y$ is a cofibration. Let $0 = \tau_0 < \cdots < \tau_k$ be a sequence discretising both X and Y. By definition $g^0 = g^{\tau_0} = \hat{g}^{\tau_0}$ is a cofibration. Since in a model category cofibrations are preserved by compositions and taking pushouts along any morphism, the indicated arrows in the following commutative diagram are cofibrations for any $a = 1, \dots, k$:



Thus, for any a in $[k]$, the morphism $g^{\tau_a} : X^{\tau_a} \to Y^{\tau_a}$ is a cofibration. Tameness can be then used to conclude that g^t is a cofibration for any t in $[0,\infty)$.

2. Let $\emptyset\colon [0,\infty) \to \mathcal{M}$ be the initial object in $\operatorname{tame}\big([0,\infty),\mathcal{M}\big)$, which is the constant functor whose value is the initial object in $\mathcal{M}$. Consider a morphism $g\colon \emptyset \to X$ in $\operatorname{tame}\big([0,\infty),\mathcal{M}\big)$. Let $0 = \tau_0 < \cdots < \tau_k$ be a sequence discretising X. Then $Q^0 = \emptyset$ and $Q^{\tau_a} = X^{\tau_{a-1}}$ for $a > 0$. Furthermore, $\hat{g}^0 = (\emptyset \to X^0)$ and $\hat{g}^{\tau_a} : Q^{\tau_a} = X^{\tau_{a-1}} \to X^{\tau_a}$ is the transition morphism in X for $a > 0$. The statement is then a direct consequence of the definition of a cofibration in $\operatorname{tame}\big([0,\infty),\mathcal{M}\big)$. □

Proof of Theorem 2.2. MC1: This is a consequence of the fact that there is a sequence that discretises all elements in a finite collection of tame functors.

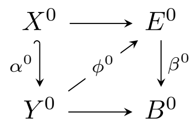
MC2 and MC3: These follows from the fact that $\mathcal{M}$ satisfies these axioms, and from the functoriality of the mediating morphism $\hat{g}$.

MC4: Consider a commutative square in $\operatorname{tame}\big([0,\infty),\mathcal{M}\big)$:

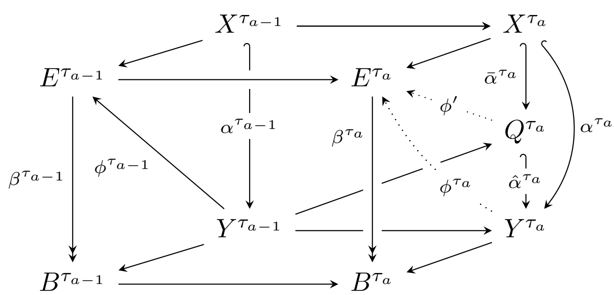
$$\begin{array}{ccc} X & \longrightarrow & E \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ Y & \longrightarrow & B \end{array}$$

where either α is a cofibration and β is a fibration and a weak equivalence, or α is a cofibration and a weak equivalence and β is a fibration. We need to show that there is a morphism $\phi\colon Y \to E$ which if added to the above square would make the obtained diagram commutative. Let us choose a sequence $0 = \tau_0 < \cdots < \tau_k$ that discretises all functors in this square. We are going to define by induction on a in $[k]$ morphisms $\phi^{\tau_a} : Y^{\tau_a} \to E^{\tau_a}$. We then use this sequence to get the desired $\phi\colon Y \to E$.

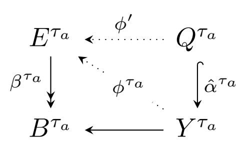
Set $\phi^0\colon Y^0 \to E^0$ to be any morphism in $\mathcal{M}$ that makes the following square commutative. It exists by the axiom MC4 in $\mathcal{M}$.



Assume $a \geqslant 1$ and that we have defined $\phi^{\tau_b}\colon Y^{\tau_b} \to E^{\tau_b}$ for $b < a$. We can then form the following commutative diagram, where the indicated arrows are cofibrations by Proposition 2.3.1:



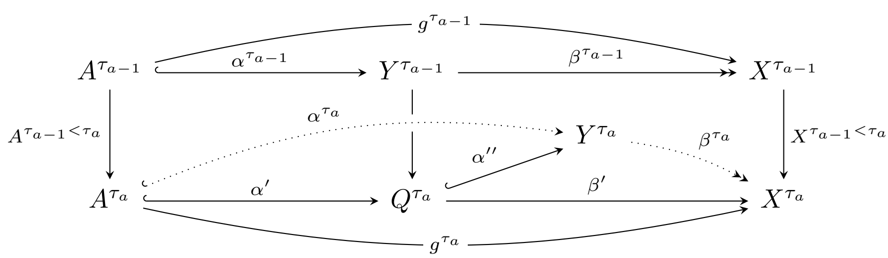
All the horizontal arrows represent the transition morphisms, $\phi'\colon Q^{\tau_a} \to E^{\tau_a}$ is induced by the universal property of a pushout, and $\phi^{\tau_a}\colon Y^{\tau_a} \to E^{\tau_a}$ is any morphism that makes the following diagram commute, whose existence is guaranteed by axiom MC4:



MC5: Consider a morphism $g\colon A \to X$ in $\mathrm{tame}\big([0,\infty),\mathcal{M}\big)$. Let us choose a sequence $0 = \tau_0 < \cdots < \tau_k$ that discretises both A and X. By induction on $a = 0,\ldots,k$, we are going to construct the appropriate factorisations $g^{\tau_a} = \beta^{\tau_a}\alpha^{\tau_a}$. Set $\alpha^0\colon A^0 \hookrightarrow Y^0$ and $\beta^0\colon Y^0 \twoheadrightarrow X^0$ to be the factorisation of $g^0\colon A \to X$, where one of α^0, β^0 is also a weak equivalence. Such a factorisation exists by MC5 in $\mathcal{M}$. Assume $a \geqslant 1$ and that we have defined $\alpha^{\tau_b}\colon A^{\tau_b} \hookrightarrow Y^{\tau_b}$ and $\beta^{\tau_b}\colon Y^{\tau_b} \twoheadrightarrow X^{\tau_b}$ for $b < a$. We can then define:

$$Q^{\tau_a} \coloneqq \mathrm{colim}\Big(\ A^{\tau_a} \xleftarrow{\ A^{\tau_{a-1}<\tau_a}\ } A^{\tau_{a-1}} \xhookrightarrow{\ \alpha^{\tau_{a-1}}\ } Y^{\tau_{a-1}}\ \Big)$$

and form the following commutative diagram:



where the left square is pushout and $\beta'\colon Q^{\tau_a} \to X^{\tau_a}$ is induced by the universal property of the pushout. The morphisms $\alpha''\colon Q^{\tau_a} \to Y^{\tau_a}$ and $\beta^{\tau_a}: Y^{\tau_a} \to X^{\tau_a}$ form the appropriate factorisation of β' into the composition of either a cofibration which is a weak equivalence and a fibration, or a cofibration and a fibration which is a weak equivalence. Set $\alpha^{\tau_a}: A^{\tau_a} \to Y^{\tau_a}$ to be the composition $\alpha''\alpha'$, and $Y^{\tau_{a-1}<\tau_a}: Y^{\tau_{a-1}} \to Y^{\tau_a}$ to be the composition of $Y^{\tau_{a-1}} \to Q^{\tau_a}$ and $\alpha''\colon Q^{\tau_a} \to Y^{\tau_a}$. Define Y to be the Kan extension along $0 = \tau_0 < \cdots < \tau_k$ of the sequence $\{Y^{\tau_{a-1}<\tau_a}\}_{a=1,\ldots,k}$ (see Section 2.1). Let $\alpha\colon A \to Y$ and $\beta\colon Y \to X$ be the natural transformations induced by the sequences of morphisms $\{\alpha^{\tau_a}: A^{\tau_a} \to Y^{\tau_a}\}_{a=0,\ldots,k}$ and $\{\beta^{\tau_a}: Y^{\tau_a} \to X^{\tau_a}\}_{a=0\ldots,k}$. By construction, α is a cofibration and β is a fibration. Furthermore, depending on the choice of the factorisations of $\beta'\colon Q^{\tau_a} \to X^{\tau_a}$, either α or β is a weak equivalence. □

2.3. Minimal factorisations

Assume $\mathcal{M}$ satisfies the minimality axiom. Consider a morphism $g\colon A \to X$ in $\mathrm{tame}\big([0,\infty),\mathcal{M}\big)$. Perform the same constructions as in the proof of MC5 but instead of taking arbitrary factorisations consider the minimal ones. In step zero, we take morphisms $\alpha^0\colon X^0 \hookrightarrow Y^0$ and $\beta^0\colon Y^0 \xrightarrow{\sim}\!\!\!\!\rightarrow X^0$ that form a minimal factorisation of $g^0\colon A \to X$. Analogously, in the a-th step we take morphisms $\alpha''\colon Q^{\tau_a} \hookrightarrow Y^{\tau_a}$ and $\beta^{\tau_a}\colon Y^{\tau_a} \xrightarrow{\sim}\!\!\!\!\rightarrow X^{\tau_a}$ which form a minimal factorisation of $\beta'\colon Q^{\tau_a} \to X^{\tau_a}$. We claim that the obtained morphisms $\alpha\colon A \hookrightarrow Y$ and $\beta\colon Y \xrightarrow{\sim}\!\!\!\!\rightarrow X$ form a minimal factorisation of $g\colon A \to X$. We just proved:

Theorem 2.4. *If the model category $\mathcal{M}$ satisfies the minimal factorisation axiom, then so does* $\mathrm{tame}\big([0,\infty),\mathcal{M}\big)$.

Corollary 2.5. *If the model category $\mathcal{M}$ satisfies the minimal factorisation axiom, then so does* $\mathrm{tame}([0,\infty)^k,\mathcal{M})$ *for any* $k = 1, 2, \ldots$.

3. Chain complexes of vector spaces

Let K be a field and $\mathbf{N} = \{0, 1, \ldots\}$ the set of natural numbers. A (non-negatively graded) chain complex of K-vector spaces is a sequence of linear functions $X = \{\delta_n\colon X_{n+1} \to X_n\}_{n\in\mathbf{N}}$ of K-vector spaces, called **differentials**, such that $\delta_n\delta_{n+1} = 0$ for all n in $\mathbf{N}$. In the notation of the differentials we often ignore their indexes and simply denote them by δ, or δ_X to indicate which chain complex is considered.

A chain complex X is called **compact** if $\bigoplus_{n\in\mathbb{N}} X_n$ is finite dimensional [1]. This happens if and only if X_n is finite dimensional for all n and X_n is trivial for $n \gg 0$.

3.1. Homology

The following vector spaces are called respectively the n-th cycles and the n-th boundaries of X:

$$Z_nX \coloneqq \begin{cases} X_0 & \text{if } n = 0,\\ \ker(\delta_{n-1}\colon X_n \to X_{n-1}) & \text{if } n \geqslant 1,\end{cases} \qquad B_nX \coloneqq \mathrm{im}(\delta_n\colon X_{n+1} \to X_n).$$

Since $\delta_n\delta_{n+1} = 0$, the n-th boundaries B_nX are vector subspaces of the n-th cycles Z_nX. The quotient Z_nX/B_nX is called the n-th **homology** of X and is denoted by H_nX. We write ZX, BX and HX to denote the non-negatively graded vector spaces $\{Z_nX\}_{n\in\mathbf{N}}$, $\{B_nX\}_{n\in\mathbf{N}}$, and $\{H_nX\}_{n\in\mathbf{N}}$ (see Section 3.7).

3.2. Model structure

A morphism of chain complexes $g\colon X \to Y$ is a sequence of linear functions $\{g_n\colon X_n \to Y_n\}_{n\in\mathbf{N}}$ such that $g_n\delta_X = \delta_Y g_{n+1}$ for all n. Such a morphism maps boundaries and cycles in X to boundaries and cycles in Y. The induced map on homologies is denoted by $Hg\colon HX \to HY$. If $Hg\colon HX \to HY$ is an isomorphism, then g is a weak equivalence. If $g_n\colon X_n \to Y_n$ is an epimorphism for all $n \geqslant 1$ (no assumption is made for $n = 0$), then g is a fibration. If $g_n\colon X_n \to Y_n$ is a monomorphism for all $n \geqslant 0$, then g is a cofibration. This choice of weak equivalences, fibrations and cofibrations defines a model structure on the category of chain complexes, denoted by Ch (see [**10**, **21**]). Consider the full subcategory of Ch given by compact chain complexes. The same choices of weak equivalences, fibrations, and cofibrations, as for Ch, define a model structure on such a subcategory, denoted by ch.

3.3. Suspension

The **suspension** of a chain complex X, denoted by SX is a chain complex such that:

$$\delta_n\colon (SX)_{n+1} \to (SX)_n = \begin{cases} X_0 \to 0 & \text{if } n = 0,\\ -\delta_{n-1}\colon X_n \to X_{n-1} & \text{if } n > 0.\end{cases}$$

Analogously, the suspension of a morphism $g\colon X \to Y$ of chain complex is a morphism $Sg\colon SX \to SX$ such that:

$$(Sg)_n\colon (SX)_n \to (SY)_n = \begin{cases} 0\colon 0 \to 0 & \text{if } n = 0,\\ g_{n-1}\colon X_{n-1} \to Y_{n-1} & \text{if } n > 0.\end{cases}$$

The assignment $g \mapsto Sg$ is a functor denoted by $S\colon \mathrm{Ch} \to \mathrm{Ch}$.

Note that $H_0SX = 0$ and H_nSX is isomorphic to $H_{n-1}X$ for all $n > 0$. Furthermore, if f is a cofibration or a weak equivalence, then so is Sf, and if f is a fibration, then Sf is a fibration if and only if f_0 is an epimorphism.

The **desuspension** of a chain complex X, denoted by $S^{-1}X$, is a chain complex such that:

$$\delta_n\colon (S^{-1}X)_{n+1} \to (S^{-1}X)_n = \begin{cases} -\delta_1\colon X_2 \to Z_1X & \text{if } n = 0,\\ -\delta_{n+1}\colon X_{n+2} \to X_{n+1} & \text{if } n > 0.\end{cases}$$

Analogously, the desuspension of a morphism $g\colon X \to Y$ of chain complex is a morphism $S^{-1}g\colon S^{-1}X \to S^{-1}X$ such that:

$$(S^{-1}g)_n\colon (S^{-1}X)_n \to (S^{-1}Y)_n = \begin{cases} g_1\colon Z_1X \to Z_1Y & \text{if } n = 0,\\ g_{n+1}\colon X_{n+1} \to Y_{n+1} & \text{if } n > 0.\end{cases}$$

The assignment $g \mapsto S^{-1}g$ is a functor denoted by $S^{-1}\colon \mathrm{Ch} \to \mathrm{Ch}$.

Note that $H_nS^{-1}X$ is isomorphic to $H_{n+1}X$. If f is a fibration, cofibration or a weak equivalence, then so is $S^{-1}f$. Furthermore, $S^{-1}SX$ is isomorphic to X, and $SS^{-1}X$ is isomorphic to X if and only if $X_0 = 0$.

We now provide some explicit constructions of chain complexes used essentially in Section 3.8 to compute the standard decomposition and the minimal representative in ch.

3.4. Cofiber sequences

Let $f\colon X \to Y$ be a morphism of chain complexes. Define a chain complex Cf, called the **cofiber** of f, a cofibration $i\colon Y \hookrightarrow Cf$, and a fibration $p\colon Cf \twoheadrightarrow SX$, as follows:

$$
\begin{array}{ccccccccccc}
Y & = & \cdots & \xrightarrow{\delta_Y} & Y_3 & \xrightarrow{\delta_Y} & Y_2 & \xrightarrow{\delta_Y} & Y_1 & \xrightarrow{\delta_Y} & Y_0 \\
\big\downarrow{\scriptstyle i} & & & & \big\downarrow{\scriptstyle \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} & & \big\downarrow{\scriptstyle \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} & & \big\downarrow{\scriptstyle \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} & & \big\downarrow{\scriptstyle 1} \\
Cf & = & \cdots & \xrightarrow{\left[\begin{smallmatrix}\delta_Y & f\\ 0 & -\delta_X\end{smallmatrix}\right]} & Y_3 \oplus X_2 & \xrightarrow{\left[\begin{smallmatrix}\delta_Y & f\\ 0 & -\delta_X\end{smallmatrix}\right]} & Y_2 \oplus X_1 & \xrightarrow{\left[\begin{smallmatrix}\delta_Y & f\\ 0 & -\delta_X\end{smallmatrix}\right]} & Y_1 \oplus X_0 & \xrightarrow{\left[\begin{smallmatrix}\delta_Y & f\end{smallmatrix}\right]} & Y_0 \\
\big\downarrow{\scriptstyle p} & & & & \big\downarrow{\scriptstyle [0\ 1]} & & \big\downarrow{\scriptstyle [0\ 1]} & & \big\downarrow{\scriptstyle [0\ 1]} & & \big\downarrow{\scriptstyle 0} \\
SX & = & \cdots & \xrightarrow{-\delta_X} & X_2 & \xrightarrow{-\delta_X} & X_1 & \xrightarrow{-\delta_X} & X_0 & \longrightarrow & 0
\end{array}
$$

The cofibration i and the fibration p form an exact sequence, called the **cofiber sequence** of f:

$$0 \longrightarrow Y \xhookrightarrow{i} Cf \xrightarrow{p} SX \longrightarrow 0$$

Consider two maps of chain complexes $f\colon X \to Y$ and $g\colon W \to Z$. Each of them leads to a cofiber sequence. A natural transformation between these exact sequences is by definition a triple of morphisms of chain complexes $S\alpha\colon SX \to SW$, $\beta\colon Y \to Z$ and $\gamma\colon Cf \to Cg$ which make the following diagram commute:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & Y & \xhookrightarrow{i} & Cf & \xrightarrow{p} & SX & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle \beta} & & \big\downarrow{\scriptstyle \gamma} & & \big\downarrow{\scriptstyle S\alpha} & & \\
0 & \longrightarrow & Z & \xhookrightarrow{i} & Cg & \xrightarrow{p} & SW & \longrightarrow & 0
\end{array}
$$

Commutativity of this diagram has two consequences. First, γ is of the form:

$$\gamma_n = \begin{cases} \beta_0\colon Y_0 \to Z_0 & \text{if } n = 0, \\ \left[\begin{smallmatrix}\beta_n & h_{n-1}\\ 0 & \alpha_{n-1}\end{smallmatrix}\right]\colon Y_n \oplus X_{n-1} \to Z_n \oplus W_{n-1} & \text{if } n > 0. \end{cases}$$

Second, the sequence of linear functions $h = \{h_n\colon X_n \to Z_{n+1}\}_{n\geqslant 0}$ satisfies the equation $\beta f - g\alpha = \delta_Z h + h\delta_X$, which means that h is a homotopy between βf and $g\alpha$. It follows that the set of natural transformations between the two cofiber sequences is in bijection with the set of triples consisting of morphisms $\alpha\colon X \to W$ and $\beta\colon Y \to Z$, and a homotopy h between βf and $g\alpha$. We illustrate such a triple in form of a diagram:

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
{\scriptstyle \alpha}\big\downarrow & \overset{h}{\searrow\!\!\searrow} & \big\downarrow{\scriptstyle \beta} \\
W & \xrightarrow{g} & Z
\end{array}
$$

The symbol $C(\alpha, \beta, h)\colon Cf \to Cg$ denotes the morphism $\gamma\colon Cf \to Cg$, corresponding to this triple (α, β, h).

In the case $h = 0$, such a diagram corresponds to a commutative square:

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
{\scriptstyle \alpha}\big\downarrow & \overset{0}{\searrow\!\!\searrow} & \big\downarrow{\scriptstyle \beta} \\
W & \xrightarrow{g} & Z
\end{array}
\quad = \quad
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
{\scriptstyle \alpha}\big\downarrow & & \big\downarrow{\scriptstyle \beta} \\
W & \xrightarrow{g} & Z
\end{array}
$$

In this case, the corresponding morphism between the cofibers is denoted simply by $C(\alpha,\beta)\colon Cf \to Cg$.

In the case the differentials δ_Z and δ_X are trivial (in all degrees), the following implication holds (homotopy commutative square is commutative):

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle\alpha}\downarrow & \overset{h}{\searrow\!\!\searrow} & \downarrow{\scriptstyle\beta} \\ W & \xrightarrow{g} & Z \end{array} \quad \text{implies} \quad \begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle\alpha}\downarrow & \overset{0}{\searrow\!\!\searrow} & \downarrow{\scriptstyle\beta} \\ W & \xrightarrow{g} & Z \end{array}$$

3.5. Comparison morphism

Let $f\colon X \to Y$ be a morphism of chain complexes. Consider the quotient morphism q: $Y \to Y/f(X)$ and define the **comparison morphism** $Cf \to Y/f(X)$ to be:

$$\begin{array}{ccccccccc} Cf & = & \cdots & \xrightarrow{\left[\begin{smallmatrix}\delta_Y & f\\ 0 & -\delta_X\end{smallmatrix}\right]} & Y_2 \oplus X_1 & \xrightarrow{\left[\begin{smallmatrix}\delta_Y & f\\ 0 & -\delta_X\end{smallmatrix}\right]} & Y_1 \oplus X_0 & \xrightarrow{[\,\delta_Y\ f\,]} & Y_0 \\ \downarrow & & & & \downarrow{\scriptstyle[\,q\ 0\,]} & & \downarrow{\scriptstyle[\,q\ 0\,]} & & \downarrow{\scriptstyle q} \\ Y/f(X) & = & \cdots & \xrightarrow{\delta} & (Y/f(X))_2 & \xrightarrow{\delta} & (Y/f(X))_1 & \xrightarrow{\delta} & (Y/f(X))_0 \end{array}$$

If f is a monomorphism, then the comparison morphism $Cf \twoheadrightarrow Y/f(X)$ is a weak equivalence.

3.6. Factorisation

The complex $C1_X$ is also denoted by CX and called the **cone** on X. Explicitly, the cofibration $i\colon X \hookrightarrow CX$ is given by:

$$\begin{array}{ccccccccccc} X & = & \cdots & \xrightarrow{\delta_X} & X_3 & \xrightarrow{\delta_X} & X_2 & \xrightarrow{\delta_X} & X_1 & \xrightarrow{\delta_X} & X_0 \\ \downarrow{\scriptstyle i} & & & & \downarrow{\scriptstyle\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} & & \downarrow{\scriptstyle\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} & & \downarrow{\scriptstyle\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} & & \downarrow{\scriptstyle 1} \\ CX & = & \cdots & \xrightarrow{\left[\begin{smallmatrix}\delta_X & 1\\ 0 & -\delta_X\end{smallmatrix}\right]} & X_3 \oplus X_2 & \xrightarrow{\left[\begin{smallmatrix}\delta_X & 1\\ 0 & -\delta_X\end{smallmatrix}\right]} & X_2 \oplus X_1 & \xrightarrow{\left[\begin{smallmatrix}\delta_X & 1\\ 0 & -\delta_X\end{smallmatrix}\right]} & X_1 \oplus X_0 & \xrightarrow{[\,\delta_X\ 1\,]} & X_0 \end{array}$$

Note that $HCX = 0$.

The complex $S^{-1}CX$ is also denoted by PX and called the **path complex** on X. We also use the symbol $p\colon PX \to X$ to denote the fibration given by the desuspension $S^{-1}p\colon S^{-1}CX \to S^{-1}SX = X$. Explicitly:

$$\begin{array}{ccccccccccc} PX & = & \cdots & \xrightarrow{\left[\begin{smallmatrix}-\delta_X & 1\\ 0 & \delta_X\end{smallmatrix}\right]} & X_4 \oplus X_3 & \xrightarrow{\left[\begin{smallmatrix}-\delta_X & 1\\ 0 & \delta_X\end{smallmatrix}\right]} & X_3 \oplus X_2 & \xrightarrow{\left[\begin{smallmatrix}-\delta_X & 1\\ 0 & \delta_X\end{smallmatrix}\right]} & X_2 \oplus X_1 & \xrightarrow{[\,-\delta_X\ 1\,]} & X_1 \\ \downarrow{\scriptstyle p} & & & & \downarrow{\scriptstyle[\,0\ 1\,]} & & \downarrow{\scriptstyle[\,0\ 1\,]} & & \downarrow{\scriptstyle[\,0\ 1\,]} & & \downarrow{\scriptstyle\delta_X} \\ X & = & \cdots & \xrightarrow{\delta_X} & X_3 & \xrightarrow{\delta_X} & X_2 & \xrightarrow{\delta_X} & X_1 & \xrightarrow{\delta_X} & X_0 \end{array}$$

Note that $HPX = 0$.

Since $HPX = 0 = HCX$, the fibration $p\colon PX \twoheadrightarrow X$ and the cofibration $i\colon X \hookrightarrow CX$ fit into the following factorisations of the morphisms $0 \to X \to 0$:

$$\begin{array}{ccccccccc} & & PX & & & & CX & & \\ & \overset{\sim}{\nearrow} & & \overset{p}{\searrow} & & \overset{i}{\nearrow} & & \overset{\sim}{\searrow} & \\ 0 & & \longrightarrow & & X & & \longrightarrow & & 0 \end{array}$$

These morphisms i and p can be used to construct explicit factorisations of arbitrary morphisms in Ch, whose existence is guaranteed by axiom MC5: any $g\colon X \to Y$ fits into a commutative diagram:

$$\begin{array}{ccccc} & \xhookrightarrow[\sim]{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} & X\oplus PY & \xrightarrow{[g\ p]} & \\ X & & \xrightarrow{\quad g \quad} & & Y \\ & \xhookrightarrow{\left[\begin{smallmatrix}i\\g\end{smallmatrix}\right]} & CX\oplus Y & \xrightarrow[\sim]{[0\ 1]} & \end{array}$$

These factorisations are natural, however in general not minimal (see Definition 1.1). To obtain minimal factorisations we cannot perform natural constructions and we will be forced to make some choices.

3.7. Graded vector spaces

A (non-negatively) graded K-vector space is by definition a sequence of K-vector spaces $V = \{V_n\}_{n\in\mathbf{N}}$. Such a graded vector space is concentrated in degree k if $V_n = 0$ for all $n \neq k$. Graded vector spaces concentrated in degree 0 are identified with vector spaces.

Let $V = \{V_n\}_{n\in\mathbf{N}}$ be a graded K-vector space. The same symbol V is also used to denote the chain complex $\{0\colon V_{n+1} \to V_n\}_{n\in\mathbf{N}}$ with the trivial differentials. In this case, $HV = V$ and hence any weak equivalence $\phi\colon V \to V$ is an isomorphism. In fact, an arbitrary chain complex X is minimal (see Definition 1.3) if and only if all its differentials are trivial. More generally, any cofibration $\alpha\colon X \hookrightarrow Y$ for which the chain complex $Y/\alpha(X)$ has all trivial differentials satisfies the following minimality condition: *any weak equivalence $\phi\colon Y \to Y$ for which $\alpha\phi = \alpha$ is an isomorphism.* To see this consider a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xhookrightarrow{\alpha} & Y & \twoheadrightarrow & Y/\alpha(X) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \phi} & & \downarrow & & \\ 0 & \longrightarrow & X & \xhookrightarrow{\alpha} & Y & \twoheadrightarrow & Y/\alpha(X) & \longrightarrow & 0 \end{array}$$

Using the long sequences of homologies for each row, we can conclude the morphism $Y/\alpha(X) \to Y/\alpha(X)$ is a weak equivalence and hence an isomorphism as $Y/\alpha(X)$ is assumed to have all differentials trivial. We can then use the exactness of the rows to get that ϕ is also an isomorphism.

To denote the n-fold suspension of K we use the symbol S^n. Explicitly, S^n is the chain complex concentrated in degree n such that $(S^n)_n = K$. For example, $S^0 = K$. The complex S^n is called the n-th **sphere**. The cone CS^n is denoted by D^{n+1} and called the $(n+1)$-st **disk**. Explicitly:

$$(D^{n+1})_k = \begin{cases} K & \text{if } k = n \text{ or } k = n+1 \\ 0 & \text{otherwise} \end{cases}, \quad \delta_k = \begin{cases} 1 & \text{if } k = n \\ 0 & \text{otherwise} \end{cases}.$$

3.8. Standard decomposition and minimal representative

Let X be a chain complex. Consider the morphisms $p\colon CBX \twoheadrightarrow SBX \leftarrow X : \delta_X$ (see Section 3.4). Axiom MC4 guarantees existence of a morphism $\phi\colon CBX \to X$

making the following diagram commutative:

$$\begin{array}{ccc} 0 & \longrightarrow & X \\ \sim\big\downarrow & \overset{\phi}{\nearrow} & \big\downarrow\delta \\ CBX & \xrightarrow{\;p\;} & SBX \end{array}$$

The restriction of any such ϕ to $i\colon BX \hookrightarrow CBX$ is the standard inclusion $BX \hookrightarrow ZX \hookrightarrow X$. This can be seen by looking at the long exact sequences of homologies applied to the rows in the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & BX & \xhookrightarrow{\;i\;} & CBX & \xrightarrow{\;p\;} & SBX & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow\phi & & \big\downarrow 1 & & \\ 0 & \longrightarrow & ZX & \hookrightarrow & X & \xrightarrow{\;\delta\;} & SBX & \longrightarrow & 0 \end{array}$$

The morphism ϕ leads therefore to a pushout square (in particular ϕ is a cofibration):

$$\begin{array}{ccc} BX & \xhookrightarrow{\;i\;} & CBX \\ \big\downarrow & & \big\downarrow\phi \\ ZX & \hookrightarrow & X \end{array}$$

Since considered coefficients are in a field and all the differentials in BX, ZX and HX are trivial, there is a morphism $s\colon HX \to ZX$, whose composition with the quotient $ZX \twoheadrightarrow HX$ is 1_{HX}. For any such s, the morphism $\begin{bmatrix} i & s \end{bmatrix} : BX \oplus HX \to ZX$ is an isomorphism. It follows that so is the morphism $\begin{bmatrix} \phi & s \end{bmatrix} : CBX \oplus HX \to X$, where the symbol s also denotes the composition of $s\colon HX \to ZX$ and the inclusion $ZX \hookrightarrow X$. We call $CBX \oplus HX$ the **standard decomposition** of the chain complex X. Since CBX has trivial homology, the morphism $s\colon HX \to X$ is a weak equivalence and hence HX is the minimal representative (see Definition 1.3) of X.

3.9. Minimal factorisations

Let $g\colon X \to Y$ be a morphism of chain complexes. To construct its minimal factorisation (see Definition 1.1) we perform the following steps:

1. Take the kernel $j\colon W \hookrightarrow X$ of $g\colon X \to Y$;
2. Choose an isomorphism $W \xrightarrow{\simeq} CBW \oplus HW$ (see Section 3.8);
3. Consider the composition:

$$W \xrightarrow{\;\simeq\;} CBW \oplus HW \xhookrightarrow{\left[\begin{smallmatrix} 1 & 0 \\ 0 & i \end{smallmatrix}\right]} CBW \oplus CHW, \qquad \alpha\colon W \hookrightarrow CBW \oplus CHW$$

4. Use axiom MC4 to construct a morphism $\phi\colon X \to CBW \oplus CHW$, which fits into the following commutative diagram:

$$\begin{array}{ccc} W & \xhookrightarrow{\;\alpha\;} & CBW \oplus CHW \\ j\big\downarrow & \overset{\phi}{\nearrow} & \big\downarrow\sim \\ X & \longrightarrow & 0 \end{array}$$

5. The morphism $\left[\begin{smallmatrix} \phi \\ g \end{smallmatrix}\right] : X \to (CBW \oplus CHW) \oplus Y$ is then a cofibration.

We are now ready to state:

Proposition 3.1. *The following factorisation is minimal:*

$$\begin{array}{ccccc} & & (CBW \oplus CHW) \oplus Y & & \\ & \overset{\left[\begin{smallmatrix}\phi\\ g\end{smallmatrix}\right]}{\nearrow} & & \underset{\sim}{\overset{[\,0\ 1\,]}{\searrow}} & \\ X & & \xrightarrow{\qquad\qquad g \qquad\qquad} & & Y \end{array}$$

Proof. Let $\psi = \left[\begin{smallmatrix}\psi_{11} & \psi_{12}\\ \psi_{21} & \psi_{22}\end{smallmatrix}\right] : (CBW \oplus CHW) \oplus Y \to (CBW \oplus CHW) \oplus Y$ be a morphism making the following diagram commutative:

$$\begin{array}{ccc} X & \xhookrightarrow{\left[\begin{smallmatrix}\phi\\ g\end{smallmatrix}\right]} & (CBW \oplus CHW) \oplus Y \\ {\scriptstyle\left[\begin{smallmatrix}\phi\\ g\end{smallmatrix}\right]}\Big\downarrow & \overset{\psi}{\nearrow} & \sim\Big\downarrow{\scriptstyle[\,0\ 1\,]} \\ (CBW \oplus CHW) \oplus Y & \underset{\sim}{\xrightarrow{[\,0\ 1\,]}} & Y \end{array}$$

Commutativity of the bottom triangle implies $\psi_{21} = 0$ and $\psi_{22} = 1$. Since W is the kernel of g, commutativity of the top triangle implies commutativity of:

$$\begin{array}{ccc} W & \xhookrightarrow{\ \alpha\ } & CBW \oplus CHW \\ {\scriptstyle\alpha}\Big\downarrow & \overset{\psi_{11}}{\nearrow} & \\ CBW \oplus CHW & & \end{array}$$

The quotient $(CBW \oplus CHW)/\alpha(W) = SHW$ has all differentials trivial. The morphism $\psi_{11} : CBW \oplus CHW \to CBW \oplus CHW$ is therefore an isomorphism (see Section 3.7). It follows that so is $\psi = \left[\begin{smallmatrix}\psi_{11} & \psi_{12}\\ 0 & 1\end{smallmatrix}\right]$ □

We just proved that ch satisfies the minimal factorisation axiom. By Theorem 2.4, it follows that $\text{tame}\big([0,\infty), \text{ch}\big)$ also satisfies the minimal factorisation axiom, and thus any tame parametrised chain complex admits a minimal cover. In Section 4, we provide a characterisation of such minimal covers.

4. Cofibrations in $\text{tame}([0,\infty), \text{ch})$

In this section we discuss cofibrations in the model category $\text{tame}\big([0,\infty), \text{ch}\big)$ as described in Theorem 2.2. According to Proposition 2.3, an object X in $\text{tame}\big([0,\infty), \text{ch}\big)$ is cofibrant if and only if the transition $X^{s<t} : X^s \to X^t$ is a monomorphism (a cofibration in ch) for every $s < t$ in $[0,\infty)$. This implies that if Y is cofibrant, and $f : X \to Y$ is a monomorphism in $\text{tame}\big([0,\infty), \text{ch}\big)$, then X is also cofibrant.

For example, the following objects are cofibrant. They are parametrised by a natural number n and an element in $\Omega := \{(s,e) \in [0,\infty) \times [0,\infty] \mid s \leqslant e\}$, where $[0,\infty]$ is the poset of non-negative reals plus ∞.

Definition 4.1. The Kan extensions (see Section 2.1) given by the data described in the following table are called **interval spheres**:

Index	$n,\ s<e=\infty$	$n,\ s=e<\infty$	$n,\ s<e<\infty$
Name	$I^n\,[s,\infty)$	$I^n\,[s,s)$	$I^n\,[s,e)$
k	0	0	1
Functor $[k]\to\mathrm{ch}$	S^n	D^{n+1}	$i\colon S^n\hookrightarrow D^{n+1}$
Inclusion $[k]\subset[0,\infty)$	s	s	$s<e$

For example, $I^2\,[5,\infty):[0,\infty)\to\mathrm{ch}$ is a functor whose value at $t<5$ is 0, and at $5\leqslant t$ is S^2. Similarly, $I^2\,[5,5):[0,\infty)\to\mathrm{ch}$ has value 0 if $t<5$, and D^3 if $5\leqslant t$. The functor $I^2\,[5,7):[0,\infty)\to\mathrm{ch}$ has three values: 0 if $t<5$, S^2 if $5\leqslant t<7$, and D^3 if $7\leqslant t$. The transition morphisms in $I^2\,[5,7)$ are either the identities, or the inclusion $0\hookrightarrow S^2$ or the inclusion $i\colon S^2\hookrightarrow D^3$.

A morphism $I^n\,[s,e)\to 0$ is a weak equivalence if and only if $s=e$. Thus, interval spheres of type $I^n\,[s,s)$ are the only interval spheres for which the chain complex $I^n\,[s,s)^t$ has trivial homology for every parameter t in $[0,\infty)$.

The main result of this section is the structure theorem (compare with [4,19,23]):

Theorem 4.2.

(1) *Any cofibrant object in* $\mathrm{tame}\big([0,\infty),\mathrm{ch}\big)$ *is isomorphic to a direct sum* $\oplus_{i=1}^{l} I^{n_i}\,[s_i,e_i)$, *where l could possibly be 0.*

(2) *If* $\oplus_{i=1}^{l} I^{n_i}\,[s_i,e_i)\cong\oplus_{j=1}^{l'} I^{n'_j}\,\big[s'_j,e'_j\big)$, *then $l=l'$ and there is a permutation σ of the set $\{1,\dots,l\}$ such that* $n_i=n'_{\sigma(i)}$, $s_i=s'_{\sigma(i)}$, *and* $e_i=e'_{\sigma(i)}$ *for any i.*

To prove Theorem 4.2, we first need to characterise cofibrations in $\mathrm{tame}\big([0,\infty),\mathrm{ch}\big)$ and explain how to enumerate morphisms out of $I^n\,[s,e)$. We start with cofibrations:

Proposition 4.3. *For every morphism $g\colon X\to Y$ in* $\mathrm{tame}\big([0,\infty),\mathrm{ch}\big)$*, the following statements are equivalent:*

1. *g is a cofibration;*
2. *$g^t\colon X^t\to Y^t$ is a monomorphism for every t in $[0,\infty)$, and $Y/g(X)$ is cofibrant;*
3. *$g^t\colon X^t\to Y^t$ is a monomorphism for every t in $[0,\infty)$ and, for all $s<t$ in $[0,\infty)$, the following is a pullback square:*

$$\begin{array}{ccc} X^s & \xrightarrow{X^{s<t}} & X^t \\ {\scriptstyle g^s}\downarrow & & \downarrow{\scriptstyle g^t} \\ Y^s & \xrightarrow{Y^{s<t}} & Y^t \end{array}$$

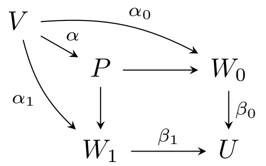
Proof. In the proof, we utilise the following fundamental linear algebra statement. Consider commutative diagrams of vector spaces:

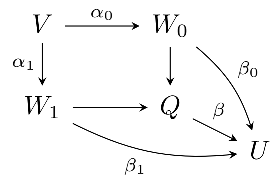
$$\begin{array}{ccc} V & \xrightarrow{\alpha_0} & W_0 \\ {\scriptstyle \alpha_1}\downarrow & & \downarrow{\scriptstyle \beta_0} \\ W_1 & \xrightarrow{\beta_1} & U \end{array}$$

It leads to two vector spaces:

$$P:=\lim\big(\ W_1\xrightarrow{\beta_1}U\xleftarrow{\beta_0}W_0\ \big),\qquad Q:=\operatorname{colim}\big(\ W_1\xleftarrow{\alpha_1}V\xrightarrow{\alpha_0}W_0\ \big)$$

and two linear functions $\alpha\colon V\to P$ and $\beta\colon Q\to U$ that make the following diagrams commutative, where the inside squares are respectively a pullback and a pushout:

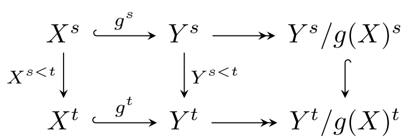




Then $\alpha\colon V \to P$ is surjective if and only if $\beta\colon Q \to U$ is injective.

1⇒2: The first part of *2* follows from Proposition 2.3.1, and the second from the fact that in a model category cofibrations are preserved by pushouts.

2⇒3: For all $s < t$ in $[0, \infty)$, we have the following commutative diagram, where the indicated arrows are cofibrations in ch:



The pullback $\lim(X^t \xhookrightarrow{g^t} Y^t \xleftarrow{Y^{s<t}} Y^s)$ is isomorphic to the kernel of the composition $Y^s \twoheadrightarrow Y^s/g(X)^s \hookrightarrow Y^t/g(X)^t$. Since the second map is an inclusion, this pullback coincides with the kernel of $Y^s \twoheadrightarrow Y^s/g(X)^s$. Consequently, the left square is a pullback.

3⇒1: Let $0 = \tau_0 < \cdots < \tau_k$ be a sequence that discretises both X and Y. According to Theorem 2.2, we need to show that $\hat{g}^{\tau_a}: P^{\tau_a} \to Y^{\tau_a}$ is a cofibration for all a. This is a consequence of the linear algebra statement given at the beginning of the proof. Indeed, the square is a pullback, its mediating morphism is the identity and thus $\hat{g}^{\tau_a}$ is a monomorphism, and so a cofibration in ch. □

4.1. Morphisms out of $I^n\,[s, \infty)$

A morphism $g\colon I^n\,[s,\infty) \to X$ leads to a linear function $g^s_n\colon I^n\,[s,\infty)^s_n = K \to X^s_n$. Let $x \coloneqq g^s_n(1)$ in X^s_n. This element satisfies the equation $\delta(x) = 0$, which means that x belongs to the cycles Z_nX^s. Choosing an element in Z_nX^s is all what is needed to describe a morphism out of $I^n\,[s,\infty)$. For any x in Z_nX^s, there is a unique morphism $I(x)\colon I^n\,[s,\infty) \to X$ such that $x = I(x)^s_n(1)$. The association $g \mapsto g^s_n(1)$ describes a bijection (in fact a linear isomorphism) between the set of morphisms $I^n\,[s,\infty) \to X$ and the set of cycles Z_nX^s.

4.2. Morphisms out of $I^n[s, e)$

Let $s \leqslant e < \infty$. A morphism $g\colon I^n\,[s,e) \to X$ leads to two functions $g^s_n\colon I^n\,[s,e)^s_n = K \to X^s_n$ and $g^e_{n+1}\colon I^n\,[s,e)^e_{n+1} = K \to X^e_{n+1}$. Define two elements $x \coloneqq g^s_n(1)$ in X^s_n and $y \coloneqq g^e_{n+1}(1)$ in X^e_{n+1}. These elements satisfy equations $\delta(x) = 0$ and $X^{s\leqslant e}_n(x) = \delta(y)$. These equations contain all the information needed to describe a morphism out of $I^n\,[s,e)$. If x in X^s_n and y in X^e_{n+1} satisfy these equations, then there is a unique morphism $I(x,y)\colon I^n\,[s,e) \to X$ such that $x = I(x,y)^s_n(1)$ and $y = I(x,y)^e_{n+1}(1)$. The association $g \mapsto \big(g^s_n(1), g^e_{n+1}(1)\big)$ describes therefore a bijection between the set of morphisms $I^n\,[s,e) \to X$ and the pullback:

$$\lim(Z_nX^s \xrightarrow{X_n^{s\leqslant e}} X_n^e \xleftarrow{\delta} X_{n+1}^e)$$

In the case $s = e$, this pullback can be identified with X_{n+1}^s. Thus, the set of morphisms $I^n\,[s,s) \to X$ is in bijection with X_{n+1}^s.

For $X = I^n\,[s,s)$, the elements 1 in $K = I^n\,[s,s)_n^s$ and 1 in $K = I^n\,[s,s)_{n+1}^e$ satisfy the required equations. The obtained morphism $I^n\,[s,e) \to I^n\,[s,s)$ is called the **standard inclusion**. Since not all of the transition morphisms of the quotient $I^n\,[s,s)\,/I^n\,[s,e)$ are monomorphisms, the standard inclusion is not a cofibration.

4.3. Cofibrations out of $I^n[s,e)$

In this paragraph we describe necessary and sufficient conditions for $g\colon I^n[s,e) \to X$ to be a cofibration. Since $I^n\,[s,e)$ is cofibrant, if there is such a cofibration, then X has to be cofibrant. Let us then make this assumption. The transition morphisms in X are therefore assumed to be cofibrations (see Proposition 2.3).

Choose $0 = \tau_0 < \cdots < \tau_k$ that discretises X and $I^n\,[s,e)$, and consider the diagrams:

$$\begin{array}{ccc} I^n\,[s,e)^{\tau_{a-1}} & \xhookrightarrow{I^n[s,e)^{\tau_{a-1}<\tau_a}} & I^n\,[s,e)^{\tau_a} \\ \downarrow{\scriptstyle g^{\tau_{a-1}}} & & \downarrow{\scriptstyle g^{\tau_a}} \\ X^{\tau_{a-1}} & \xhookrightarrow{X^{\tau_{a-1}<\tau_a}} & X^{\tau_a} \end{array}$$

By Proposition 4.3, since X is cofibrant, $g\colon I^n\,[s,e) \to X$ is a cofibration if and only if the diagram above is pullback for all $a = 1,\ldots,k$. These diagrams are pullbacks if in every homological degree h they are pullbacks of vector spaces. They can fail to be so only if the transition morphism $I^n\,[s,e)_h^{\tau_{a-1}<\tau_a}$ is not the identity, which happens in two cases: (i) $\tau_{a-1} < s = \tau_a$ and $h = n$, or (ii) $e < \infty$, $\tau_{a-1} < e = \tau_a$, and $h = n+1$. In both of these cases, the diagram above becomes:

$$\begin{array}{ccc} 0 & \longrightarrow & K \\ \downarrow & & \downarrow{\scriptstyle g_h^{\tau_a}} \\ X_h^{\tau_{a-1}} & \xhookrightarrow{X_h^{\tau_{a-1}<\tau_a}} & X_h^{\tau_a} \end{array}$$

and hence it is a pullback if and only if $g_h^{\tau_a}(1)$ is not in the image of $X_h^{\tau_{a-1}<\tau_a}$. We have just proven:

Proposition 4.4. *Let X be an object in* $\mathrm{tame}\big([0,\infty),\mathrm{ch}\big)$.

1. *Let n be a natural number, s be in $[0,\infty)$, and x be in Z_nX^s. Then the morphism $I(x)\colon I^n\,[s,\infty) \to X$ (see Section 4.1) is a cofibration if and only if X is cofibrant and x is not in the image of $X_n^{t<s}\colon X_n^t \to X_n^s$ for any $t < s$.*
2. *Let n be a natural number, $s \leqslant e$ be in $[0,\infty)$, and x in Z_nX^s and y in X_{n+1}^e be such that $X_n^{s\leqslant e}(x) = \delta(y)$. Then $I(x,y)\colon I^n\,[s,e) \to X$ (see Section 4.2) is a cofibration if and only if X is cofibrant, x is not in the image of $X_n^{t<s}\colon X_n^t \to X_n^s$ for any $t < s$, and y is not in the image of $X_{n+1}^{t<e}\colon X_{n+1}^t \to X_{n+1}^e$ for any $t < e$.*

We are now ready for:

Proof of Theorem 4.2. *(2)* This is a consequence the fact that $I^n\,[s,e)$ is indecomposable, for all n and $s \leqslant e$ (see [**2**, **3**]).

(1) Let X in $\mathrm{tame}\big([0,\infty),\mathrm{ch}\big)$ be cofibrant. Choose a sequence $0=\tau_0<\cdots<\tau_k$ discretising X. The morphism $X^{\tau_{a-1}<\tau_a}\colon X^{\tau_{a-1}} \to X^{\tau_a}$ is then a cofibration in **ch** for every $a=1,\ldots,k$.

Assume first all the differentials in X^t are trivial for all t. In this case, X is isomorphic to $\bigoplus_{n\geqslant 0} X_n$. Let $l_n^0 \coloneqq \dim X_n^0$ and $l_n^{\tau_a} \coloneqq \dim\operatorname{coker}(X_n^{\tau_{a-1}<\tau_a}\colon X_n^{\tau_{a-1}} \to X_n^{\tau_a})$ for $a=1,\ldots,k$. Then X_n is isomorphic to $\bigoplus_{a=0}^{k}\bigoplus_{j=1}^{l_n^{\tau_a}} I^n\,[\tau_a,\infty)$ and consequently X is isomorphic to:

$$\bigoplus_{n\geqslant 0}\bigoplus_{a=0}^{k}\bigoplus_{j=1}^{l_n^{\tau_a}} I^n\,[\tau_a,\infty)\,.$$

Assume now there is a non-trivial differential in X and set:

(a) n to be the smallest natural number for which $\delta\colon X_{n+1}^t \to X_n^t$ is non trivial for some t. This assumption implies $X_n^t = Z_nX^t$ for any t.

(b) e to be the smallest τ_a for which $\delta\colon X_{n+1}^{\tau_a} \to X_n^{\tau_a}$ is non trivial.

(c) s to be the smallest τ_a such that $\tau_a \leqslant e$ and for which the following intersection contains a non zero element:

$$\operatorname{im}\big(X_n^{\tau_a\leqslant e}\colon Z_nX^{\tau_a} = X_n^{\tau_a} \hookrightarrow X_n^e\big) \cap \operatorname{im}\big(\delta\colon X_{n+1}^e \to X_n^e\big) \neq 0$$

We claim that these choices imply X is isomorphic to $I^n\,[s,e)\oplus X'$. We can then apply the same strategy to X'. If X' has a non-trivial differential, we split out of X' another direct summand of the form $I^{n'}\,[s',e')$ for $s'\leqslant e'$ in $[0,\infty)$. Tameness guarantees that this process eventually terminates and we end up with an object with all the differentials being trivial, which we can decompose as described above and the theorem would be proven.

It remains to show our claim that X is isomorphic to $I^n\,[s,e)\oplus X'$. For that, we make some choices:

1. Choose a non zero vector v in the intersection from step (c) above.
2. Choose x in $X_n^s = Z_nX^s$ and y in X_{n+1}^e such that: $X_n^{s\leqslant e}(x) = v = \delta(y)$.
3. Consider the morphism $I(x,y)\colon I^n\,[s,e) \to X$ (see Section 4.2).

The reason why we made all these choices is to ensure $I(x,y)\colon I^n\,[s,e)\to X$ is a cofibration (see Proposition 4.4.2).

Let $\phi\colon X \to I^n\,[s,s)$ be a morphism that fits into the following commutative diagram, where the top horizontal morphism is the standard inclusion (see Section 4.2). Existence of such a ϕ is guaranteed by axiom MC4:

$$\begin{array}{ccc} I^n\,[s,e) & \longrightarrow & I^n\,[s,s) \\ {\scriptstyle I(x,y)}\downarrow & \overset{\phi}{\nearrow} & \downarrow\sim \\ X & \longrightarrow & 0 \end{array}$$

If $t<e$, then the differential $\delta\colon X_{n+1}^t \to X_n^t$ is trivial. Thus, for any $s\leqslant t<e$, the linear function $\phi_{n+1}^t\colon X_{n+1}^t \to I^n\,[s,e)_{n+1}^t$ has to be trivial. It follows that $\phi\colon X \to$

$I^n[s,s)$ factors through the standard inclusion:

$$\begin{array}{ccc} & & I^n[s,e) \\ & \overset{\psi}{\nearrow} & \downarrow i \\ X & \xrightarrow{\ \phi\ } & I^n[s,s) \end{array}$$

The composition $I^n[s,e) \xhookrightarrow{I(x,y)} X \xrightarrow{\psi} I^n[s,e)$ is therefore the identity and consequently X is isomorphic to a direct sum $I^n[s,e) \oplus X'$. □

5. Betti diagrams of objects

Let X be a cofibrant object in $\mathrm{tame}\big([0,\infty),\mathrm{ch}\big)$. According to Theorem 4.2 there are unique functions $\{\beta_n X\colon \Omega \to \{0,1,\ldots\}\}_{n=0,1,\ldots}$, called **Betti diagrams** of X, such that X is isomorphic to:

$$\bigoplus_n \bigoplus_{(s,e)\in\Omega} \left(I^n[s,e)\right)^{\beta_n X(s,e)}$$

Betti diagrams have finite support: the set $\mathrm{supp}(\beta_n X) := \{(s,e)\in\Omega \mid \beta_n X(s,e)\neq 0\}$ is finite for every n. Thus, to describe an isomorphism type of a cofibrant object X in $\mathrm{tame}\big([0,\infty),\mathrm{ch}\big)$, a sequence of functions $\{\beta_n X\colon \Omega\to\{0,1,\ldots\}\}_{n=0,1,\ldots}$ with finite supports needs to be specified. Such functions are also called persistence diagrams (see [**11**]). Betti diagrams are complete invariants of cofibrant objects in $\mathrm{tame}\big([0,\infty),\mathrm{ch}\big)$ and they play a fundamental role in persistence and TDA.

In this section, we explain various ways of assigning Betti diagrams to arbitrary objects (not only cofibrant) in $\mathrm{tame}\big([0,\infty),\mathrm{ch}\big)$. For such general objects one should not expect these invariants to be complete. Our strategy is to approximate arbitrary objects by cofibrant objects and use Theorem 4.2 to extract Betti diagrams from the obtained approximations.

Minimal representatives (Definition 1.3) and minimal covers (Definition 1.1) are the most fundamental constructions that convert an arbitrary object in $\mathrm{tame}\big([0,\infty),\mathrm{ch}\big)$ into a cofibrant one. This leads to two invariants of an isomorphism class of X in $\mathrm{tame}\big([0,\infty),\mathrm{ch}\big)$ which are called **minimal Betti diagrams** and **Betti diagrams**:

$$\{\beta_n X'\colon \Omega\to\{0,1,\ldots\}\}_{n=0,1,\ldots} \xleftarrow{\text{minimal Betti diagrams}} X \xrightarrow{\text{Betti diagrams}} \{\beta_n \mathrm{cov}(X)\colon \Omega\to\{0,1,\ldots\}\}_{n=0,1,\ldots}$$

where X' is a minimal representative of X and $\mathrm{cov}(X)\to X$ is a minimal cover of X. We also use the symbols $\beta_n^{\min}X$ and $\beta_n X$ to denote respectively $\beta_n X'$ and $\beta_n\mathrm{cov}(X)$. Moreover, if X and Y are weakly equivalent, then $\beta_n^{\min}X = \beta_n^{\min}Y$.

To understand relationship between these invariants, we first characterise minimal objects (see Definition 1.3) in $\mathrm{tame}\big([0,\infty),\mathrm{ch}\big)$. Let X be cofibrant in $\mathrm{tame}\big([0,\infty),\mathrm{ch}\big)$. Consider its decomposition into a direct sum of interval spheres (Theorem 4.2). If this decomposition contains a component of the form $I^n[s,s)$, then by projecting it away, we obtain a self weak equivalence of X which is not an isomorphism. Thus, if X is minimal, its decomposition cannot contain such components, which is equivalent to having $\beta_n X(s,s)=0$ for all n and all s. This implication can be reversed:

Proposition 5.1.

1. *An object X in $\operatorname{tame}\big([0,\infty),\mathrm{ch}\big)$ is minimal if and only if it is cofibrant and $\beta_n X(s,s)=0$ for all natural numbers n and all s in $[0,\infty)$.*
2. *A morphism $c\colon X\to Y$ is a minimal cover if and only if X is cofibrant, c is a weak equivalence and a fibration, and no direct summand of X of the form $I^n[s,s)$, for some n and s, is in the kernel of c (is mapped via c to 0).*

Proof. We start with describing how a self weak equivalence of an object X in $\operatorname{tame}\big([0,\infty),\mathrm{ch}\big)$ leads to its decomposition as a direct sum. Let $f\colon X\to X$ be a weak equivalence. Since all objects in $\operatorname{tame}\big([0,\infty),\mathrm{ch}\big)$ are compact, there is a natural number l for which $f^l=f^{l+k}$ for all $k\geqslant 0$. We can then form the following commutative diagram where the top horizontal morphism is an isomorphism:

$$\begin{array}{ccccccccc} & \nearrow\!\!\!\!\nearrow & \operatorname{im}(f^l) & \longrightarrow & & \nearrow\!\!\!\!\nearrow & \operatorname{im}(f^l) & \searrow & \\ X & & \xrightarrow{\quad f^l \quad} & & X & & \xrightarrow{\quad f^l \quad} & & X \end{array}$$

Commutativity of this diagram, and the facts that the top horizontal morphism is an isomorphism and f^l is a weak equivalence, have two consequences: X is isomorphic to a direct sum $\operatorname{im}(f^l)\oplus\ker(f^l)$, and the morphisms $X\twoheadrightarrow\operatorname{im}(f^l)$ and $\ker(f^l)\to 0$ are weak equivalences.

1. Assume X is cofibrant and $\beta_n X(s,s)=0$ for all n and s. Let $f\colon X\to X$ be a self weak equivalence and l be such that X is isomorphic to $\operatorname{im}(f^l)\oplus\ker(f^l)$ and the morphism $\ker(f^l)\to 0$ is a weak equivalence. According to Theorem 4.2, $\ker(f^l)$ is a direct sum of interval spheres of the form $I^n[s,s)$. Since by the assumption X does not have such components, $\ker(f^l)=0$ and consequently f^l and hence f are isomorphisms. That proves statement *1.*

2. Consider a morphism $c\colon X\to Y$ such that X is cofibrant, c is a weak equivalence and a fibration. If the kernel of c contains a direct summand of X of the form $I^n[s,s)$, then by projecting it away, we would obtain a weak equivalence $f\colon X\to X$ such that $cf=c$ and which is not an isomorphism, preventing c to be a minimal cover.

Assume now that the kernel of c does not contain any direct summand of X of the form $I^n[s,s)$. Consider a weak equivalence $f\colon X\to X$ such that $cf=c$. Choose l for which X is isomorphic to $\operatorname{im}(f^l)\oplus\ker(f^l)$ and the morphism $\ker(f^l)\to 0$ is a weak equivalence. As before, Theorem 4.2 ensures that $\ker(f^l)$ is a direct sum of interval spheres of the form $I^n[s,s)$. Since $\ker(f^l)$ is in $\ker(c)$ and it is a direct summand of X, the assumption implies $\ker(f^l)$ is trivial, and as before f is an isomorphism. □

Corollary 5.2. *Let X be an object in $\operatorname{tame}\big([0,\infty),\mathrm{ch}\big)$ (not necessarily cofibrant).*

1. *A cofibrant object X' in $\operatorname{tame}\big([0,\infty),\mathrm{ch}\big)$ is a minimal representative of X if and only if it is weakly equivalent to X and $\beta_n X'(s,s)=0$ for all natural numbers n and all s in $[0,\infty)$.*
2. *Let X' be the minimal representative of X and $\operatorname{cov}(X)$ its minimal cover. Then $\beta_n X'(s,e)=\beta_n\operatorname{cov}(X)(s,e)$ for all $s<e$.*

According to the above corollary, the minimal Betti diagrams and the Betti diagrams may differ only on the diagonal $\Delta\coloneqq\{(s,s)\in\Omega\}\subset\Omega$. The minimal Betti diagrams ignore the diagonal by assigning 0 to all its elements (see Proposition 5.1), reflecting the fact that minimal representative does not contain any component with trivial homology. The Betti diagrams on the other hand do not ignore the diagonal and retain information about components of the cover $\text{cov}(X)$ that have trivial homology.

For certain objects in $\text{tame}\big([0,\infty),\text{ch}\big)$, the minimal Betti diagrams and the Betti diagrams coincide and provide a complete set of invariants:

Proposition 5.3. *Assume X and Y in $\text{tame}\big([0,\infty),\text{ch}\big)$ are such that all the differentials of X^t and Y^t are trivial for all t in $[0,\infty)$. Then:*

1. *X and Y are isomorphic if and only if they are weakly equivalent.*
2. *X and Y are isomorphic if and only if their minimal Betti diagrams are equal.*
3. *The minimal cover and the minimal representative of X are isomorphic.*
4. *The minimal Betti diagrams and Betti diagrams of X coincide.*

Proof. Statement *1* is a consequence of the fact that X and Y are isomorphic to their respective homologies. Statement *2* follows from statement *1*. To show statement *3*, choose a minimal representative X' of X and a weak equivalence $f\colon X'\to X$. Since X is isomorphic to its homology, f is a fibration and hence it is also a minimal cover of X. Finally statement *4* is a consequence of statement *1*. □

5.1. Tame $[0,\infty)$-parametrised vector spaces

We regard **tame $[0,\infty)$-parametrised vector spaces**, also known as persistence modules, as objects in $\text{tame}\big([0,\infty),\text{ch}\big)$ whose values are concentrated only in degree 0 for all parameters t in $[0,\infty)$ (see Section 3.7). Such objects in $\text{tame}\big([0,\infty),\text{ch}\big)$ satisfy the assumption of Proposition 5.3 and hence their isomorphism types are uniquely determined by their Betti diagrams. Furthermore, minimal representatives and minimal covers of such objects coincide. It follows that for a tame $[0,\infty)$-parametrised vector space X, we have $\beta_n X(s,s)=0$ for all n and s (see Proposition 5.1). Since homology in positive degrees of X is trivial, we also get $\beta_n X=0$ for all $n>0$. Thus, the isomorphism type of X is uniquely determined by its 0-th Betti diagram $\beta_0 X\colon \Omega\to\{0,1,\ldots\}$. In this case $\beta_0 X$ coincides with the usual persistence diagram of X [11].

6. Betti diagrams of morphisms

In this section we explain various ways of assigning Betti diagrams to a morphism $g\colon X\to Y$ in $\text{tame}\big([0,\infty),\text{ch}\big)$.

6.1. Minimal factorisations

If $g\colon X\to Y$ is a cofibration, then the quotient $Y/g(X)$ is cofibrant and we can take its Betti diagrams $\beta_n\left(Y/g(X)\right)$. If $g\colon X\to Y$ is not a cofibration, we can consider its minimal factorisation (see Definition 1.1):

$$\begin{array}{ccccc} & & A & & \\ & \overset{\alpha}{\nearrow} & & \overset{\beta}{\underset{\sim}{\searrow}} & \\ X & & \xrightarrow{\quad g \quad} & & Y \end{array}$$

and assign to g the Betti diagrams $\beta_n\left(A/\alpha(X)\right)$ of the quotient $A/\alpha(X)$. These Betti diagrams are invariants of the isomorphism type of g, and in the case $X = 0$ recover the Betti diagrams of Y discussed in Section 5.

6.2. The cover of the cofiber

Instead of taking the minimal factorisation of g, we can apply the cofiber construction (see Section 3.4) parameterwise, to obtain an exact sequence in $\text{tame}\big([0,\infty),\text{ch}\big)$:

$$0 \longrightarrow Y \xrightarrow{\ i\ } Cg \xrightarrow{\ p\ } SX \longrightarrow 0$$

We can then assign to g the Betti diagrams $\beta_n\text{cov}(Cg)$ of the minimal cover $\text{cov}(Cg)$ of the cofiber Cg. The diagrams $\beta_n\text{cov}(Cg)$ depend on the isomorphism type of g, and as before, in the case $X = 0$, recover the Betti diagrams of Y discussed in Section 5.

6.3. The cofiber of the covers

We can extract a cofibrant object out of $g\colon X \to Y$ yet in another way. Use axiom MC4 to choose a morphism g' that fits into the following commutative square, where the vertical morphisms denote the minimal covers:

$$\begin{array}{ccc} \text{cov}(X) & \xrightarrow{\ g'\ } & \text{cov}(Y) \\ {\scriptstyle c_X}\downarrow{\scriptstyle \sim} & & {\scriptstyle \sim}\downarrow{\scriptstyle c_Y} \\ X & \xrightarrow{\quad g \quad} & Y \end{array}$$

Since $\text{cov}(X)$ and $\text{cov}(Y)$ are cofibrant, then so is the cofiber Cg', and hence we can take its Betti diagrams $\beta_n Cg'$. Although in this construction we made a choice of g', the obtained Betti diagrams do not depend on it and hence provide invariants of the isomorphism type of g. To prove this independence, consider the morphism $C(c_X, c_Y)\colon Cg' \to Cg$, induced by the commutativity of the square above (see Section 3.4). It fits into the following commutative diagram with exact rows, where the indicated morphisms are weak equivalences, cofibrations, and fibrations:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \text{cov}(Y) & \overset{i}{\hookrightarrow} & Cg' & \overset{p}{\twoheadrightarrow} & S\text{cov}(X) & \longrightarrow & 0 \\ & & {\scriptstyle c_Y}\downarrow{\scriptstyle \sim} & & {\scriptstyle \sim}\downarrow{\scriptstyle C(c_X,c_Y)} & & {\scriptstyle \sim}\downarrow{\scriptstyle Sc_X} & & \\ 0 & \longrightarrow & Y & \xrightarrow{\ i\ } & Cg & \overset{p}{\twoheadrightarrow} & SX & \longrightarrow & 0 \end{array}$$

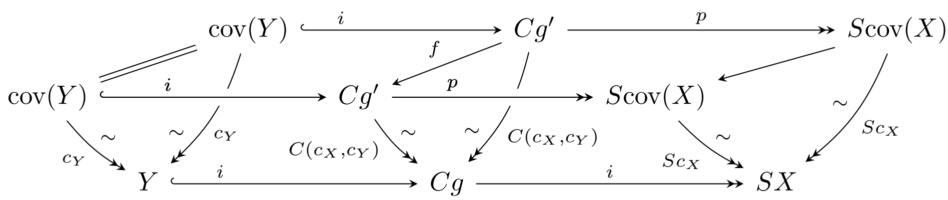
Proposition 6.1. *The following factorisation is minimal:*

$$\begin{array}{ccccc} & & Cg' & & \\ & \overset{i}{\nearrow} & & \overset{C(c_X,c_Y)}{\underset{\sim}{\searrow}} & \\ \text{cov}(Y) & & \xrightarrow{\quad ic_Y \quad} & & Cg \end{array}$$

Furthermore, if X is cofibrant, then $C(c_X, c_Y)\colon Cg' \to Cg$ is a minimal cover.

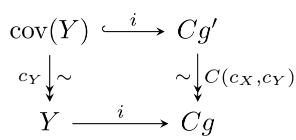
Since the morphism $ic_Y\colon \mathrm{cov}(Y) \to Cg$ does not depend on g', neither does its minimal factorisation. Therefore, Proposition 6.1 implies that the isomorphism type of Cg' does not depend on the choice of g' and consequently neither $\beta_n Cg'$.

Proof of Proposition 6.1. Consider a self equivalence $f\colon Cg' \to Cg'$ of the factorisation. It fits into the following commutative diagram:



The induced morphism $S\mathrm{cov}(X) \to S\mathrm{cov}(X)$ is then a weak equivalence and hence an isomorphism as $Sc_X\colon S\mathrm{cov}(X) \to SX$ is the minimal cover. The morphism f is therefore an isomorphism as well.

Assume X is cofibrant. We are going to use the criteria from Proposition 5.1 to argue that in this case $C(c_X, c_Y)$ is a minimal cover. Since $\mathrm{cov}(X) = X$, the following square is a pullback:



It follows that the kernel of $C(c_X, c_Y)$ coincides with the kernel of c_Y. Consider a direct summand of Cg' of the form $I^n[s,s)$, for some n and s, that belongs to the kernel of $C(c_X, c_Y)$. Then this summand belongs also to the kernel of c_Y. In particular, it is included in $\mathrm{cov}(Y)$ and hence it is also a direct summand of $\mathrm{cov}(Y)$. That contradicts the criteria from Proposition 5.1 applied to c_Y. We can conclude that such summands do not exist and hence, according to the same criteria, $C(c_X, c_Y)$ is a minimal cover. $\square$

With a morphism $g\colon X \to Y$ in $\mathrm{tame}\big([0,\infty), \mathrm{ch}\big)$, we have associated four cofibrant objects: $A/\alpha(X)$ (see Section 6.1), $\mathrm{cov}(Cg)$ (see Section 6.2), Cg' (see Section 6.3), and the minimal representative of $\mathrm{cov}(Cg)$. These cofibrant objects lead to Betti diagrams $\beta_n\,(A/\alpha(X))$, $\beta_n \mathrm{cov}(Cg)$, $\beta_n Cg'$, and $\beta_n^{\min}\mathrm{cov}(Cg)$. Since all these cofibrant objects are weakly equivalent to each other, all these Betti diagrams agree for all (s,e) in Ω such that $s<e$. They may have different values only on the diagonal $\Delta \subset \Omega$.

6.4. Commutative ladders

A **commutative ladder** is by definition an object in $\mathrm{tame}\big([0,\infty), \mathrm{ch}\big)$ whose values at all parameters are chain complexes which are non trivial only in degrees zero and one. For example, $I^0[s,s)$ is a commutative ladder. Similarly, so is the Kan extension of $D^1 \to 0$ with respect to a sequence $s<e$ of elements in $[0,\infty)$ (see Section 2.1). A tame $[0,\infty)$-parametrised vector space is also an example of a commutative ladder. In general, however, in contrast to tame $[0,\infty)$-parametrised vector spaces, the minimal Betti diagrams of a commutative ladder can fail to be equal to its Betti

diagrams. For example, the minimal representative of $I^0\,[s,s)$ is trivial, however its minimal cover is $I^0\,[s,s)$. Furthermore, again in contrast to tame vector spaces, Betti diagrams are not complete invariants of commutative ladders. For example, $I^0\,[s,s)$ and the Kan extension of $D^1 \to 0$ with respect to a sequence $s < e$ of elements in $[0,\infty)$ have isomorphic minimal covers and hence same Betti diagrams.

If $f\colon X \to Y$ is a morphism of tame $[0,\infty)$-parametrised vector spaces, then its cofiber Cf is a commutative ladder. Any commutative ladder Z is the cofiber of its differential $\delta\colon Z_1 \to Z_0$, where we regard Z_1 and Z_0 as $[0,\infty)$-parametrised vector spaces. Thus, we can extract from Z various Betti diagrams assigned to the morphism $\delta\colon Z_1 \to Z_0$. For example, we can consider the Betti diagrams of the quotient of the cofibration in the minimal factorisation of the differential $\delta\colon Z_1 \to Z_0$ (see Section 6.1). We can also apply to δ the procedure described in Section 6.3 to obtain another sequence of Betti diagrams. Thus, a commutative ladder leads to four sequences of Betti diagrams. These Betti diagrams are not arbitrary. Let $\beta_n\colon \Omega \to \{0,1,\ldots\}$ be any of these Betti diagrams extracted from a commutative ladder. Since for $n \geqslant 1$, there are no non-trivial morphisms from the interval sphere $I^n\,[s,s)$ into any commutative ladder, $\beta_n(s,s) = 0$ for $n \geqslant 1$ and s in $[0,\infty)$. As values of commutative ladders have no homology in degrees strictly bigger than 1, we then also get $\beta_n = 0$, for $n > 1$.

7. Zigzags

7.1. Discrete zigzags

Throughout this section, k is assumed to be a positive natural number. Elements of the set $\{r,l\}$ are called directions, r stands for right and l for left. Let $c = (c_1,\ldots,c_k)$ be a sequence of directions i.e., elements of $\{r,l\}$. Such a sequence determines a poset structure "$\to$" on $\{0,1,\ldots,k\}$ where, for $a < b$ in $\{0,1,\ldots,k\}$, $a \leftarrow b$ if $c_{a+1} = \cdots = c_b = l$, and $a \to b$ if $c_{a+1} = \cdots = c_b = r$. This poset is denoted by $[k]_c$ and the sequence c is called its **profile**. A profile c consisting of only r's is called standard and the induced poset structure on $\{0,1,\ldots,k\}$ is denoted by $[k]$. Here are graphical illustrations of $[4]_c$ for 3 different profiles:

$$0 \leftarrow 1 \leftarrow 2 \to 3 \to 4, \qquad 0 \to 1 \to 2 \to 3 \to 4, \qquad 0 \leftarrow 1 \to 2 \leftarrow 3 \to 4.$$

To define a functor $X\colon [k]_c \to \mathrm{Ch}$, the following needs to be specified:

- $k+1$ chain complexes X^a for every a in $\{0,1,\ldots,k\}$,
- k chain morphisms: $X^{a-1\to a}\colon X^{a-1} \to X^a$ for every a such that $c_a = r$, and $X^{a\to a-1}\colon X^a \to X^{a-1}$ for every a such that $c_a = l$.

A **discrete zigzag** is by definition a functor of the form $X\colon [k]_c \to \mathrm{Ch}$ for some k and some profile c.

7.2. Straightening zigzags

Choose a profile $c = (c_1,\ldots,c_k)$. For a in $\{0,1,\ldots,k\}$, define its weight w_a to be the size of the set $\{k \mid k \leqslant a \text{ and } c_k = l\}$. The weight of a is the number of l directions in c whose indexes are not bigger than a. Thus, $w_0 = 0$, and $w_1 = 1$ if and only if $c_1 = l$.

In this paragraph we are going to explain how to convert a zigzag $X\colon [k]_c \to \mathrm{Ch}$, indexed by the poset $[k]_c$, into a functor $\overline{X}\colon [k] \to \mathrm{Ch}$ indexed by the standard poset $[k]$. For a in $\{0,1,\dots,k\}$ define:

$$\overline{X}^a = \begin{cases} S^{w_a}X^a & \text{if } a<k \text{ and } c_{a+1}=r,\\ S^{w_a}CX^{a+1\to a} & \text{if } a<k \text{ and } c_{a+1}=l,\\ S^{w_k}X^k & \text{if } a=k. \end{cases}$$

Thus, for $a<k$, the value $\overline{X}^a$ depends on the direction c_{a+1}. If $c_{a+1}=r$, then $\overline{X}^a$ is the w_a suspension of X^a. If $c_{a+1}=l$, then $\overline{X}^a$ is the w_a suspension of the cofiber $CX^{a+1\to a}$ (see Section 3.4).

Next we are going to define morphisms $\overline{X}^{a-1<a}\colon \overline{X}^{a-1}\to\overline{X}^a$ for every a in $\{1,\dots,k\}$. These morphisms depend on the directions c_a and c_{a+1} for $a<k$, and c_k for $a=k$, and are defined as follows, using the morphisms i and p as described in Section 3.4:

- Assume either $a<k$, $c_a=r$ and $c_{a+1}=r$, or $a=k$ and $c_k=r$. Then:
  - $w_a = w_{a-1}$,
  - $\overline{X}^{a-1} = S^{w_{a-1}}X^{a-1} = S^{w_a}X^{a-1}$,
  - $\overline{X}^a = S^{w_a}X^a$.

  The morphism $\overline{X}^{a-1<a}\colon \overline{X}^{a-1}\to\overline{X}^a$ is set to be:

$$\begin{array}{ccc} \overline{X}^{a-1} & \xrightarrow{\overline{X}^{a-1<a}} & \overline{X}^a \\ \| & & \| \\ S^{w_a}X^{a-1} & \xrightarrow{S^{w_a}X^{a-1\to a}} & S^{w_a}X^a \end{array}$$

- If $c_a=r$ and $c_{a+1}=l$, then:
  - $w_a = w_{a-1}$,
  - $\overline{X}^{a-1} = S^{w_{a-1}}X^{a-1} = S^{w_a}X^{a-1}$,
  - $\overline{X}^a = S^{w_a}CX^{a+1\to a}$.

  The morphism $\overline{X}^{a-1<a}\colon \overline{X}^{a-1}\to\overline{X}^a$ is set to be the composition:

$$\begin{array}{ccccc} \overline{X}^{a-1} & & \xrightarrow{\overline{X}^{a-1<a}} & & \overline{X}^a \\ \| & & & & \| \\ S^{w_a}X^{a-1} & \xrightarrow{S^{w_a}X^{a-1\to a}} & S^{w_a}X^a & \xhookrightarrow{S^{w_a}i} & S^{w_a}CX^{a+1\to a} \end{array}$$

- Assume either $a<k$, $c_a=l$ and $c_{a+1}=r$, or $a=k$ and $c_k=l$. Then:
  - $w_a = w_{a-1}+1$,
  - $\overline{X}^{a-1} = S^{w_{a-1}}CX^{a\to a-1}$,
  - $\overline{X}^a = S^{w_a}X^a = S^{w_{a-1}}SX^a$.

  The morphism $\overline{X}^{a-1<a}\colon \overline{X}^{a-1}\to\overline{X}^a$ is set to be:

$$\begin{array}{ccc} \overline{X}^{a-1} & \xrightarrow{\overline{X}^{a-1<a}} & \overline{X}^a \\ \| & & \| \\ S^{w_{a-1}}CX^{a\to a-1} & \xrightarrow{S^{w_{a-1}}p} & S^{w_{a-1}}SX^a = S^{w_a}X^a \end{array}$$

- If $c_a = l$ and $c_{a+1} = l$, then:
  - $w_a = w_{a-1} + 1$,
  - $\overline{X}^{a-1} = S^{w_{a-1}} C X^{a \to a-1}$,
  - $\overline{X}^{a} = S^{w_a} C X^{a+1 \to a} = S^{w_{a-1}} S C X^{a+1 \to a}$.

  The morphism $\overline{X}^{a-1<a} \colon \overline{X}^{a-1} \to \overline{X}^{a}$ is set to be the composition:

$$\begin{array}{ccc} \overline{X}^{a-1} & \xrightarrow{\qquad\qquad\overline{X}^{a-1<a}\qquad\qquad} & \overline{X}^{a} \\ \| & & \| \\ S^{w_{a-1}} C X^{a \to a-1} & \xrightarrow{S^{w_{a-1}}p} S^{w_{a-1}} S X^{a} = S^{w_a} X^{a} \xhookrightarrow{S^{w_a} i} & S^{w_a} C X^{a+1 \to a} \end{array}$$

For example, consider the following discrete zigzags of chain complexes:

$$X = \left(X_0 \xleftarrow{f_1} X_1 \xleftarrow{f_2} X_2 \xrightarrow{f_3} X_3 \xrightarrow{f_4} X_4 \right)$$

$$Y = \left(Y_0 \xrightarrow{g_1} Y_1 \xrightarrow{g_2} Y_2 \xleftarrow{g_3} Y_3 \xleftarrow{g_4} Y_4 \right)$$

Then:

$$\overline{X} = \left(\begin{array}{ccccccccc} Cf_1 & \xrightarrow{\qquad\qquad} & SCf_2 & \overset{p}{\twoheadrightarrow} & S^2X_2 & \xrightarrow{S^2 f_3} & S^2X_3 & \xrightarrow{S^2 f_4} & S^2X_4 \\ & \underset{p}{\searrow\!\!\!\!\searrow} \; SX_1 \; \overset{Si}{\hookrightarrow\!\!\nearrow} & & & & & & & \end{array} \right)$$

$$\overline{Y} = \left(\begin{array}{ccccccc} Y_0 \xrightarrow{g_1} Y_1 & \xrightarrow{\qquad\qquad} & Cg_3 & \xrightarrow{\qquad\qquad} & SCg_4 & \overset{Sp}{\twoheadrightarrow} & S^2Y_4 \\ & \underset{g_2}{\searrow} \; Y_2 \; \overset{i}{\hookrightarrow\!\!\nearrow} & & \underset{p}{\searrow\!\!\!\!\searrow} \; SY_3 \; \overset{Si}{\hookrightarrow\!\!\nearrow} & & & \end{array} \right)$$

7.3. Morphisms between straightened zigzags

Consider a natural transformation $f \colon X \to Y$ between two discrete zigzags $X, Y \colon [k]_c \to \mathrm{Ch}$. It is a sequence of morphisms $f = \{f^a \colon X^a \to Y^a\}_{0 \leqslant a \leqslant k}$ for which the following squares commute for all a in $\{1, \ldots, k\}$:

if $c_a = r$	if $c_a = l$
$\begin{array}{ccc} X^{a-1} & \xrightarrow{X^{a-1\to a}} & X^a \\ {\scriptstyle f^{a-1}}\downarrow & & \downarrow{\scriptstyle f^a} \\ Y^{a-1} & \xrightarrow{Y^{a-1\to a}} & Y^a \end{array}$	$\begin{array}{ccc} X^{a-1} & \xleftarrow{X^{a\to a-1}} & X^a \\ {\scriptstyle f^{a-1}}\downarrow & & \downarrow{\scriptstyle f^a} \\ Y^{a-1} & \xleftarrow{Y^{a\to a-1}} & Y^a \end{array}$

The following morphisms form a natural transformation denoted by $\overline{f} \colon \overline{X} \to \overline{Y}$:

$$\overline{f}^a \colon \overline{X}^a \to \overline{Y}^a = \begin{cases} S^{w_a} f^a & \text{if } a < m \text{ and } c_{a+1} = r, \\ S^{w_a} C(f^a) & \text{if } a < m \text{ and } c_{a+1} = l, \\ S^{w_k} f^k & \text{if } a = k. \end{cases}$$

The association $f \mapsto \overline{f}$ defines an additive functor. This functor is faithful i.e., it is injective on the set of morphisms. Furthermore, it commutes with direct sums, since taking suspensions and cofiber sequences commute with direct sums. In general, however, this functor fails to be full i.e., surjective on the set of morphisms. To understand this failure we enumerate all natural transformations of the form $\overline{X} \to \overline{Y}$, using the following algorithm. Choose an arbitrary natural transformation $g\colon \overline{X} \to \overline{Y}$.

- Let $g^k\colon S^{w_k}X^k \to S^{w_k}Y^k$ be the k-th component of g. Define $\widehat{g}^k\colon X^k \to Y^k$ to be $S^{-w_k}g^k$ (see Section 3.3).

For $1 \leqslant a \leqslant k-1$,

- Assume $c_{a+1} = r$. Let $g^a\colon S^{w_a}X^a \to S^{w_a}Y^a$ be the a-th component of g. Define $\widehat{g}^a\colon X^a \to Y^a$ to be $S^{-w_a}g^a$. Since g is a natural transformation, the following square commutes:

$$\begin{array}{ccc} X^a & \xrightarrow{X^{a\to a+1}} & X^{a+1} \\ {\scriptstyle \widehat{g}^a}\downarrow & & \downarrow{\scriptstyle \widehat{g}^{a+1}} \\ Y^a & \xrightarrow{Y^{a\to a+1}} & Y^{a+1} \end{array}$$

- Assume $c_{a+1} = l$. Let $g^a\colon S^{w_a}CX^{a+1\to a} \to S^{w_a}CY^{a+1\to a}$ be the a-th component of g. Recall the arguments in Section 3.4. Since g is a natural transformation, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X^a & \xhookrightarrow{i} & CX^{a+1\to a} & \xrightarrow{p} & SX^{a+1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle S^{-w_a}g^a} & & \downarrow{\scriptstyle S\widehat{g}^{a+1}} & & \\ 0 & \longrightarrow & Y^a & \xhookrightarrow{i} & CY^{a+1\to a} & \xrightarrow{p} & SY^{a+1} & \longrightarrow & 0 \end{array}$$

Define $\widehat{g}^a\colon X^a \to Y^a$ to be the left vertical morphism in this diagram. This diagram leads to a homotopy commutative square with a choice of a homotopy:

$$\begin{array}{ccc} X^a & \xleftarrow{X^{a+1\to a}} & X^{a+1} \\ {\scriptstyle \widehat{g}^a}\downarrow & \overset{h_{a+1}}{\swarrow} & \downarrow{\scriptstyle \widehat{g}^{a+1}} \\ Y^a & \xleftarrow[Y^{a+1\to a}]{} & Y^{a+1} \end{array}$$

By applying this algorithm, we obtain a bijection between the set of natural transformations $\overline{X} \to \overline{Y}$ and the set of pairs consisting of a sequence of morphisms $\{\widehat{g}^a\colon X^a \to Y^a\}_{1\leqslant a\leqslant k}$ and a sequence of homotopies $\{h^a\colon X^a \to Y^{a-1} \mid 1 \leqslant a \leqslant k \text{ and } c_a = l\}$ such that, for all a in $\{1,\ldots,k\}$:

if $c_a = r$	if $c_a = l$
$\begin{array}{ccc} X^{a-1} & \xrightarrow{X^{a-1\to a}} & X^a \\ {\scriptstyle \widehat{g}^{a-1}}\downarrow & & \downarrow{\scriptstyle \widehat{g}^a} \\ Y^{a-1} & \xrightarrow{Y^{a-1\to a}} & Y^a \end{array}$	$\begin{array}{ccc} X^{a-1} & \xleftarrow{X^{a\to a-1}} & X^a \\ {\scriptstyle \widehat{g}^{a-1}}\downarrow & \overset{h_a}{\swarrow} & \downarrow{\scriptstyle \widehat{g}^a} \\ Y^{a-1} & \xleftarrow[Y^{a\to a-1}]{} & Y^a \end{array}$

Here is a consequence of this enumeration:

Corollary 7.1. *Let $X\colon [k]_c \to \mathrm{ch}$ be a discrete zigzag of compact chain complexes. Assume that X^a has trivial differentials for all a in $\{0,\ldots,k\}$. Then $\overline{X}$ is indecomposable if and only if X is indecomposable.*

Proof. Since the functor $X \mapsto \overline{X}$ commutes with the direct sum and is faithful, if X is decomposable, then so is $\overline{X}$. If $\overline{X}$ is decomposable, then there is an idempotent morphism $g\colon \overline{X} \to \overline{X}$ which is not an isomorphism. As the differentials of X^a's are trivial, the morphisms $\{\widehat{g}_a\}_{1\leqslant a\leqslant k}$ form an idempotent natural transformation $X \to X$ (see Section 3.4), which is not an isomorphism. Consequently, X is also decomposable. □

Definition 7.2. An object in $\mathrm{tame}\big([0,\infty),\mathrm{ch}\big)$ is called a **zigzag** if it is isomorphic to the Kan extension, along some sequence $\tau_0 < \cdots < \tau_k$ in $[0,\infty)$, of a functor of the form $\overline{X}$ for some discrete zigzag $X\colon [k]_c \to \mathrm{vect}_K \subset \mathrm{ch}$ whose values are concentrated in degree 0. Such a zigzag in $\mathrm{tame}\big([0,\infty),\mathrm{ch}\big)$ is called an **incarnation** of X.

We think about $\mathrm{tame}\big([0,\infty),\mathrm{ch}\big)$ as an ambient category containing various incarnations of discrete zigzags of the form $X\colon [k]_c \to \mathrm{vect}_K \subset \mathrm{ch}$ indexed by posets $[k]_c$ for different k's and different profiles c. Important properties of discrete zigzags are reflected well by their incarnations. For example, according to Corollary 7.1, a discrete zigzag $X\colon [k]_c \to \mathrm{vect}_K \subset \mathrm{ch}$ is indecomposable if and only if all (equivalently any) of its incarnations are indecomposable in $\mathrm{tame}\big([0,\infty),\mathrm{ch}\big)$. We can also use the category $\mathrm{tame}\big([0,\infty),\mathrm{ch}\big)$ and its morphisms to compare discrete zigzags indexed by different posets. Furthermore, the model structure on $\mathrm{tame}\big([0,\infty),\mathrm{ch}\big)$ can be utilised to extract invariants of discrete zigzags through taking minimal representatives and minimal covers of their incarnations. However, the minimal cover is not a complete invariant for zigzags. Consider the following non-isomorphic discrete zigzags:

$$X:\ K \xrightarrow{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} K^2 \xleftarrow{\ 1\ } K^2 \xrightarrow{[1\ 0]} K, \qquad Y:\ K \xrightarrow{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} K^2 \xleftarrow{\ 1\ } K^2 \xrightarrow{[0\ 1]} K.$$

The minimal covers of their incarnations $L\overline{X}$ and $L\overline{Y}$ along a sequence $\tau_0 < \tau_1 < \tau_2 < \tau_3$ in $[0,\infty)$ coincide: $\mathrm{cov}(L\overline{X}) \cong \mathrm{cov}(L\overline{Y}) \cong I^0\,[\tau_0,\tau_1)^2 \oplus I^1\,[\tau_2,\tau_3) \oplus I^1\,[\tau_2,\infty)$. As a consequence, neither the minimal representative is a complete invariant for zigzags.

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