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STRONG VERSION OF SNAKE LEMMA IN EXACT CATEGORIES

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(*communicated by Claude Cibils*)

Abstract

We prove the strong version of Snake Lemma, and its equivalence to the other two versions of Snake Lemma, in weakly idempotent complete exact categories.

1. Introduction

An exact category in this paper is in the sense of Quillen [**15**, §2]. Keller [**12**, Appendix] has simplified its axioms; see also [**6**, Appendix]. For homological algebra in exact categories we refer to Bühler [**4**].

Snake Lemma is a fundamental and widely used theorem in homological algebra, which has interests in various categories (see e.g. [**3**], [**5**], [**11**], [**18**]). In his book [**10**, 1.6, p.4], Iversen has given a strong version of Snake Lemma in a category $\mathcal{A}$ which has zero object, kernels and cokernels, such that the canonical morphism $\operatorname{Coim} f \longrightarrow \operatorname{Im} f$ is always an isomorphism, but $\mathcal{A}$ is not necessarily an additive category. For convenience we call such a category *a generalized abelian category.*

The aim of this paper is to prove the strong version of Snake Lemma in weakly idempotent complete exact categories, by using the simple version of Snake Lemma. Note that the simple version and the classic version of Snake Lemma in weakly idempotent complete exact categories have been proved in [**4**]. Hence the three versions of Snake Lemma are equivalent in weakly idempotent complete exact categories. Since the morphism category of a weakly idempotent complete exact category is again a weakly idempotent complete exact category, all the versions of Snake Lemma, as well as other diagram lemmas on exactness, are functorial. Examples of weakly idempotent complete exact categories which are not generalized abelian can be found in [**4**, 13.2] from functional analysis. We also provide a class of such examples from representation theory of algebras via Gorenstein-projective modules.

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2. Weakly idempotent complete exact category

2.1. Exact category

Let $\mathcal{A}$ be an additive category. An *exact pair* (i, d) is a sequence of morphisms $X \xrightarrow{i} Y \xrightarrow{d} Z$ in $\mathcal{A}$ such that i is a kernel of d, and d is a cokernel of i. The following definition given by Keller is equivalent to the original one in Quillen [**15**, §2].

Definition 2.1 ([**12**, Appendix A])**.** An exact category is a pair $(\mathcal{A}, \mathcal{E})$, where $\mathcal{A}$ is an additive category, and $\mathcal{E}$ is a class of exact pairs satisfying the axioms (E0), (E1), (E2) and (E2$^{\rm op}$), where an exact pair $(i, d) \in \mathcal{E}$ is called a *conflation*, i is called an *inflation*, and d is called a *deflation*.

(E0) $\mathcal{E}$ is closed under isomorphisms, and Id_0 is a deflation.

(E1) The composition of two deflations is a deflation.

(E2) For each deflation $d\colon Y \longrightarrow Z$ and each morphism $f\colon Z' \longrightarrow Z$, there is a pullback square

$$\begin{array}{ccc} Y' & \overset{d'}{\dashrightarrow} & Z' \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{d} & Z \end{array} \tag{1}$$

such that d' is a deflation.

(E2$^{\rm op}$) For each inflation $i\colon X \longrightarrow Y$ and each morphism $f\colon X \longrightarrow X'$, there is a pushout square

$$\begin{array}{ccc} X & \xrightarrow{i} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ X' & \overset{i'}{\dashrightarrow} & Y' \end{array} \tag{2}$$

such that i' is an inflation.

Fact 2.2. *Let $(\mathcal{A}, \mathcal{E})$ be an exact category. Then*

(1) *The composition of inflations is inflation.*

(2) *All the identity morphisms are deflations and inflations. For any object X, the morphism $0 \longrightarrow X$ is an inflation and the morphism $X \longrightarrow 0$ is a deflation.*

(3) *A morphism f is an isomorphism if and only if f is a deflation and f is an inflation.*

(4) *The opposite category $(\mathcal{A}^{\rm op}, \mathcal{E}^{\rm op})$ of an exact category $(\mathcal{A}, \mathcal{E})$ is again an exact category.*

Lemma 2.3 ([**4**, 2.15])**.** *Let $(\mathcal{A}, \mathcal{E})$ be an exact category.*

(1) *Let (1) be a pullback square with d a deflation. If f is an inflation, then so is f'.*

(1′) *Let (2) be a pushout square with i an inflation. If f is a deflation, then so is f'.*

2.2. Exactness in exact categories

A morphism $f\colon X \longrightarrow Y$ in an exact category $(\mathcal{A}, \mathcal{E})$ is *admissible*, if f has *a deflation-inflation-decomposition* as $f = \widetilde{f}\overline{f}$, as the diagram

$$\begin{array}{ccccc} X & & \xrightarrow{\quad f \quad} & & Y \\ & \overline{f} \searrow & & \nearrow \widetilde{f} & \\ & & \bullet & & \end{array}$$

shows, where $\overline{f}$ is a deflation and $\widetilde{f}$ is an inflation.

Clearly, the zero morphisms, deflations and inflations are admissible.

Let $f : X \longrightarrow Y$ be an admissible morphism with deflation-inflation-decomposition $f = \widetilde{f}\overline{f}$. Since $\overline{f}$ is a deflation, the kernel $\operatorname{Ker}\overline{f}$ of $\overline{f}$ exists. Since $f = \widetilde{f}\overline{f}$ and $\widetilde{f}$ is a monomorphism, the kernel of f also exists and $\operatorname{Ker} f = \operatorname{Ker}\overline{f}$. Dually, $\operatorname{Coker} f$ exists and $\operatorname{Coker} f = \operatorname{Coker}\widetilde{f}$. Namely, the kernel and the cokernel of an admissible morphism always exist.

A sequence of admissible morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ is said to be $\mathcal{E}$-*exact at* Y, or *exact at* Y, provided that for the deflation-inflation-decompositions $f = \widetilde{f}\overline{f}$ and $g = \widetilde{g}\overline{g}$, one has that $(\widetilde{f}, \overline{g})$ is a conflation. See the diagram

$$\begin{array}{ccccccccc} X & & \xrightarrow{\quad f \quad} & & Y & & \xrightarrow{\quad g \quad} & & Z \\ & \overline{f} \searrow & & \nearrow \widetilde{f} & & \overline{g} \searrow & & \nearrow \widetilde{g} & \\ & & \bullet & & & & \bullet & & \end{array}$$

By definition, a sequence of admissible morphisms

$$X_s \xrightarrow{f_s} \cdots \longrightarrow X_{i-1} \xrightarrow{f_{i-1}} X_i \xrightarrow{f_i} X_{i+1} \longrightarrow \cdots \xrightarrow{f_{t-1}} X_t$$

$(t \geqslant s+2)$ is *an exact sequence*, provided that it is exact at each X_j, $s+1 \leqslant j \leqslant t-1$, i.e., for the deflation-inflation-decompositions $f_j = d_j i_j$, one has $(i_{j-1}, d_j) \in \mathcal{E}$ for $s+1 \leqslant j \leqslant t-1$.

In particular, a sequence of admissible morphisms $0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$ is an exact sequence, provided that $(f, g) \in \mathcal{E}$.

Lemma 2.4 ([**4**, 2.12])**.** *Let $(\mathcal{A}, \mathcal{E})$ be an exact category.*

(1) *Consider the commutative square $(\mathcal{A}, \mathcal{E})$*

$$\begin{array}{ccc} B' & \xrightarrow{d'} & C' \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow{d} & C \end{array}$$

with deflations d and d'. Then the following are equivalent.

   (i) *The square is a pullback square.*

   (ii) *The sequence* $0 \to B' \xrightarrow{\left(\begin{smallmatrix} d' \\ -f' \end{smallmatrix}\right)} C' \oplus B \xrightarrow{(f,d)} C \to 0$ *is an exact sequence.*

   (iii) *The square is a pullback - pushout square.*

(iv) *There is a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B' & \xrightarrow{d'} & C' & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \xrightarrow{d} & C & \longrightarrow & 0 \end{array}$$

with exact rows.

(1′) *Consider the commutative square in* $(\mathcal{A}, \mathcal{E})$

$$\begin{array}{ccc} A & \xrightarrow{i} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ A' & \xrightarrow{i'} & B' \end{array}$$

with inflations i *and* i'. *Then the following are equivalent.*

(i′) *The square is a pushout square.*

(ii′) *The sequence* $0 \to A \xrightarrow{\binom{i}{-f}} B \oplus A' \xrightarrow{(f', i')} B' \to 0$ *is an exact sequence.*

(iii′) *The square is a pushout - pullback square.*

(iv′) *There is a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{i} & B & \longrightarrow & C & \longrightarrow & 0 \\ & & {\scriptstyle f}\downarrow & & {\scriptstyle f'}\downarrow & & \| & & \\ 0 & \longrightarrow & A' & \xrightarrow{i'} & B' & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

with exact rows.

The unique difference between Lemma 2.4 and the corresponding result in abelian categories is that, for example in (1), the condition that d' is also assumed to be a deflation. This is to guarantee that the morphisms $\binom{d'}{-f'}$ and (f, d) are admissible.

2.3. Weakly idempotent complete exact categories

In order to study various diagram lemmas on exactness in an exact category, one needs

Lemma 2.5 ([**6**, Appendix]; [**4**, 7.2, 7.6])**.** *Let* $\mathcal{A}$ *be an exact category. Then the following are equivalent*:

(i) *Any splitting epimorphism in* $\mathcal{A}$ *has a kernel.*

(ii) *Any splitting monomorphism in* $\mathcal{A}$ *has a cokernel.*

(iii) *If* de *is a deflation, then so is* d.

(iv) *If* ki *is an inflation, then so is* i.

In fact, the assertions (i) and (ii) are equivalent for any additive category (see [**4**, 7.1]). Following [**4**] and [**17**, 1.11.5], we call an exact category satisfying the above equivalent conditions in Lemma 2.5 *a weakly idempotent complete* exact category.

2.4. The morphism category of an exact category

Let $\mathcal{A}$ be an additive category. Recall that an object in the morphism category $\mathrm{Mor}(\mathcal{A})$ of $\mathcal{A}$ is a triple (X, α, Y), or simply α, where $\alpha\colon X \longrightarrow Y$ is a morphism in $\mathcal{A}$; and a morphism $(X, \alpha, Y) \longrightarrow (X', \alpha', Y')$ in $\mathrm{Mor}(\mathcal{A})$ is a pair (f, g) of morphisms with $f\colon X \longrightarrow X'$ and $g\colon Y \longrightarrow Y'$ in $\mathcal{A}$, such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X' & \xrightarrow{\alpha'} & Y' \end{array}$$

commutes. The following fact is well-known and the proof is straightforward.

Lemma 2.6. *Let $\mathcal{A}$ be an additive category.*

(1) *Let $(f_1, f_2)\colon \beta \longrightarrow \alpha$ and $(d_1, d_2)\colon \gamma \longrightarrow \alpha$ be two morphisms in the morphism category* $\mathrm{Mor}(\mathcal{A})$*. If f_1 and d_1 admit a pullback in $\mathcal{A}$, and f_2 and d_2 admit a pullback in $\mathcal{A}$, then (f_1, f_2) and (d_1, d_2) admit a pullback in* $\mathrm{Mor}(\mathcal{A})$*.*

(1′) *Let $(f_1, f_2)\colon \alpha \longrightarrow \beta$ and $(i_1, i_2)\colon \alpha \longrightarrow \gamma$ be two morphisms in the morphism category* $\mathrm{Mor}(\mathcal{A})$*. If f_1 and i_1 admit a pushout in $\mathcal{A}$, and f_2 and i_2 admit a pushout in $\mathcal{A}$, then (f_1, f_2) and (i_1, i_2) admit a pushout in* $\mathrm{Mor}(\mathcal{A})$*.*

Let $(\mathcal{A}, \mathcal{E})$ be an exact category. Denote by $\mathcal{E}_{\mathrm{Mor}}$ the class of a pair $((i, i'), (d, d'))$ of morphisms in $\mathrm{Mor}(\mathcal{A})$, such that both (i, d) and (i', d') are conflations, i.e., $(i, d),\ (i', d') \in \mathcal{E}$. Thus, one has the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & Y \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i'} \\ X' & \xrightarrow{\alpha'} & Y' \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle d'} \\ X'' & \xrightarrow{\alpha''} & Y'' \end{array}$$

with $(i, d),\ (i', d') \in \mathcal{E}$. The proof of the following fact is straightforward.

Lemma 2.7. *If $(\mathcal{A}, \mathcal{E})$ is an exact category, then so is* $(\mathrm{Mor}(\mathcal{A}), \mathcal{E}_{\mathrm{Mor}})$*; and if $(\mathcal{A}, \mathcal{E})$ is a weakly idempotent complete exact category, then so is* $(\mathrm{Mor}(\mathcal{A}), \mathcal{E}_{\mathrm{Mor}})$*.*

3. The strong version of Snake Lemma with applications

3.1. The strong version of Snake Lemma in weakly idempotent complete exact categories

We will prove the strong version of Snake Lemma from the simple version, in weakly idempotent complete exact categories.

Lemma 3.1 (The simple version of Snake Lemma, [**4**, 8.13])**.** *Consider the commutative diagram in a weakly idempotent complete exact category $(\mathcal{A}, \mathcal{E})$*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X_1 & \xrightarrow{x_1} & X_2 & \xrightarrow{x_2} & X_3 & \longrightarrow & 0 \\ & & {\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle f_3} & & \\ 0 & \longrightarrow & Y_1 & \xrightarrow{y_1} & Y_2 & \xrightarrow{y_2} & Y_3 & \longrightarrow & 0 \end{array} \qquad (3)$$

with exact rows and vertical admissible morphisms. Then one has a canonical sequence *of admissible morphisms*

$$0 \longrightarrow \operatorname{Ker} f_1 \longrightarrow \operatorname{Ker} f_2 \longrightarrow \operatorname{Ker} f_3 \longrightarrow \operatorname{Coker} f_1 \longrightarrow \operatorname{Coker} f_2 \longrightarrow \operatorname{Coker} f_3 \longrightarrow 0$$

such that it is exact.

Theorem 3.2 (The strong version of Snake Lemma)**.** *Assume the following diagram with exact rows and vertical admissible morphisms in a weakly idempotent complete exact category* $(\mathcal{A}, \mathcal{E})$

$$\begin{array}{ccccccccc} X_0 & \xrightarrow{x_0} & X_1 & \xrightarrow{x_1} & X_2 & \xrightarrow{x_2} & X_3 & \xrightarrow{x_3} & X_4 \\ {\scriptstyle f_0}\downarrow & & {\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow & & {\scriptstyle f_3}\downarrow & & \downarrow{\scriptstyle f_4} \\ Y_0 & \xrightarrow{y_0} & Y_1 & \xrightarrow{y_1} & Y_2 & \xrightarrow{y_2} & Y_3 & \xrightarrow{y_3} & Y_4 \end{array}$$

commutes, where f_0 *is a deflation and* f_4 *is an inflation. Then there is a canonical sequence of admissible morphisms*

$$\operatorname{Ker} f_1 \longrightarrow \operatorname{Ker} f_2 \longrightarrow \operatorname{Ker} f_3 \longrightarrow \operatorname{Coker} f_1 \longrightarrow \operatorname{Coker} f_2 \longrightarrow \operatorname{Coker} f_3$$

such that it is exact.

Moreover, if x_1 *is an inflation, then so is the induced morphism* $\operatorname{Ker} f_1 \longrightarrow \operatorname{Ker} f_2$*; and if* y_2 *is a deflation, then so is the induced morphism* $\operatorname{Coker} f_2 \longrightarrow \operatorname{Coker} f_3$*.*

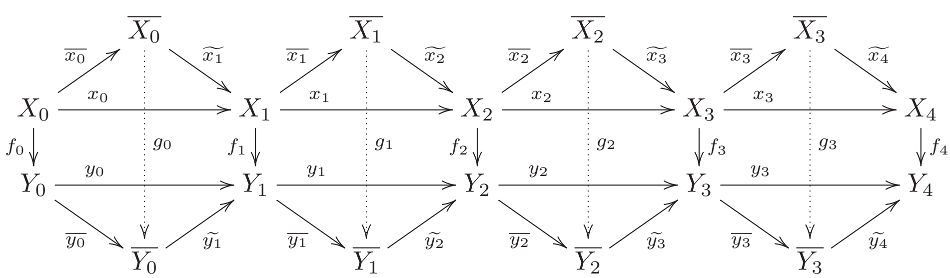
Proof. **Step 1.** Consider the deflation-inflation-decompositions

$$x_i = \widetilde{x_{i+1}}\,\overline{x_i}, \quad y_i = \widetilde{y_{i+1}}\,\overline{y_i}, \quad 0 \leqslant i \leqslant 3.$$

Thus, $\overline{x_0}$ and $\overline{y_0}$ are deflations, $(\widetilde{x_i}, \overline{x_i})$ are conflations for $1 \leqslant i \leqslant 3$, and $\widetilde{x_4}$ and $\widetilde{y_4}$ are inflations.

It is clear that there are morphisms g_i for $0 \leqslant i \leqslant 3$, such that all the trapezoids in the following diagram commute, i.e., one has

$$\widetilde{y_{i+1}}\,g_i = f_{i+1}\widetilde{x_{i+1}}, \quad g_i\overline{x_i} = \overline{y_i}f_i, \quad 0 \leqslant i \leqslant 3.$$

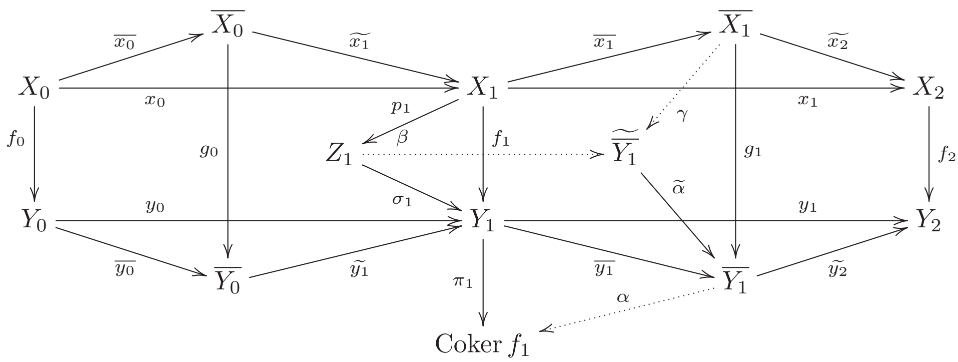


Step 2. It is clear that g_0 is a deflation. Dually, g_3 is an inflation.

In fact, by assumption f_0 and $\overline{y_0}$ are deflations, so is $g_0\overline{x_0} = \overline{y_0}f_0$. Thus g_0 is a deflation, by Lemma 2.5(iii).

Step 3. We claim that g_1 and g_2 are admissible morphisms.

We will only show that g_1 is admissible. By duality, g_2 is also admissible. By assumption $\overline{y_0}f_0$ is epic. Since $\pi_1\widetilde{y_1}\overline{y_0}f_0 = \pi_1 y_0 f_0 = \pi_1 f_1 x_0 = 0$, one has $\pi_1\widetilde{y_1} = 0$. Since $(\widetilde{y_1}, \overline{y_1})$ is a conflation, there is a unique morphism $\alpha\colon \overline{Y_1} \longrightarrow \operatorname{Coker} f_1$ such that $\pi_1 = \alpha\overline{y_1}$. See the following diagram. Since π_1 is a deflation, by Lemma 2.5(iii), α is a deflation, say, with conflation $(\widetilde{\alpha}, \alpha)$.



Let $f_1 = \sigma_1 p_1$ is the deflation-inflation-decomposition. Since $\alpha\overline{y_1}\sigma_1 = \pi_1\sigma_1 = 0$, there is a unique morphism $\beta\colon Z_1 \longrightarrow \widetilde{\overline{Y_1}}$ such that $\widetilde{\alpha}\beta = \overline{y_1}\sigma_1$. Thus one gets the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z_1 & \xrightarrow{\sigma_1} & Y_1 & \xrightarrow{\pi_1} & \operatorname{Coker} f_1 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\overline{y_1}} & & \| & & \\ 0 & \longrightarrow & \widetilde{\overline{Y_1}} & \xrightarrow{\widetilde{\alpha}} & \overline{Y_1} & \xrightarrow{\alpha} & \operatorname{Coker} f_1 & \longrightarrow & 0. \end{array}$$

By Lemma 2.4(1′), the left square of the diagram above is in particular a pullback square, hence β is a deflation, by (E2). Since

$$\widetilde{y_2}\widetilde{\alpha}\beta p_1\widetilde{x_1}\overline{x_0} = \widetilde{y_2}\overline{y_1}\sigma_1 p_1\widetilde{x_1}\overline{x_0} = y_1 f_1 x_0 = f_2 x_1 x_0 = 0,$$

and since $\overline{x_0}$ is epic and $\widetilde{y_2}\widetilde{\alpha}$ monic, one has $\beta p_1\widetilde{x_1} = 0$. Since $(\widetilde{x_1}, \overline{x_1})$ is a conflation, there is a unique morphism $\gamma\colon \overline{X_1} \longrightarrow \widetilde{\overline{Y_1}}$ such that $\gamma\overline{x_1} = \beta p_1$. Note that γ is a deflation, by Lemma 2.5(iii). Thus

$$\widetilde{y_2} g_1\overline{x_1} = \widetilde{y_2}\overline{y_1} f_1 = \widetilde{y_2}\overline{y_1}\sigma_1 p_1 = \widetilde{y_2}\widetilde{\alpha}\beta p_1 = \widetilde{y_2}\widetilde{\alpha}\gamma\overline{x_1}.$$

Since $\overline{x_1}$ is epic and $\widetilde{y_2}$ is monic, it follows that $g_1 = \widetilde{\alpha}\gamma$ is admissible.

Step 4. Since g_0 is a deflation, $\operatorname{Coker} g_0 = 0$. Applying Lemma 3.1 to the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \overline{X_0} & \xrightarrow{\widetilde{x_1}} & X_1 & \xrightarrow{\overline{x_1}} & \overline{X_1} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle g_0} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle g_1} & & \\ 0 & \longrightarrow & \overline{Y_0} & \xrightarrow{\widetilde{y_1}} & Y_1 & \xrightarrow{\overline{y_1}} & \overline{Y^1} & \longrightarrow & 0 \end{array}$$

one gets $\operatorname{Coker} f_1 \cong \operatorname{Coker} g_1$ and the exact sequence

$$0 \longrightarrow \operatorname{Ker} g_0 \longrightarrow \operatorname{Ker} f_1 \longrightarrow \operatorname{Ker} g_1 \longrightarrow 0. \tag{4}$$

Step 4’. Dually, one has $\operatorname{Ker} g_2 \cong \operatorname{Ker} f_3$ and an exact sequence

$$0 \longrightarrow \operatorname{Coker} g_1 \longrightarrow \operatorname{Coker} f_3 \longrightarrow \operatorname{Coker} g_3 \longrightarrow 0. \tag{5}$$

Step 5. Applying Lemma 3.1 to the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \overline{X_1} & \xrightarrow{\widetilde{x_2}} & X_2 & \xrightarrow{\overline{x_2}} & \overline{X_2} & \longrightarrow & 0 \\ & & {\scriptstyle g_1}\downarrow & & {\scriptstyle f_2}\downarrow & & {\scriptstyle g_2}\downarrow & & \\ 0 & \longrightarrow & \overline{Y_1} & \xrightarrow{\widetilde{y_2}} & Y_2 & \xrightarrow{\overline{y_2}} & \overline{Y_2} & \longrightarrow & 0 \end{array}$$

one gets an exact sequence

$$0 \to \operatorname{Ker} g_1 \to \operatorname{Ker} f_2 \to \operatorname{Ker} g_2 \to \operatorname{Coker} g_1 \to \operatorname{Coker} f_2 \to \operatorname{Coker} g_2 \to 0. \qquad (6)$$

Connecting (4) and (5) with (6), one gets the desired exact sequence

$$\begin{array}{ccccccccccccc} \operatorname{Ker} f_1 & & \longrightarrow & & \operatorname{Ker} f_2 & \longrightarrow & \operatorname{Ker} f_3 & \longrightarrow & \operatorname{Coker} f_1 & \longrightarrow & \operatorname{Coker} f_2 & \longrightarrow & \operatorname{Coker} f_3 \\ & \searrow & & \nearrow & & & & & & \searrow & & \nearrow & \\ & & \operatorname{Ker} g_1 & & & & & & & & \operatorname{Coker} g_2 & & \end{array}$$

This completes the proof. □

3.2. Consequences of the strong version of Snake Lemma

Corollary 3.3 (the classic version of Snake Lemma, [**4**, 8.15])**.** *Assume that the following diagram in a weakly idempotent complete exact category $(\mathcal{A}, \mathcal{E})$ commutes, with exact rows and admissible vertical morphisms.*

$$\begin{array}{ccccccccc} & & X_1 & \xrightarrow{x_1} & X_2 & \xrightarrow{x_2} & X_3 & \longrightarrow & 0 \\ & & {\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow & & {\scriptstyle f_3}\downarrow & & \\ 0 & \longrightarrow & Y_1 & \xrightarrow{y_1} & Y_2 & \xrightarrow{y_2} & Y_3 & & \end{array}$$

Then there is the canonical sequence of admissible morphisms

$$\operatorname{Ker} f_1 \longrightarrow \operatorname{Ker} f_2 \longrightarrow \operatorname{Ker} f_3 \longrightarrow \operatorname{Coker} f_1 \longrightarrow \operatorname{Coker} f_2 \longrightarrow \operatorname{Coker} f_3$$

such that it is exact.

Moreover, if x_1 is an inflation, then so is the induced morphism $\operatorname{Ker} f_1 \longrightarrow \operatorname{Ker} f_2$*; if y_2 is a deflation, then so is the induced morphism* $\operatorname{Coker} f_2 \longrightarrow \operatorname{Coker} f_3$*.*

Proof. One has the following diagram with $\mathcal{E}$-exact rows and admissible vertical morphisms:

$$\begin{array}{ccccccccc} \operatorname{Ker} x_1 & \xrightarrow{\sigma} & X_1 & \xrightarrow{x_1} & X_2 & \xrightarrow{x_2} & X_3 & \longrightarrow & 0 \\ \downarrow & & {\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow & & {\scriptstyle f_3}\downarrow & & \downarrow \\ 0 & \longrightarrow & Y_1 & \xrightarrow{y_1} & Y_2 & \xrightarrow{y_2} & Y_3 & \xrightarrow{\pi} & \operatorname{Coker} y_2 \end{array}$$

Since $y_1 f_1 \sigma = 0$ and y_1 is an monomorphism, it follows that $f_1 \sigma = 0$. Dually, one has $\pi f_3 = 0$. That is, the diagram above commutes. Thus, the desired exact sequence follows from Theorem 3.2, the strong version of Snake Lemma. □

Corollary 3.4 (Four Lemma)**.** *Let $(\mathcal{A}, \mathcal{E})$ be a weakly idempotent complete exact category.*

(1) *Assume that the following diagram in* $(\mathcal{A}, \mathcal{E})$ *commutes, with* $\mathcal{E}$*-exact rows*

$$\begin{array}{ccccccc} X_1 & \xrightarrow{x_1} & X_2 & \xrightarrow{x_2} & X_3 & \xrightarrow{x_3} & X_4 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} \\ Y_1 & \xrightarrow{y_1} & Y_2 & \xrightarrow{y_2} & Y_3 & \xrightarrow{y_3} & Y_4 \end{array}$$

where f_1 *is a deflation,* f_2 *and* f_4 *are inflations, and* f_3 *is admissible. Then* f_3 *is an inflation.*

(2) *Assume that the following diagram in* $(\mathcal{A}, \mathcal{E})$ *commutes, with* $\mathcal{E}$*-exact rows*

$$\begin{array}{ccccccc} X_1 & \xrightarrow{x_1} & X_2 & \xrightarrow{x_2} & X_3 & \xrightarrow{x_3} & X_4 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} \\ Y_1 & \xrightarrow{y_1} & Y_2 & \xrightarrow{y_2} & Y_3 & \xrightarrow{y_3} & Y_4 \end{array}$$

where f_4 *is an inflation,* f_1 *and* f_3 *are deflations, and* f_2 *is admissible. Then* f_2 *is a deflation.*

Proof. By duality one only needs to prove (1). From the **Step 2** in the proof of Theorem 3.2, one has the following commutative diagram

$$\begin{array}{ccccccccc} X_1 & \longrightarrow & X_2 & \longrightarrow & X_3 & \longrightarrow & \operatorname{Coker} x_2 & \longrightarrow & 0 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle g_3} & & \downarrow \\ Y_1 & \longrightarrow & Y_2 & \longrightarrow & Y_3 & \longrightarrow & \operatorname{Coker} y_2 & \longrightarrow & 0 \end{array}$$

with g_3 an inflation (this needs that f_4 is an inflation). Since by assumption f_2 is an inflation, by applying Theorem 3.2 one sees that $0 \longrightarrow \operatorname{Ker} f_3 \longrightarrow 0$ is an exact sequence, i.e., f_3 is an inflation. □

Corollary 3.5 (Five Lemma, [**4**, 8.9])**.** *Assume that the following diagram in a weakly idempotent complete exact category* $(\mathcal{A}, \mathcal{E})$ *commutes, with* $\mathcal{E}$*-exact rows and admissible vertical morphisms.*

$$\begin{array}{ccccccccc} X_1 & \longrightarrow & X_2 & \longrightarrow & X_3 & \longrightarrow & X_4 & \longrightarrow & X_5 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_5} \\ Y_1 & \longrightarrow & Y_2 & \longrightarrow & Y_3 & \longrightarrow & Y_4 & \longrightarrow & Y_5 \end{array}$$

(1) *If* f_1 *is a deflation, and* f_2 *and* f_4 *are inflations, then* f_3 *is an inflation.*

(2) *If* f_5 *is an inflation, and* f_2 *and* f_4 *are deflations, then* f_3 *is a deflation.*

(3) *If* f_1, f_2, f_4, f_5 *are isomorphisms, then so is* f_3*.*

Proof. The assertions (1) and (2) follow from Corollary 3.4(1) and Corollary 3.4(2), respectively. By Fact 2.2(3), the assertion (3) follows from of (1) and (2). □

3.3. Remarks

Remark 3.6. From the proofs above, the simple version (of Snake Lemma) $\Longrightarrow$ the strong version $\Longrightarrow$ the classic version $\Longrightarrow$ the simple version, i.e., the three versions are equivalent.

Remark 3.7. The strong version of Snake Lemma, and all the other exact diagram lemmas, are functorial: this can be precisely seen in $\mathrm{Mor}(\mathcal{A})$ (cf. Lemma 2.7).

Remark 3.8. Let $\mathcal{A}$ be a small abelian category. Freyd–Mitchell Embedding Theorem (see e.g. [**8**, p.150]) asserts that there is a fully faithful and exact functor $F\colon \mathcal{A} \longrightarrow R$-Mod for a ring R, so that F reflects the exactness. By this, Snake Lemma in an abelian category can be seen from the one in module category. See [**19**], [**9**], [**16**]. Since one needs Snake Lemma quite early, it is also reasonable to have an independent proof. See in [**10**] and [**14**].

Let $(\mathcal{A}, \mathcal{E})$ be a small exact category. Gabriel–Quillen Embedding Theorem ([**12**, A.2]; [**17**, A.7.1]) claims that there is a fully faithful exact functor $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ for an abelian category $\mathcal{B}$, such that $\mathrm{Im}\, F$ is closed under extensions, and F reflects the exactness (i.e., if $0 \longrightarrow FX \xrightarrow{Ff} FY \xrightarrow{Fg} FZ \longrightarrow 0$ is exact in $\mathcal{B}$, then $(f, g) \in \mathcal{E}$). However, Snake Lemma in an exact category is not a direct consequence of the one in abelian categories and Gabriel–Quillen Embedding Theorem.

For example, consider the simple version of Snake Lemma in a weakly idempotent complete exact category $(\mathcal{A}, \mathcal{E})$. Using Gabriel–Quillen Embedding Theorem and applying F to diagram (3), by Snake Lemma in abelian category, one gets a canonical exact sequence in $\mathcal{B}$

$$0 \to \mathrm{Ker}\, Ff_1 \to \mathrm{Ker}\, Ff_2 \to \mathrm{Ker}\, Ff_3 \to \mathrm{Coker}\, Ff_1 \to \mathrm{Coker}\, Ff_2 \to \mathrm{Coker}\, Ff_3 \to 0.$$

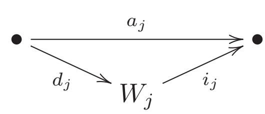
Since all f_j are admissible and F is exact, one indeed has $\mathrm{Ker}\, Ff_j \cong F\,\mathrm{Ker}\, f_j$ and $\mathrm{Coker}\, Ff_j \cong F\,\mathrm{Coker}\, f_j$, i.e., one has a canonical exact sequence in $\mathcal{B}$ (the existence of a_j is by F being full)

$$\begin{aligned} 0 \longrightarrow F\,\mathrm{Ker}\, f_1 &\xrightarrow{Fa_1} F\,\mathrm{Ker}\, f_2 \xrightarrow{Fa_2} F\,\mathrm{Ker}\, f_3 \\ &\xrightarrow{Fa_3} F\,\mathrm{Coker}\, f_1 \xrightarrow{Fa_4} F\,\mathrm{Coker}\, f_2 \xrightarrow{Fa_5} F\,\mathrm{Coker}\, f_3 \longrightarrow 0. \end{aligned} \tag{7}$$

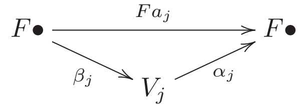
So we get a sequence of morphisms in $(\mathcal{A}, \mathcal{E})$

$$0 \to \mathrm{Ker}\, f_1 \xrightarrow{a_1} \mathrm{Ker}\, f_2 \xrightarrow{a_2} \mathrm{Ker}\, f_3 \xrightarrow{a_3} \mathrm{Coker}\, f_1 \xrightarrow{a_4} \mathrm{Coker}\, f_2 \xrightarrow{a_5} \mathrm{Coker}\, f_3 \to 0. \tag{8}$$

But, F reflects the exactness does not imply that (8) is exact in $(\mathcal{A}, \mathcal{E})$. The reason is that each a_j is not yet known to be admissible in $(\mathcal{A}, \mathcal{E})$. Indeed, if it is, with deflation-inflation-decomposition $a_j = i_j d_j$ (this is true if $(\mathcal{A}, \mathcal{E})$ is abelian; but it needs to be proved in an exact category)



then by the exactness of F we would have the epimorphism-monomorphism-decompositions $Fa_j = Fi_j Fd_j$, and by the exactness of (7) and the fact that F reflects the exactness, one indeed gets the exactness of (8). Although Fa_j has the epimorphism-monomorphism-decomposition $Fa_j = \beta_j \alpha_j$



but the problem is that whether or not V_j is in $\operatorname{Im} F$.

In fact, the proof of that all a_j are admissible in $(\mathcal{A}, \mathcal{E})$ needs heavily the axioms of an exact category. This explains that Snake Lemma in weakly idempotent complete exact categories needs a direct proof.

4. Weakly idempotent complete exact categories and Iversen's exact categories

A category with zero object, kernels and cokernels, such that the canonical morphism $\operatorname{Coim} f \longrightarrow \operatorname{Im} f$ is an isomorphism for each morphism f, is called *an exact category* by Iversen in his book [**10**, Definition 1.1]. To avoid confusions and for convenience, we call such a category *a generalized abelian category.* Thus, a category is abelian if and only if it is additive and generalized abelian. Typical examples of non-abelian generalized abelian categories are the category of cyclic groups and the category of A-modules with dimension lower than a fixed integer, where A is an algebra over a field.

Iversen [**10**, 1.6, p.4] has given the strong version of Snake Lemma in generalized abelian categories. To justify the necessity of the strong version of Snake Lemma in weakly idempotent complete exact categories, one needs examples of weakly idempotent complete exact categories which are not generalized abelian. Such examples can be found in [**4**, 13.2] from functional analysis. Below we will provide a class of such examples from representation theory of algebras.

Gorenstein-projective modules are introduced by Auslander and Bridger [**1**] (under the name of *modules of* G*-dimension zero*). We quickly recall an equivalent definition from Enochs and Jenda [**7**]. Let A be an Artin algebra, and A-mod the category of finitely generated left A-modules. *A complete A-projective resolution* is an exact sequence

$$P^\bullet: \qquad \cdots \longrightarrow P^{-1} \longrightarrow P^0 \xrightarrow{d^0} P^1 \longrightarrow \cdots$$

of finitely generated projective (left) A-modules, such that $\operatorname{Hom}_A(P^\bullet, A)$ is again an exact sequence. A left A-module M is *Gorenstein-projective*, if there is a complete A-projective resolution $P^\bullet$ such that $M \cong \operatorname{Ker} d^0$. Let $\mathcal{P}(A)$ be the full subcategory of A-mod consisting of projective modules, and $\mathcal{GP}(A)$ the full subcategory of A-mod consisting of Gorenstein-projective modules. Then $\mathcal{P}(A) \subseteq \mathcal{GP}(A) \subseteq {}^\perp A = \{X \in A\text{-mod} \mid \operatorname{Ext}^i(X, A) = 0, \ \forall \ i \geqslant 1\}$ and $\mathcal{GP}(A)$ is an extension-closed subcategory of A-mod, thus the embedding of $\mathcal{GP}(A)$ into A-mod gives an exact structure on $\mathcal{GP}(A)$, and $\mathcal{GP}(A)$ is a weakly idempotent complete exact category.

The dominant dimension $\operatorname{dom.dim} M$ of a module M ([**2**, p.211]) is the maximal integer t (or ∞) such that in a minimal injective resolution $0 \longrightarrow M \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \cdots \longrightarrow I_t \longrightarrow \cdots$ of M, all the modules I_j with $j < t$ are projective. Here one sets $\operatorname{dom.dim} M = 0$ if the injective envelope I_0 of M is not projective. Thus, if M is projective-injective, then $\operatorname{dom.dim} M = \infty$.

The following result is due to Kong.

Lemma 4.1 ([**13**, 2.1])**.** *Let A be an Artin algebra. Then $\mathcal{GP}(A)$ is an abelian category if and only if* $\operatorname{inj.dim}{}_A A \leqslant 2$ *and* $\operatorname{dom.dim}{}_A A \geqslant 2$.

By this result one gets a class of examples of weakly idempotent complete exact categories which are not generalized abelian.

Corollary 4.2.

(1) *Let A be an Artin algebra. Then $\mathcal{GP}(A)$ is a weakly idempotent complete exact category; and it is not generalized abelian if and only if either* $\operatorname{inj.dim}_A A \geqslant 3$ *or* $\operatorname{dom.dim}_A A \leqslant 1$.

(2) *Let A be the path algebra of a finite acyclic quiver Q. Then $\mathcal{P}(A)$ is a weakly idempotent complete exact category and not generalized abelian.*

Proof. Since $\mathcal{GP}(A)$ is additive, the assertion (1) follows immediately from Lemma 4.1.

(2) In this case $\mathcal{P}(A) = \mathcal{GP}(A)$.

If Q is not of type A_n with linear orientation, then there are no non-zero projective-injective A-modules. Thus $\operatorname{dom.dim}_A A = 0$. It follows from (1) that $\mathcal{P}(A) = \mathcal{GP}(A)$ is a weakly idempotent complete exact category and not generalized abelian.

If A is the path algebra of the quiver $1 \longrightarrow \cdots \longrightarrow n$ $(n \geqslant 1)$, then $\operatorname{dom.dim}_A A = 1$. Again by (1), $\mathcal{P}(A) = \mathcal{GP}(A)$ is a weakly idempotent complete exact category and not generalized abelian. However, in this case we give a direct justification, only using the definition.

Write the connection of paths from right to left. Then $P(1) = I(n)$ is the unique left indecomposable projective-injective module with $\operatorname{rad} P(1) = P(2)$. Consider the embedding $\sigma\colon P(2) \longrightarrow P(1)$. In $\mathcal{P}(A)$, it is clear that $\operatorname{Ker}\sigma = (0 \longrightarrow P(2))$, and that $\operatorname{Coker}(0 \longrightarrow P(2)) = (\operatorname{Id}_{P(2)}\colon P(2) \longrightarrow P(2))$. So

$$\operatorname{Coim}\sigma = \operatorname{Coker}(\operatorname{Ker}\sigma) = (\operatorname{Id}_{P(2)}\colon P(2) \longrightarrow P(2))$$

or simply $\operatorname{Coim}\sigma = P(2)$. On the other hand, in $\mathcal{P}(A)$, $\operatorname{Coker}\sigma = (P(1) \longrightarrow 0)$, and $\operatorname{Ker}(P(1) \longrightarrow 0) = (\operatorname{Id}_{P(1)}\colon P(1) \longrightarrow P(1))$. So

$$\operatorname{Im}\sigma = \operatorname{Ker}(\operatorname{Coker}\sigma) = (\operatorname{Id}_{P(1)}\colon P(1) \longrightarrow P(1))$$

or simply $\operatorname{Im}\sigma = P(1)$. Thus $\operatorname{Coim}\sigma \ncong \operatorname{Im}\sigma$ in $\mathcal{P}(A)$, which shows that $\mathcal{P}(A)$ is not generalized abelian. □

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