Homology, Homotopy and Applications, vol. 23(2), 2021, pp.141–149

MONOIDS OF SELF-MAPS OF TOPOLOGICAL SPHERICAL SPACE FORMS

DAISUKE KISHIMOTO AND NOBUYUKI ODA

(*communicated by Donald M. Davis*)

Abstract

A topological spherical space form is the quotient of a sphere by a free action of a finite group. In general, their homotopy types depend on specific actions of a group. We show that the monoid of homotopy classes of self-maps of a topological spherical space form is determined by the acting group and the dimension of the sphere, not depending on a specific action.

1. Introduction

Let X be a pointed space, and let $\mathcal{M}(X)$ denote the pointed homotopy set $[X, X]$. Then $\mathcal{M}(X)$ is a monoid under the composition of maps. The monoid $\mathcal{M}(X)$ is obviously fundamental for understanding the space X. Invertible elements of $\mathcal{M}(X)$ form a group, which is the group of self-homotopy equivalences of X, denoted by $\mathcal{E}(X)$. The groups of self-homotopy equivalences have been intensely studied so that there are a lot of results on them. There is a comprehensive survey on them [**8**]. However, despite its importance, not much is known about the monoids of self-maps $\mathcal{M}(X)$, and in particular, there are only two cases that we know an explicit description of $\mathcal{M}(X)$: the case X is a sphere or a complex projective spaces. Notice that $\mathcal{M}(X)$ has not been determined even in the case X is a real projective space or a lens space.

A topological spherical space form is, by definition, the quotient space of a sphere by a free action of a finite group. Then real projective spaces and lens spaces are typical examples of such. We refer to [**4**] for details about topological spherical space forms. The purpose of this paper is to determine the monoids of self-maps of topological spherical space forms.

We recall basic facts about free actions of finite groups on spheres. Let G be a finite group acting freely on S^n. Then it is well known that

$$H^{2n+2}(BG; \mathbb{Z}) \cong \mathbb{Z}/|G|. \tag{1}$$

If n is even, then G must be a cyclic group of order 2, and so S^n/G is homotopy equivalent to $\mathbb{R}P^n$. Suppose n is odd. Then every orientation reversing self-map of S^n has a fixed point by the Lefschetz fixed-point theorem, and so the action of G on S^n is orientation-preserving. Hence S^n/G is an oriented compact connected manifold.

Received April 20, 2020; published on May 12, 2021.
2010 Mathematics Subject Classification: 55Q05.
Key words and phrases: monoid of self-maps, topological spherical space form, equivariant Hopf theorem.
Article available at `http://dx.doi.org/10.4310/HHA.2021.v23.n2.a8`

First, we state the main theorem in odd dimension. Let G be a finite group acting freely on S^{2n+1}. We introduce a new monoid out of a finite group G. Let $\alpha \in \mathrm{End}(G)$. By (1), the induced map $\alpha_*\colon H^{2n+2}(BG;\mathbb{Z}) \to H^{2n+2}(BG;\mathbb{Z})$ is identified with an element of $\mathbb{Z}/|G|$, which gives rise to a monoid homomorphism

$$d\colon \mathrm{End}(G) \longrightarrow (\mathbb{Z}/|G|)_\times \tag{2}$$

where $(\mathbb{Z}/m)_\times$ denotes the monoid of integers mod m under multiplication. Let M_α be a subset $d(\alpha) + |G|\mathbb{Z}$ of $\mathbb{Z}$. Then we can define a new monoid by

$$M(G,n) = \coprod_{\alpha \in \mathrm{End}(G)} M_\alpha$$

such that the product of $x \in M_\alpha$ and $y \in M_\beta$ is $xy \in M_{\alpha\beta}$. Clearly, the identity element of $M(G,n)$ is $1 \in M_1$.

Now we are ready to state the main theorem in odd dimension.

Theorem 1.1. *Let G be a finite group acting freely on S^{2n+1}. Then there is an isomorphism*

$$\mathcal{M}(S^{2n+1}/G) \cong M(G,n).$$

Here is an important remark. It is well known that lens spaces of the same dimension with the same π_1 can have different homotopy types. See [**6**, Theorem VI]. Then different free actions of the same finite group G on the same sphere S^{2n+1} can produce topological spherical sphere forms of different homotopy types. However, Theorem 1.1 implies that the monoid of self-maps does not distinguish actions.

When G is abelian, the map d in (2) is explicitly given in terms of the order of G and the integer n, where G acts freely on S^{2n+1}. Then a more precise description of $\mathcal{M}(S^{2n+1}/G)$ is available, which will be shown in Section 3. For example, one gets:

Corollary 1.2. $\mathcal{M}(\mathbb{R}P^{2n+1}) \cong \mathbb{Z}_\times$.

Next, we state the main theorem in even dimension. As mentioned above, each of topological spherical forms of dimension $2n$ is of the homotopy type of $\mathbb{R}P^{2n}$. Then the even dimensional case is covered by the following.

Theorem 1.3. *Let $M = \mathbb{Z}_\times/\sim$ where $x \sim y$ for $x, y \in \mathbb{Z}_\times$ if $x \equiv y \equiv 0$ or $2 \bmod 4$. Then*

$$\mathcal{M}(\mathbb{R}P^{2n}) \cong M.$$

Remark 1.4. In [**5**] McGibbon claimed that $\mathcal{M}(\mathbb{R}P^{2n})$ has cardinality two, but this is false as pointed out by Fred Cohen. This was fixed in [**3**] by Iriye, Matsushita and the first author, but the monoid structure was not considered.

Finally, we present two corollaries of Theorem 1.1. The monoid of self-maps is not abelian in general. See [**1**] for instance. But by Theorem 1.3, $\mathcal{M}(\mathbb{R}P^{2n})$ is abelian. Moreover, if G is abelian, implying G is cyclic, then $M(G,n)$ is so, hence $\mathcal{M}(S^{2n+1}/G)$ by Theorem 1.1. Thus one gets:

Corollary 1.5. *Let G be a finite abelian group acting freely on S^{2n+1}. Then $\mathcal{M}(S^{2n+1}/G)$ is abelian.*

Let $E(G,n)$ denote the group of invertible elements of $M(G,n)$. Then by Theorem 1.1, $\mathcal{E}(S^{2n+1}/G) \cong E(G,n)$. Clearly, $E(G,n)$ consists of ± 1 in M_α for $\alpha \in \mathrm{Aut}(G)$, that is,

$$E(G,n) = \coprod_{\alpha \in \mathrm{Aut}(G)} M_\alpha \cap \{\pm 1\}.$$

If $|G| > 2$, then $|M_\alpha \cap \{\pm 1\}| \leqslant 1$, implying $E(G,n) \cong \{\alpha \in \mathrm{Aut}(G) \mid d(\alpha) = \pm 1\}$. On the other hand, for $|G| \leqslant 2$, $E(G,n) = M_1 = \{\pm 1\} = C_2$. Thus we obtain the following, which reproves the result of Smallen [**9**] and Plotnick [**7**], where their results seem to exclude the case $|G| \leqslant 2$.

Corollary 1.6. *Let G be a finite group acting freely on S^{2n+1}. Then*

$$\mathcal{E}(S^{2n+1}/G) \cong \begin{cases} \{\alpha \in \mathrm{Aut}(G) \mid d(\alpha) = \pm 1\} & |G| \geqslant 3, \\ C_2 & |G| \leqslant 2. \end{cases}$$

Acknowledgments

The first author was supported by JSPS KAKENHI No. 17K05248.

2. Mapping degree

In this section, we characterize self-maps of odd dimensional topological spherical space forms in terms of mapping degrees and the induced maps on the fundamental groups.

Recall that we can define the mapping degree of a map $f\colon X \to Y$, denoted by $\deg(f)$, in the following cases:

1. X is an n-dimensional CW-complex with $H_n(X;\mathbb{Z}) \cong \mathbb{Z}$ and Y is an $(n-1)$-connected space with $\pi_n(Y) \cong \mathbb{Z}$.
2. X and Y are oriented compact connected manifolds of dimension n.

In particular, we can define the mapping degrees of self-maps of topological spherical space forms of odd dimension.

Let G be a group, and let X, Y be G-spaces. We denote the set of G-equivariant homotopy classes of G-equivariant maps from X to Y by $[X,Y]_G$. The following can be easily deduced from the equivariant Hopf degree theorem [**10**, Theorem 8.4.1].

Lemma 2.1. *Let G be a finite group. Let X be a free G-complex of dimension n such that $H_n(X;\mathbb{Z}) \cong \mathbb{Z}$, and let Y be an $(n-1)$-connected G-space with $\pi_n(Y) \cong \mathbb{Z}$. Then the map*

$$[X,Y]_G \to \mathbb{Z}, \quad [f] \mapsto \deg(f)$$

is injective.

The following lemma is proved by Olum [**6**, Theorem IIIc] in the case that $|G|$ is odd, where we impose nothing on $|G|$.

Lemma 2.2. *Let G be a finite group acting freely on S^{2n+1}. Then for $f, g\colon S^{2n+1}/G \to S^{2n+1}/G$, the following are equivalent:*

1. *f and g are homotopic;*
2. *$\pi_1(f) = \pi_1(g)$ and $\deg(f) = \deg(g)$.*

Proof. Suppose that (2) holds. Let $\tilde{f}, \tilde{g} \colon S^{2n+1} \to S^{2n+1}$ be lifts of f, g, respectively. Let X be a sphere S^{2n+1} equipped with a G-action which is the composite of $\pi_1(f) = \pi_1(g)$ and a given G-action on S^{2n+1}. Then $\tilde{f}, \tilde{g}$ are G-equivariant maps $S^{2n+1} \to X$. Since the projection $S^{2n+1} \to S^{2n+1}/G$ is injective in H_{2n+1}, $\deg(\tilde{f}) = \deg(f) = \deg(g) = \deg(\tilde{g})$. Then by applying Lemma 2.1 to $[S^{2n+1}, X]_G$, we obtain that $\tilde{f}$ and $\tilde{g}$ are G-equivariantly homotopic. Thus f and g are homotopic, and so (2) implies (1). Clearly, (1) implies (2). Therefore the proof is complete. □

3. Proof of Theorem 1.1

Lemma 3.1. *Let $S^n \to E \to B$ be a fibration such that $H^{n+1}(B; \mathbb{Z}) \cong \mathbb{Z}/m$ and the transgression $\tau \colon H^n(S^n; \mathbb{Z}) \to H^{n+1}(B; \mathbb{Z})$ is surjective. If there is a homotopy commutative diagram*

$$\begin{array}{ccccc} S^n & \longrightarrow & E & \longrightarrow & B \\ {\scriptstyle f}\downarrow & & \downarrow & & \downarrow{\scriptstyle g} \\ S^n & \longrightarrow & E & \longrightarrow & B \end{array}$$

such that $g^* = k \colon H^{n+1}(B; \mathbb{Z}) \to H^{n+1}(B; \mathbb{Z})$, then

$$\deg(f) \equiv k \bmod m.$$

Proof. Let u denote a generator of $H^n(S^n; \mathbb{Z}) \cong \mathbb{Z}$. Then by naturality

$$\deg(f)\tau(u) = \tau(\deg(f)u) = \tau(f^*(u)) = g^*(\tau(u)) = k\tau(u)$$

and since $\tau(u)$ is of order m, $\deg(f) \equiv k \bmod m$, as claimed. □

Lemma 3.2. *Let G be a finite group acting freely on S^{2n+1}, and let α be any endomorphism of G. Then for any integer k with $k \equiv d(\alpha) \bmod |G|$, there is $f \colon S^{2n+1}/G \to S^{2n+1}/G$ such that*

$$\deg(f) = k \quad \text{and} \quad \pi_1(f) = \alpha.$$

Proof. Let $u \colon BG \to K(\mathbb{Z}, 2n+2)$ denote a generator of $H^{2n+2}(BG; \mathbb{Z}) \cong \mathbb{Z}/|G|$, and let F denote the homotopy fiber of u. Then F is the second stage Postnikov tower of S^{2n+1}/G, and so there is a natural map $g \colon S^{2n+1}/G \to F$. Let F^{2n+1} and X denote the $(2n+1)$-skeleton of F and the homotopy fiber of the canonical map $F^{2n+1} \to BG$, respectively. By considering the Serre spectral sequence associated to a homotopy fibration $X \to F^{2n+1} \to BG$, one gets

$$H^*(X; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & * = 0, 2n+1, \\ 0 & * \neq 0, 2n+1. \end{cases}$$

Moreover, X is simply connected since the map $F^{2n+1} \to BG$ is an isomorphism in π_1. Then $X \simeq S^{2n+1}$. Now there is a homotopy commutative diagram

$$\begin{array}{ccccc} S^{2n+1} & \longrightarrow & S^{2n+1}/G & \longrightarrow & BG \\ \| & & \downarrow & & \| \\ S^{2n+1} & \longrightarrow & F^{2n+1} & \longrightarrow & BG \end{array}$$

where rows are homotopy fibrations. Then $S^{2n+1}/G \simeq F^{2n+1}$, and so the map $g\colon S^{2n+1}/G \to F$ is identified with the inclusion of the $(2n+1)$-skeleton.

Let α and k be as in the statement. Then there is a homotopy commutative diagram

$$\begin{array}{ccccc} F & \longrightarrow & BG & \xrightarrow{u} & K(\mathbb{Z}, 2n+2) \\ {\scriptstyle \widetilde{\alpha}}\downarrow & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle k} \\ F & \longrightarrow & BG & \xrightarrow{u} & K(\mathbb{Z}, 2n+2). \end{array}$$

Then $\widetilde{\alpha}^* = k$ on $H^{2n+1}(F;\mathbb{Z}) \cong \mathbb{Z}$ and $\pi_1(\widetilde{\alpha}) = \alpha$. Thus the restriction of $\widetilde{\alpha}$ to S^{2n+1}/G, which is identified with the $(2n+1)$-skeleton of F, is the desired map. □

Proof of Theorem 1.1. For $\alpha \in \mathrm{End}(G)$, let

$$N_\alpha = \{f \in \mathcal{M}(S^{2n+1}/G) \mid \pi_1(f) = \alpha \text{ and } \deg(f) \equiv d(\alpha) \bmod |G|\}.$$

By Lemmas 3.1 and 3.2, $\mathcal{M}(S^{2n+1}/G) = \coprod_{\alpha\in\mathrm{End}(G)} N_\alpha$ as a set. For $f \in N_\alpha$ and $g \in N_\beta$, one has

$$\deg(fg) = \deg(f)\deg(g), \qquad \pi_1(fg) = \pi_1(f)\pi_1(g). \tag{3}$$

By Lemmas 2.2, 3.1 and 3.2, the map

$$N_\alpha \to M_\alpha, \quad f \mapsto \deg(f)$$

is well-defined and bijective. Then one gets a bijection $\mathcal{M}(S^{2n+1}/G) \to M(G,n)$. Moreover, this map is a monoid homomorphism by (3). Thus the proof is complete. □

We describe M_α in $M(G,n)$ when G is cyclic. Let C_m denote a cyclic group of order m, and consider a free action of C_m on S^{2n+1}. Since

$$H^*(BC_m;\mathbb{Z}) = \mathbb{Z}[x]/(mx), \quad |x| = 2,$$

the map $d\colon \mathrm{End}(C_m) \to (\mathbb{Z}/m)_\times$ is given by $d(\alpha) = \alpha^{n+1}$, where $\alpha \in \mathrm{End}(C_m)$ is assumed to be an element of $(\mathbb{Z}/m)_\times$ through a natural isomorphism $\mathrm{End}(C_m) \cong (\mathbb{Z}/m)_\times$. Then

$$M_r = r^{n+1} + m\mathbb{Z}$$

for $r \in (\mathbb{Z}/m)_\times \cong \mathrm{End}(G)$. This gives us, for example, an explicit description of the monoid of self-maps of a lens space. From the description of M_r above, one can see that there is an isomorphism $M(C_2,n) \cong \mathbb{Z}_\times$, hence Corollary 1.2.

4. Proof of Theorem 1.3

First, we set notation that we are going to use in this section. Let $p\colon S^n \to \mathbb{R}P^n$ denote the universal covering, and let $q\colon \mathbb{R}P^n \to S^n$ be the pinch map onto the top cell. Let $j\colon \mathbb{R}P^{n-1} \to \mathbb{R}P^n$ denote the inclusion. We collect well known facts about real projective spaces that we are going to use. See [**2**] for the proof.

Lemma 4.1. *Let n be an integer $\geqslant 2$.*

1. *The mod 2 cohomology of $\mathbb{R}P^n$ is given by*
$$H^*(\mathbb{R}P^n;\mathbb{Z}/2) = \mathbb{Z}/2[w]/(w^{n+1}), \quad |w| = 1.$$
2. *$q_*\colon H_n(\mathbb{R}P^n;\mathbb{Z}/2) \to H_n(S^n;\mathbb{Z}/2)$ is an isomorphism.*
3. *The composite*
$$S^n \xrightarrow{p} \mathbb{R}P^n \xrightarrow{q} S^n$$
is of degree $1 + (-1)^{n+1}$.
4. *There is a cofibration*
$$S^{n-1} \xrightarrow{p} \mathbb{R}P^{n-1} \xrightarrow{j} \mathbb{R}P^n.$$

First, we determine $\mathcal{M}(\mathbb{R}P^{2n})$ as a set in a way different from [**3**]. Since $\pi_1(\mathbb{R}P^{2n}) \cong \mathbb{Z}/2$ and $\mathrm{End}(\mathbb{Z}/2) \cong (\mathbb{Z}/2)_\times$, we have the following decomposition

Lemma 4.2. *For $r = 0, 1$, Let $M_r = \{f \in \mathcal{M}(\mathbb{R}P^{2n}) \mid \pi_1(f) = r\}$. Then*
$$\mathcal{M}(\mathbb{R}P^{2n}) = M_0 \sqcup M_1.$$

Let $f \in M_0$. Then f lifts to S^{2n}, implying that f factors as the composite
$$\mathbb{R}P^{2n} \xrightarrow{q} S^{2n} \xrightarrow{k} S^{2n} \xrightarrow{p} \mathbb{R}P^{2n} \tag{4}$$
for some integer $k \in \mathbb{Z}$. Thus $M_0 = p_* \circ q^*(\pi_{2n}(S^{2n}))$. There is an exact sequence of pointed sets
$$[\mathbb{R}P^{2n}, C_2] \longrightarrow [\mathbb{R}P^{2n}, S^{2n}] \xrightarrow{p_*} \mathcal{M}(\mathbb{R}P^{2n})$$
induced from the covering $C_2 \to S^{2n} \xrightarrow{p} \mathbb{R}P^{2n}$. Since $[\mathbb{R}P^{2n}, C_2] = *$, one sees that $p_*\colon [\mathbb{R}P^{2n}, S^{2n}] \to \mathcal{M}(\mathbb{R}P^{2n})$ is injective by considering the action of $[\mathbb{R}P^{2n}, C_2]$ on $[\mathbb{R}P^{2n}, S^{2n}]$. On the other hand, there is a diagram
$$\begin{array}{ccccc}
\pi_{2n}(S^{2n}) & & & & \\
\downarrow{\scriptstyle \Sigma q^*} & & & & \\
[\Sigma\mathbb{R}P^{2n-1}, S^{2n}] & \xrightarrow{\Sigma p^*} & \pi_{2n}(S^{2n}) & \xrightarrow{q^*} & [\mathbb{R}P^{2n}, S^{2n}] \\
\downarrow{\scriptstyle \Sigma j^*} & & & & \\
[\Sigma\mathbb{R}P^{2n-2}, S^{2n}] & & & &
\end{array}$$
in which the column and the row are exact sequences of pointed sets induced from the cofibration in Lemma 4.1 (4). Since $[\Sigma\mathbb{R}P^{2n-2}, S^{2n}] = *$, the map $\Sigma q^*\colon \pi_{2n}(S^{2n}) \to [\Sigma\mathbb{R}P^{2n-1}, S^{2n}]$ is surjective. For $k \in \mathbb{Z}$, let $a_{2k} \in \mathcal{M}(\mathbb{R}P^{2n})$ be the composite (4). Then one gets the following by Lemma 4.1 (3).

Proposition 4.3. *$M_0 = \{a_0, a_2\}$ such that $a_{4k} = a_0$ and $a_{4k+2} = a_2$ for each $k \in \mathbb{Z}$.*

Lemma 4.4. *For each $l \in \mathbb{Z}$, there is a unique $b_{2l+1} \in M_1$ which lifts to a map $S^{2n} \to S^{2n}$ of degree $2l+1$.*

Proof. First, we reproduce the construction of the map b_{2l+1} in [**3**]. Let $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. Consider the antipodal action of C_2 on S^1 and the canonical free action of C_2 on S^0. Then the diagonal C_2-action on $S^1 * \underbrace{S^0 * \cdots * S^0}_{2n-1}$ is identified with the antipodal action on S^{2n}, where $S^1 * \underbrace{S^0 * \cdots * S^0}_{2n-1} = S^{2n}$. Define $f_l \colon S^1 \to S^1$ by $f_l(z) = z^{2l+1}$ for $z \in S^1$. Since f_l is a C_2-map of degree $2l+1$, the map

$$f_l * \underbrace{1 * \cdots * 1}_{2n-1} \colon S^1 * \underbrace{S^0 * \cdots * S^0}_{2n-1} = S^{2n} \to S^1 * \underbrace{S^0 * \cdots * S^0}_{2n-1} = S^{2n}$$

is a C_2-map of degree $2l+1$. Then we get $b_{2l+1} \in M_1$.

Next, we show the uniqueness of b_{2l+1}. Let $b'_{2l+1} \in M_1$ be a map which lifts to a map $S^{2n} \to S^{2n}$ of degree $2l+1$. Clearly, this lift is a C_2-map. Then the lifts of b_{2l+1} and b'_{2l+1} are C_2-maps $S^{2n} \to S^{2n}$ of the same degree $2l+1$, and so by Lemma 2.1, these lifts are C_2-equivariantly homotopic. Thus b_{2l+1} and b'_{2l+1} are homotopic, completing the proof. □

Now we are ready to determine M_1.

Proposition 4.5. $M_1 = \{b_{2l+1} \mid l \in \mathbb{Z}\}$.

Proof. The inclusion $M_1 \supset \{b_{2l+1} \mid l \in \mathbb{Z}\}$ follows from Lemma 4.4. Let $f \in M_1$. Then f lifts to a map $g \colon S^{2n} \to S^{2n}$. Since $\pi_1(f) = 1$, $f^* = 1$ on $H^1(\mathbb{R}P^n; \mathbb{Z}/2)$. Then there is a homotopy commutative diagram

$$\begin{array}{ccc} S^{2n} & \xrightarrow{g} & S^{2n} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ \mathbb{R}P^{2n} & \xrightarrow{f} & \mathbb{R}P^{2n} \\ \downarrow & & \downarrow \\ \mathbb{R}P^{\infty} & = & \mathbb{R}P^{\infty} \end{array}$$

in which columns are homotopy fibrations. Since the action of $\pi_1(\mathbb{R}P^\infty)$ on $H^*(S^{2n}; \mathbb{Z}/2)$ is trivial and the transgression $\tau \colon H^{2n}(S^{2n}; \mathbb{Z}/2) \to H^{2n+1}(\mathbb{R}P^\infty; \mathbb{Z}/2)$ is an isomorphism, we can apply a mod 2 cohomology analog of Lemma 3.1 to get that g is of odd degree. Thus we obtain the inclusion $M_1^n \subset \{b_{2l+1} \mid l \in \mathbb{Z}\}$, completing the proof. □

Next, we determine the monoid structure of $\mathcal{M}(\mathbb{R}P^{2n})$.

Lemma 4.6. *In $\mathcal{M}(\mathbb{R}P^{2n})$,*

$$a_{2k}a_{2k'} = a_0, \qquad a_{2k}b_{2l+1} = b_{2l+1}a_{2k} = a_{2k(2l+1)}, \qquad b_{2l+1}b_{2l'+1} = b_{(2l+1)(2l'+1)}.$$

Proof. By definition, $a_{2k}a_{2k'}$ is the composite

$$\mathbb{R}P^{2n} \xrightarrow{q} S^{2n} \xrightarrow{k'} S^{2n} \xrightarrow{p} \mathbb{R}P^{2n} \xrightarrow{q} S^{2n} \xrightarrow{k} S^{2n} \xrightarrow{p} \mathbb{R}P^{2n}.$$

Then the first equality follows from Lemma 4.1 (3).

There is a homotopy commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & & & S^{2n} & \xrightarrow{2l+1} & S^{2n} \\
 & & & & & \diagup\!\!\diagup & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\
\mathbb{R}P^{2n} & \xrightarrow{q} & S^{2n} & \xrightarrow{k} & S^{2n} & \xrightarrow{p} & \mathbb{R}P^{2n} & \xrightarrow{b_{2l+1}} & \mathbb{R}P^{2n}.
\end{array}$$

The composite of the bottom maps is $b_{2l+1}a_{2k}$ and the composite around the upper perimeter is $a_{2k(2l+1)}$. Then one gets $b_{2l+1}a_{2k} = a_{2k(2l+1)}$.

As in the proof of Lemma 4.4, one can construct a map $b_{2l+1}\colon \mathbb{R}P^{2n-1} \to \mathbb{R}P^{2n-1}$ which lifts to a map $S^{2n-1} \to S^{2n-1}$ of degree $2l+1$ and is a restriction of $b_{2l+1}\colon \mathbb{R}P^{2n} \to \mathbb{R}P^{2n}$. Then by Lemma 4.1 (4), there is a homotopy commutative diagram:

$$\begin{array}{ccccccc}
S^{2n-1} & \xrightarrow{p} & \mathbb{R}P^{2n-1} & \xrightarrow{j} & \mathbb{R}P^{2n} & \xrightarrow{q} & S^{2n} \\
{\scriptstyle 2l+1}\downarrow & & \downarrow{\scriptstyle b_{2l+1}} & & \downarrow{\scriptstyle b_{2l+1}} & & \downarrow{\scriptstyle 2l+1} \\
S^{2n-1} & \xrightarrow{p} & \mathbb{R}P^{2n-1} & \xrightarrow{j} & \mathbb{R}P^{2n} & \xrightarrow{q} & S^{2n}.
\end{array}$$

Thus $a_{2k}b_{2l+1} = p \circ k \circ q \circ b_{2l+1} = p \circ k \circ (2l+1) \circ q = a_{2k(2l+1)}$.

Clearly, $b_{2l+1}b_{2l'+1}$ belongs to M_1 and lifts to a map $S^{2n} \to S^{2n}$ of degree $(2l+1)(2l'+1)$. Then the third equality holds. □

Proof of Theorem 1.3. By Propositions 4.3 and 4.5 and Lemma 4.6, one gets $\mathcal{M}(\mathbb{R}P^{2n}) = \{a_{2k}, b_{2l+1} \mid k = 0, 1 \text{ and } l \in \mathbb{Z}\}$ such that for $i, j = 0, 2$,

$$a_i a_j = a_0, \qquad a_i b_{2l+1} = b_{2l+1} a_i = a_i, \qquad b_{2l+1} b_{2l'+1} = b_{(2l+1)(2l'+1)}.$$

Clearly, the map

$$f\colon \mathcal{M}(\mathbb{R}P^{2n}) \to M, \qquad f(a_{2i}) = 2i \qquad (i = 0, 1), \qquad f(b_{2l+1}) = 2l+1 \quad (l \in \mathbb{Z})$$

is well defined. Furthermore, it is bijective and a monoid homomorphism. Thus the proof is complete. □

References

[1] C. Costoya and A. Viruel, Every finite group is the group of self-homotopy equivalences of an elliptic space, Acta Math. **213**, no. 1 (2014), 49–62.

[2] A. Hatcher, Algebraic Topology, Cambridge University Press, Cambridge, 2002.

[3] K. Iriye, D. Kishimoto, and T. Matsushita, Relative phantom maps, Algebr. Geom. Topol. **19** (2019), no. 1, 341–362.

[4] I. Hambleton, Topological spherical space forms, Handbook of group actions, Vol. II, 151–172, Adv. Lect. Math. **32**, Int. Press, Somerville, MA, 2015.

[5] C.A. McGibbon, Self-maps of projective spaces, Trans. Amer. Math. Soc. **271** (1982), 325–346.

[6] P. Olum, Mappings of manifolds and the notion of degree, Ann. of Math. **58**, no. 3 (1953), 458–480.

[7] S. Plotnick, Homotopy equivalences and free modules, Topology **21**, no. 1 (1982), 91–99.

[**8**] J. Rutter, Spaces of homotopy self-equivalences: A survey, Lecture Notes in Mathematics **162**, Springer-Verlag, Berlin, 1997.

[**9**] D. Smallen, The group of self-equivalences of certain complexes, Pacific J. Math. **54** (1974), 269–276.

[**10**] T. tom Dieck, Transformation Groups and Representation Theory, Lecture Notes in Mathematics **766**, Springer-Verlag, Berlin, 1979.

Daisuke Kishimoto `kishi@math.kyoto-u.ac.jp`

Department of Mathematics, Kyoto University, Kyoto, 606-8502, Japan

Nobuyuki Oda `odanobu@fukuoka-u.ac.jp`

Department of Applied Mathematics, Faculty of Science, Fukuoka University, Fukuoka, 814-0180, Japan