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HIGHER STRUCTURE IN THE UNSTABLE ADAMS SPECTRAL SEQUENCE

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Abstract

We describe a variant construction of the unstable Adams spectral sequence for a space $\mathbf{Y}$, associated to any free simplicial resolution of $H^*(\mathbf{Y}; R)$, for $R = \mathbb{F}_p$ or $\mathbb{Q}$. We use this construction to describe the differentials and filtration in the spectral sequence in terms of appropriate systems of higher cohomology operations.

1. Introduction

The original Adams spectral sequence of [**Ada**] calculates the stable homotopy groups of a space $\mathbf{Y}$ at a prime p, starting with its $\mathbb{F}_p$-cohomology. Later, several unstable versions of this were proposed (see [**Cu, Re, MP, BC, BK1**]), all shown in [**BK2**, X, 6] to agree for reasonable spaces $\mathbf{Y}$. There are also variants for computing $\pi_* \operatorname{map}(\mathbf{X}, \mathbf{Y})$, as well as for more general coefficients, but for simplicity we restrict attention here to the original version, for coefficients in $R = \mathbb{F}_p$ or $\mathbb{Q}$.

The E_2-terms of both the stable and unstable spectral sequences for $\mathbf{Y}$ can be identified as certain graded Ext groups associated to $H^*(\mathbf{Y}; R)$, equipped with an action of the (stable or unstable) primary R-cohomology operations (cf. [**Ada, BK1**]). These can be computed from any resolution $V_\bullet$ of $H^*(\mathbf{Y}; R)$, in an appropriate category of Θ_R-algebras (for $R = \mathbb{F}_p$: these are modules or unstable algebras, respectively, over the mod p Steenrod algebra).

We show here how, as in the stable case, the unstable Adams spectral sequence for $\mathbf{Y}$ can be obtained from a realization of any such algebraic resolution $V_\bullet \to H^*(\mathbf{Y}; R)$ by a cosimplicial space $\mathbf{W}^\bullet$, constructed inductively through successive approximations $\mathbf{W}^\bullet_{[n]}$ which we think of as forming an unstable Adams resolution of $\mathbf{Y}$.

1.1. Systems of higher cohomology operations

In [**BS2**], we showed how this construction of $\mathbf{W}^\bullet_{[n]}$ can be used to define certain "universal higher cohomology operations" associated to each R-good space $\mathbf{Y}$, which can be used to distinguish it from other spaces $\mathbf{Z}$ having $H^*(\mathbf{Z}; R) \cong H^*(\mathbf{Y}; R)$ (as Θ_R-algebras). We further proved there that similar higher operations can also be used to distinguish among homotopy classes of maps $f_0, f_1 \colon \mathbf{X} \to \mathbf{Y}$ between R-good

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spaces which induce the same map $f_0^* = f_1^* \colon H^*(\mathbf{Y}; R) \to H^*(\mathbf{X}; R)$ in cohomology (see [**BS2**, 5]).

Our second goal in this paper is to show that analogous higher operations define the differentials in the unstable Adams spectral sequence for $\mathbf{Y}$, as well as the filtration index of each element in $\pi_* \mathbf{Y}$.

The notion of secondary operations in homotopy theory goes back at least to the 1950's, when Massey products, Toda brackets, and Adem's secondary cohomology operations first appeared (in [**Mas, T1, T2, Ade**]). Since then, there have been several attempts to give general definitions of higher order operations (see [**Sp, Mau, W, BM, BJT1**]), but none have been completely satisfactory.

Rather than trying to give one definition covering all variants, we will describe the basic properties we expect of a general n-th order homotopy operation $\langle\langle X \rangle\rangle_n$ for $n \geqslant 2$:

(a) It serves as the final obstruction to rectifying a homotopy-commutative diagram $X \colon I \to \operatorname{ho} \mathcal{M}$ – or equivalently, making it ∞-homotopy commutative – where $\mathcal{M}$ is a pointed simplicial model category and I is a suitable finite directed diagram of length $n+1$ (see [**BM**] or [**BJT3**]).

(b) Its *value*, for an appropriate choice of *initial data* $G^{(n-1)}$, is a homotopy class in $[X(v_{\text{init}}), \Omega^{n-1} X(v_{\text{fin}})]$, where v_{init} is weakly initial in I and v_{fin} is weakly terminal (cf. [**BM**, 2.1]). This value is zero (the operation *vanishes*) if and only if the diagram can be rectified for this choice of initial data.

(c) It has an associated system of lower order operations $(\langle\langle X|_{I_k} \rangle\rangle_k)_{k=2}^{n-1}$, corresponding to the filtration of I by initial (or final) segments I_k of length $k+1$. The initial data $G^{(k)}$ for $\langle\langle X \rangle\rangle_{k+1}$ is determined by a rectification of $X|_{I_k}$ made fibrant or cofibrant (in an appropriate model category structure on $\mathcal{M}^{I_k}$) – thus assuming in particular that $\langle\langle X|_{I_k} \rangle\rangle_k$ vanishes.

(d) Two n-th order operations (for different indexing categories I and J) are *equivalent* if the corresponding rectification problems are equivalent – so there is a bijective correspondence of the initial data for the two, and the resulting value for one vanishes if and only if it does so for the other.
We say that they are *strongly equivalent* if the correspondence induces a bijection of values in $[X(v_{\text{init}}), \Omega^{n-1} X(v_{\text{fin}})]$ (so in particular the two diagrams have the same initial and final objects in $\mathcal{M}$).

(e) When $\mathcal{M}$ is a category of spaces or spectra, we say that $\langle\langle X \rangle\rangle_n$ is an n-th order *R-cohomology* operation, and that $(\langle\langle X|_{I_k} \rangle\rangle_k)_{k=2}^{n}$ is a *system of higher R-cohomology operations*, if $Y(v)$ is an R-GEM for all $v \in \operatorname{Obj}(J) \setminus \{v_{\text{init}}\}$, for some strongly equivalent system associated to $Y \colon J \to \operatorname{ho} \mathcal{M}$.
Note that the spaces in the chosen diagram $X \colon I \to \operatorname{ho} \mathcal{M}$ itself need not all be R-GEMs – only those in some equivalent system. See last paragraph in 3.2 below (but compare [**BBS**]).

1.2. Main results

This paper continues the project begun in [**BS2**], intended to show how higher cohomology operations serve as a unifying setting for describing finer homotopy invariants of (the R-completion of) topological spaces.

This principle is applied here to the unstable Adams spectral sequence, but in fact Theorems B and C below apply equally to the stable Adams spectral sequence (since the former agrees with the latter in the stable range).

In Section 3 we recall from [**BS2**] how to associate to any CW resolution $V_\bullet$ of the Θ_R-algebra $H^*(\mathbf{Y};R)$ (for any R-good space $\mathbf{Y}$) an *unstable Adams resolution*: that is to say, a cosimplicial space $\mathbf{W}^\bullet$, obtained as the limit of a tower of fibrations:

$$\cdots \longrightarrow \mathbf{W}^\bullet_{[n]} \xrightarrow{\pi_{[n]}} \mathbf{W}^\bullet_{[n-1]} \xrightarrow{\pi_{[n-1]}} \mathbf{W}^\bullet_{[n-2]} \longrightarrow \cdots \longrightarrow \mathbf{W}^\bullet_{[0]}. \tag{1}$$

Each stage $\mathbf{W}^\bullet_{[n]}$ in this tower realizes the corresponding skeleton $\operatorname{sk}_n V_\bullet$ of the algebraic resolution $V_\bullet$; this in turn is obtained from $\operatorname{sk}_{n-1} V_\bullet$ by attaching a free Θ_R-algebra $\overline{V}_n$ by a suitable map (as in the usual construction of a CW complex). We can realize $\overline{V}_n$ by an R-GEM $\overline{\mathbf{W}}^n$.

In Section 5 we then show:

Theorem A. *The homotopy spectral sequence for the tower (1) coincides with the usual unstable Adams spectral sequence for* $\mathbf{Y}$*.*

See Theorem 5.6.

In Section 6, we associate to any such unstable Adams resolution $\mathbf{W}^\bullet$ of $\mathbf{Y}$ a sequence of higher cohomology operations $\langle\langle - \rangle\rangle_r \colon \pi_{k+n}\overline{\mathbf{W}}^n \to \pi_{k+n+r-1}\overline{\mathbf{W}}^{n+r}$ (see Definition 6.3), and show:

Theorem B. *Each value* $\langle\langle \gamma \rangle\rangle_r \in \pi_{k+n+r-1}\overline{\mathbf{W}}^{n+r} = E_1^{n+r,k+n+r-1}$ *of the* r*-th order operation* $\langle\langle - \rangle\rangle_r$ *represents the result of applying the differential* d_r *to the element of* $E_r^{n,k+n}$ *represented by* $\gamma \in \pi_{k+n}\overline{\mathbf{W}}^n = E_1^{n,k+n}$*.*

See Theorem 6.4.

Finally, in Section 7 we produce another sequence of higher cohomology operations $\langle\langle - \rangle\rangle'_r \colon \pi_k\mathbf{Y} \to \pi_{k+n}\overline{\mathbf{W}}^n$, and prove

Theorem C. *For any* $0 \neq \gamma \in \pi_k\mathbf{Y}$*, the operation* $\langle\langle \gamma \rangle\rangle'_{n-1}$ *vanishes while* $\langle\langle \gamma \rangle\rangle'_n \neq 0$ *if and only if* γ *is represented in the unstable Adams spectral sequence in filtration* n *by the value of* $\langle\langle \gamma \rangle\rangle'_n \in \pi_k\Omega^n\overline{\mathbf{W}}^n$*.*

See Theorem 7.5.

A simple example of the secondary cohomology operation associated to an element in Adams filtration 1 is given in 7.6.

Notation 1.1. The category of finite ordered sets and order-preserving maps will be denoted by Δ (cf. [**May**, 2]), with objects $[\mathbf{n}] = [0 < 1 < \cdots n]$ $(n \in \mathbb{N})$, so a *cosimplicial object* $A^\bullet$ in a category $\mathcal{C}$ is a functor $\Delta \to \mathcal{C}$, and a *simplicial object* $A_\bullet$ in $\mathcal{C}$ is a functor $\Delta^{\mathrm{op}} \to \mathcal{C}$. Write $c\mathcal{C} = \mathcal{C}^\Delta$ for the category of cosimplicial objects in $\mathcal{C}$, and $s\mathcal{C} = \mathcal{C}^{\Delta^{\mathrm{op}}}$ for that of simplicial objects. There is a natural embedding $c(-)^\bullet \colon \mathcal{C} \to c\mathcal{C}$ (the constant cosimplicial object), and similarly $c(-)_\bullet \colon \mathcal{C} \to s\mathcal{C}$.

If Δ_+ denotes the subcategory of injective maps in Δ, a functor $\Delta_+ \to \mathcal{C}$ will be called a *restricted cosimplicial object*.

A *chain complex* in a pointed category $\mathcal{C}$ is a sequence of maps $\partial_n \colon A_n \to A_{n-1}$ with $\partial_n \circ \partial_{n+1} = 0$ for each $n \geqslant 1$. The category of such will be denoted by $\mathsf{Ch}_{\mathcal{C}}$. The category of *cochain complexes* in $\mathcal{C}$, defined dually, is denoted by $\mathsf{Ch}^{\mathcal{C}}$.

The category of simplicial sets will be denoted by $\mathcal{S} = s\mathsf{Set}$, and that of pointed simplicial sets (here called simply "spaces") by $\mathcal{S}_* = s\mathsf{Set}_*$. Write $\mathrm{map}_{\mathcal{C}}(\mathbf{X}, \mathbf{Y})$ for the standard function complex in a simplicial model category $\mathcal{C}$ (see [**GJ**, I, 1.5] or [**GS**, 4.2]).

The *half-smash* of $\mathbf{X}$ and $\mathbf{Y}$, where $(\mathbf{Y}, y) \in \mathcal{S}_*$ is pointed, but $\mathbf{X} \in \mathcal{S}$ is not, is denoted by $\mathbf{X} \ltimes \mathbf{Y} := (\mathbf{X} \times \mathbf{Y})/(\mathbf{X} \times \{y\}) \in \mathcal{S}_*$. In particular, the (unreduced) *cone* on $\mathbf{X}$ is $C\mathbf{X} := \mathbf{X} \ltimes [0, 1]$, where the interval $[0, 1]$ has base point 1, while the *reduced cone* on $\mathbf{Y}$ is $\bar{C}\mathbf{Y} := \mathbf{Y} \wedge [0, 1]$.

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2. Background

We first recall some background material on cohomology algebras, their resolutions, and the realizations of these resolutions.

Definition 2.1. For any ring R and limit cardinal λ, let $\Theta_R = \Theta_R^\lambda$ denote (a skeleton of) the full subcategory of $\mathrm{ho}\,\mathcal{S}_*$ spanned by finite products of objects of the form $\{\mathbf{K}(V, n) : n \in \mathbb{N}_{>0}\}$, where V is an R-module generated by a set of cardinality $< \lambda$. This is a (multi-sorted) *theory*, in the sense of Lawvere (see [**L**] and [**E**]). A product-preserving functor $\Gamma \colon \Theta_R \to \mathsf{Set}_*$ will be called a Θ_R*-algebra* (cf. [**Bor**, 5.6]). Since each $\mathbf{B} \in \Theta_R$ is an R-module object, all Θ_R-algebras take values in R-modules, and their category will be denoted by Θ_R-Alg.

In particular, a Θ_R-algebra Γ is *realizable* if it is represented by a space $\mathbf{Y} \in \mathcal{S}_*$, with $\Gamma\{\mathbf{K}(R, n)\} := [\mathbf{Y}, \mathbf{K}(R, n)]$. By abuse of notation, we denote such a Γ by $H^*(\mathbf{Y}; R)$. Thus $H^*(\mathbf{Y}; R)$ is just the R-cohomology algebra of $\mathbf{Y}$, equipped with the action of the primary R-cohomology operations.

A Θ_R-algebra of the form $H^*(\mathbf{B}; R)$ for $\mathbf{B} \in \Theta_R^\lambda$ is called *free*. Note that this definition depends on our choice of cardinal λ (cf. [**BS2**, 1.25]).

Example 2.2. If $\lambda = \aleph_0$ and $R = \mathbb{F}_p$, Θ_R consists of finite R-GEMs, and a Θ_R-algebra is an unstable algebra over the mod p Steenrod algebra, as in [**Sc**, 1.4]. When $R = \mathbb{Q}$, a Θ_R-algebra is just a graded-commutative $\mathbb{Q}$-algebra.

2.1. Algebraic resolutions

As in [**Q**, II, 4], there is a model category structure on the category $s\Theta_R$-Alg of simplicial Θ_R-algebras, so there is a notion of a free simplicial resolution $V_\bullet$ of a Θ_R-algebra Γ.

We shall be interested in a particular kind, known as a *CW-resolution* (cf. [**Bl**, 3.10]), defined as follows

Definition 2.3. Recall that for any simplicial object $V_\bullet$ over a complete pointed category $\mathcal{M}$, the n-th *Moore chains* object ($n \geqslant 0$) is

$$C_n V_\bullet := \bigcap_{i=1}^{n} \operatorname{Ker}(d_i)$$

with differential $\partial_n := d_0$ satisfying $\partial_n \circ \partial_{n+1} = 0$. The n-th *Moore cycles* object is $Z_n V_\bullet := \operatorname{Ker}(\partial_n)$.

If $\mathcal{M}$ is cocomplete, the n-th *latching object* $L_n V_\bullet$ is defined to be $\operatorname{colim}_{\theta\colon [\mathbf{k}]\to[\mathbf{n}]} V_k$ (where the colimit is taken over the maps in Δ^{op} from $[\mathbf{k}]$ to $[\mathbf{n}]$ (see 1.1), with $k < n$). This is equipped with the obvious canonical map $\delta\colon V_n \to L_n V_\bullet$ (see [**GJ**, VII, 1]).

Definition 2.4. We say that $V_\bullet \in s\mathcal{M}$ is a *CW object* if it is equipped with a *CW basis* $(\overline{V}_n)_{n=0}^{\infty}$ in $\mathcal{M}$ such that $V_n = \overline{V}_n \amalg L_n V_\bullet$, and $d_i|_{\overline{V}_n} = 0$ for $1 \leqslant i \leqslant n$. In this case $\overline{\partial}_0^{V_n} := d_0|_{\overline{V}_n} : \overline{V}_n \to V_{n-1}$ is called the attaching map for $\overline{V}_n$. By the simplicial identities $\overline{\partial}_0^{V_n}$ factors as $\overline{\partial}_0^{V_n} : \overline{V}_n \to Z_{n-1} V_\bullet \subset V_{n-1}$.

In this case we have an explicit description

$$L_n V_\bullet := \coprod_{0\leqslant k\leqslant n}\ \coprod_{0\leqslant i_1<\cdots<i_{n-k-1}\leqslant n-1} \overline{V}_k$$

for its n-th latching object, in which the iterated degeneracy map $s_{i_{n-k-1}} \cdots s_{i_2} s_{i_1}$, restricted to the basis $\overline{V}_k$, is the inclusion into the copy of $\overline{V}_k$ indexed by $(i_1, \ldots, i_{n-k-1})$.

In particular, if in $\mathcal{M} = \Theta_R\text{-}\mathsf{Alg}$ we set $Z_{-1} V_\bullet := \Gamma \in \mathcal{M}$ and require that $\overline{V}_n$ be free and that $\overline{\partial}_0^{V_n} : \overline{V}_n \to Z_{n-1} V_\bullet$ be surjective for each $n \geqslant 0$, we call the resulting augmented free simplicial Θ_R-algebra $V_\bullet \to \Gamma$ a *CW resolution* (compare [**GS**, 4]).

Definition 2.5. Dually, for a cosimplicial object $V^\bullet$ over a cocomplete pointed category $\mathcal{M}$ and $n \geqslant 0$, the n-th *Moore cochains* object is

$$C^n V^\bullet := \operatorname{Cof}\left(\coprod_{i=1}^{n-1} V^{n-1} \xrightarrow{\perp_i d^i} V^{n-1} \right),$$

with differential $\delta^{n-1}\colon C^{n-1}V^\bullet \to C^n V^\bullet$ induced by d^0_{n-1}, and structure map $v^n\colon V^n \to C^n V^\bullet$. We denote the cofiber of δ^{n-1} by $Z^n V^\bullet$, with structure map $w^n\colon C^n V^\bullet \to Z^n V^\bullet$.

If $\mathcal{M}$ is complete, the n-th *matching object* for $V^\bullet$ is

$$M^n V^\bullet := \lim_{\phi\colon [\mathbf{n}]\to[\mathbf{k}]} V^k,$$

where ϕ ranges over the surjective maps $[\mathbf{n}] \to [\mathbf{k}]$ in Δ, with the obvious natural map $\zeta^n\colon V^n \to M^{n-1}V^\bullet$ (through which all codegeneracies factor). If $\mathcal{M}$ is a model category, we say $V^\bullet$ is *(Reedy) fibrant* if each map ζ^n is a fibration (see [**BK2**, X, 4]).

2.2. Cosimplicial resolutions

Let $\mathbf{W}^\bullet$ be a weak R-resolution of $\mathbf{Y}$ (see [**Bou**, 6.1]) – that is, a cosimplicial space with each $\mathbf{W}^n$ an R-GEM, equipped with a coaugmentation $\varepsilon\colon \mathbf{Y} \to \mathbf{W}^\bullet$ which is an R-equivalence (cf. [**Bou**, §3.2]). We assume for simplicity that $\mathbf{W}^\bullet$ is Reedy fibrant(see [**H**, §15.3]), so $\operatorname{Tot} \mathbf{W}^\bullet \simeq \widehat{\mathbf{Y}}$, the R-completion of $\mathbf{Y}$, by [**Bou**, Theorem 6.5]. Such a resolution may be constructed functorially using a suitable monad, as in [**BK2**, I, §2] (see also [**BS1**, §2–3])

3. Realizing CW resolutions

Our main technical tool in this paper is the construction of appropriate cosimplicial resolutions of an (R-good) space $\mathbf{Y}$, realizing a given algebraic resolution of $H^*(\mathbf{Y}; R)$. These are the unstable analogue of the Adams resolution of a space or spectrum (see, e.g., [**Ra**, §2.2]).

3.1. Cosimplicial CW resolutions

In [**BS2**, §2] we showed that, given a space $\mathbf{Y}$ with $\Gamma := H^*(\mathbf{Y}; R)$, any CW resolution $V_\bullet \in s\Theta_R\text{-}\mathsf{Alg}$ of Γ with CW basis $(\overline{V}_n)_{n=0}^\infty$ (§ 2.4) can be realized by a coaugmented cosimplicial space $\mathbf{Y} \to \mathbf{W}^\bullet$. This $\mathbf{W}^\bullet$ is the limit of a tower of Reedy fibrant and cofibrant $\mathbf{Y}$-coaugmented cosimplicial spaces:

$$\cdots \longrightarrow \mathbf{W}^\bullet_{[n]} \xrightarrow{\pi_{[n]}} \mathbf{W}^\bullet_{[n-1]} \xrightarrow{\pi_{[n-1]}} \mathbf{W}^\bullet_{[n-2]} \longrightarrow \cdots \longrightarrow \mathbf{W}^\bullet_{[0]}, \tag{2}$$

in $c\mathcal{S}_* = \mathcal{S}_*^\Delta$, with each $\pi_{[n]}$ a Reedy fibration.

The passage from $\mathbf{W}^\bullet_{[n-1]}$ to $\mathbf{W}^\bullet_{[n]}$ is as follows:

(a) Choose an R-GEM $\overline{\mathbf{W}}^n$ realizing the free Θ_R-algebra $\overline{V}_n$ (this is possible because of our choice of λ in § 2.1).

(b) The n-th attaching map $\overline{\partial}_0\colon \overline{V}_n \to C_{n-1}V^\bullet$ defines a unique map

$$\phi\colon \overline{V}_n \otimes S^{n-1} \to C_* \operatorname{sk}_{n-1} V_\bullet$$

of chain complexes in $\Theta_R\text{-}\mathsf{Alg}$, where $\overline{V} \otimes S^{n-1}$ is the chain complex with $\overline{V}$ in dimension $n-1$, and 0 elsewhere.
Evidently, one can realize $\overline{V} \otimes S^{n-1}$ by a cochain complex in $\mathcal{S}_*$; we choose a realization D^* which is a Reedy fibrant cochain complex in $\mathcal{S}_*$ in the sense of [**BS2**, §2.4(i)], by setting

$$D^k := P\Omega^{n-k-2}\overline{\mathbf{W}}^n \tag{3}$$

for each $k \geqslant 0$, where $P\Omega^{-1}\overline{\mathbf{W}}^n := \overline{\mathbf{W}}^n$ and $P\Omega^k\overline{\mathbf{W}}^n := *$ for $k < -1$. The differential is $\iota \circ p$, where $p\colon P\mathbf{X} \to \mathbf{X}$ is the appropriate path fibration and $\iota\colon \Omega\mathbf{X} \to P\mathbf{X}$ is the inclusion.

(c) Note that the Moore cochains define a functor $C^*\colon \mathcal{S}_*^{\Delta_+} \to \mathsf{Ch}^{\mathcal{S}_*}$ into the category of cochain complexes of spaces (§ 1.1), with right adjoint $\mathcal{E}$, so if we can realize ϕ by a cochain map $'F\colon C^*\mathbf{W}^\bullet_{[n-1]} \to D^*$ (see Proposition 3.2 below), it induces $\widetilde{F}\colon \mathcal{U}\mathbf{W}^\bullet_{[n-1]} \to \mathcal{E}D^*$ (where $\mathcal{U}\colon \mathcal{S}_*^\Delta \to \mathcal{S}_*^{\Delta_+}$ is the forgetful functor – see § 1.1)

Taking the fiber of $\widetilde{F}$ in $\mathcal{S}_*^{\Delta_+}$ yields a restricted cosimplicial space $\widetilde{\mathbf{W}}^\bullet_{[n]}$ with

$$\widetilde{\mathbf{W}}^r_{[n]} = \mathbf{W}^r_{[n-1]} \times P\Omega^{n-r-1}\overline{\mathbf{W}}^n. \tag{4}$$

(d) We add the missing codegeneracies to form a full cosimplicial space $\widehat{\mathbf{W}}^\bullet_{[n]}$, as follows: set $M^r\widehat{\mathbf{W}}^\bullet_{[n]} := M^r\mathbf{W}^\bullet_{[n-1]} \times \widehat{\mathbf{M}}^r_{[n]}$, where

$$\widehat{\mathbf{M}}^r_{[n]} := \prod_{0\leqslant k\leqslant r} \ \prod_{0\leqslant i_1<\cdots<i_k\leqslant r} P\Omega^{n+k-r-1}\overline{\mathbf{W}}^n \tag{5}$$

for each $r \geqslant 0$. We then set

$$\begin{aligned} \widehat{\mathbf{W}}^r_{[n]} :=\ & \widetilde{\mathbf{W}}^r_{[n]} \times \widehat{\mathbf{M}}^{r-1}_{[n]} = \mathbf{W}^r_{[n-1]} \times \widehat{\mathbf{M}}^{r-1}_{[n]} \times P\Omega^{n-r-1}\overline{\mathbf{W}}^n \\ =\ & \mathbf{W}^r_{[n-1]} \times \prod_{0\leqslant k\leqslant r} \ \prod_{0\leqslant i_1<\cdots<i_k\leqslant r-1} P\Omega^{n+k-r}\overline{\mathbf{W}}^n, \end{aligned} \tag{6}$$

and the codegeneracy map $s^t\colon \widehat{\mathbf{W}}^{r+1}_{[n]} \to \widehat{\mathbf{W}}^r_{[n]}$ is defined into the factor $Q := P\Omega^{n+k-r-1}\overline{\mathbf{W}}^n$ of $\widehat{\mathbf{W}}^r_{[n]}$ indexed by the k-tuple $I = (i_1, \ldots, i_k)$ by projecting $\widehat{\mathbf{W}}^{r+1}_{[n]}$ onto the copy of Q indexed by the unique $(k+1)$-tuple $J = (j_1, \ldots, j_{k+1})$ satisfying the cosimplicial identity $s^I \circ s^t = s^J$.

This defines a functor $\mathcal{F}\colon \mathcal{S}_*^{\Delta_+} \to \mathcal{S}_*^{\Delta}$ ("add codegeneracies"), with $\mathcal{F}(\widetilde{\mathbf{W}}^\bullet_{[n]}) := \widehat{\mathbf{W}}^\bullet_{[n]}$, right adjoint to $\mathcal{U}\colon \mathcal{S}_*^{\Delta} \to \mathcal{S}_*^{\Delta_+}$. By adjunction we therefore have a map

$$\mathbf{F}_{[n-1]}\colon \mathbf{W}^\bullet_{[n-1]} \longrightarrow \mathcal{F}\mathcal{E}D^* =: \mathbf{D}^\bullet_{[n]}$$

determined by $'F\colon C^*\mathbf{W}^\bullet_{[n-1]} \to D^*$ and the codegeneracies.

Note that (assuming the objects $\overline{\mathbf{W}}^n$ are all fibrant) the cosimplicial space $\widehat{\mathbf{W}}^\bullet_{[n]}$ we have constructed is Reedy fibrant, and from (6) we see that the dimensionwise projection defines a Reedy fibration $r_{[n]}\colon \widehat{\mathbf{W}}^\bullet_{[n]} \to \mathbf{W}^\bullet_{[n-1]}$. We write $i_{[n]}\colon \mathbf{\Sigma D}^\bullet_{[n]} \hookrightarrow \widehat{\mathbf{W}}^\bullet_{[n]}$ for the inclusion of the fiber $\mathbf{\Sigma D}^\bullet_{[n]} := \mathcal{F}\mathcal{E}\widetilde{\Sigma}D^*$ of $r_{[n]}$.

Here $\widetilde{\Sigma}D^*$ is the obvious Reedy fibrant cochain complex in $\mathcal{S}_*$ realizing $\overline{V} \otimes S^n$. Note that the unique non-zero coface map into the non-codegenerate part $P\Omega^{n-k-1}\overline{\mathbf{W}}^n$ of $\mathbf{\Sigma D}^k_{[n]}$ is d^1, not d^0.

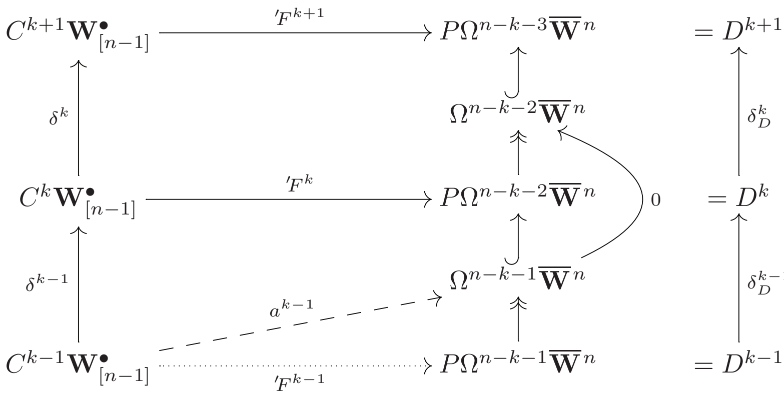
(e) Finally, we factor $* \to \widehat{\mathbf{W}}^\bullet_{[n]}$ as a cofibration followed by trivial (Reedy) fibration $q_{[n]}\colon \mathbf{W}^\bullet_{[n]} \xrightarrow{\simeq} \widehat{\mathbf{W}}^\bullet_{[n]}$, so $\pi_{[n]} := r_{[n]} \circ q_{[n]}$ is the required Reedy fibration of (2).

Remark 3.1. In step (b), we construct the map $'F\colon C^*\mathbf{W}^\bullet_{[n-1]} \to D^*$ by a downward induction on the dimension $k \leqslant n-1$, starting with

$$'F^{n-1}\colon C^{n-1}\mathbf{W}^\bullet_{[n-1]} \to D^{n-1} = \overline{\mathbf{W}}^n$$

which exists by [**BS2**, Lemma 2.19].

At the k-th stage in the induction we have $'F^{k+1}$ and $'F^k$ in the following diagram:



(7)

We see that $'F^k$ induces a map a^{k-1} as indicated, which must be nullhomotopic in order for $'F^{k-1}$ to exist. In fact, we have:

Proposition 3.2. *Let $R = \mathbb{F}_p$ or $\mathbb{Q}$ and $\Gamma = H^*(\mathbf{Y}; R)$, and let $V_\bullet \to \Gamma$ be a CW resolution. Assume we have an $(n-1)$-stage coaugmented realization $\mathbf{Y} \to \mathbf{W}^\bullet_{[n-1]}$ of $V_\bullet$ as above, with D^* a Reedy fibrant cochain complex, as well as a cochain map $'F\colon C^*\mathbf{W}^\bullet_{[n-1]} \to D^*$ as in (7), defined in degrees $\geqslant k$. Then one can modify the choice of $'F^k$ so that a^{k-1} defined as above is nullhomotopic.*

Proof. This follows from the proof of [**BS2**, Theorem A.11]. □

Corollary 3.3. *For $\Gamma = H^*(\mathbf{Y}; R)$, any CW resolution $V_\bullet \to \Gamma$ as above is realizable by a coaugmented cosimplicial space $\mathbf{Y} \to \mathbf{W}^\bullet$ obtained as a limit of a tower (2) as above.*

3.2. Higher cohomology operations

We think of a^{k-1} as the value of a $(n-k)$-th order cohomology operation; [**BS2**, Theorem A.11(b)] then shows that, given $\mathbf{Y}$, this higher order operation vanishes, so F exists.

However, given another R-good space $\mathbf{Z}$ with $H^*(\mathbf{Z}; R) \cong H^*(\mathbf{Y}; R)$, we can try to construct a coaugmentation $\varepsilon\colon \mathbf{Z} \to \mathbf{W}^\bullet$ inducing a weak equivalence to $\operatorname{Tot} \mathbf{W}^\bullet \simeq \widehat{\mathbf{Y}}$: this is possible if and only if $\mathbf{Z}$ and $\mathbf{Y}$ are R-equivalent. This will be carried out by inductively attempting to produce successive lifts $\varepsilon_{[k]}\colon \mathbf{Z} \to \mathbf{W}^\bullet_{[k]}$, starting with the obvious $\varepsilon_{[0]}\colon \mathbf{Z} \to \mathbf{W}^\bullet_{[0]} = c(\overline{\mathbf{W}}^0)^\bullet$.

Given $\varepsilon_{[n-1]}$, consider the composite ξ of

$$\mathbf{Z} \xrightarrow{\varepsilon_{[n-1]}} \mathbf{W}^0_{[n-1]} \xrightarrow{'F^0} P\Omega^{n-2}\overline{\mathbf{W}}^n \xrightarrow{p} \Omega^{n-2}\overline{\mathbf{W}}^n \xrightarrow{\iota} P\Omega^{n-3}\overline{\mathbf{W}}^n, \tag{8}$$

which represents the component of the iterated coface map $d^1 \circ d^0 \circ \varepsilon_{[n-1]}$ from $\mathbf{Z}$ into $P\Omega^{n-3}\overline{\mathbf{W}}^n$. Since $d^1 \circ d^0 \circ \varepsilon_{[n-1]} = d^2 \circ d^1 \circ \varepsilon_{[n-1]}$ and $d^2 = 0$ into the factor $P\Omega^{n-3}\overline{\mathbf{W}}^n$, we see that ξ is zero. Since ι is monic, this means that $p \circ {'F^0} \circ \varepsilon_{[n-1]}$ is already zero, so $'F^0 \circ \varepsilon_{[n-1]}$ factors through the fiber $\Omega^{n-1}\overline{\mathbf{W}}^n$ of the path fibration p.

We denote the resulting map by $a^{-1}\colon \mathbf{Z} \to \Omega^{n-1}\overline{\mathbf{W}}^n$. This is the obstruction to lifting $\varepsilon_{[n-1]}$ to $\varepsilon_{[n]}$ (see [**BS2**, Lemma 4.5]).

Since all but the first map in (8) are R-GEMs, from the discussion in [**BS2**, ¯4] we see that $[a^{-1}]$ can indeed be interpreted as the value of an appropriate n-th order cohomology operation.

Note that in (7) the various spaces $C^j\mathbf{W}^\bullet_{[n-1]}$ are not R-GEMs. Nevertheless, that realization problem which we solve by the vanishing of the classes a^{k-1}, for the chain map $\phi\colon \overline{V}_n \otimes S^{n-1} \to C_*\,\mathrm{sk}_{n-1}\,V_\bullet$ of ¯ 3.1(b), is equivalent to that of realizing the n-skeletal augmented simplicial object $\mathrm{sk}_n\,V_\bullet \to \Gamma$ in $s\Theta_R$-Alg, which can be thought of as an n-skeletal augmented cosimplicial object in $c\,\mathrm{ho}\,\mathcal{S}_*$, for which indeed all but one object are R-GEMs.

Remark 3.4. Note that because we are mapping into R-GEMs, from the universal property of the R-completion we see that the value of $[a^{-1}]$ depends only on the R-type of $\mathbf{Z}$ (that is, up to zigzags of maps inducing isomorphisms in $H^*(-;R)$). In particular, since the spaces in (2) are all coaugmented out of $\mathbf{Y}$, all these higher operations indeed vanish for $\mathbf{Y}$ – and thus also for its R-completion $\widehat{\mathbf{Y}} \simeq \mathrm{Tot}\,\mathbf{W}^\bullet$. Thus $\widehat{\mathbf{Y}}$ too is coaugmented into (2), and thus into $\mathbf{W}^\bullet$.

This is a somewhat unusual situation, since $\mathrm{Tot}\,\mathbf{W}^\bullet$ is a homotopy limit, and we would not generally expect a map $\mathbf{\Delta}^\bullet \to \mathbf{W}^\bullet$ to lift through the natural map $\lim \to \mathrm{holim}$ to an actual cone for $\mathbf{W}^\bullet$ – that is, to a map $* \to \mathbf{W}^\bullet$.

4. The homotopy spectral sequence of a cosimplicial space

For any fibrant pointed cosimplicial space $\mathbf{W}^\bullet$, Bousfield and Kan construct a spectral sequence as follows:

4.1. The Tot tower

In the version of [**BK2**, X, ¯6], this is just the homotopy spectral sequence of the tower of fibrations:

$$\cdots \to \mathrm{Tot}_{n+1}\,\mathbf{W}^\bullet \xrightarrow{q^{n+1}} \mathrm{Tot}_n\,\mathbf{W}^\bullet \xrightarrow{q^n} \mathrm{Tot}_{n-1}\,\mathbf{W}^\bullet \to \cdots \to \mathrm{Tot}_{-1}\,\mathbf{W}^\bullet = *, \quad (9)$$

with (homotopy) limit $\mathrm{Tot}\,\mathbf{W}^\bullet$.

Recall that $\mathrm{Tot}\,\mathbf{W}^\bullet := \mathrm{map}_{c\mathcal{S}}(\mathbf{\Delta}^\bullet, \mathbf{W}^\bullet)$ (the simplicial enrichment of $c\mathcal{S}$), where $\mathbf{\Delta}^\bullet$ is the cosimplicial space with $\mathbf{\Delta}^k$ (the standard k-simplex) in dimension k, and similarly $\mathrm{Tot}_n\,\mathbf{W}^\bullet := \mathrm{map}_{c\mathcal{S}}(\mathrm{sk}_n\,\mathbf{\Delta}^\bullet, \mathbf{W}^\bullet)$. One should think of a map $\mathbf{Z} \to \mathrm{Tot}\,\mathbf{W}^\bullet$ as an ∞-homotopy commutative diagram mapping $\mathbf{Z}$ into the Δ-indexed diagram $\mathbf{W}^\bullet$.

We shall use $\Delta[k]$ as an alternative notation for the standard k-simplex in $\mathcal{S}$, when we think of it as representing k-simplices in simplicial sets. In particular, a k-simplex in $\mathrm{Tot}_n\,\mathbf{W}^\bullet$ is a sequence of maps $f^m\colon\ \mathrm{sk}_n\,\mathbf{\Delta}^m \times \Delta[k] \to \mathbf{W}^m$ $(m = 0, 1, \dots)$ such that

$$f^j \circ (\mathrm{sk}_n\mathbf{\Delta}(\phi) \times \mathrm{Id}) = \mathbf{W}(\phi) \circ f^m\colon\ \mathrm{sk}_n\mathbf{\Delta}^m \times \Delta[k] \to \mathbf{W}^j \quad (10)$$

for every morphism $\phi\colon [m] \to [j]$ in Δ. Therefore, for each $k \geqslant 1$ we have $f^{n-k} = s^I_{\mathbf{W}} \circ f^n \circ d^I_{\mathbf{\Delta}}$, where $d^I_{\mathbf{\Delta}} = d^{i_k}_{\mathbf{\Delta}} \circ \cdots d^{i_1}_{\mathbf{\Delta}}$ is an iterated coface map of $\mathbf{\Delta}$, and $s^I_{\mathbf{W}}$ is the corresponding iterated codegeneracy map of $\mathbf{W}^\bullet$ (since $s^I \circ d^I = \mathrm{Id}$). Moreover, since

$\mathrm{sk}_n\,\boldsymbol{\Delta}^N = \mathrm{colim}_{i\leqslant n}\,\boldsymbol{\Delta}^i$ for $N > n$, the map f^N is determined by the (compatible) maps f^i for $i \leqslant n$. Thus $(\mathrm{Tot}_n\,\mathbf{W}^\bullet)_k \subseteq \mathrm{Hom}(\boldsymbol{\Delta}^n \times \Delta[k],\, \mathbf{W}^n)$ and in fact

$$\mathrm{Tot}_n\,\mathbf{W}^\bullet \subseteq \mathrm{map}_{\mathcal{S}}(\boldsymbol{\Delta}^n,\, \mathbf{W}^n), \tag{11}$$

where the subspace is the limit given by (10).

Note that because $\mathrm{sk}_n\,\boldsymbol{\Delta}^{n+1} = \partial\boldsymbol{\Delta}^{n+1}$ and each of the coface maps $d^i\colon \mathbf{W}^n \to \mathbf{W}^{n+1}$ has a retraction, the compatibility conditions mean that also

$$\mathrm{Tot}_n\,\mathbf{W}^\bullet \subseteq \mathrm{map}_{\mathcal{S}}(\partial\boldsymbol{\Delta}^{n+1},\, \mathbf{W}^{n+1}). \tag{12}$$

As in [**BK2**, X, 6.3], the map q^n of (9) fits into a fibration sequence

$$\Omega^n N^n \mathbf{W}^\bullet \xhookrightarrow{\iota^n} \mathrm{Tot}_n\,\mathbf{W}^\bullet \xrightarrow{q^n} \mathrm{Tot}_{n-1}\,\mathbf{W}^\bullet, \tag{13}$$

where

$$N^n\mathbf{W}^\bullet := \mathbf{W}^n \cap \mathrm{Ker}(s^0) \cap \cdots \cap \mathrm{Ker}(s^{n-1})$$

(and thus $\Omega^n N^n \mathbf{W}^\bullet = N^n \Omega^n \mathbf{W}^\bullet$).

Furthermore, by (11) and (12), the sequence (13) is just the restriction of the fibration sequence:

$$\mathrm{map}_{\mathcal{S}_*}(\boldsymbol{\Delta}^n/\partial\boldsymbol{\Delta}^n,\, \mathbf{W}^n) \xrightarrow{p_n^*} \mathrm{map}_{\mathcal{S}}(\boldsymbol{\Delta}^n, \mathbf{W}^n) \xrightarrow{\iota_n^*} \mathrm{map}_{\mathcal{S}}(\partial\boldsymbol{\Delta}^n, \mathbf{W}^n)$$

induced by the cofibration sequence

$$\partial\boldsymbol{\Delta}^n \xrightarrow{\iota_n} \boldsymbol{\Delta}^n \xrightarrow{p_n} \boldsymbol{\Delta}^n/\partial\boldsymbol{\Delta}^n.$$

Remark 4.1. Since $\mathbf{W}^\bullet$ is pointed, $\mathrm{map}_{\mathcal{S}}(\boldsymbol{\Delta}^n,\, \mathbf{W}^n)$ has a chosen basepoint, and an element in $\pi_k\,\mathrm{Tot}_n\,\mathbf{W}^\bullet$ is represented by a suitable pointed map $f\colon \mathbf{S}^k \to \mathrm{map}_{\mathcal{S}}(\boldsymbol{\Delta}^n,\, \mathbf{W}^n)$, or by its (pointed) adjoint $\hat{f}\colon \boldsymbol{\Delta}^n \ltimes \mathbf{S}^k \to \mathbf{W}^n$ (cf. ¯ 1.1). Note that the maps into $\mathbf{W}^j$ $(0 \leqslant j < n)$ are encoded by maps into the appropriate codegeneracies in $\mathbf{W}^n$.

Thus a class $\alpha \in \pi_k\Omega^n N^n\mathbf{W}^\bullet \subseteq \pi_k\,\mathrm{Tot}_n\,\mathbf{W}^\bullet$ is represented by $a\colon \partial\boldsymbol{\Delta}^{n+1} \times \Delta[k] \to \mathbf{W}^{n+1}$ which vanishes on $\partial\boldsymbol{\Delta}^{n+1} \times \partial\Delta[k]$. Such an α represents an element $\gamma \in \pi_k\,\mathrm{Tot}\,\mathbf{W}^\bullet$ if and only if $j^n_\#(\alpha)$ lifts to all levels of (9), where $j^n_\#\colon \pi_k\Omega^n N^n\mathbf{W}^\bullet \to \pi_k\,\mathrm{Tot}_n\,\mathbf{W}^\bullet$ is induced by the inclusion. The successive obstructions to lifting $j^n(\alpha)$ represent the differentials in the spectral sequence.

4.2. The spectral sequence

The E_1-exact couple of the homotopy spectral sequence for the Tot tower (9) may be presented as in Figure 4.1, with $E_1^{n,n+k} := \pi_k\Omega^n N^n\mathbf{W}^\bullet$ and d_1-differential given by

$$d_1^{n,n+k} = \delta^n \circ j^n = \sum_{t=0}^{n-1} (-1)^t d^t_\#\colon \pi_{k+n}N^n\mathbf{W}^\bullet \longrightarrow \pi_{k+n}N^{n+1}\mathbf{W}^\bullet \tag{14}$$

by [**BK2**, X, 6.3] again.

$$\begin{array}{ccccccc}
\pi_{k+1}\operatorname{Tot}_n \mathbf{W}^\bullet & \xrightarrow{\delta^n} & \pi_k\Omega^{n+1}N^{n+1}\mathbf{W}^\bullet & \xrightarrow{j^{n+1}} & \pi_k \operatorname{Tot}_{n+1}\mathbf{W}^\bullet & \xrightarrow{\delta^{n+1}} & \pi_{k-1}\Omega^{n+2}N^{n+2}\mathbf{W}^\bullet \\
\Big\downarrow{\scriptstyle q^n} & & & & \Big\downarrow{\scriptstyle q^{n+1}} & & \\
\pi_{k+1}\operatorname{Tot}_{n-1}\mathbf{W}^\bullet & \xrightarrow{\delta^{n-1}} & \pi_k\Omega^{n}N^{n}\mathbf{W}^\bullet & \xrightarrow{j^{n}} & \pi_k \operatorname{Tot}_{n}\mathbf{W}^\bullet & \xrightarrow{\delta^{n}} & \pi_{k-1}\Omega^{n+1}N^{n+1}\mathbf{W}^\bullet
\end{array}$$

Figure 4.1: Exact couple for Tot tower

5. The unstable Adams spectral sequence

From now on we restrict attention to the case $R = \mathbb{F}_p$ (although most results are valid also for $R = \mathbb{Q}$). When $\mathbf{W}^\bullet$ is an $\mathbb{F}_p$-resolution of a p-good space $\mathbf{Y}$, the homotopy spectral sequence of Section 4 is the unstable Adams spectral sequence of [**BK2**], converging to $\pi_*\widehat{\mathbf{Y}}$ (where $\widehat{\mathbf{Y}} := R_\infty\mathbf{Y}$ is the p-completion, equipped with the natural $H^*(-;R)$-equivalence $\eta\colon \mathbf{Y} \to \widehat{\mathbf{Y}}$). In this case we can say a little more about the structure of the spectral sequence:

5.1. Using the CW structure

From now on, we assume that $\mathbf{W}^\bullet$ has been constructed as in § 3.1 to realize a given CW resolution $V_\bullet$ of $\Gamma = H^*(\mathbf{Y};R)$. From (4) and (6), and the fact that $q_{[n]}$ is an acyclic Reedy fibration, we see that the maps $\pi_{[n+1]}\colon \mathbf{W}^\bullet_{[n+1]} \to \mathbf{W}^\bullet_{[n]}$ in (2) induce weak equivalences in Tot_k for all $0 \leqslant k \leqslant n$. Since $\mathbf{W}^\bullet_{[n]}$ is n-coskeletal, we have a tower of fibrations:

$$\cdots \operatorname{Tot}_{n+1}\mathbf{W}^\bullet_{[n+1]} \xrightarrow{(\pi_{[n+1]})_*} \operatorname{Tot}_n \mathbf{W}^\bullet_{[n]} \xrightarrow{(\pi_{[n]})_*} \cdots \to \operatorname{Tot}_1\mathbf{W}^\bullet_{[1]} \xrightarrow{(\pi_{[1]})_*} \operatorname{Tot}_0\mathbf{W}^\bullet_{[0]}, \tag{15}$$

obtained by combining (9) and (2).

In order to better understand the tower (15), we recall a (somewhat simplified) version of a construction introduced in [**BS2**, §5.10]:

Definition 5.1. For each $n \geqslant 1$ and $1 \leqslant k \leqslant n+1$, the n-th *folding polytope* $\mathcal{P}^n_k$ is obtained from a union of k disjoint n-simplices $\mathbf{\Delta}^n_{(n-k+1)}, \ldots, \mathbf{\Delta}^n_{(n)}$ by identifying the j-th facets of $\mathbf{\Delta}^n_{(n-j)}$ and $\mathbf{\Delta}^n_{(n-j-1)}$ for each $0 \leqslant j \leqslant n$. See Figure 6.2 below for an example.

Remark 5.2. By induction on $1 \leqslant k \leqslant n$ we readily see that $\mathcal{P}^n_k$ is PL-equivalent to an n-ball, so its boundary $\partial\mathcal{P}^n_k$ is PL-equivalent to an $(n-1)$-sphere.

Lemma 5.3. *For* $\mathbf{W}^\bullet$, $\mathbf{D}^\bullet_{[n]}$, *and* $\mathbf{\Sigma D}^\bullet_{[n]}$ *as in § 3.1,* $\operatorname{Tot}\mathbf{D}^\bullet_{[n]} \simeq \Omega^{n-1}\overline{\mathbf{W}}^n$ *and* $\operatorname{Tot}\mathbf{\Sigma D}^\bullet_{[n]} \simeq \Omega^n\overline{\mathbf{W}}^n$.

Proof. Since $\mathbf{D}^\bullet_{[n]}$ is $(n-1)$-coskeletal, $\operatorname{Tot}\mathbf{D}^\bullet_{[n]} = \operatorname{Tot}_{n-1}\mathbf{D}^\bullet_{[n]}$. Moreover, by § 4.1, and § 3.1(d), for any $\mathbf{Z} \in \mathcal{S}_*$, a pointed map $g\colon \mathbf{Z} \to \operatorname{Tot}_{n-1} \mathbf{D}^{\bullet}_{[n]}$ is completely determined by a sequence of maps $g^j : \mathbf{\Delta}^j \ltimes \mathbf{Z} \to P\Omega^{n-j-2}\overline{\mathbf{W}}^n$ $(0 \leqslant j \leqslant n-1)$, making

$$\begin{array}{ccc}
\mathbf{\Delta}^j \ltimes \mathbf{Z} & \xrightarrow{\quad g^j \quad} & P\Omega^{n-j-2}\overline{\mathbf{W}}^n \\
\delta^0 \uparrow \cdots \uparrow \delta^j & & \iota\circ p = d^0 \uparrow \cdots \uparrow 0 = d^i \ (i \geqslant 1) \\
\mathbf{\Delta}^{j-1} \ltimes \mathbf{Z} & \xrightarrow{\quad g^{j-1} \quad} & P\Omega^{n-j-1}\overline{\mathbf{W}}^n
\end{array} \tag{16}$$

commute for each $0 < j \leqslant n-1$ (where the coface maps δ^i on the left are induced by those of $\mathbf{\Delta}^\bullet$). See Remark 4.1 and (3).

Note that the cone functor in $\mathcal{S}_*$ is left adjoint to P, so if we include each $P\Omega^i\overline{\mathbf{W}}^n$ into $P^{i+1}\overline{\mathbf{W}}^n$, and identify $C\mathbf{\Delta}^j$ with $\mathbf{\Delta}^{j+1}$, we see that for each $0 \leqslant j \leqslant n-1$ the adjoint of g^j is a map $\widetilde{g}^j : \mathbf{\Delta}^{n-1} \ltimes \mathbf{Z} \to \overline{\mathbf{W}}^n$. We arrange the adjunction between g^j and $\widetilde{g}^j$ in such a way that the coface maps $\delta^0, \dots, \delta^{n-j-3}$ of $\mathbf{\Delta}^{n-1}$ correspond to the loop directions of $P\Omega^{n-j-2}\overline{\mathbf{W}}^n$ (counted outwards from $\overline{\mathbf{W}}^n$), and δ^{n-j-2} corresponds to the path direction. Finally, as long as $j > 0$, the remaining $j+1$ coface maps into $\mathbf{\Delta}^{n-1}$ are the original coface maps of $\mathbf{\Delta}^j$, re-indexed by $n-j-1$.

The fact that (16) commutes implies that these adjoints satisfy the relations

$$\widetilde{g}^j \circ \delta^i = \begin{cases} \widetilde{\iota p g^j} & \text{for} \quad i = n-j-2 \quad \text{and } j < n-1 \\ \widetilde{\iota p g^{j-1}} & \text{for} \quad i = n-j-1 \quad \text{and } j > 0 \\ 0 & \text{otherwise.} \end{cases} \tag{17}$$

By Definition 5.1, the maps $\widetilde{g}^j$ thus induce a single map $\widetilde{g}\colon \mathcal{P}^{n-1}_n \ltimes \mathbf{Z} \to \overline{\mathbf{W}}^n$ Moreover, (17) also implies that $\widetilde{g}|_{\partial\mathcal{P}^{n-1}_n \ltimes \mathbf{Z}} = 0$, so $\widetilde{g}$ factors uniquely through a map $(\mathcal{P}^{n-1}_n/\partial\mathcal{P}^{n-1}_n) \wedge \mathbf{Z} \to \overline{\mathbf{W}}^n$. By Remark 5.2, $\mathcal{P}^{n-1}_n/\partial\mathcal{P}^{n-1}_n$ is a PL $(n-1)$-sphere, so setting $\mathbf{Z} = \mathbf{S}^i$ we see that $\operatorname{Tot}\mathbf{D}^{\bullet}_{[n-1]}$ is weakly equivalent to $\Omega^{n-1}\overline{\mathbf{W}}^n$.

Similarly, $\mathbf{\Sigma D}^{\bullet}_{[n]}$ is n-coskeletal, so $\operatorname{Tot}\mathbf{\Sigma D}^{\bullet}_{[n]} = \operatorname{Tot}_n \mathbf{\Sigma D}^{\bullet}_{[n]}$, and a map of simplicial sets $g\colon \mathbf{Z} \to \operatorname{Tot}_n \mathbf{\Sigma D}^{\bullet}_{[n]}$ is determined (via the codegeneracies) by maps $g^j : \mathbf{\Delta}^j \ltimes \mathbf{Z} \to P\Omega^{n-j-1}\overline{\mathbf{W}}^n$ making the following diagram commute:

$$\begin{array}{ccc}
\mathbf{\Delta}^j \ltimes \mathbf{Z} & \xrightarrow{\quad g^j \quad} & P\Omega^{n-j-1}\overline{\mathbf{W}}^n \\
\delta^0 \uparrow \cdots \uparrow \delta^j & & \iota\circ p = d^1 \uparrow \cdots \uparrow 0 = d^i \ (i \neq 1) \\
\mathbf{\Delta}^{j-1} \ltimes \mathbf{Z} & \xrightarrow{\quad g^{j-1} \quad} & P\Omega^{n-j}\overline{\mathbf{W}}^n.
\end{array}$$

See § 3.1(d).

Taking adjoints $\widetilde{g}^j : \mathbf{\Delta}^n \ltimes \mathbf{Z} \to \overline{\mathbf{W}}^n$ $(0 \leqslant j \leqslant n)$ as above, (17) is replaced by:

$$\widetilde{g}^j \circ \delta^i = \begin{cases} \widetilde{\iota p g^j} & \text{for} \quad i = n-j-1 \quad \text{and } j < n \\ \widetilde{\iota p g^{j-1}} & \text{for} \quad i = n-j+1 \quad \text{and } j > 0 \\ 0 & \text{otherwise,} \end{cases} \tag{18}$$

and as before we deduce that

$$\operatorname{Tot}_n \mathbf{\Sigma D}^{\bullet}_{[n]} = \operatorname{Tot}\mathbf{\Sigma D}^{\bullet}_{[n]} \simeq \Omega^n\overline{\mathbf{W}}^n. \tag{19}$$

□

Remark 5.4. We see from (18) that the folding polytopes used to show that (19) holds are different from those defined in § 5.1, since we need to identify the j-th facet of $\mathbf{\Delta}^n_{(n-j+1)}$ with the $(j-1)$-facet of $\mathbf{\Delta}^n_{(n-j)}$ for each $0 < j \leqslant k$. We denote this variant by $\widehat{\mathcal{P}}^n_k$, which we called a *modified* folding polytope. See Figure 7.3 below for an example.

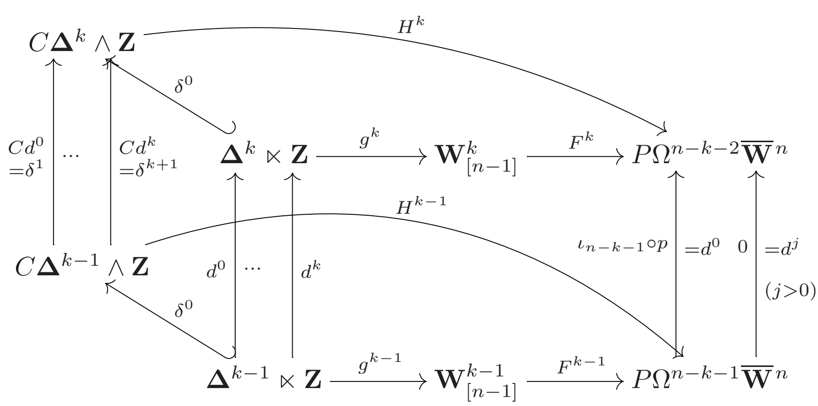
Proposition 5.5. *For $\mathbf{W}^\bullet$ as above, the sequence of maps of § 3.1(d) induce a quasi-fibration sequence*

$$\operatorname{Tot}_n \mathbf{\Sigma D}^\bullet_{[n]} \xrightarrow{\operatorname{Tot} i_{[n]}} \operatorname{Tot}_n \widehat{\mathbf{W}}^\bullet_{[n]} \xrightarrow{\operatorname{Tot} r_{[n]}} \operatorname{Tot}_{n-1} \mathbf{W}^\bullet_{[n-1]} \xrightarrow{\operatorname{Tot} \mathbf{F}_{[n-1]}} \operatorname{Tot}_{n-1} \mathbf{D}^\bullet_{[n]}. \quad (20)$$

Proof. As noted in § 3.1, $\mathbf{\Sigma D}^\bullet_{[n]} \xrightarrow{i_{[n]}} \widehat{\mathbf{W}}^\bullet_{[n]} \xrightarrow{r_{[n]}} \mathbf{W}^\bullet_{[n-1]}$ is a Reedy fibration sequence of Reedy fibrant cosimplicial sets, and $\mathbf{W}^\bullet_{[n-1]}$ is $(n-1)$-coskeletal, so applying Tot yields exactness at the left three terms of (20).

For the right three terms, note that for any pointed space $\mathbf{Z}$ a map $g\colon \mathbf{Z} \to \operatorname{Tot}_{n-1} \mathbf{W}^\bullet_{[n-1]}$ is described by $g^k\colon \mathbf{\Delta}^k \ltimes \mathbf{Z} \to \mathbf{W}^k_{[n-1]}$ for $0 \leqslant k \leqslant n-1$, as in the proof of Lemma 5.3. Moreover, the reduced cone on the half-smash: $\bar{C}(\mathbf{X} \ltimes \mathbf{Y})$ (where $\mathbf{Y} \in \mathcal{S}_*$ is pointed, but $\mathbf{X} \in \mathcal{S}$ is not) is isomorphic to $C\mathbf{X} \wedge \mathbf{Y}$ (the smash product with the unreduced cone on $\mathbf{X}$ – cf. § 1.1).

So a nullhomotopy $H\colon \mathbf{F}_{[n-1]} \circ g \sim 0$ is determined by a sequence of maps $H^k\colon \mathbf{\Delta}^k \ltimes \mathbf{Z} \to P\Omega^{n-k-1}\overline{\mathbf{W}}^n$ for $1 \leqslant k \leqslant n$, and the following diagram must commute for each k, as in (16):



(21)

Here we think of $\mathbf{\Delta}^k$ as the (unreduced) cone $C\mathbf{\Delta}^{k-1}$, with $\delta^0\colon \mathbf{\Delta}^{k-1} \to \mathbf{\Delta}^k$ the inclusion of the base, and δ^j the cone on $d^{j-1}\colon \mathbf{\Delta}^{k-2} \to \mathbf{\Delta}^{k-1}$ for $1 \leqslant j \leqslant k$. We write $F^k = \mathbf{F}^k_{[n-1]}\colon \mathbf{W}^k_{[n-1]} \to P\Omega^{n-k-2}\overline{\mathbf{W}}^n$ for the composite

$$\mathbf{W}^k_{[n-1]} \xrightarrow{v^k} C^k \mathbf{W}^\bullet_{[n-1]} \xrightarrow{'F^k} P\Omega^{n-k-2}\overline{\mathbf{W}}^n. \quad (22)$$

See § 2.5 and § 3.1(c).

The maps H^k must satisfy:

$$H^k \circ \delta^0 = F^k \circ g^k, \;\; H^k \circ \delta^1 = \iota_{n-k-1} \circ p \circ H^{k-1}, \text{ and } H^k \circ \delta^j = 0 \text{ for } j \geqslant 2, \quad (23)$$

where $\iota_r\colon \Omega^r\overline{\mathbf{W}}^n \hookrightarrow P\Omega^r\overline{\mathbf{W}}^n$ is the inclusion and p is the path fibration. Moreover,

$$H^1 \circ \delta^1 = 0, \tag{24}$$

since H is a nullhomotopy.

On the other hand, a lift of g to $h\colon \mathbf{Z} \to \operatorname{Tot}_n \widehat{\mathbf{W}}^\bullet_{[n]}$, is given by a sequence of maps $h^k\colon \mathbf{\Delta}^k \ltimes \mathbf{Z} \to P\Omega^{n-k-2}\overline{\mathbf{W}}^n$ for $0 \leqslant k \leqslant n$, with

$$h^k \circ \delta^0 = F^k \circ g^k, \qquad h^k \circ \delta^1 = \iota_{n-k-1} \circ p \circ h^{k-1}, \qquad \text{and } h^k \circ \delta^j = 0 \quad \text{for } j \geqslant 2.$$

Thus, given H, we may set $h^k := H^k$ for $1 \leqslant k \leqslant n$. By (24), we then have $\iota_{n-1} \circ p \circ h^0 = 0$ so h^0 must factor uniquely as $\mathbf{Z} \xrightarrow{\varphi} \Omega^n\overline{\mathbf{W}}^n \xrightarrow{\iota_n} P\Omega^n\overline{\mathbf{W}}^n$ (since ι_{n-1} is monic).

From the description of $g\colon \mathbf{Z} \to \operatorname{Tot} \mathbf{\Sigma D}^\bullet_{[n]}$ in the proof of Lemma 5.3, we see that for any $0 \leqslant m \leqslant n$, any class in $[\mathbf{Z}, \operatorname{Tot} \mathbf{\Sigma D}^\bullet_{[n]}]$ may be represented by a collection of maps $(\widetilde{g}_k\colon \mathbf{\Delta}^n \ltimes \mathbf{Z} \to \overline{\mathbf{W}}^n)_{k=0}^n$ with $\widetilde{g}_k = 0$ for $k \neq m$ and $\widetilde{g}_m|_{\partial\mathbf{\Delta}^n\ltimes\mathbf{Z}} = 0$. For $m = 0$, this shows that the choices for the lift h, given H, are uniquely determined by the image under $(i_{[n]})_*$ of $[g] \in [\mathbf{Z}, \operatorname{Tot}_n \mathbf{\Sigma D}^\bullet_{[n]}]$. This completes the proof by showing the exactness at $\operatorname{Tot}_{n-1} \widehat{\mathbf{W}}^\bullet_{[n-1]}$ in applying $[\mathbf{Z}, -]$ to (20). ☐

Theorem 5.6. *For $\mathbf{W}^\bullet$ constructed as in ¯ 3.1, the spectral sequence associated to the tower of fibrations (15) agrees from the E_2-term on with the unstable Adams spectral sequence of [**BK1**, ¯4].*

Proof. Because each of the cosimplicial spaces $\mathbf{W}^\bullet$, $\mathbf{W}^\bullet_{[n]}$, and $\widehat{\mathbf{W}}^\bullet_{[n]}$ is Reedy fibrant, and the maps $\pi_{[n]}$, $q_{[n]}$, and $r_{[n]}$ of ¯ 3.1 are Reedy fibrations, we have trivial fibrations

$$N^n\mathbf{W}^\bullet \xrightarrow{\simeq} N^n\mathbf{W}^\bullet_{[n]} \xrightarrow{\simeq} N^n\widehat{\mathbf{W}}^\bullet_{[n]} \xrightarrow{\simeq} \overline{\mathbf{W}}^n \tag{25}$$

for each $n \geqslant 0$, since from (6) we see that $N^n\widehat{\mathbf{W}}^\bullet_{[n]} = \prod_{r\geqslant n} P\Omega^{r-n-1}\overline{\mathbf{W}}^r$.

Moreover, by [**BK2**, X, 6.3(ii)] we have

$$\pi_*\overline{\mathbf{W}}^n \cong \pi_*N^n\mathbf{W}^\bullet \cong N^n\pi_*\mathbf{W}^\bullet \cong C^n\pi_*\mathbf{W}^\bullet$$

using the dual of [**BJT2**, Lemma 2.11] for the graded cosimplicial abelian group $\pi_*\mathbf{W}^\bullet$.

Finally, from the fact that $H^*(\mathbf{W}^\bullet; R) \cong V_\bullet$ (a free Θ_R-algebra resolution of $\Gamma = H^*(\mathbf{Y}; R)$), and that, as in [**BK2**, X, ¯7], the d_1-differential of (14) reduces to $d^0_\#$, coming from the CW attaching map of $V_\bullet$ (¯ 2.4), we conclude from [**BK2**, X, ¯6.4] that we have a natural isomorphism between our E_2-term and $\pi^*\pi_*\mathbf{W}^\bullet$, which is isomorphic in turn to that of the unstable Adams spectral sequence by [**BK1**, ¯10.2]. ☐

6. Differentials in the unstable Adams spectral sequence

In order to describe the differentials in the homotopy spectral sequence for the tower (15), we associate to every (n, k) slot in the spectral sequence a sequence of r-th order cohomology operations $\langle\langle - \rangle\rangle_r\colon \pi_{k+n}\overline{\mathbf{W}}^n \to \pi_{k+n+r-1}\overline{\mathbf{W}}^{n+r}$ for $r \geqslant 1$, as described in ¯ 1.1.

6.1. Differentials and higher cohomology operations

These operations are constructed inductively by a sequence of choices, starting with (but independent of) a representative of $\gamma \in \pi_{k+n}\overline{\mathbf{W}}^n$, called the *data* for $\langle\langle\gamma\rangle\rangle_r$. In particular, for each $r \geqslant 1$, $\langle\langle\gamma\rangle\rangle_{r+1}$ is defined only if $\langle\langle\gamma\rangle\rangle_r$ vanishes, and the data for the former includes a choice of a nullhomotopy $\mathbf{H}_{[n+r]}$ for the latter value.

The choice of $\mathbf{H}_{[n+r]}$ defines a certain ∞-homotopy commutative diagram (in the form of a map $\widehat{\mathbf{G}}_{[n+r]} \colon \mathbf{\Delta}^\bullet \ltimes \mathbf{S}^k \to \widehat{\mathbf{W}}^\bullet_{[n+r]}$), which we then make cofibrant (as a map $\mathbf{G}_{[n+r]} \colon \mathbf{\Delta}^\bullet \ltimes \mathbf{S}^k \to \mathbf{W}^\bullet_{[n+r]}$), yielding an appropriate value for $\langle\langle\gamma\rangle\rangle_{r+1}$.

6.2. The inductive construction

We want to associate to every (n,k) slot in the spectral sequence for $\mathbf{W}^\bullet$ a sequence of r-th order cohomology operations $\langle\langle - \rangle\rangle_r \colon \pi_{k+n}\overline{\mathbf{W}}^n \to \pi_{k+n+r-1}\overline{\mathbf{W}}^{n+r}$ for $r \geqslant 1$, as described in § 1.1.

We start by representing $\gamma \in E_1^{n,k+n} = \pi_{k+n}\overline{\mathbf{W}}^n = \pi_k\Omega^n\overline{\mathbf{W}}^n$ by a map $h \colon \mathbf{S}^k \to \Omega^n N^n \mathbf{W}^\bullet$, using (25). Postcomposing h with the inclusion $\iota^n \colon \Omega^n N^n \mathbf{W}^\bullet \hookrightarrow \operatorname{Tot}_n \mathbf{W}^\bullet_{[n]}$ from (13) and the identification $\operatorname{Tot}_n \mathbf{W}^\bullet_{[n]} \cong \operatorname{Tot} \mathbf{W}^\bullet_{[n]}$ (cf. § 3.1), we obtain $h' \colon \mathbf{S}^k \to \operatorname{Tot} \mathbf{W}^\bullet_{[n]}$ and so by adjunction

$$\mathbf{G}_{[n]} \colon \mathbf{\Delta}^\bullet \ltimes \mathbf{S}^k \longrightarrow \mathbf{W}^\bullet_{[n]}.$$

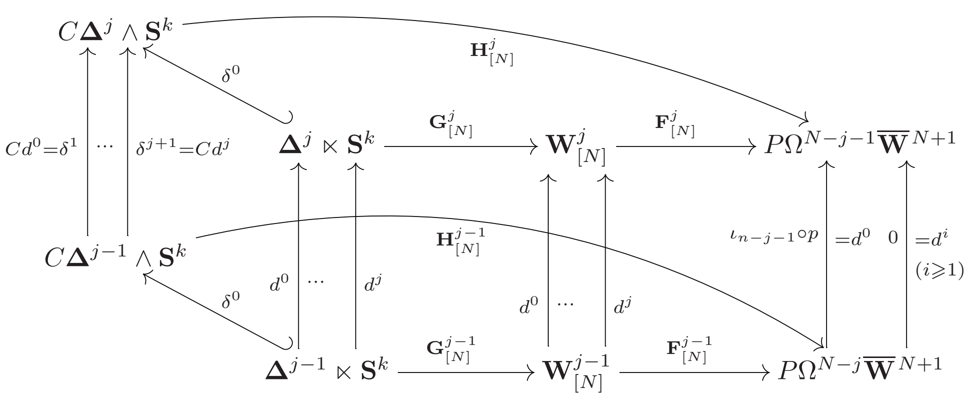
By (13), we may assume that $\mathbf{G}^i_{[n]} \colon \mathbf{\Delta}^i \ltimes \mathbf{S}^k \to \mathbf{W}^i_{[n]}$ is zero for $i < n$.

At the r-th stage, let $N := n + r - 1$, and assume by induction that we have lifted γ (that is, $\mathbf{G}_{[n]}$) along (2) to $\mathbf{G}_{[N]} \colon \mathbf{\Delta}^\bullet \ltimes \mathbf{S}^k \to \mathbf{W}^\bullet_{[N]}$, again with $\mathbf{G}^j_{[N]} \colon \mathbf{\Delta}^j \ltimes \mathbf{S}^k \to \mathbf{W}^j_{[N]}$ equal to zero for $j < n$. By Proposition 5.5, $\mathbf{G}_{[N]}$ can be lifted to $\widehat{\mathbf{G}}_{[N+1]}$ (and thus to $\mathbf{G}_{[N+1]}$), up to homotopy, if and only if $\mathbf{F}_{[N]} \circ \mathbf{G}_{[N]} \sim 0$. We wish to identify the obstruction to the existence of a nullhomotopy

$$\mathbf{H}_{[N]} \colon C\mathbf{\Delta}^\bullet \wedge \mathbf{S}^k \longrightarrow \mathbf{D}^\bullet_{[N+1]}$$

as an r-th order cohomology operation.

Note that $\mathbf{H}_{[N]}$ is completely determined by its projection on the non-codegenerate factors of $\mathbf{D}^\bullet_{[N+1]}$, namely, $\mathbf{H}^j_{[N]} \colon C\mathbf{\Delta}^j \wedge \mathbf{S}^k \to P\Omega^{N-j-1}\overline{\mathbf{W}}^{N+1}$ (cf. (3) and (21)):



with $\mathbf{F}_{[N]}$ given by (22).

As in the proof of Lemma 5.3 (see also [**BS2**, §5]), by adjointing each of the path or loop directions of $P\Omega^{N-j-1}\overline{\mathbf{W}}^{N+1}$ to a cone direction on $\mathbf{\Delta}^j$, we can replace $\mathbf{H}^j_{[N]}$ by $\widetilde{H}^j_{[N]}\colon \mathbf{\Delta}^{N+1} \ltimes \mathbf{S}^k \to \overline{\mathbf{W}}^{N+1}$ for $n \leqslant j \leqslant N$ (since $\mathbf{G}^j_{[N]} = 0$ for $j < n$, we may assume the same for $\mathbf{H}^j_{[N]}$, and thus $\widetilde{H}^j_{[N]}$).

We retain the conventions of the proof of Proposition 5.5: thus the first facet of $\mathbf{\Delta}^{N+1}$ is the base of the cone $C\mathbf{\Delta}^j$, the facets $1, \dots, N-j-1$ of $\mathbf{\Delta}^N$ correspond to the loop directions of $P\Omega^{N-j-1}\overline{\mathbf{W}}^{N+1}$ (counted outwards from $\overline{\mathbf{W}}^{N+1}$), with the $(N-j)$-th facet corresponding to the path direction. As long as $j > 0$, the next $j+1$ faces of $\mathbf{\Delta}^{N+1}$ are the original faces of $\mathbf{\Delta}^j$, re-indexed by $N-j$.

The fact that $\mathbf{H}_{[N]}$ is a map of cosimplicial spaces then translates into the following conditions:

$$\widetilde{H}^j_{[N]} \circ \delta^i = \begin{cases} \widetilde{F^j G^j} & \text{for } i = 0 \\ \widetilde{\iota p H^j} & \text{for } i = N-j \text{ and } j < N \\ \widetilde{\iota p H^{j-1}} & \text{for } i = N-j+1 \text{ and } j > 0 \\ 0 & \text{otherwise,} \end{cases} \tag{26}$$

for each $n \leqslant j \leqslant N$.

Thus we see that $\mathbf{H}_{[N]}$ defines a map $\widetilde{H}_{[N]}\colon \mathcal{P}^{N+1}_r \ltimes \mathbf{S}^k \to \overline{\mathbf{W}}^{N+1}$ (cf. § 5.1), since the maps $\widetilde{H}^j_{[N]}$ on $\mathbf{\Delta}^{N+1}_{(j)} \wedge \mathbf{S}^k$ agree on the identified facets.

Note that from (26) we see that the map $\widetilde{H}_{[N]}$, when restricted to $\partial\mathcal{P}^{N+1}_r \ltimes \mathbf{S}^k$, depends only on the given map $\mathbf{F}_{[N]}$ and the chosen lift $\mathbf{G}_{[N]}$, so we may denote it by

$$\Phi'_{(F,G)}\colon \partial\mathcal{P}^{N+1}_r \ltimes \mathbf{S}^k \longrightarrow \overline{\mathbf{W}}^{N+1}.$$

Moreover, $\Phi'_{(F,G)}$ is zero on $\{v\} \times \mathbf{S}^k$ for each of the cone vertices v of $\mathbf{\Delta}^{N+1}_{(j)}$ in $\mathcal{P}^{N+1}_r$ (because our maps were defined on the smash product with the cone). Thus $\Phi'_{(F,G)}$ induces a map

$$\Phi_{(F,G)}\colon \partial\mathcal{P}^{N+1}_r \wedge \mathbf{S}^k \longrightarrow \overline{\mathbf{W}}^{N+1}.$$

Its domain is a topological $(N+k)$-sphere.

Example 6.1. The boundaries of the three constituent tetrahedra of $\mathcal{P}^3_3$, split open, are illustrated in Figure 6.2, which also shows how each facet is mapped under $\widetilde{H}^j_{[3]}\colon \mathbf{\Delta}^3_{(j)} \to \overline{\mathbf{W}}^3$, and which facets are identified in $\mathcal{P}^3_3$ (dotted arrows). Here $*$ on a facet means that the facet maps to the base point.

Lemma 6.2. *Given maps* $\mathbf{F}_{[N]}$ *and* $\mathbf{G}_{[N]}$ *as above,* $\Phi_{(F,G)}\colon \partial\mathcal{P}^{N+1}_r \wedge \mathbf{S}^k \to \overline{\mathbf{W}}^{N+1}$ *is nullhomotopic if and only if* $\Psi'_{(F,G)}\colon \partial\mathcal{P}^{N+1}_r \ltimes \mathbf{S}^k \to \overline{\mathbf{W}}^{N+1}$ *extends to* $\widetilde{H}_{[N]}\colon \mathcal{P}^{N+1}_r \ltimes \mathbf{S}^k \to \overline{\mathbf{W}}^{N+1}$*, implying the existence of* $\mathbf{H}_{[N]}\colon \mathbf{F}_{[N]} \circ \mathbf{G}_{[N]} \sim 0$ *– and therefore of a lift of* $\mathbf{G}_{[N]}$ *to* $\mathbf{G}_{[N+1]}$*.*

Proof. As noted in § 5.2, the cone $C\partial\mathcal{P}^{N+1}_r$ is homeomorphic to $\mathcal{P}^{N+1}_r$, and the quotient map $q\colon \partial\mathcal{P}^{N+1}_r \ltimes \mathbf{S}^k \to \partial\mathcal{P}^{N+1}_r \wedge \mathbf{S}^k$ extends naturally to $q'\colon \mathcal{P}^{N+1}_r \ltimes \mathbf{S}^k \to$ $\to C\partial\mathcal{P}^{N+1}_r \wedge \mathbf{S}^k$. Thus a nullhomotopy for $\Phi_{(F,G)}$ defines an extension of $\Phi'_{(F,G)}$ to $\widehat{H}_{[N]}\colon \mathcal{P}^{N+1}_r \times \mathbf{S}^k \to \overline{\mathbf{W}}^{N+1}$. Restricting $\widetilde{H}_{[N]}$ to each of the $(N+1)$-simplices

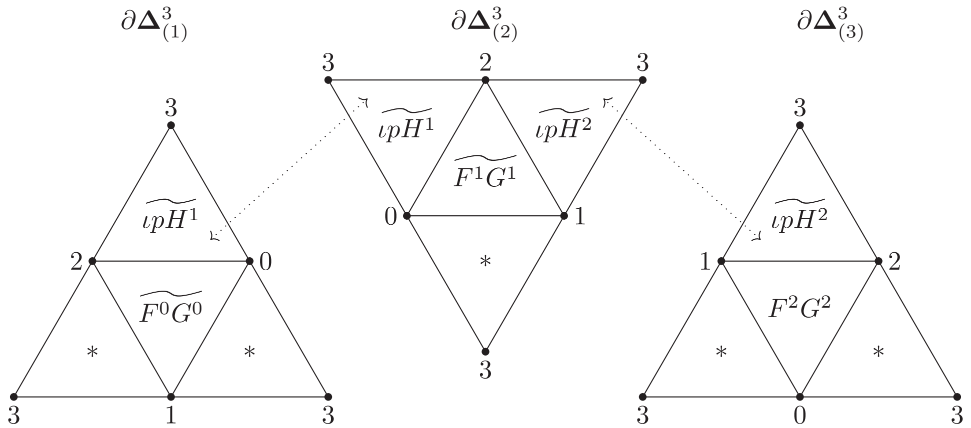


Figure 6.2: The three tetrahedra of $\mathcal{P}^3_3$ mapped to $\overline{\mathbf{W}}^3$

$\mathbf{\Delta}^{N+1}_{(j)}$ of $\mathcal{P}^{N+1}_r$ defines a collection of maps $\widetilde{H}^j_{[N]}$ $(j = 1, \ldots, N+1)$ satisfying (26) (with $\iota p H^j$ defined by restricting $\widetilde{H}_{[N]}$ to the appropriate facets gluing the $(N+1)$-simplices together). As in the proof of Proposition 5.5, the nullhomotopy $\mathbf{H}_{[N]}$ defines a lift of $\mathbf{G}_{[N]}$ to $\widehat{\mathbf{G}}_{[N+1]} \colon \mathbf{\Delta}^\bullet \ltimes \mathbf{S}^k \to \widehat{\mathbf{W}}^\bullet_{[N+1]}$.

If we assume by induction that $\mathbf{G}^i_{[N]} \colon \mathbf{\Delta}^i \ltimes \mathbf{S}^k \to \mathbf{W}^i_{[N]}$ is zero for $i < n$, we may assume the same for the maps $\mathbf{H}^i_{[N]}$, and thus for $\widehat{\mathbf{G}}^i_{[N+1]}$. We then use the left lifting property in the Reedy model category of cosimplicial spaces to obtain the required lift:

$$\begin{array}{ccc} \mathrm{sk}_{n-1}\,\mathbf{\Delta}^\bullet \ltimes \mathbf{S}^k & \xrightarrow{\quad 0 \quad} & \mathbf{W}^\bullet_{[N+1]} \\ {\scriptstyle \mathrm{inc}} \big\downarrow & \overset{\mathbf{G}_{[N+1]}}{\nearrow} & \big\downarrow {\scriptstyle \simeq\ q_{[N+1]}} \\ \mathbf{\Delta}^\bullet \ltimes \mathbf{S}^k & \xrightarrow[\widehat{\mathbf{G}}_{[N+1]}]{} & \widehat{\mathbf{W}}^\bullet_{[N+1]}, \end{array}$$

again with $\mathbf{G}^i_{[N+1]} = 0$ for $i < n$. □

Definition 6.3. Assume given an (R-good) space $\mathbf{Y}$, a CW resolution $V_\bullet \in s\Theta_R\text{-}\mathsf{Alg}$ of $H^*(\mathbf{Y}; R)$, and a realization $\mathbf{W}^\bullet$ of $V_\bullet$ constructed by suitable choices of maps $\mathbf{F}_{[n]} \colon \mathbf{W}^\bullet_{[n]} \to \mathbf{D}^\bullet_{[n+1]}$ for each $n \geqslant 0$, as in § 3.1. For each pair (k, n) we then have a sequence of higher cohomology operations

$$\langle\langle - \rangle\rangle_r \colon \pi_{k+n}\overline{\mathbf{W}}^n \to \pi_{N+k}\overline{\mathbf{W}}^{N+1} = \pi_{k+n+r-1}\overline{\mathbf{W}}^{n+r}$$

for $r \geqslant 1$ and $N = n + r - 1$, which serve as obstructions to lifting $\gamma \in \pi_k \Omega^n \overline{\mathbf{W}}^n$ to $\mathbf{G}_{[n+r+1]} \colon \mathbf{\Delta}^\bullet \ltimes \mathbf{S}^k \to \mathbf{W}^\bullet_{[n+r+1]}$. The *data* for $\langle\langle - \rangle\rangle_r$ consists of compatible choices of lifts $\mathbf{G}_{[i]} \colon \mathbf{\Delta}^\bullet \ltimes \mathbf{S}^k \to \mathbf{W}^\bullet_{[i]}$ for $n \leqslant i \leqslant N$ as above, and the *value* of $\langle\langle \gamma \rangle\rangle_r$ associated to this data is the class of $[\Phi_{(F,G)}] \in \pi_{N+k}\overline{\mathbf{W}}^{N+1}$ constructed in § 6.2. The *indeterminacy* of $\langle\langle \gamma \rangle\rangle_r$ is the subset of $\pi_{N+k}\overline{\mathbf{W}}^{N+1}$ consisting of all possible values, for all (compatible) choices of the data $\mathbf{G}_{[i]}$ (with the maps $(\mathbf{F}_{[n]}\colon \mathbf{W}^{\bullet}_{[n]} \to \mathbf{D}^{\bullet}_{[n+1]})_{n=0}^{\infty}$ fixed). We say that the operation *vanishes* if there is a choice of the data $\mathbf{G}_{[i]}$ with value zero.

From the description above we deduce:

Theorem 6.4. *Each value $\langle\langle\gamma\rangle\rangle_r \in \pi_{k+n+r-1}\overline{\mathbf{W}}^{n+r} = E_1^{n+r,k+n+r-1}$ of the r-th order operation $\langle\langle-\rangle\rangle_r$ represents the result of applying the differential d_r to the element of $E_r^{n,k+n}$ represented by $\gamma \in \pi_{k+n}\overline{\mathbf{W}}^n = E_1^{n,k+n}$. The indeterminacy for $\langle\langle-\rangle\rangle_r$ is the same as that for the differential (as a map $E_1 \to E_1$).*

Remark 6.5. More generally, we can think of the maps $\mathbf{F}_{[n+r]}\colon \mathbf{W}^{\bullet}_{[n+r]} \to \mathbf{D}^{\bullet}_{[n+r+1]}$ of § 3.1(d) as providing a template for a system of higher operations

$$\langle\langle-\rangle\rangle_r\colon [\mathbf{Z},\, \Omega^n\overline{\mathbf{W}}^n] \longrightarrow [\mathbf{Z},\, \Omega^{n+r-1}\overline{\mathbf{W}}^{n+r}]$$

for $r = 2, 3, \ldots$ (here we had $\mathbf{Z} = \mathbf{S}^k$). By a “template” we mean that we can fix the maps $\mathbf{F}_{[i]}$ once and for all, while the choices of the liftings $\mathbf{G}_{[n+r]}$ may need to be changed if a particular value of $\langle\langle\gamma\rangle\rangle_r$ is non-zero.

From Theorem 6.4, and the fact that the unstable Adams spectral sequence is unique up to isomorphism, it follows that *any* choice of this template yields equivalent operations, in the sense that vanishing for one choice implies their vanishing for any other choice of the maps $\mathbf{F}_{[i]}$ – for the right choice of data $\mathbf{G}_{[i]}$.

7. Filtration in the unstable Adams spectral sequence

We now consider the question of determining the filtration index of a non-zero element in the unstable Adams spectral sequence. Here we assume that the coaugmented cosimplicial space $\mathbf{Y} \to \mathbf{W}^{\bullet}$ is constructed as in § 3 for an R-good space $\mathbf{Y}$, with R-completion $\widehat{\mathbf{Y}} \simeq \operatorname{Tot}\mathbf{W}^{\bullet}$, so we require that

$$\pi_k \operatorname{Tot}\mathbf{W}^{\bullet} = \lim_n\ \pi_k \operatorname{Tot}\mathbf{W}^{\bullet}_{[n]}.$$

If we write $p_{[n]}\colon \operatorname{Tot}\mathbf{W}^{\bullet} \to \operatorname{Tot}_n \mathbf{W}^{\bullet}_{[n]}$ for the appropriate structure map in (15), the *filtration index* for $0 \neq \gamma \in \pi_k \operatorname{Tot}\mathbf{W}^{\bullet}$ is the least n for which $(p_{[n]})_*\gamma \neq 0$. Thus $(\pi_{[n]} \circ p_{[n]})_*\gamma = (p_{[n-1]})_*\gamma = 0$, so $(p_{[n]})_*\gamma$ lifts to some element $\alpha \in \pi_k\Omega^n\overline{\mathbf{W}}^n = E_1^{n,n-k}$, by Proposition 5.5. This α represents γ in the spectral sequence.

In general, γ is represented by a map of cosimplicial spaces $\Gamma\colon \mathbf{\Delta}^{\bullet} \ltimes \mathbf{S}^k \to \mathbf{W}^{\bullet}$, as in § 4.1. However, because $\widehat{\mathbf{Y}}$ is R-good, the R-completion $\widehat{\mathbf{Y}} \simeq \operatorname{Tot}\mathbf{W}^{\bullet}$ is coaugmented to $\mathbf{W}^{\bullet}$ by Remark 3.4. Therefore (replacing $\mathbf{Y}$ by $\widehat{\mathbf{Y}}$, if necessary), we may assume for simplicity that γ is represented by a map $g\colon \mathbf{S}^k \to \mathbf{Y}$. The map Γ thus factors in cosimplicial dimension j as the composite Γ^j of

$$\mathbf{\Delta}^j \ltimes \mathbf{S}^k \xrightarrow{\ q\ } \mathbf{S}^k \xrightarrow{\ g\ } \mathbf{Y} \xrightarrow{\ D^j\ } \mathbf{W}^j, \tag{27}$$

where D^j is the unique iterated coface map starting from the coaugmentation, say:

$$D^j := d^j \circ \cdots d^1 \circ \varepsilon. \tag{28}$$

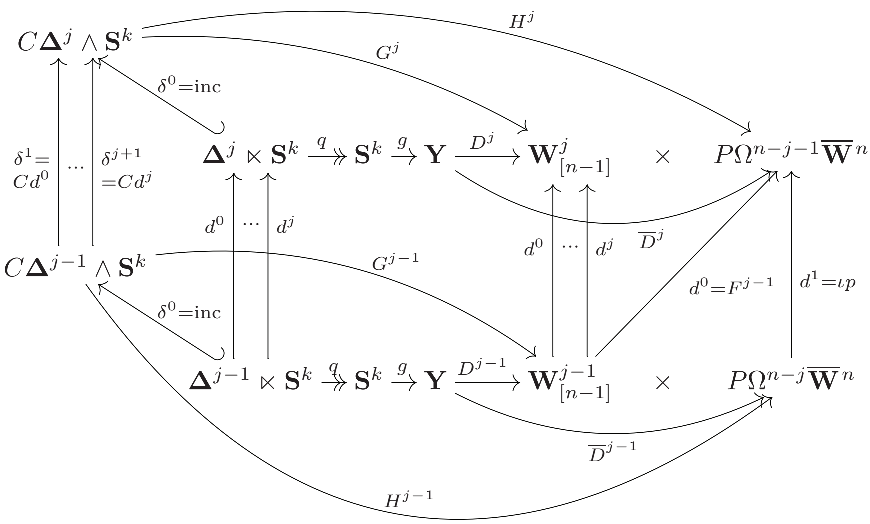
For $\mathbf{W}^{\bullet} = \widehat{\mathbf{W}}^{\bullet}_{[n]}$, we denote the projection of D^j onto the factor $P\Omega^{n-j-1}\overline{\mathbf{W}}^n$ by $\overline{D}^j = \overline{D}^j_{[n]}$.

7.1. The inductive process

We start with the map Γ^0 given by $\varepsilon_{[0]} \circ g\colon \mathbf{S}^k \to \mathbf{W}^0_{[0]}$. If this is non-zero, g has filtration index 0 (which means it is "visible to R-cohomology", since $\varepsilon_{[0]}$ encodes all cohomology classes of $\mathbf{Y}$). Otherwise, we have a nullhomotopic map of cosimplicial spaces $\Gamma_{[0]}\colon \mathbf{\Delta}^\bullet \ltimes \mathbf{S}^k \to \mathbf{W}^\bullet_{[0]}$ (of the simple kind given by (27), and we can choose a nullhomotopy $\mathbf{G}_{[0]}$ for it.

In the induction step, until we reach the filtration index, assume we have $\Gamma_{[n-1]}\colon \mathbf{\Delta}^\bullet \ltimes \mathbf{S}^k \to \mathbf{W}^\bullet_{[n-1]}$ as in (27), with a nullhomotopy $\mathbf{G}_{[n-1]}\colon C\mathbf{\Delta}^\bullet \wedge \mathbf{S}^k \to \mathbf{W}^\bullet_{[n-1]}$. The 0-th coface maps $\delta^0\colon \mathbf{\Delta}^j \hookrightarrow \mathbf{\Delta}^{j+1} = C\mathbf{\Delta}^j$ fit together to define a map of cosimplicial spaces $\mathrm{inc}\colon \mathbf{\Delta}^\bullet \hookrightarrow C\mathbf{\Delta}^\bullet$, with $\mathbf{G}_{[n-1]} \circ \mathrm{inc} = \Gamma_{[n-1]}$.

From (5) we see that to lift $\mathbf{G}_{[n-1]}$ to $\widehat{\mathbf{W}}^\bullet_{[n]}$, for each $0 < j \leqslant n$ we must choose maps $H^j = H^j_{[n]}\colon C\mathbf{\Delta}^j \wedge \mathbf{S}^k \to P\Omega^{n-j-1}\overline{\mathbf{W}}^n$ making the following diagram commute:



(29)

where $q\colon \mathbf{\Delta}^j \ltimes \mathbf{S}^k \to \mathbf{S}^k$ is the projection. Note that we have chosen a different indexing for the coface maps under the identification of $C\mathbf{\Delta}^j$ with $C\mathbf{\Delta}^{j+1}$.

The maps H^j must satisfy:

$$\begin{aligned} &H^j \circ Cd^0 = F^{j-1} \circ G^{j-1}, && H^j \circ Cd^1 = \iota \circ p \circ H^{j-1}, \\ &H^j|_{\mathbf{\Delta}^j \ltimes \mathbf{S}^k} = \overline{D}^j \circ g \circ q, \qquad \text{and} && H^k \circ Cd^i = 0 \text{ for } i \geqslant 2, \end{aligned} \tag{30}$$

for $0 < j \leqslant n$, as in (23).

Using the conventions of § 6.2, we let $\widetilde{H}^j = \widetilde{H}^j_{[n]}\colon \mathbf{\Delta}^{n+1} \to \overline{\mathbf{W}}^n$ denote the appropriate adjoint of $H^j = H^j_{[n]}$, and deduce from (30) that for each $0 \leqslant j \leqslant n$:

$$\widetilde{H}^j \circ \delta^i = \begin{cases} \widetilde{\overline{D}^j gq} & \text{for } i = 0 \\ \widetilde{\iota p H^j} & \text{for } i = n-j \text{ and } j \leqslant n-1 \\ \widetilde{F^{j-1}G^{j-1}} & \text{for } i = n-j+1 \text{ and } j > 0 \\ \widetilde{\iota p H^{j-1}} & \text{for } i = n-j+2 \text{ and } j > 0 \\ 0 & \text{otherwise,} \end{cases} \tag{31}$$

as in (26).

For $j=0$ we may let $H^0\colon C\mathbf{S}^k \to P\Omega^{n-1}\overline{\mathbf{W}}^n$ be the tautological nullhomotopy of $F^{-1}\varepsilon g\colon \mathbf{S}^k \to P\Omega^{n-1}\overline{\mathbf{W}}^n$ (so that its adjoint $\widetilde{H}^0\colon \mathbf{\Delta}^{n+1} \to \overline{\mathbf{W}}^n$, depicted on the left in Figure 7.3, has 2-facet $\widetilde{\iota p H^0} = \widetilde{F^{-1}\varepsilon g}$).

As before, we use (31) to glue the maps $\widetilde{H}^j$ and define a map $\widetilde{H}: \widehat{\mathcal{P}}^{n+1}_{n+2} \ltimes \mathbf{S}^k \to \overline{\mathbf{W}}^n$ (see Remark 5.4). Its restriction to $\partial\widehat{\mathcal{P}}^{n+1}_{n+2} \ltimes \mathbf{S}^k$ again depends only on $\mathbf{F} = \mathbf{F}_{[n]}$ and the chosen nullhomotopy $\mathbf{G} = \mathbf{G}_{[n-1]}$, so we may denote it by $\Psi'_{(F,G)}\colon \partial\widehat{\mathcal{P}}^{n+1}_{n+2} \ltimes \mathbf{S}^k \to \overline{\mathbf{W}}^n$. Moreover, $\Psi'_{(F,G)}$ is zero on $\{v\} \times \mathbf{S}^k$ for any cone vertex v of $\widehat{\mathcal{P}}^{n+1}_n$, so it induces a map $\Psi_{(F,G)}\colon \partial\widehat{\mathcal{P}}^{n+1}_{n+2} \wedge \mathbf{S}^k \to \overline{\mathbf{W}}^n$ from a PL $(n+k)$-sphere.

Remark 7.1. If $j \geqslant 2$, the projection of d^j onto $P\Omega^{n-j-1}\overline{\mathbf{W}}^n$ is zero, so $\overline{D}^j_{[n]} = 0$ by (28), and thus $\widetilde{H}^j_{[n]} \circ \delta^{n+1} = 0$ by (31). On the other hand, $\widetilde{H}^1_{[n]} \circ \delta^{n+1} = \widetilde{F^0 \circ \varepsilon \circ g \circ q}$, and $\widetilde{H}^0_{[n]} \circ \delta^{n+1} = \widetilde{F^{-1} \circ g}$. Note that it is consistent with the description in § 3.1 to think of $g\colon \mathbf{S}^k \to \mathbf{Y}$ as G^{-1}, so more suggestively we may write this last as $\widetilde{F^{-1} \circ G^{-1}}$.

Example 7.2. The boundaries of the three constituent tetrahedra of $\widehat{\mathcal{P}}^3_3$ are given in Figure 7.3, including their identifications, showing how the facets are mapped to $\overline{\mathbf{W}}^2$ under $\widetilde{H}^j_{[2]}$.

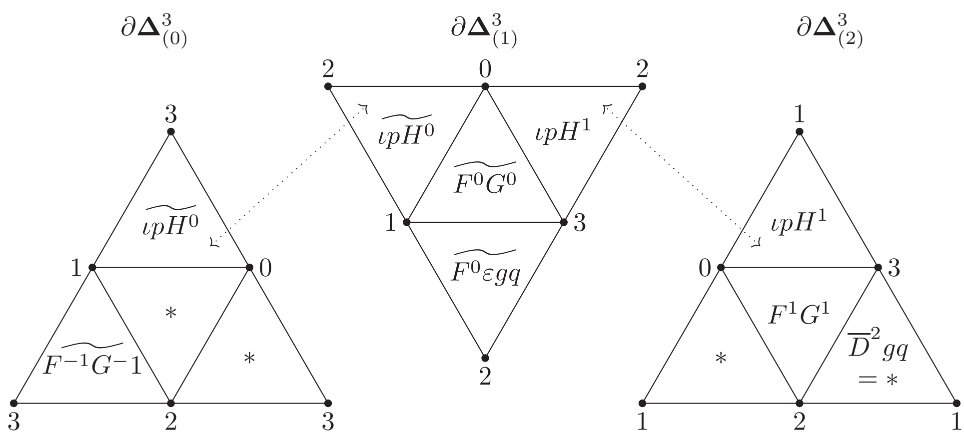


Figure 7.3: The three tetrahedra of $\widehat{\mathcal{P}}^3_3$, mapping to $\overline{\mathbf{W}}^3$

Note that all 2-simplices of the boundary $\partial\widehat{\mathcal{P}}^3_3$ map to $\overline{\mathbf{W}}^3$ by zero or by a map of the form $\widetilde{F^j G^j}$ $(-1 \leqslant j \leqslant 1)$, except for one mapping by $\widetilde{F^0 \varepsilon g q}$. However, the fact that we precompose this map with the projection $q\colon \mathbf{\Delta}^1 \ltimes \mathbf{S}^k \to \mathbf{S}^k$ indicates that this 2-simplex in $\partial\widehat{\mathcal{P}}^3_3$ is *degenerate* (that is, it may be collapsed to its top edge, the 0-face of $\widetilde{F^0 G^0}$).

As in Lemma 6.2 we deduce:

Lemma 7.3. *Given maps* $\mathbf{F}_{[n]}$ *and a nullhomotopy* $\mathbf{G}_{[n-1]}$ *as above,* $\Psi_{(F,G)}\colon \partial\mathcal{P}^{n+1}_{n+1}\wedge \mathbf{S}^k \to \overline{\mathbf{W}}^n$ *is nullhomotopic if and only if* $\Psi'_{(F,G)}\colon \partial\mathcal{P}^{N+1}_r \ltimes \mathbf{S}^k \to \overline{\mathbf{W}}^{N+1}$ *extends to* $\widehat{H}_{[N]}\colon \mathcal{P}^{N+1}_r \ltimes \mathbf{S}^k \to \overline{\mathbf{W}}^{N+1}$*, implying existence of a nullhomotopy* $\mathbf{H}_{[N]}$ *– and therefore of a lift of* $\mathbf{G}_{[N]}$ *to* $\mathbf{G}_{[N+1]}$.

Definition 7.4. By analogy with § 6.3, given a space $\mathbf{Y}$, a CW resolution $V_\bullet \in s\Theta_R\text{-}\mathsf{Alg}$ of $H^*(\mathbf{Y};R)$, and maps $\mathbf{F}_{[n]}\colon \mathbf{W}^\bullet_{[n]} \to \mathbf{D}^\bullet_{[n+1]}$ $(n \geqslant 0)$, yielding a realization $\mathbf{W}^\bullet$ of $V_\bullet$ as in § 3.1, we have a new sequence of higher cohomology operations $\langle\langle - \rangle\rangle'_r\colon \pi_k\mathbf{Y} \to \pi_{k+n}\overline{\mathbf{W}}^n$ for each pair (k,n), which serve as obstructions to representing $\gamma \in \pi_k\mathbf{Y}$ in higher Adams filtration by lifting nullhomotopies $\mathbf{G}_{[i]}$ $(0 \leqslant i < n)$ – the *data* for $\langle\langle - \rangle\rangle'_r$ – to a nullhomotopy $\mathbf{G}_{[n]}$.

The *value* of $\langle\langle \gamma \rangle\rangle'_r$ associated to this data is the class of $[\Psi_{(F,G)}] \in \pi_{n+k}\overline{\mathbf{W}}^n$ constructed in § 7.1. The *indeterminacy* of $\langle\langle \gamma \rangle\rangle'_r$ is the subset of $\pi_{n+k}\overline{\mathbf{W}}^n$ consisting of all possible values, for all (compatible) choices of the data $\mathbf{G}_{[i]}$ (with the maps $(\mathbf{F}_{[n]}\colon \mathbf{W}^\bullet_{[n]} \to \mathbf{D}^\bullet_{[n+1]})_{n=0}^\infty$ again fixed).

Remark 6.5 as to the independence of the operations $\langle\langle - \rangle\rangle'_r$ from the "template" maps $\mathbf{F}_{[n]}\colon \mathbf{W}^\bullet_{[n]} \to \mathbf{D}^\bullet_{[n+1]}$ applies here too, *mutatis mutandis*, because of the following

Theorem 7.5. *For any* $0 \neq \gamma \in \pi_k\mathbf{Y}$*, the operation* $\langle\langle \gamma \rangle\rangle'_{n-1}$ *vanishes while* $\langle\langle \gamma \rangle\rangle'_n \neq 0$ *if and only if* γ *is represented in the unstable Adams spectral sequence in filtration* n *by the value of* $\langle\langle \gamma \rangle\rangle'_n$ *in* $\pi_k\Omega^n\overline{\mathbf{W}}^n = E_1^{n,n-k}$.

Proof. By Lemma 7.3 $\langle\langle \gamma \rangle\rangle'_{n-1}$ vanishes (for some choice of nullhomotopy $\mathbf{G}_{[n-1]}$) if and only if we have a lift of $\mathbf{G}_{[n-1]}$ to a nullhomotopy $\mathbf{G}_{[n]}$, so γ has Adams filtration $\geqslant n$. Assume $\langle\langle \gamma \rangle\rangle'_n$ is represented by $\Psi_{(F,G)}\colon \partial\widehat{\mathcal{P}}^{n+1}_n \wedge \mathbf{S}^k \to \overline{\mathbf{W}}^n$, induced by $\widetilde{H}\colon \widehat{\mathcal{P}}^{n+1}_n \ltimes \mathbf{S}^k \to \overline{\mathbf{W}}^n$ as in § 7.1.

For each $0 \leqslant j \leqslant n$ denote $F^{j-1}\circ G^{j-1}$ by $\phi^j\colon \mathbf{\Delta}^j \ltimes \mathbf{S}^k \to P\Omega^{n-j-1}\overline{\mathbf{W}}^n$. For $j = 0$ we let $C\mathbf{\Delta}^{-1}\wedge \mathbf{S}^k := \mathbf{S}^k$ and $\mathbf{W}^{-1}_{[n-1]} := \mathbf{Y}$ (compare § 3.1(b)), with $G^{-1}\colon C\mathbf{\Delta}^{-1}\wedge \mathbf{S}^k \to \mathbf{W}^{-1}_{[n-1]}$ equal to g as in § 7.1.

Using the conventions of § 7.1 for $j \geqslant 1$, we write δ^0 for the inclusion of the base $\mathbf{\Delta}^j \ltimes \mathbf{S}^k \hookrightarrow C\mathbf{\Delta}^j \wedge \mathbf{S}^k$, and δ^{i+1} for $Cd^i\colon C\mathbf{\Delta}^{j-1}\wedge \mathbf{S}^k :\to C\mathbf{\Delta}^j \wedge \mathbf{S}^k$ $(0 \leqslant i \leqslant j)$. From (29) and (30) we see that for each $j \geqslant 1$ and $0 \leqslant i \leqslant j$, we have

$$\phi^{j+1}\circ\delta^{i+1} := F^j \circ G^j \circ Cd^i = F^j \circ d^i \circ G^{j-1} = d^0 \circ d^i \circ G^{j-1} = d^{i+1}\circ d^0 \circ G^{j-1} \tag{32}$$

into $P\Omega^{n-j-2}\overline{\mathbf{W}}^n$, which is $d^1 \circ F^{j-1}\circ G^{j-1} = d^1 \circ \phi^j$ when $i = 0$. When $i \geqslant 1$, the projection of d^{i+1} onto $P\Omega^{n-j-2}\overline{\mathbf{W}}^n$ vanishes (cf. § 3.1), so by (32) both $\phi^{j+1}\circ\delta^{i+1}$ and $d^{i+1}\circ\phi^j$ are zero there.

Similarly, by (28) we have

$$\phi^{j+1}\circ\delta^0 = F^j \circ G^j|_{\mathbf{\Delta}^j\ltimes\mathbf{S}^k} = F^j \circ D^j \circ g = d^0 d^j \cdots \varepsilon \circ g = d^{j+1}d^j \cdots \varepsilon \circ g = 0$$

into $P\Omega^{n-j-2}\overline{\mathbf{W}}^n$, and also $d^0|_{P\Omega^{n-j-1}\overline{\mathbf{W}}^n}$ is zero.

Finally, for $j = 0$ we write δ^1 for the inclusion of the base $\mathbf{S}^k$ into $C\mathbf{S}^k = \mathbf{\Delta}^1 \wedge \mathbf{S}^k$, and δ^0 for the map collapsing $\mathbf{S}^k$ to the cone point, so we have:

$$\phi^1 \circ \delta^i := F^0 \circ G^0 \circ \delta^i = \begin{cases} \iota p \circ F^{-1} \circ G^{-1} = d^1\phi^0 & \text{if } i = 1 \\ 0 = d^0\phi^0 & \text{if } i = 0 \end{cases}$$

into $P\Omega^{n-2}\overline{\mathbf{W}}^n$.

Thus for each $j \geqslant 0$ we have:

$$d^i \circ \phi^j = \phi^{j+1} \circ \delta^i \qquad \text{for all } 0 \leqslant i \leqslant j+1$$

(into $P\Omega^{n-j-2}\overline{\mathbf{W}}^n$).

Therefore, by the proof of Lemma 5.3, the maps $\phi^j\colon \mathbf{\Delta}^j \ltimes \mathbf{S}^k \to P\Omega^{n-j-1}\overline{\mathbf{W}}^n$ uniquely determine a pointed map $\phi\colon \mathbf{S}^k \to \operatorname{Tot} \mathbf{\Sigma D}^\bullet_{[n]}$. The inclusion $i_{[n]}\colon \mathbf{\Sigma D}^\bullet_{[n]} \hookrightarrow \widehat{\mathbf{W}}^\bullet_{[n]}$ allows us to think of ϕ as a map $(i_{[n]})_*\phi = \varphi\colon \mathbf{S}^k \to \operatorname{Tot}_n \widehat{\mathbf{W}}^\bullet_{[n]}$, which maps by zero to the factor $\mathbf{W}^j_{[n-1]}$ of $\widehat{\mathbf{W}}^j_{[n]}$ in (6).

Let $t^j\colon \mathbf{\Delta}^j \times I \to C\mathbf{\Delta}^j \cong \mathbf{\Delta}^{j+1}$ be the map collapsing the top of the prism, $\mathbf{\Delta}^j \times \{1\}$, to a point, and let $\sigma^0\colon \mathbf{\Delta}^{j+1} \to \mathbf{\Delta}^j$ be the 0-the codegeneracy map of $\mathbf{\Delta}^\bullet$.

We then define a pointed homotopy $K\colon I \ltimes \mathbf{S}^k \longrightarrow \operatorname{Tot}_n \widehat{\mathbf{W}}^\bullet_{[n]}$ by setting $K^j\colon (\mathbf{\Delta}^j \times I) \ltimes \mathbf{S}^k \to \widehat{\mathbf{W}}^j_{[n]}$ equal to:

$$(\mathbf{\Delta}^j \times I) \ltimes \mathbf{S}^k \xrightarrow{(G^j\circ t^j)\top(X^j)\top(F^{j-1}\circ G^{j-1}\circ\sigma^0\circ t^j)} \mathbf{W}^j_{[n-1]} \times \widehat{\mathbf{M}}^{r-1}_{[n]} \times P\Omega^{n-j-1}\overline{\mathbf{W}}^n$$

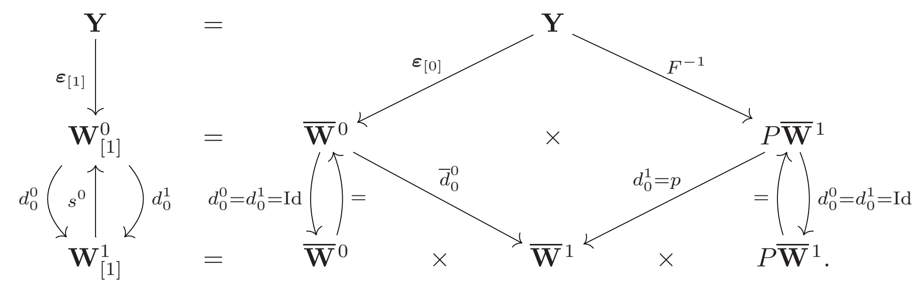
for each $1 \leqslant j \leqslant n$, where X^j is determined by the codegeneracies on G^j and $F^{j-1} \circ G^{j-1} \circ \sigma^0 \circ t^j$. When $j = 0$ we have $\Gamma^0_{[n]} = \varphi^0$ into $P\Omega^{n-1}\overline{\mathbf{W}}^n$, with the third map the identity homotopy.

The restriction $K \circ \iota_0$ to the 0-end of I (the base of each prism $\mathbf{\Delta}^j \times I$) is the given map $\Gamma_{[n]}\colon C\mathbf{\Delta}^\bullet \ltimes \mathbf{S}^k \to \widehat{\mathbf{W}}^\bullet_{[n]}$, since $K^j \circ \iota_0 = D^j \circ g \circ q\colon \mathbf{\Delta}^j \ltimes \mathbf{S}^k \to \widehat{\mathbf{W}}^j_{[n]}$ by (29).

On the other hand, $K \circ \iota_1$ is zero into each factor $\mathbf{W}^j_{[n-1]}$, since G^j is a nullhomotopy, while the component into $P\Omega^{n-j-1}\overline{\mathbf{W}}^n$ is $F^{j-1} \circ G^{j-1} = \phi^j$. Thus $K \circ \iota_1$ is $\varphi\colon \mathbf{S}^k \to \operatorname{Tot}_n \widehat{\mathbf{W}}^\bullet_{[n]}$, showing that $[\phi] \in \pi_k \operatorname{Tot} \mathbf{\Sigma D}^\bullet_{[n]}$ (the value of $\langle\langle\gamma\rangle\rangle'_{n-1}$) indeed represents $\Gamma_{[n]}$ (the lift of γ to the n-th level in (15). ∎

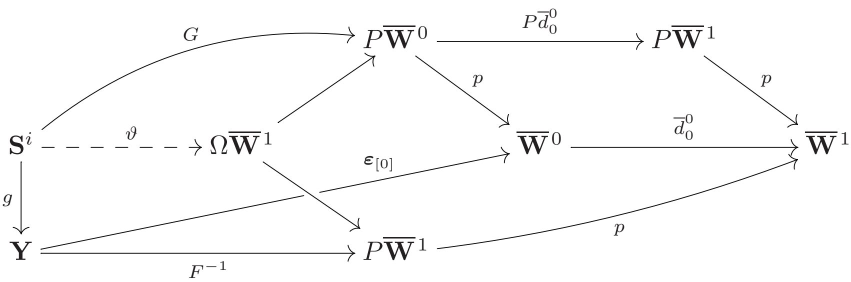
7.2. The one-dimensional case

From the description in 3.1 we have



where p is the path fibration, so F^{-1} is a nullhomotopy for $a^{-1} := \overline{d}^0_0 \circ \varepsilon_{[0]}$, in the notation of (7).

If $\gamma \in \pi_i \widehat{\mathbf{Y}}$ has filtration index $\geqslant 1$, it can be represented by a map $g\colon \mathbf{S}^i \to \mathbf{Y}$ with $\boldsymbol{\varepsilon}_{[0]} \circ g \sim *$. Choosing a nullhomotopy $G\colon \mathbf{S}^i \to P\overline{\mathbf{W}}^0$ (with $p \circ G = \boldsymbol{\varepsilon}_{[0]} \circ g$), we obtain a solid commutating diagram



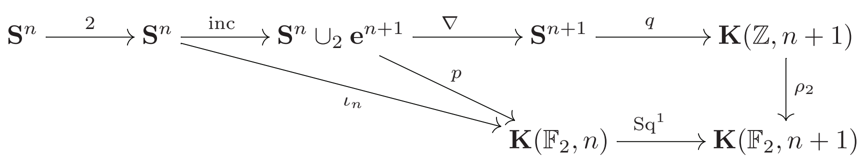
(33)

where the outer right pentagon is Cartesian, thus defining ϑ into the pullback, which is a (non-standard model for) $\Omega\overline{\mathbf{W}}^1$. In fact, $[\vartheta]$ is (one value of) the Toda bracket $\langle \overline{d}^0_0, \boldsymbol{\varepsilon}_{[0]}, g\rangle \subseteq \pi_{i+1}\overline{\mathbf{W}}^1$, and it is non-zero if and only if g has filtration 1, since $\Omega\overline{\mathbf{W}}^1 = \Omega N^1\mathbf{W}^\bullet_{[1]}$, and $j^1[\vartheta]$ is the lift of g to $\mathrm{Tot}_1\,\mathbf{W}^\bullet \simeq \mathrm{Tot}_1\,\mathbf{W}^\bullet_{[1]}$ (cf. ¯ 5.1) in Figure 4.1.

Combining this with Theorem 7.5, we see that γ has filtration index 1 if and only if there is a secondary cohomology operation which acts on it non-trivially (that is, cannot be made to vanish for any choice of nullhomotopies).

More generally, γ has filtration index n if and only if there is an $(n+1)$-th order cohomology operation which acts on it non-trivially (where n is necessarily the highest order for which this is possible).

Example 7.6. Consider the cofibration sequence $\mathbf{S}^n \xrightarrow{2} \mathbf{S}^n \xrightarrow{\mathrm{inc}} \mathbf{S}^n \cup_2 \mathbf{e}^{n+1} \xrightarrow{\nabla} \mathbf{S}^{n+1}$ for the mod 2 Moore space, fitting into the homotopy commutative diagram



(34)

where p and q are the appropriate Postnikov fibrations and ρ_2 is the reduction mod 2 map.

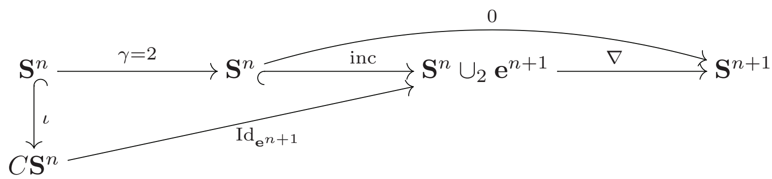
Therefore, we can realize a free simplicial resolution of the Θ_R-algebra $H^*(\mathbf{S}^n;\mathbb{F}_2)$ by $\mathbf{Y} = \mathbf{S}^n \to \mathbf{W}^\bullet$ in cosimplicial dimensions $\leqslant 1$:

$$\mathbf{S}^n \xrightarrow{\iota_n} \mathbf{K}(\mathbb{F}_2, n) \xrightarrow{\mathrm{Sq}^1} \mathbf{K}(\mathbb{F}_2, n+1)$$

For simplicity we have omitted all factors $\mathbf{K}(\mathbb{F}_2, i)$ for $i > n+1$. Assuming that $n \geqslant 3$, this means that we are in the stable range, so we can also omit the codegeneracies, the contractible path space factors, and thus the coface maps other than d^0, which were shown in (33).

Since the map $\gamma = 2$ in $\pi_n\mathbf{S}^n$ is invisible in $H^*(-;\mathbb{F}_2)$, it has Adams filtration $\geqslant 1$.

However, the diagram



shows that one value of the Toda bracket $\langle \nabla, \mathrm{inc}, \gamma \rangle$ is the pinch map $\Sigma\mathbf{S}^n \cong C\mathbf{S}^n/\mathbf{S}^n \xrightarrow{\cong} \mathbf{S}^{n+1}$, which has degree 1. Post-composing with $\rho_2 \circ q$ of (34) we obtain the value $\iota_{n+1} \in \pi_{n+1}\mathbf{K}(\mathbb{F}_2, n+1)$ for the associated secondary cohomology operation $\langle \mathrm{Sq}^1, \iota_n, \gamma \rangle$ (see [**T2**, Ch. 1]), with indeterminacy

$$\{0\} = 2 \cdot \pi_{n+1}\mathbf{K}(\mathbb{F}_2, n+1) + \mathrm{Sq}^1_* \, \pi_n \mathbf{K}(\mathbb{F}_2, n).$$

This shows that in fact γ has filtration index 1, as expected. Note that as often happens, we do not calculate with higher order operations from scratch, using any available information to deduce the value. Nevertheless, it is sometimes possible to produce infinite families of non-trivial higher operations: for an example, see [**BBS**, §8], where for each $n \geqslant 1$, the Hopf map $g_n \colon \mathbf{S}^{2n+1} \to \mathbb{C}\mathbf{P}^n$ is described as the value of an explicit n-th order operation involving only Whitehead products.

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