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A REMARK ON THE DOUBLE COMPLEX OF A COVERING FOR SINGULAR COHOMOLOGY

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Abstract

Given an open covering of a paracompact topological space X, there are two natural ways to construct a map from the cohomology of the nerve of the covering to the cohomology of X. One of them is based on a partition of unity, and is more topological in nature, while the other one relies on the double complex associated to an open covering, and has a more algebraic flavour. In this paper we prove that these two maps coincide.

1. Introduction

Let X be a paracompact space, and let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open covering of X. We denote by $N(\mathcal{U})$ the *nerve* of $\mathcal{U}$, i.e. the simplicial set having I as set of vertices, in which a finite subset $\{i_0, \ldots, i_k\} \subseteq I$ spans a simplex if and only if $U_{i_0} \cap \ldots \cap U_{i_k} \neq \varnothing$. As usual, we endow the geometric realization $|N(\mathcal{U})|$ of $N(\mathcal{U})$ with the weak topology associated to the natural CW structure of $|N(\mathcal{U})|$.

Any partition of unity $\Phi = \{\varphi_i \colon X \to \mathbb{R}\}_{i\in I}$ subordinate to $\mathcal{U}$ induces a map

$$f_\Phi \colon X \to |N(\mathcal{U})| , \qquad f_\Phi(x) = \sum_{i\in I} \varphi_i(x) \cdot i .$$

Moreover, the homotopy class of f_Φ does not depend on the chosen partition of unity Φ. Indeed, if Ψ is another partition of unity, then we have a well-defined homotopy $tf_\Psi + (1-t)f_\Phi$ between f_Ψ and f_Φ. Therefore, if R is any ring with unity, the map f_Φ induces a map

$$f^* = f_\Phi^* \colon H^*(|N(\mathcal{U})|, R) \to H^*(X, R) ,$$

which does not depend on the choice of Φ. Throughout this paper, we fix a ring with unity R, and for any topological space Y we denote by $C^*(Y) = C^*(Y, R)$ (resp. $H^*(Y) = H^*(Y, R)$) the singular cochain complex (resp. the singular cohomology algebra) of Y with coefficients in R.

There is another natural way to define a map from the (simplicial) cohomology of $N(\mathcal{U})$ to the singular cohomology of X, using a double complex associated to the covering $\mathcal{U}$. The idea of this construction goes back at least to the paper of Weil on the de Rham Theorem [**Wei52**], where the author dealt with differential forms in

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Section 2, and with singular chains in Section 3. Let $C^{*,*}(\mathcal{U})$ be the double complex associated to the covering $\mathcal{U}$, i.e. for every $(p,q) \in \mathbb{N}^2$ let

$$C^{p,q}(\mathcal{U}) = \prod_{\underline{i} \in I_p} C^q(U_{\underline{i}}) \ ,$$

where I_p denotes the set of ordered $(p+1)$-tuples $(i_0, \ldots, i_p) \in I^{p+1}$ such that $U_{\underline{i}} := U_{i_0} \cap \ldots \cap U_{i_p} \neq \varnothing$ (in particular, $I_0 = \{i \in I \mid U_i \neq \varnothing\}$). We refer the reader to Section 2 for the precise definition of this double complex.

To the double complex $C^{*,*}(\mathcal{U})$ there is associated the *total complex* T^*, and we have maps

$$\alpha_X \colon H^*(X) \to H^*(T^*) \ , \qquad \beta \colon H^*(N(\mathcal{U})) \to H^*(T^*)$$

from the singular cohomology of X to the cohomology of T^* and from the simplicial cohomology of $N(\mathcal{U})$ to the cohomology of T^*. Moreover, the map α turns out to be an isomorphism (see Section 2).

Let now $\nu \colon H^*(|N(\mathcal{U})|) \to H^*(N(\mathcal{U}))$ be the canonical isomorphism between the simplicial cohomology of $N(\mathcal{U})$ and the singular cohomology of its geometric realization (see Section 3). By setting $\eta = \alpha_X^{-1} \circ \beta \circ \nu$ we have thus defined a map

$$\eta \colon H^*(|N(\mathcal{U})|) \to H^*(X) \ .$$

The main result of this paper shows that the maps f^* and η coincide:

Theorem 1.1. *The maps*

$$f^* \colon H^*(|N(\mathcal{U})|) \to H^*(X) \, , \quad \eta \colon H^*(|N(\mathcal{U})|) \to H^*(X)$$

coincide.

We were motivated to study whether the maps f^* and η should coincide by the fact that our result may be exploited to provide an interpretation of a result by Ivanov on the bounded cohomology of topological spaces. In his pioneering paper [**Gro82**], Gromov introduced the notion of *bounded* singular cochains. For any topological space X, bounded cochains (with real coefficients) provide a subcomplex $C_b^*(X)$ of $C^*(X)$. The cohomology of the complex $C_b^*(X)$ is denoted by $H_b^*(X)$, and defines the *bounded cohomology* of X. The inclusion of bounded cochains into ordinary cochains induces the *comparison map* $c^* \colon H_b^*(X) \to H^*(X)$.

A subset U of X is *amenable* if, for every $x_0 \in U$, the image of the map $\pi_1(U, x_0) \to \pi_1(X, x_0)$ induced by the inclusion is amenable, and a covering of X is amenable if each of its elements is amenable. Gromov's Vanishing Theorem [**Gro82**] asserts that, if X admits an open amenable covering $\mathcal{U}$ of multiplicity n, then the comparison map $c^m \colon H_b^m(X) \to H^m(X)$ is null for every $m \geqslant n$ (if X is a closed n-dimensional manifold, then the vanishing of the comparison map in degree n has strong implications on the vanishing of several interesting invariants of X, such as the simplicial volume, the minimal volume, the volume entropy).

Gromov's Vanishing Theorem was generalized and made more precise by Ivanov in [**Iva87, Iva**], where it is shown that, if $\mathcal{U}$ is a nice covering of X, then there exists a map $\theta^*\colon H^*_b(X) \to H^*_b(|N(\mathcal{U})|)$ which makes the following diagram commute

$$\begin{tikzcd} H^*_b(X) \arrow[rr, "c^*"] \arrow[dr, "\theta^*"'] & & H^*(X) \\ & H^*_b(|N(\mathcal{U})|) \arrow[ur, "\eta"'] & \end{tikzcd}$$

(we refer the reader to [**Iva**] for the definition of nice covering; for example, any open covering of a hereditary paracompact space is nice).

Our result implies that, in the diagram above, the map η can be replaced by the map f^*. This provides a clear topological interpretation of the well-known concept that amenable spaces are somewhat invisible to bounded cohomology: in fact, the map $f\colon X \to |N(\mathcal{U})|$ shrinks each element of $\mathcal{U}$ into a contractible subset of $|N(\mathcal{U})|$, thus trivializing its topology, while the commutativity of the diagram implies that the information carried by $|N(\mathcal{U})|$ still suffices to completely determine singular cohomology classes admitting a bounded representative. For recent alternative proofs of the fact that the comparison map c^* factors through f^* (which, however, do not deal with the question whether $f^* = \eta$ in general) we refer the reader to [**FM**, Chapter 6] and [**LS**].

2. The double complex associated to an open covering

Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open covering of the topological space X. We now thoroughly describe the horizontal and the vertical differentials of the double complex $C^{*,*}(\mathcal{U})$ defined in the introduction, also fixing the notation we will need later.

If $\varphi \in C^{p,q}(\mathcal{U})$ and $\underline{i} \in I_p$, then we denote by $\varphi_{\underline{i}}$ the projection of φ on $C^q(U_{\underline{i}})$. For every $(p,q) \in \mathbb{N}^2$ we denote by

$$\delta_v^{p,q}\colon C^{p,q}(\mathcal{U}) \to C^{p,q+1}(\mathcal{U})$$

the "vertical" differential which restricts to the usual differential $C^q(U_{\underline{i}}) \to C^{q+1}(U_{\underline{i}})$ for every $\underline{i} \in I_p$, and by

$$\delta_h^{p,q}\colon C^{p,q}(\mathcal{U}) \to C^{p+1,q}(\mathcal{U})$$

the "horizontal" differential such that, for every $\underline{i} = (i_0, \dots, i_{p+1}) \in I_{p+1}$ and every $\varphi \in C^{p,q}(\mathcal{U})$,

$$(\delta_h^{p,q}(\varphi))_{\underline{i}} = \sum_{k=0}^{p+1} (-1)^k \big(\varphi_{(i_0,\dots,\hat{i}_k,\dots,i_{p+1})}\big)|_{U_{\underline{i}}} \ . \tag{1}$$

We augment the double complex $C^{*,*}(\mathcal{U})$ as follows. We define $C_q^{\mathcal{U}}$ as the subcomplex of the singular chain complex $C_q(X)$ generated (over R) by those singular simplices $s\colon \Delta^q \to X$ such that $s(\Delta^q)$ is contained in U_i for some $i \in I$. We then set $C^{-1,q}(\mathcal{U}) = C^q_{\mathcal{U}} = \mathrm{Hom}(C_q^{\mathcal{U}}, R)$. The usual boundary maps of the complex $C_*^{\mathcal{U}}$ induce dual coboundary maps, which endow $C^*_{\mathcal{U}}$ with the structure of a complex. The inclusion of the complex $C_*^{\mathcal{U}}$ in the full complex of singular chains induces a map of complexes $\widetilde{\gamma}\colon C^*(X) \to C^*_{\mathcal{U}}$. It is known that the map γ induced in cohomology is an isomorphism (see e.g. [**Hat02**, Proposition 2.21]) and we will identify the singular cohomology of X with the cohomology of the complex $C^q_{\mathcal{U}}$ via γ. The augmentation maps $\delta^{-1,q}\colon C^{-1,q}(\mathcal{U}) \to C^{0,q}(\mathcal{U})$ are defined by setting, for every $i \in I_0$,

$$(\delta^{-1,q}(\varphi))_i = \varphi|_{U_i} \ .$$

In order to define the augmentation of the vertical complexes, we consider the Cech complex given by $C^{p,-1}(\mathcal{U}) = \check{C}^p(\mathcal{U}) = \prod_{\underline{i}\in I_p} R$, with boundary maps defined as in formula (1). We then define the augmentation maps $\delta^{p,-1}\colon C^{p,-1}(\mathcal{U}) \to C^{p,0}(\mathcal{U})$ by setting

$$(\delta^{p,-1}(\varphi))_{\underline{i}}(s) = \varphi_{\underline{i}} \ \in \ R$$

for every $\varphi \in C^{p,-1}(\mathcal{U})$, every $\underline{i} = (i_0, \ldots, i_p) \in I_p$ and every singular simplex $s\colon \Delta^0 \to U_{i_0} \cap \ldots \cap U_{i_p}$.

Remark 2.1. The complex $\check{C}^*(\mathcal{U})$ computes the Cech cohomology of the covering $\mathcal{U}$ with coefficients in the *constant* presheaf R. Such cohomology, which is usually denoted by $\check{H}(\mathcal{U})$, is tautologically isomorphic to the simplicial cohomology of the nerve $N(\mathcal{U})$. It is costumary to rather study the Cech cohomology of $\mathcal{U}$ with coefficients in the *locally constant* sheaf R. However this cohomology does not always coincide with the cohomology of $N(\mathcal{U})$. They coincide, for example, under the assumption that every $U_{\underline{i}}$, $i \in I_p$, $p \in \mathbb{N}$, is path connected.

It is well known that, for every $q \in \mathbb{N}$, the row

$$0 \longrightarrow C^{-1,q}(\mathcal{U}) \xrightarrow{\delta_h^{-1,q}} C^{0,q}(\mathcal{U}) \xrightarrow{\delta_h^{0,q}} \cdots \xrightarrow{\delta_h^{p-1,q}} C^{p,q}(\mathcal{U}) \xrightarrow{\delta_h^{p,q}} \cdots \tag{2}$$

of the augmented double complex introduced above is exact. A proof of the analogous statement for singular chains can be found, for example, in [**BT82**, Proposition 15.2], and the statement for cochains follows immediately.

As a consequence, the cohomology groups of the complex $C^{-1,*}$ are isomorphic to the cohomology of the *total complex* T^* associated to the double complex. Recall that T^* is defined by setting

$$T^n = \bigoplus_{\substack{(p,q)\in\mathbb{N}^2 \\ p+q=n}} C^{p,q}(\mathcal{U})$$

with differential $\delta^n\colon T^n \to T^{n+1}$ given by $\delta^n = \bigoplus_{p+q=n}(\delta_h^{p,q} + (-1)^p\delta_v^{p,q})$. The augmentation maps induce morphisms of complexes $\widetilde{\alpha}^*\colon C^*_{\mathcal{U}} \to T^*$ and $\widetilde{\beta}^*\colon \check{C}^* \to T^*$ and we denote by α, β the maps induced by α^*, β^* on cohomology. Since the rows (2) are exact, α is an isomorphism in every degree and the map $\alpha \circ \gamma\colon H^*(X) \to H^*(T^*)$ is the isomorphism α_X defined in the introduction. We define $\zeta = \alpha^{-1} \circ \beta$ and $\eta = \alpha_X^{-1} \circ \beta \circ \nu$.

The notation introduced so far is summarized in the following diagram:

$$\begin{tikzcd}
H^*(X) \arrow[r, "\simeq", "\gamma"'] \arrow[rr, bend left, "\alpha_X"] & H^*(C^q_{\mathcal{U}}) \arrow[r, "\simeq", "\alpha"'] & H^*(T) \\
& & \check{H}^*(\mathcal{U}) = H^*(N(\mathcal{U})) \arrow[u, "\beta"'] \arrow[ul, "\zeta"] \\
& & H^*(|N(\mathcal{U})|) \ . \arrow[u, "\nu"'] \arrow[uull, "\eta"]
\end{tikzcd}$$

When we want to stress the dependence of these constructions on the covering $\mathcal{U}$ we write $\alpha_{\mathcal{U}}$, $\beta_{\mathcal{U}}$, etc.

3. The case of a simplicial complex

In this section we analyze the double complex associated to the open star covering of the geometric realization $X = |S|$ of a simplicial complex S. Let I be the vertex set of S. We consider the open covering $\mathcal{U}^* = \{U_i^*\}_{i\in I}$ of $|S|$ given by the open stars of the vertices, i.e., for every $i \in I$ we set $U_i = \{x \in |S| : x_i > 0\}$, where x_i denotes the barycentric coordinate of the point x relative to the vertex i. Observe that the simplical complexes $N(\mathcal{U}^*)$ and S on the set of vertices I are equal and we will identify them. Hence, in this case $\eta_{\mathcal{U}^*} \colon H^*(|S|) \to H^*(|S|)$. Notice also that in this case all intersections $U_{\underline{i}}^*$ are contractible, hence, also the columns of the augmented double complex are exact. As a consequence, β and ζ are isomorphisms. The next proposition shows that the map η is the identity in this case.

Proposition 3.1. *If S is a simplicial complex and $\mathcal{U}^*$ is the covering described above then $\eta = Id$.*

To prove this proposition we will perform a computation by describing a lift of ζ at the level of cochains. To simplify the computations we will use *alternating* cochains, whose definition is recalled below.

Construction of $\widetilde{\zeta}$

We start by describing a lift

$$\widetilde{\zeta} \colon \check{C}(\mathcal{U}) \to C^{-1,p}(\mathcal{U}) = C^p_{\mathcal{U}}$$

of the map ζ at the level of cochains. We first construct chain homotopies

$$K^{p,q} \colon C^{p,q}(\mathcal{U}) \to C^{p-1,q}(\mathcal{U})\,, \quad p \geqslant 0\,, \quad q \geqslant 0\ .$$

For each singular simplex s with image contained in some open subset U_i we fix an index $i(s)$ such that $\operatorname{Im} s \subseteq U_{i(s)}$. For all $\varphi \in C^{p,q}(\mathcal{U})$ and for all singular simplices s with image contained in $U_{\underline{i}}$ for some $\underline{i} \in I_{p-1}$, $p \geqslant 0$, we define

$$\big(K^{p,q}(\varphi)_{\underline{i}}\big)(s) = \varphi_{i(s),\underline{i}}(s)$$

(when $p = 0$ there is no index $\underline{i}$ and we just take $s \in C_q^{\mathcal{U}}$). It is easy to check that $\delta_h^{p-1,q}K^{p,q} + K^{p+1,q}\delta_h^{p,q} = \mathrm{Id}$ for every $p \geqslant 0$, $q \geqslant 0$. Hence, if we define

$$\widetilde{\zeta} = (-1)^{\frac{p(p+1)}{2}}\, K^{0,p} \circ \delta_v^{0,p-1} \circ K^{1,p-1} \circ \cdots \circ K^{p-1,1} \circ \delta_v^{p-1,0} \circ K^{p,0} \circ \delta_v^{p,-1}$$

then for every cocycle $\varphi \in \check{C}^p(\mathcal{U})$ we have $\zeta([\varphi]) = [\widetilde{\zeta}(\varphi)]$ in $H^p(C^*_{\mathcal{U}})$.

Singular and algebraic simplices

Let us now recall the construction of the isomorphism ν between the simplicial cohomology $H^*(S)$ of S and the singular cohomology $H^*(|S|)$ of its geometric realization. Let $C_*(S)$ be the chain complex of simplicial chains on S, i.e. let C_p be the free

R-module with basis

$$\{(i_0, \ldots, i_p) \in I^{p+1} \mid \{i_0, \ldots, i_p\} \text{ is a simplex of } S\} ,$$

and let $C^*(S)$ be the dual chain complex of $C_*(S)$. Elements of the basis just described are usually called *algebraic* simplices.

For any algebraic simplex $\sigma = (i_0, \ldots, i_p)$ of S, one defines the singular simplex $\langle \sigma \rangle \colon \Delta^p \to |S|$ by setting

$$\langle \sigma \rangle (t_0, \ldots, t_p) = t_0 i_0 + \cdots + t_p i_p .$$

The map $\sigma \mapsto \langle \sigma \rangle$ extends to a chain map $C_*(S) \to C_*(|S|)$, whose dual map $\widetilde{\nu} \colon C^*(|S|) \to C^*(S)$ induces the isomorphism $\nu \colon H^*(|S|) \to H^*(S)$ (see e.g. [**Hat02**] Theorem 2.27). We write $\nu_S, \widetilde{\nu}_S$ when we want to stress the dependence on the simplicial complex.

Alternating cochains

To compute $\zeta \circ \nu$ it is convenient to use *alternating* cochains. Let $\mathfrak{S}_{p+1}$ be the permutation group of $\{0, \ldots, p\}$. We say that a simplicial cochain $\varphi \in C^p(S)$ is alternating if $\varphi(i_{\tau(0)}, \ldots, i_{\tau(p)}) = \varepsilon(\tau)\varphi(i_0, \ldots, i_p)$ for every $\tau \in \mathfrak{S}_{p+1}$, and $\varphi(i_0, \ldots, i_p) = 0$ whenever $i_j = i_{j'}$ for some $j \neq j'$. Alternating cochains form a subcomplex of the complex of cochains which is homotopy equivalent to the full complex (see e.g. [**ES52**, Chapter VI, Section 6, Theorems 6.9 and 6.10]).

Alternating cochains may be defined also in the context of singular homology as follows. For every $\tau \in \mathfrak{S}_{p+1}$ denote by $\rho_\tau \colon \Delta^p \to \Delta^p$ the affine automorphism of Δ^p defined by $\rho_\tau(t_0, \ldots, t_p) = (t_{\tau(0)}, \ldots, t_{\tau(p)})$. If X is a topological space, we say that a singular cochain $\varphi \in C^p(X)$ is *alternating* if $\varphi(s \circ \rho_\tau) = \varepsilon(\tau)\, \varphi(s)$ for every $\tau \in \mathfrak{S}_{p+1}$ and every singular simplex $s \colon \Delta^p \to X$, and $\varphi(s) = 0$ for every singular simplex s such that $s = s \circ \rho_\tau$ for an odd permutation $\tau \in \mathfrak{S}_{p+1}$. Alternating singular cochains form a subcomplex of the complex of singular cochains which is homotopy equivalent to the full complex (see e.g. [**Bar95**]). Moreover, the map $\widetilde{\nu}$ introduced above sends alternating singular cochains to alternating simplicial cochains, and both the homotopy maps $K^{p,q}$ and the vertical differential send alternating cochains to alternating ones.

We want to compute $\widetilde{\zeta}(\varphi)$ on singular simplices of the form $\langle \sigma \rangle$, as σ varies among the algebraic simplices of S. However, simplices of S are not contained in any U_i^*. We will then make use of the barycentric subdivision S' of S, together with a suitable simplicial approximation of the identity $S' \to S$. Let I' be the set of vertices of S'. This set is in bijective correspondence with the set of simplices of S: for $i' \in I'$ we denote by $\Delta_{i'}$ the simplex of S of which i' is the barycenter; in the opposite direction, if Δ is a simplex of S we denote by i'_Δ its barycenter. The p-simplices of S' are then the subsets $\{i'_0, \ldots, i'_p\}$ where $\Delta_{i'_0} \subset \cdots \subset \Delta_{i'_p}$.

If for every simplex Δ of S we denote by $b_\Delta \in |S|$ the geometric barycenter of Δ then the map $b \colon |S'| \to |S|$ defined by $b(\sum_\Delta t_\Delta i'_\Delta) = \sum_\Delta t_\Delta b_\Delta$ is a homeomorphism, and we will identify the geometric realizations of S' and S via this map. We construct a second map from $|S'|$ to $|S|$ as follows. We fix an auxiliary total ordering on I, and we define a simplicial map $g \colon S' \to S$ by setting

$$g(i') = \max \Delta_{i'}$$

for every vertex i' of S'. The geometric realization $|g|\colon |S| = |S'| \to |S|$ of g is homotopic to b via the homotopy $tb + (1-t)|g|$, $t \in [0,1]$.

We may define the map i used to construct the homotopies $K^{p,q}$ in such a way that, for every algebraic simplex $\sigma' = (i'_0, \dots, i'_p)$ of $C_*(S')$,

$$i(\langle \sigma' \rangle) = \min\{g(i'_0), \dots, g(i'_p)\} \ .$$

For simplicity, we will denote $i(\langle\sigma'\rangle)$ by $i(\sigma')$. With this choice, the singular simplex $\langle\sigma'\rangle$ is supported in $U^*_{i(\sigma')}$ as required in the definition of the map i.

Let $\alpha = (\alpha_{\underline{i}}) \in C^{h,k}(\mathcal{U}^*)$ and let $\sigma' = (i'_0, \dots, i'_{k+1}) \in C_{k+1}(U^*_{\underline{i}})$, $\underline{i} \in I_h$, be an algebraic $(k+1)$-simplex of S'. If $\partial_h\sigma' = (i'_0, \dots, \widehat{i'_h}, \dots, i'_{k+1})$ denotes the algebraic h-th face of σ', then

$$\big(\delta_v^{h-1,k} K^{h,k}(\alpha)\big)(\langle\sigma'\rangle) = \sum_{h=0}^{k+1} (-1)^h \big(K^{h,k}(\alpha)\big)_{\underline{i}}(\langle\partial_h\sigma'\rangle) = \sum_{h=0}^{k+1} (-1)^h \alpha_{i(\partial_h\sigma'),\underline{i}}(\langle\partial_h\sigma'\rangle) \ . \tag{3}$$

Lemma 3.2. *Let φ be an alternating cocycle in $C^p(N(\mathcal{U}^*)) = \check{C}^p(\mathcal{U}^*)$, and let $\sigma' \in C_p(S')$ be an algebraic simplex. Then*

$$\big(\widetilde{\zeta}(\varphi)\big)(\langle\sigma'\rangle) = \varphi(g_*(\sigma')) \ ,$$

where $g_*\colon C_p(S') \to C_p(S)$ is the map induced by $g\colon S' \to S$.

Proof. Let $\sigma' = (i'_0, \dots, i'_p)$ and set $\Delta_\ell = \Delta_{i'_\ell}$ for $\ell = 0, \dots, p$ and $i_\ell = g(i'_\ell)$. Recall that simplices of S' correspond to comparable subsets of a simplex of S. Moreover, since φ is alternating, both $g^*(\varphi)$ and $\widetilde{\zeta}(\varphi)$ are alternating, thus in order to check that the equality of the statement holds we may assume that

$$\Delta_0 \subsetneq \Delta_1 \cdots \subsetneq \Delta_p \ .$$

By definition we have $i_\ell = \max \Delta_\ell$, hence in particular $i_0 \leqslant i_1 \leqslant \cdots \leqslant i_p$. Since φ is alternating, this implies at once that

$$\varphi(g_*(\sigma')) = \begin{cases} \varphi_{i_0,i_1,\dots,i_p} & \text{if } i_0 < \cdots < i_p, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Let us now compute $\big(\widetilde{\zeta}(\varphi)\big)(\langle\sigma'\rangle)$. For every algebraic simplex $\tau'_k \in C_k(S')$, we write $\tau'_{k-1} < \tau'_k$ if τ'_{k-1} is an algebraic face of τ'_k, i.e. if there exists $h = 0, \dots, k$ such that $\tau'_{k-1} = \partial_h \tau'_k$. By iterating (3) we get

$$\big(\widetilde{\zeta}(\varphi)\big)(\langle\sigma'\rangle) = (-1)^{\frac{p(p+1)}{2}} \sum_{\sigma'_0 < \cdots < \sigma'_p = \sigma'} \pm \varphi_{i(\sigma'_0), i(\sigma'_1), \dots, i(\sigma'_p)} \ . \tag{5}$$

Let now $\sigma'_0 < \cdots < \sigma'_p$ be a fixed descending sequence of faces of σ'. Since the map i is given by taking a minimum we have $i(\sigma'_0) \geqslant i(\sigma'_1) \geqslant \cdots \geqslant i(\sigma'_p)$ and all these elements belong to the set $\{i_0, \dots, i_p\}$. Hence if $\varphi_{i(\sigma'_0), i(\sigma'_1), \dots, i(\sigma'_p)} \neq 0$ we have $i_0 < \cdots < i_p$ and $i(\sigma'_\ell) = i_{p-\ell}$ for every ℓ. In particular $\big(\widetilde{\zeta}(\varphi)\big)(\langle\sigma'\rangle)$ agrees with $\varphi(g_*(\sigma'))$ in the second case of formula (4).

Assume now $i_0 < \cdots < i_p$. As just observed, if $\varphi_{i(\sigma'_0),i(\sigma'_1),\ldots,i(\sigma'_p)} \neq 0$ then $i(\sigma'_\ell) = i_{p-\ell}$ for every ℓ, and this readily implies that the unique non-trivial addend in the right-hand sum in (5) corresponds to the sequence

$$\overline{\sigma}'_0 = (i'_p), \quad \overline{\sigma}'_1 = (i'_{p-1}, i'_p), \quad \ldots \quad , \overline{\sigma}'_p = (i'_0, \ldots, i'_{p-1}, i'_p) \ .$$

In particular, for every $j = 0, \ldots, p-1$ we have $\overline{\sigma}'_j = (-1)^0 \partial_0 \overline{\sigma}'_{j+1}$. Hence

$$\big(\widetilde{\zeta}(\varphi)\big)(\langle\sigma'\rangle) = (-1)^{\frac{p(p+1)}{2}} \varphi_{i(\overline{\sigma}'_0),i(\overline{\sigma}'_1),\ldots,i(\overline{\sigma}'_p)} = (-1)^{\frac{p(p+1)}{2}} \varphi_{i_p,i_{p-1},\ldots,i_0} = \varphi_{i_0,i_1,\ldots,i_p}$$

settling also the first case in formula (4). □

Before proving the proposition we notice that the map $C_*(S) \to C_*(|S|)$ constructed above does not factor through $C_*^{\mathcal{U}^*}$ because no positive-dimensional simplex of S is contained in U_i^* for any $i \in I$. However the analogous map from $C_*(S')$ to $C_*(|S|)$ does. Hence the map $\widetilde{\nu}_{S'} \colon C^*(|S'|) \to C^*(S')$ factors as $\widetilde{\nu}_{S'} = \widetilde{\mu} \circ \widetilde{\gamma}$, where $\widetilde{\gamma} \colon C^*(|S'|) \to C^*_{\mathcal{U}^*}$ is the map defined in Section 2, and $\widetilde{\mu} \colon C^*_{\mathcal{U}^*} \to C^*(S')$. We denote by $\mu \colon H^*(C^*_{\mathcal{U}^*}) \to H^*(S)$ the map induced by $\widetilde{\mu}$ on cohomology.

Proof of Proposition 3.1. Being $\nu_{S'} \colon H^*(|S|) = H^*(|S'|) \to H^*(S')$ injective and $|g|$ homotopic to the identity, in order to prove the proposition it is sufficient to show that $\nu_{S'} \circ \eta = \nu_{S'} \circ |g|^*$. Now recall that $\eta = \gamma^{-1} \circ \zeta \circ \nu_S$, hence $\nu_{S'} \circ \eta = \mu \circ \zeta \circ \nu_S$. Hence it is enough to prove that $\mu(\zeta(\nu_S(c))) = \nu_{S'}(|g|^*(c))$ for all $c \in H^p(|S|)$ or, equivalently, that

$$\widetilde{\mu}(\widetilde{\zeta}(\widetilde{\nu}_S(\psi)))(\sigma') = \widetilde{\nu}_{S'}(|g|^*(\psi))(\sigma'),$$

where $\psi \in C^p(|S|)$ is a cocycle and σ' is any algebraic simplex of S'. Moreover, as observed above we can choose ψ to be alternating. However, if we set $\phi = \widetilde{\nu}_S(\psi)$, then

$$\mu(\widetilde{\zeta}(\widetilde{\nu}_S(\psi)))(\sigma') = \mu(\widetilde{\zeta}(\phi))(\sigma') = \big(\widetilde{\zeta}(\varphi)\big)(\langle\sigma'\rangle) \ ,$$
$$\widetilde{\nu}_{S'}(|g|^*(\psi))(\sigma') = (|g|^*(\psi))(\langle\sigma'\rangle) = \psi(|g|_*(\langle\sigma'\rangle)) = \varphi(g_*(\sigma')) \ ,$$

hence the conclusion follows from Lemma 3.2. □

4. Proof of Theorem 1.1

We can now prove Theorem 1.1 stated in the introduction. We first notice that the construction of η is compatible with continuous maps in the following sense.

Lemma 4.1. *Let $h \colon Y \to Z$ be a continuous map, and let $\mathcal{V} = \{V_i\}_{i \in I}$, $\mathcal{W} = \{W_i\}_{i \in I}$ be open coverings of Y, Z, respectively, such that $h(V_i) \subseteq W_i$ for every $i \in I$. The identity of the set I extends to a simplicial map $N(h) \colon N(\mathcal{V}) \to N(\mathcal{W})$, and in particular it induces a continuous map $\hat{h} \colon |N(\mathcal{V})| \to |N(\mathcal{W})|$. Then the following diagram commutes:*

$$\begin{array}{ccc} H^*(|N(\mathcal{W})|) & \xrightarrow{\hat{h}^*} & H^*(|N(\mathcal{V})|) \\ \Big\downarrow{\scriptstyle \eta_{\mathcal{W}}} & & \Big\downarrow{\scriptstyle \eta_{\mathcal{V}}} \\ H^*(Z) & \xrightarrow{h^*} & H^*(Y) \ . \end{array}$$

Proof. By considering the restriction of h to the open subset V_i the map h induces a morphism $\{h^{p,q}\}$ between the double complex associated to $\mathcal{W}$ and the double complex associated to $\mathcal{V}$ and between their augmentations. Hence we have $\zeta_{\mathcal{V}} \circ N(h)^* = h^* \circ \zeta_{\mathcal{W}} \colon H^*(N(\mathcal{W})) \to H^*(C^*_{\mathcal{V}})$. We also have $\widetilde{\gamma}_{\mathcal{V}} \circ h^* = h^{-1,*} \circ \widetilde{\gamma}_{\mathcal{W}}$ and by the definition of the map ν we have $\nu_{\mathcal{V}} \circ h^* = N(h)^* \circ \nu_{\mathcal{W}}$. By the definition of η, these three commutations imply the commutativity claimed in the lemma. □

We can now conclude the proof of our main theorem. Let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open covering of the space X, let $N(\mathcal{U})$ be the nerve of $\mathcal{U}$, and let $\mathcal{U}^* = \{U_i^*\}_{i\in I}$ be the open covering of $|N(\mathcal{U})|$ given by the open stars of the vertices of $N(\mathcal{U})$. Let $f_\Phi \colon X \to |N(\mathcal{U})|$ be the map associated to a partition of unity subordinate to $\mathcal{U}$ as described in the introduction. We would like to apply the previous lemma to the coverings $\mathcal{U}$ of X and $\mathcal{U}^*$ of $|N(\mathcal{U})|$ and to the map $h = f_\Phi$, but the containment $f_\Phi(U_i) \subseteq U_i^*$ does not hold in general. Therefore, we consider the covering $\widetilde{\mathcal{U}} = \{\widetilde{U}_i\}_{i\in I}$ of X defined by $\widetilde{U}_i = f_\Phi^{-1}(U_i^*)$ for every $i \in I$.

We can now apply Lemma 4.1 to the map $h = f_\Phi$ and to the coverings $\mathcal{V} = \widetilde{\mathcal{U}}$ and $\mathcal{W} = \mathcal{U}^*$. Since $\widetilde{U}_i \subseteq U_i$ for every $i \in I$, Lemma 4.1 also applies to the case when $h = i_X$ is the identity map of X, and to the coverings $\mathcal{V} = \widetilde{\mathcal{U}}$ and $\mathcal{W} = \mathcal{U}$. Hence we obtain the following commutative diagrams:

$$\begin{tikzcd} H^*(|N(\mathcal{U}^*)|) \arrow[r, "\hat{f}_\Phi^*"] \arrow[d, "\eta_{\mathcal{U}^*}"'] & H^*(|N(\widetilde{\mathcal{U}})|) \arrow[d, "\eta_{\widetilde{\mathcal{U}}}"] \\ H^*(|N(\mathcal{U})|) \arrow[r, "f_\Phi^*"] & H^*(X) \end{tikzcd} \qquad \begin{tikzcd} H^*(|N(\mathcal{U})|) \arrow[r, "\hat{i}_X^*"] \arrow[d, "\eta_{\mathcal{U}}"'] & H^*(|N(\widetilde{\mathcal{U}})|) \arrow[d, "\eta_{\widetilde{\mathcal{U}}}"] \\ H^*(X) \arrow[r, equal] & H^*(X) \end{tikzcd} .$$

As already noticed in the previous section the simplicial complexes $N(\mathcal{U})$ and $N(\mathcal{U}^*)$ with set of vertices I are equal and, by construction, so are the simplicial maps $N(i_X)$ and $N(f_\Phi)$ from $N(\widetilde{\mathcal{U}})$ to $N(\mathcal{U}^*) = N(\mathcal{U})$. In particular $\hat{f}_\Phi^* = \hat{i}_X^*$. Finally by Proposition 3.1 $\eta_{\mathcal{U}^*}$ is the identity. Hence

$$f_\Phi^* = f_\Phi^* \circ \eta_{\mathcal{U}^*} = \eta_{\widetilde{\mathcal{U}}} \circ \hat{f}_\Phi^* = \eta_{\widetilde{\mathcal{U}}} \circ \hat{i}_X^* = \eta_{\mathcal{U}} \ ,$$

which proves the theorem.

References

[Bar95] M. Barr, *Oriented singular homology*, Theory Appl. Categ. **1** (1995), 1–9.

[BT82] R. Bott and L. W. Tu, *Differential forms in algebraic topology*, Graduate Texts in Mathematics, vol. 82, Springer-Verlag, 1982.

[ES52] S. Eilenberg and N. Steenrod, *Foundations of algebraic topology*, Princeton University Press, Princeton, New Jersey, 1952.

[FM] R. Frigerio and M. Moraschini, *Gromov's theory of multicomplexes with applications to bounded cohomology and simplicial volume*, arXiv:1808. 07307, to appear in Mem. Amer. Math. Soc.

[Gro82] M. Gromov, *Volume and bounded cohomology*, Inst. Hautes Études Sci. Publ. Math. **56** (1982), 5–99.

[**Hat02**] A. Hatcher, *Algebraic topology*, Cambridge University Pres, Cambridge, 2002.

[**Iva**] N. V. Ivanov, *Notes on the bounded cohomology theory*, arXiv:1708.05150.

[**Iva87**] ______, *Foundation of the theory of bounded cohomology*, J. Soviet. Math. **37** (1987), 1090–1114.

[**LS**] C. Löh and R. Sauer, *Bounded cohomology of amenable covers via classifying spaces*, Enseign. Math. **66** (2020), 151–172.

[**Wei52**] A. Weil, *Sur les théorèmes de de Rham*, Comment. Math. Helv. **26** (1952), 119–145.

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