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HOMOLOGY OF QUANTUM LINEAR GROUPS

A. KAYGUN AND S. SÜTLÜ

(*communicated by Jonathan M. Rosenberg*)

Abstract

For every $n \geqslant 1$, we calculate the Hochschild homology of the quantum monoids $M_q(n)$, and the quantum groups $GL_q(n)$ and $SL_q(n)$ with coefficients in a 1-dimensional module coming from a modular pair in involution.

Introduction

In this article we calculate the Hochschild homology of the quantum monoids $M_q(n)$, and quantum groups $GL_q(n)$ and $SL_q(n)$, [**17, 24**], with coefficients in a 1-dimensional module ${}_{f_q^{-1}}k$ coming from a modular pair in involution (MPI) defined on $GL_q(n)$ and $SL_q(n)$ when q is not a root of unity. To this end, we present an effective algorithm that calculates the explicit classes, and generates the corresponding Betti sequences. We also show that these homologies with coefficients in the MPI are direct summands of the Hochschild homologies of $GL_q(n)$ and $SL_q(n)$ with coefficients in themselves.

To achieve our goal, we introduce a general Hochschild-Serre type spectral sequence for flat algebra extensions of the form $Q \subseteq P$ through which calculating the homology of P reduces to calculating the homology of Q and the Q-relative homology of P. We then calculate the homology of $M_q(n)$ by using the lattice of extensions $M_q(a,b) \subseteq M_q(n)$ with $1 \leqslant a,b \leqslant n$, and related reductions in homology. Since $GL_q(n)$ is the localization of $M_q(n)$ at the quantum determinant, it is almost immediate to obtain the homology of $GL_q(n)$ from that of $M_q(n)$ by a localization in homology [**20**, 1.1.17]. Calculating the homology of $SL_q(n)$ becomes immediate since $GL_q(n) = SL_q(n) \otimes k[\mathcal{D}_q, \mathcal{D}_q^{-1}]$ as algebras where $\mathcal{D}_q$ is the quantum determinant [**19**, Proposition].

The Hochschild and cyclic homology of $SL_q(2)$ are calculated in [**22, 26**] using a specific resolution for $SL_q(2)$. The Hochschild homology of $SL_q(2)$ with coefficients in itself twisted by the modular automorphism is calculated in [**5**]. The Hochschild cohomology of $SL_q(n)$ with coefficients in ${}_\eta k_\eta$ (Definition 3.1 and Proposition 3.3) is computed in [**6**] using the Koszul approach similar to [**26**]. In [**6**] the authors also show that Hochschild homology and cohomology of $SL_q(n)$ satisfy Poincare duality. The Hochschild homology of the restricted dual $\mathcal{O}_h(G)$ of the quantum universal enveloping algebra $U_h(\mathfrak{g})$, on the other hand, is described in [**3**] for a semi-simple Lie group G with its Lie algebra $\mathfrak{g}$.

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In [14] we calculated the homology of several algebras including of $M_q(2)$. However, calculating homologies of $M_q(n), GL_q(n), SL_q(n)$ etc. for $n > 2$ are beyond the reach of the methods we developed in [14]. We needed new tools and techniques to deal with more general types of extensions as we observed in [12]. Thus the method we develop in Section 1 in this paper evolved out of [14] and [12] with a different set of assumptions.

One has to note that one can use the ordinary Tor functor instead of the Hochschild homology functor since, with the exception of Sections 3.4 through 3.6, the coefficient modules we primarily use are all 1-dimensional. However, on Section 3 onward we preferred to state our results in terms of the Hochschild homology for uniformity, to smooth over few subtle technical difficulties we encounter in our calculations, and ultimately we would like to use the results of this paper to calculate the Hopf-cyclic cohomology of $GL_q(n)$ and $SL_q(n)$ in the future.

Here, we would like to mention two articles by Kadison [8, 9] which we were not aware until our anonymous referee pointed them out, for which we are grateful. The results of the first article [8] are relevant for algebra extensions $Q \subseteq P$ where Q is *amenable*, i.e. has Hochschild homological dimension 0. We worked with such extensions in the context of smash biproducts in [14], but we do not work with amenable algebras in this paper. On the other hand, the main hypothesis of the second article [9] amounts to a degeneration of the spectral sequence we use in this paper akin to a generation required in [10, 11, 12]. However, we do not assume such a degeneration in this paper.

Plan of the paper

After developing a Hochschild-Serre type spectral sequence for flat algebra extensions $Q \subseteq P$ in Section 1, we review some basic facts and definitions about quantum monoids $M_q(n)$, and groups $GL_q(n)$ and $SL_q(n)$ in Section 2. Then in Section 3 we calculate the Hochschild homology of $M_q(n)$, $GL_q(n)$ and $SL_q(n)$ for every $n \geqslant 1$ with coefficients in a 1-dimensional module ${}_{f_{q,n}^{-1}}k$ defined using a modular pair in involution $(f_{q,n}^{-1}, 1)$ for the Hopf algebras $GL_q(n)$ and $SL_q(n)$. In Section 4, we explicitly write our calculations for the cases $n = 2, 3, 4$.

Notation and conventions

We fix a ground field k and an element $q \in k^\times$ which is not a root of unity. All unadorned tensor products $\otimes$ are over k. All algebras are assumed to be unital and associative, but not necessarily commutative or finite dimensional over k. We use $\langle X \rangle$ to denote the two-sided ideal generated by a subset X of elements in an algebra, and $Span_k(X)$ for the k-vector space spanned by a subset X of elements in a vector space. For a fixed vector space V, we use $\Lambda^*(V)$ to denote the exterior algebra over V, but used only as a vector space. For a fixed algebra A, we use $\mathrm{CB}_*(A)$ and $\mathrm{CH}_*(A)$ respectively for the bar complex and the Hochschild complex of the algebra A. Also, for a left A-module Y and a right A-module X, we use $\mathrm{CB}_*(X, A, Y)$ for the two sided bar construction [21, Sect. II.2.3]. We use $H_*(A)$ to denote the Hochschild homology of A. If the complexes, and therefore, homology includes coefficients other than the algebra A, we will indicate this using a pair (A, M).

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1. A Hochschild-Serre type spectral sequence

In this section, we assume P and Q are unital associative algebras, together with a fixed morphism of algebras $\varphi\colon Q \to P$.

1.1. The mapping cylinder algebra

Let $Z := P \oplus Q$ be the unital associative algebra given by the product

$$(p, q) \cdot (p', q') = (pp' + p\varphi(q') + \varphi(q)p', qq')$$

for any $(p, q), (p', q') \in Z$. The unit of the product is $(0, 1) \in Z$. Accordingly, $Q \subseteq Z$ is a unital subalgebra and $P \subseteq Z$ is an ideal. Moreover, since P is a unital algebra, the homology of its bar complex vanishes in any positive degree. As such, $P \subseteq Z$ is an H-unital ideal, [27].

Remark 1.1. Note that the mapping cylinder Z of a morphism of unital algebras $\varphi\colon Q \to P$ is isomorphic to the direct product algebra $P \times Q$ via the isomorphism $\hat{\varphi}\colon Z \to P \times Q$ given by

$$\hat{\varphi}(p, q) = (p + \varphi(q), q).$$

However, in $P \times Q$ neither P nor Q are unital subalgebras, but they are embedded as H-unital ideals. One can treat Q as a unital subalgebra of $P \times Q$ if we use the image of $\hat{\varphi}$ restricted to Q. In the sequel, we use the mapping cylinder Z and the obvious embedding $q \mapsto (0, q)$ (which is now a monomorphism of unital algebras) instead of the product ring $P \times Q$ and the embedding $\hat{\varphi}|_Q\colon Q \to P \times Q$ since the former makes the spectral sequence arguments we present in this section easier to construct.

If we consider the chain of algebra morphisms $Z \xrightarrow{\hat{\varphi}} P \times Q \xrightarrow{\pi_1} P$, one can think of every P-module as a Z-module. Let X be a right P-module, and Y a left P-module which can both can be considered as Z-modules via

$$x \triangleleft (p, q) := x \triangleleft p + x \triangleleft \varphi(q), \qquad (p, q) \triangleright y = p \triangleright y + \varphi(q) \triangleright y$$

for any $x \in X$, any $y \in Y$, and any $(p, q), (p', q') \in Z$.

1.2. A filtration on the two-sided bar construction

Let us now consider the bar complex $\mathrm{CB}_*(Z)$, and the two-sided bar construction $\mathrm{CB}_*(X, Z, Y) := X \otimes_Z \mathrm{CB}_*(Z) \otimes_Z Y$ of the mapping cylinder algebra Z with coefficients in a right P-module X and a left P-module Y.

We consider first the increasing filtration on $\mathrm{CB}_*(X,Z,Y)$ given by

$$G^i_{i+j} = \sum_{n_0+\cdots+n_i=j} X\otimes P^{\otimes n_0}\otimes Z\otimes\cdots\otimes P^{\otimes n_{i-1}}\otimes Z\otimes P^{\otimes n_i}\otimes Y \subseteq \mathrm{CB}_{i+j}(X,Z,Y).$$

The bar differential interacts with the filtration as $d(G^i_{i+j})\subseteq G^i_{i+j-1}$. Then the associated graded complex is given by

$$E^0_{i,j} := G^i_{i+j}/G^{i-1}_{i+j} \cong \bigoplus_{n_0+\cdots+n_i=j} X\otimes P^{\otimes n_0}\otimes Q\otimes\cdots\otimes P^{\otimes n_{i-1}}\otimes Q\otimes P^{\otimes n_i}\otimes Y, \tag{1.1}$$

with induced differentials. However, since we cannot reduce the number of Q's in the quotient complex, one can think of the quotient complex as a graded product of the two-sided bar constructions with induced differentials

$$E^0_{i,*}=\mathrm{CB}_*(X,P,Q)\otimes_Q \underbrace{\mathrm{CB}_*(Q,P,Q)\otimes_Q\cdots\otimes_Q \mathrm{CB}_*(Q,P,Q)}_{(i-1)\text{-times}}\otimes_Q \mathrm{CB}_*(Q,P,Y),\ \ i\geqslant 1.$$

where P acts on Q via 0, and for $i=0$

$$E^0_{0,*} := \mathrm{CB}_*(X,P,Y).$$

Accordingly, the E^1-term is given by

$$E^1_{i,j} = H_{i+j}(E^0_{i,*};d_0) = \begin{cases}\mathrm{Tor}^P_j(X,Y) & \text{if } i=0,\\ 0 & \text{if } i\neq 0.\end{cases}$$

The spectral sequence then degenerates, and we arrive at the following result.

Proposition 1.2. *Given two algebras P and Q, together with an algebra morphism $\varphi\colon Q\to P$, let Z be the mapping cylinder algebra, X a right P-module, and Y be a left P-module. Then,* $\mathrm{Tor}^Z_n(X,Y)\cong \mathrm{Tor}^P_n(X,Y)$ *for all $n\geqslant 0$.*

Remark 1.3. The result we obtain in Proposition 1.2, and later in Proposition 1.4 are trivial change of ring results as in [**1**, Proposition VI.4.1.2]. However, the method we use itself here is quite useful, and will allow us to write Proposition 1.5 and Proposition 1.6 which, in turn, relates the absolute homology with the relative homology using a spectral sequence.

1.3. A filtration on the Hochschild homology

It is possible to adapt this setting to the Hochschild homology complex. To this end, given a P-bimodule M, we start with the increasing filtration

$$G^i_{i+j} = \sum_{n_0+\cdots+n_i=j} M\otimes P^{\otimes n_0}\otimes Z\otimes\cdots\otimes P^{\otimes n_{i-1}}\otimes Z\otimes P^{\otimes n_i} \subseteq \mathrm{CH}_{i+j}(Z,M)$$

of $\mathrm{CH}_*(Z,M)$. It follows from the observation $b(G^i_{i+j})\subseteq G^i_{i+j-1}$ that the Hochschild complex $\mathrm{CH}_*(Z,M)$ is a filtered differential complex. The associated graded complex is then

$$E^0_{i,j} := G^i_{i+j}/G^{i-1}_{i+j} = \bigoplus_{n_0+\cdots+n_i=j} M\otimes P^{\otimes n_0}\otimes Q\otimes\cdots\otimes P^{\otimes n_{i-1}}\otimes Q\otimes P^{\otimes n_i}$$

together with the induced differential $b_0\colon E^0_{i,j}\longrightarrow E^0_{i,j-1}$ given similar to that of (1.1) where P acts on Q by 0. Hence, the first page of the associated differential complex appears to be

$$E^1_{i,j} = H_{i+j}(E^0_{i,*}; b_0) = \begin{cases} H_j(P, M) & \text{if } i = 0, \\ 0 & \text{if } i \neq 0. \end{cases}$$

The spectral sequence then degenerates, and we arrive at the following result.

Proposition 1.4. *Given two algebras P and Q, together with an algebra morphism $\varphi\colon Q \to P$, let Z be the mapping cylinder algebra. Then, $H_n(Z, M) \cong H_n(P, M)$ for any $n \geqslant 0$, and any P-bimodule M.*

1.4. A second filtration on the two-sided bar construction

Let P and Q be two algebras as above, but this time we assume $\varphi\colon Q \to P$ is a left (or right) flat algebra morphism. In other words, P is flat as a left (resp. right) Q-module via $\varphi\colon Q \to P$. Let us now consider the increasing filtration

$$F^i_{i+j} = \sum_{n_0+\cdots+n_i=j} X \otimes Q^{\otimes n_0} \otimes Z \otimes \cdots \otimes Q^{\otimes n_{i-1}} \otimes Z \otimes Q^{\otimes n_i} \otimes Y \subseteq \mathrm{CB}_{i+j}(X, Z, Y)$$

on the two-sided bar construction $\mathrm{CB}_*(X, Z, Y)$. Since $d(F^i_{i+j}) \subseteq F^i_{i+j-1}$, the two-sided bar construction $\mathrm{CB}_*(X, Z, Y)$ becomes a filtered differential complex; whose associated differential graded complex is

$$E^0_{i,j} = F^i_{i+j}/F^{i-1}_{i+j} = \bigoplus_{n_0+\cdots+n_i=j} X \otimes Q^{\otimes n_0} \otimes P \otimes \cdots \otimes Q^{\otimes n_{i-1}} \otimes P \otimes Q^{\otimes n_i} \otimes Y,$$

together with the induced differentials $d_0\colon E^0_{i,j} \longrightarrow E^0_{i,j-1}$ coming from the two-sided bar construction $\mathrm{CB}_*(X, Z, Y)$. As in the case of our first filtration, one can view the resulting complex as a graded multi-product of bar constructions

$$\mathrm{CB}_*(X, Q, P) \otimes_P \underbrace{\mathrm{CB}_*(P, Q, P) \otimes_P \cdots \otimes_P \mathrm{CB}_*(P, Q, P)}_{(i-1)\text{-times}} \otimes_P \mathrm{CB}_*(P, Q, Y). \quad (1.2)$$

In view of the assumption (that P is flat as a Q-module), the E^1-term is given by

$$E^1_{i,j} = H_{i+j}(E^0_{i,*}; d_0) = \begin{cases} \mathrm{Tor}^Q_j(X \otimes_Q \underbrace{P \otimes_Q \cdots \otimes_Q P}_{i\text{-times}}, Y) & \text{if } P \text{ is flat as a left } Q\text{-module}, \\ \mathrm{Tor}^Q_j(X, \underbrace{P \otimes_Q \cdots \otimes_Q P}_{i\text{-times}} \otimes_Q Y) & \text{if } P \text{ is flat as a right } Q\text{-module}. \end{cases}$$

Keeping in mind that this spectral sequence converges to the Tor-groups of the mapping cylinder algebra Z, which are in turn identified with the Tor-groups of the algebra P in the previous subsection, we obtain the result which may be summarized in the following proposition.

Proposition 1.5. *Given two algebras P and Q, together with the left (resp. right) flat algebra morphism $\varphi\colon Q \to P$, let X a right P-module and Y be a left P-module. Then, there is a spectral sequence such that*

$$E^1_{i,j} = \mathrm{Tor}^Q_j(X \otimes_Q \underbrace{P \otimes_Q \cdots \otimes_Q P}_{i\text{-times}}, Y) \Rightarrow \mathrm{Tor}^P_{i+j}(X, Y),$$

$$\Big(\textit{resp. } E^1_{i,j} = \mathrm{Tor}^Q_j(X, \underbrace{P \otimes_Q \cdots \otimes_Q P}_{i\text{-times}} \otimes_Q Y) \Rightarrow \mathrm{Tor}^P_{i+j}(X, Y)\Big).$$

1.5. A second filtration on the Hochschild homology

We can present the arguments of the previous subsection in terms of the Hochschild homology as well.

To this end, we begin with the increasing filtration

$$F^i_{j+i} = \sum_{n_0+\cdots+n_i=j} M \otimes Q^{\otimes n_0} \otimes Z \otimes \cdots \otimes Q^{\otimes n_{i-1}} \otimes Z \otimes Q^{\otimes n_i},$$

that satisfies $b(F^i_{i+j}) \subseteq F^i_{i+j-1}$. Then the associated graded complex is

$$E^0_{i,j} = F^i_{j+i}/F^{i-1}_{j+i} = \bigoplus_{n_0+\cdots+n_i=j} M \otimes Q^{\otimes n_0} \otimes P \otimes \cdots \otimes Q^{\otimes n_{i-1}} \otimes P \otimes Q^{\otimes n_i}.$$

Passing to the homology with respect to $b_0\colon E^0_{i,j} \longrightarrow E^0_{i,j-1}$, an analogue of (1.2), we arrive at the first page of the spectral sequence which is given by

$$E^1_{i,j} = H_{i+j}(E^0_{i,*}; b_0) = H_j\Big(Q, M \otimes_Q \underbrace{P \otimes_Q \cdots \otimes_Q P}_{i\text{-times}}\Big)$$

that converges to $H_*(Z, M)$ which is isomorphic to $H_*(P, M)$ by Proposition 1.4. Thus, we have proved the following proposition.

Proposition 1.6. *Given two algebras P and Q, together with the left (resp. right) flat algebra morphism $\varphi\colon Q \to P$, let M be a P-bimodule. Then there is a spectral sequence whose first page is given by*

$$E^1_{i,j} = H_j(Q, M \otimes_Q \underbrace{P \otimes_Q \cdots \otimes_Q P}_{i\text{-times}})$$

that converges to the Hochschild homology $H_*(P, M)$.

2. Quantum linear groups

2.1. The algebra of quantum matrices $M_q(n, m)$

Let n and m be two positive integers. Following [7, Lemma 2.10], we define $M_q(n, m)$ as the associative algebra on nm generators x_{ij} where $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$. These generators are subject to the following relations

$$x_{j\ell}x_{i\ell} = q\, x_{i\ell}x_{j\ell} \qquad \text{for all } 1 \leqslant i < j \leqslant n \text{ and } 1 \leqslant \ell \leqslant m, \tag{2.1}$$

$$x_{\ell j}x_{\ell i} = q\, x_{\ell i}x_{\ell j} \qquad \text{for all } 1 \leqslant i < j \leqslant m \text{ and } 1 \leqslant \ell \leqslant n, \tag{2.2}$$

$$x_{\ell i}x_{kj} = x_{kj}x_{\ell i} \qquad \text{for all } 1 \leqslant k < \ell \leqslant n \text{ and } 1 \leqslant i < j \leqslant m, \tag{2.3}$$

$$x_{ki}x_{\ell j} - x_{\ell j}x_{ki} = (q^{-1} - q)\, x_{kj}x_{\ell i} \qquad \text{for all } 1 \leqslant k < \ell \leqslant n \text{ and } 1 \leqslant i < j \leqslant m. \tag{2.4}$$

For convenience, we are going to use $M_q(n)$ for $M_q(n, n)$. We also use the following convention: for $a \leqslant n$ and $b \leqslant m$ when we write $M_q(a, b) \subseteq M_q(n, m)$ we mean that we use the subalgebra generated by x_{ij} for $1 \leqslant i \leqslant a$ and $1 \leqslant j \leqslant b$ in $M_q(n, m)$. Notice that these generators are subject to the same relations, and therefore, the canonical map $M_q(a, b) \to M_q(n, m)$ is injective.

It follows from (2.1) and (2.2) that all column or row subalgebras

$$\mathrm{Col}_\ell := \langle x_{i\ell} \mid 1 \leqslant i \leqslant n \rangle, \qquad \mathrm{Row}_\ell := \langle x_{\ell j} \mid 1 \leqslant j \leqslant n \rangle$$

are isomorphic to the quantum affine n-space k_q^n which is defined as the k-algebra

$$M_q(1,n) \cong M_q(n,1) \cong k_q^n := k\{x_1, \ldots, x_n\}/\langle x_j x_i - q\, x_i x_j \mid i < j \rangle.$$

See [**4**, Subsect. 3.1] for the multiparametric version. Next, we note from [**24**, Thm. 3.5.1] and [**17**, Prop. 9.2.6] that

$$\mathcal{B} = \left\{ \prod_{1 \leqslant i,j \leqslant n} x_{ij}^{t_{ij}} \mid t_{ij} \geqslant 0 \right\}$$

is a vector space basis of $M_q(n)$, with respect to any fixed order of the generators.

2.2. The bialgebra structure on $M_q(n)$

The algebra $M_q(n)$ of quantum matrices is a bialgebra whose comultiplication $\Delta\colon M_q(n) \to M_q(n) \otimes M_q(n)$ is given by

$$\Delta(x_{ij}) := \sum_k x_{ik} \otimes x_{kj}$$

and whose counit $\varepsilon\colon M_q(n) \to k$ is given by

$$\varepsilon(x_{ij}) = \delta_{ij}.$$

Let us note that we suppress the summations in the expression of a coproduct. However, here it serves merely to highlight the summation over the middle indices of the matrix generators.

2.3. The quantum determinant

Let S_n be the group of permutations of the set $\{1, \ldots, n\}$, and let $\ell(\sigma) \in \mathbb{N}$ be the length of $\sigma \in S_n$. Let also $I := \{i_1, \ldots, i_m\}$ and $J := \{j_1, \ldots, j_m\}$ be two subsets of $\{1, \ldots, n\}$ such that $i_1 < \cdots < i_m$ and $j_1 < \cdots < j_m$. Then, the element

$$\mathcal{D}_{IJ} := \sum_{\sigma \in S_m} (-q)^{\ell(\sigma)}\, x_{i_{\sigma(1)} j_1} \cdots x_{i_{\sigma(m)} j_m} = \sum_{\sigma \in S_m} (-q)^{\ell(\sigma)}\, x_{i_1 j_{\sigma(1)}} \cdots x_{i_m j_{\sigma(m)}} \in M_q(m)$$

is called the *quantum m-minor determinant* as defined in [**17**, Sect. 9.2.2] and [**24**, Sect. 4.1].

On one extreme we have $\mathcal{D}_{IJ} = x_{ij}$ for $I = \{i\}$ and $J = \{j\}$. On the other extreme, if we let $I = J = \{1, \ldots, n\}$ we get the *quantum determinant* which is denoted by $\mathcal{D}_q$. The quantum determinant is in the center of $M_q(n)$. Moreover, if q is not a root of unity, then the center of $M_q(n)$ is generated by the quantum determinant. For this result see [**17**, Prop. 9.9], [**24**, Thm. 4.6.1], or [**23**].

2.4. The quantum general linear group $GL_q(n)$

The quantum group $GL_q(n)$ is obtained by adjoining $\mathcal{D}_q^{-1}$ to the bialgebra $M_q(n)$. More precisely,

$$GL_q(n) = \frac{M_q(n)[t]}{\langle t\mathcal{D}_q - 1 \rangle}.$$

Let us note from this definition that $M_q(n)$ is a subalgebra of $GL_q(n)$. In terms of generators and relations, $GL_q(n)$ is the algebra generated by n^2+1 generators x_{ij} and t with $i,j \in \{1,\dots,n\}$, satisfying the same relations as (2.1)–(2.4), and

$$\begin{aligned} \mathcal{D}_q t &= t\mathcal{D}_q = 1, \\ x_{ij} t &= t x_{ij}. \end{aligned}$$

On the other hand, $GL_q(n)$ is the localization $M_q(n)_{\mathcal{D}_q}$ of $M_q(n)$ with respect to $\mathcal{D}_q$ as in [**24**, Sect. 5.3] and [**20**, Prop. 1.1.17]. As such, the bialgebra structure on $M_q(n)$ extends uniquely to $GL_q(n)$ [**24**, Lemma 5.3.1] since the quantum determinant $\mathcal{D}_q$ is a group-like element. Furthermore, $GL_q(n)$ is a Hopf algebra with the antipode $S\colon GL_q(n) \to GL_q(n)$ given by

$$S(x_{ij}) := (-q)^{j-i} A_{ji} \mathcal{D}_q^{-1}, \quad S(\mathcal{D}_q^{-1}) := \mathcal{D}_q,$$

where $A_{ij} := \mathcal{D}_{IJ}$ with $I = \{1,\dots,n\} - \{i\}$, and $J = \{1,\dots,n\} - \{j\}$. The matrix

$$\left[q^{i-j} A_{ij}\right]_{1 \leqslant i,j \leqslant n}$$

is called the *quantum cofactor matrix* of $[x_{ij}]_{1 \leqslant i,j \leqslant n}$, and the Hopf algebra $GL_q(n)$ is called the *quantum general linear group*.

2.5. The quantum special linear group $SL_q(n)$

Next, we recall briefly the quantum version of the special linear group. It is given as the quotient space

$$SL_q(n) := \frac{GL_q(n)}{\langle \mathcal{D}_q - 1\rangle} = \frac{M_q(n)}{\langle \mathcal{D}_q - 1\rangle},$$

which happens to be a Hopf algebra with the bialgebra structure induced from $GL_q(n)$, or from $M_q(n)$, and the antipode $S\colon SL_q(n) \to SL_q(n)$ is given by

$$S(x_{ij}) := q^{j-i} A_{ji}$$

induced from $GL_q(n)$. The Hopf algebra $SL_q(n)$ is called the *quantum special linear group.*

Actually, one can write $GL_q(n)$ as a direct product of $SL_q(n)$ and the Laurent polynomial ring over the quantum determinant $k[\mathcal{D}_q, \mathcal{D}_q^{-1}]$.

Proposition 2.1 ([**19**, Proposition]). *There is an isomorphism of algebras of the form*

$$GL_q(n) \cong SL_q(n) \otimes k[\mathcal{D}_q, \mathcal{D}_q^{-1}].$$

It is worth mentioning that the isomorphism in Proposition 2.1 is an algebra isomorphism, and not an isomorphism of Hopf algebras.

2.6. Modular pairs of involution for $GL_q(n)$ and $SL_q(n)$

Below, we shall calculate the Hochschild homologies of the Hopf algebras $GL_q(n)$ and $SL_q(n)$, with a one dimensional coefficient module that arises from a *modular pair in involution* (MPI). Let us recall from [**16**], see also [**2**] for the original cohomological version, that given a Hopf algebra H, a pair (δ,σ) consisting of an algebra homomorphism $\delta\colon H \to k$ and a group-like element $\sigma \in H$ is called an MPI if

$$\delta(\sigma) = 1, \qquad \widetilde{S}_\sigma^2 = \mathrm{Id},$$

where

$$\widetilde{S}_\sigma(h) := \sigma S(h_{(1)})\delta(h_{(2)})$$

for any $h \in H$.

Proposition 2.2. *Let $f_{q,n}^{-1}\colon GL_q(n) \to k$ (resp. $f_{q,n}^{-1}\colon SL_q(n) \to k$) be given by $f_{q,n}^{-1}(x_{ij}) := \delta_{ij}q^{(n+1)-2i}$. Then, $(f_{q,n}^{-1}, 1)$ is a MPI for the Hopf algebra $GL_q(n)$, (resp. for the Hopf algebra $SL_q(n)$).*

Proof. We will give the proof for $GL_q(n)$. The proof for the case of $SL_q(n)$ is similar, and therefore, is omitted. It is given in [**24**, Lemma 5.4.1] that $f_{q,n}\colon GL_q(n) \to k$ given by

$$f_{q,n}(x_{ij}) = \delta_{ij}q^{2i-(n+1)}$$

is an algebra homomorphism, i.e. a character. Then, its convolution inverse $f_{q,n}^{-1}\colon GL_q(n) \to k$ is also a character. The claim then follows from the observation that

$$\widetilde{S}^2 = f_{q,n} * S^2 * f_{q,n}^{-1}$$

and that, by [**24**, Thm. 5.4.2], $S^2 = f_{q,n}^{-1} * \mathrm{Id} * f_{q,n}$ where $*$ is the convolution multiplication on the set of characters of Hopf algebras. □

Remark 2.3. As for $GL_q(n)$, there is a second choice of MPI. It follows at once from the fact that

$$\begin{aligned} f_{q,n}(\mathcal{D}_q) &= f_{q,n}\left(\sum_{\sigma\in S_n} (-q)^{\ell(\sigma)}\, x_{1\sigma(1)}\cdots x_{n\sigma(n)}\right) \\ &= \sum_{\sigma\in S_n} (-q)^{\ell(\sigma)}\, f_{q,n}(x_{1\sigma(1)})\cdots f_{q,n}(x_{n\sigma(n)}) = q^{n(n+1)-2(1+\cdots+n)} = 1, \end{aligned}$$

and that $\mathcal{D}_q \in GL_q(n)$ is central, the pair $(f_{q,n}^{-1}, \mathcal{D}_q)$ is a MPI.

3. Hochschild homology of $M_q(n)$

3.1. Homology of quantum matrices $M_q(n,m)$

Given a sequence $(q_1,\ldots,q_n)$ of scalars in k, and let $\alpha\colon M_q(n) \to k$ be the character given by

$$\alpha(x_{ij}) = \delta_{ij}q_i.$$

Accordingly, the counit

$$\varepsilon(x_{ij}) = \delta_{ij},$$

the characters

$$f_{q,n}(x_{ij}) = \delta_{ij}q^{2i-1-n} \quad \text{and} \quad f_{q,n}^{-1}(x_{ij}) = \delta_{ij}q^{n-2i+1}$$

of Proposition 2.2, and finally the character $\eta\colon M_q(n,m) \to k$ given by

$$\eta(x_{ij}) = 0, \tag{3.1}$$

for $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$ correspond to the sequences

$$\begin{aligned} \varepsilon &\leftrightarrow (1, \ldots, 1), \\ f_{q,n} &\leftrightarrow (q^{-n+1}, q^{-n+3}, \ldots, q^{n-1}), \\ f_{q,n}^{-1} &\leftrightarrow (q^{n-1}, q^{n-3}, \ldots, q^{-n+1}), \\ \eta &\leftrightarrow (0, \ldots, 0). \end{aligned}$$

Let, now, $\alpha, \beta\colon M_q(n,m) \to k$ be two characters given by two sequences of scalars as defined above. Let also ${}_\alpha k_\beta$ denote the $M_q(n,m)$-bimodule k with the actions given by

$$x_{ij} \triangleright 1 = \alpha(x_{ij}) \quad \text{and} \quad 1 \triangleleft x_{ij} = \beta(x_{ij})$$

for any $x_{ij} \in M_q(n,m)$. In addition, the absence of a subscript such as ${}_\alpha k$ or k_α indicates that the action on the unspecified side is given by the counit.

The following result gives us the license to consider only the 1-dimensional bimodules whose right action is given by the counit.

Proposition 3.1. *Given any characters α and β defined by a sequence of scalars $(q^{a_1}, \ldots, q^{a_n})$ and $(q^{b_1}, \ldots, q^{b_n})$ as defined above, there is an automorphism $\nu_\alpha\colon M_q(n,m) \to M_q(n,m)$ so that ${}_{\alpha^{-1}\beta}k$ coincides with ${}_\beta k_\alpha$, the latter being equipped with the ν_α-twisted $M_q(n,m)$ action.*

Proof. We define

$$\nu_\alpha(x_{ij}) := q^{-a_i} x_{ij}$$

and observe that the relations (2.1) through (2.4) are invariant under this action. The action of $M_q(n,m)$ twisted by ν_α is defined as

$$1 \blacktriangleleft x_{ij} = \alpha(\nu_\alpha(x_{ij})) = \delta_{ij} q^{-a_i} q^{a_i} = \delta_{ij}$$

and the left action is given as

$$x_{ij} \blacktriangleright 1 = \beta(\nu_\alpha(x_{ij})) = \delta_{ij} q^{b_i} q^{-a_i}$$

for every generator x_{ij}. $\square$

Lemma 3.2. *The Hochschild homology of the quantum affine n-space $M_q(n,1) \cong M_q(1,n)$ with coefficients in ${}_\eta k_\eta$ is given by*

$$H_\ell(M_q(1,n),\ {}_\eta k_\eta) \cong k^{\oplus \binom{n}{\ell}}.$$

Proof. Setting $Q := k[x_{11}] \subseteq M_q(1,n) =: P$, we have

$$\begin{aligned} H_*(M_q(1,n), {}_\eta k_\eta) \Leftarrow E^1_{i,j} &= H_j(k[x_{11}], {}_\eta k_\eta \otimes_{k[x_{11}]} \underbrace{M_q(1,n) \otimes_{k[x_{11}]} \cdots \otimes_{k[x_{11}]} M_q(1,n)}_{i\text{-many}}) \\ &\cong H_j(k[x_{11}], {}_\eta k_\eta \otimes \underbrace{M_q(1,n-1) \otimes \cdots \otimes M_q(1,n-1)}_{i\text{-many}}), \end{aligned}$$

where the $k[x_{11}]$ action is still given by η on the coefficient complex. Thus, the E^1-term of the spectral sequence splits as

$$\begin{aligned} E^1_{i,j} &= H_j(k[x_{11}],\ {}_\eta k_\eta) \otimes \underbrace{M_q(1,n-1) \otimes \cdots \otimes\ M_q(1,n-1)}_{i\text{-many}} \\ &\cong \mathrm{CH}_i(M_q(1,n-1),\ {}_\eta k_\eta) \otimes H_j(k[x_{11}],\ {}_\eta k_\eta) \end{aligned}$$

since the action of $M_q(1,n-1)$ on $H_j(k[x_{11}],k)$ is again given by η. On the other hand for $k[x_{11}]$ we have

$$H_j(k[x_{11}],\ {}_\eta k_\eta) = \begin{cases} k & \text{if } j = 0,1, \\ 0 & \text{otherwise.} \end{cases}$$

Then we see that

$$H_\ell(M_q(1,n),\ {}_\eta k_\eta) \cong H_\ell(M_q(1,n-1),\ {}_\eta k_\eta) \oplus H_{\ell-1}(M_q(1,n-1),\ {}_\eta k_\eta).$$

The result follows from recursion. ☐

Let $\Lambda^*(X)$ denote the exterior algebra generated by a set X of indeterminates. The following result follows from an easy dimension counting.

Proposition 3.3. *We have isomorphisms of vector spaces of the form*

$$H_\ell(M_q(n,m), {}_\eta k_\eta) \cong \Lambda^\ell(x_{ij} \mid 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m)$$

for every $m,n \geqslant 1$ and $\ell \geqslant 0$.

Proof. Let prove this by induction on n. For $n = 1$ the result is given by Lemma 3.2. Assume we have the prescribed result for n. Consider the extension $M_q(n,m) \subseteq M_q(n+1,m)$ with the canonical embedding. Then by Proposition 1.6 we get

$$\begin{aligned} &H_*(M_q(n+1,m),\ {}_\eta k_\eta) \\ &\quad \Leftarrow E^1_{i,j} = H_j(M_q(n,m), \underbrace{M_q(n+1,m) \otimes_{M_q(n,m)} \cdots \otimes_{M_q(n,m)} M_q(n+1,m)}_{i\text{-times}}) \\ &\qquad\qquad \cong H_j(M_q(n,m), \mathrm{CH}_i(M_q(1,m),\ {}_\eta k_\eta)) \\ &\qquad\qquad \cong H_j(M_q(n,m),\ {}_\eta k_\eta) \otimes \mathrm{CH}_i(M_q(1,m),\ {}_\eta k_\eta) \end{aligned}$$

since $M_q(n,m)$ acts by η on the coefficient complex. Accordingly, we get the E^2-page

$$E^2_{i,j} = H_j(M_q(n,m),\ {}_\eta k_\eta) \otimes H_i(M_q(1,m),\ {}_\eta k_\eta)$$

in which all differentials are 0, and therefore we obtain

$$H_\ell(M_q(n+1,m), {}_\eta k_\eta) \cong \bigoplus_{i+j=\ell} H_j(M_q(n,m),\ {}_\eta k_\eta) \otimes H_i(M_q(1,m),\ {}_\eta k_\eta).$$

Then we also see that

$$\begin{aligned} \dim_k H_\ell(M_q(n+1,m), {}_\eta k_\eta) &= \sum_{\ell_1+\ell_2=\ell} \dim_k H_{\ell_1}(M_q(n,m), {}_\eta k_\eta) \cdot \dim_k H_{\ell_2}(M_q(1,m), {}_\eta k_\eta) \\ &= \sum_{\ell_1+\ell_2=\ell} \binom{nm}{\ell_1}\binom{m}{\ell_2} = \binom{(n+1)m}{\ell} \end{aligned}$$

as we wanted to show. ☐

Remark 3.4. The results of Lemma 3.2 and Proposition 3.3 are well-known in the context of Koszul duality [**26, 6**] since the algebras $M_q(n,m)$ are quadratic, and therefore, are Koszul for every $n,m \geqslant 1$. Thus $H_*(M_q(n,m), {}_\eta k_\eta) \cong \mathrm{Tor}_*^{M_q(n,m)}(k_\eta, {}_\eta k) \cong \mathrm{Ext}^*_{M_q(n,m)}(k_\eta, k_\eta)$ is isomorphic to the Koszul dual $M_q(n,m)^!$. Since $M_q(n,m)$ and $k[x_{ij} \mid 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m]$ are isomorphic as graded vector spaces, we get that $M_q(n,m)^!$ and $\Lambda^*(x_{ij} \mid 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m)$ are isomorphic as graded vector spaces. See [**26, 6**] for details. Note that this argument works only for ${}_\eta k_\eta$, and fails to work with other one dimensional coefficients.

3.2. Homology of $M_q(n)$

In this subsection we compute the Hochschild homology of the algebra $M_q(n)$ of quantum matrices with coefficients in $\alpha = (q^{a_1}, \ldots, q^{a_n})$, where $a_1, \ldots, a_n \in \mathbb{Z}$.

To this end, we begin with the extension $\mathrm{Row}_n \subseteq M_q(n)$ that yields, in the relative complex,

$$ {}_\alpha k \otimes_{\mathrm{Row}_n} \underbrace{M_q(n) \otimes_{\mathrm{Row}_n} \cdots \otimes_{\mathrm{Row}_n} M_q(n)}_{i\text{-times}} \cong {}_\alpha k \otimes \underbrace{M_q(n-1,n) \otimes \cdots \otimes M_q(n-1,n)}_{i\text{-times}}. $$

Hence, the E^1-page of the spectral sequence is

$$ E^1_{i,j} \cong H_j(\mathrm{Row}_n, \mathrm{CH}_i(M_q(n-1,n), \ {}_\alpha k)). $$

We then note that the elements $x_{ni} \in \mathrm{Row}_n$, for $i \neq n$, act on the coefficient complex via η, whereas x_{nn} act via a scalar q^{a_n} on the left. On the right, the action of x_{nn} is via another scalar determined by the total degree of the terms in x_{in} in $\mathrm{CH}_i(M_q(n-1,n), {}_\alpha k)$ for $1 \leqslant i \leqslant n-1$. Accordingly,

$$ E^1_{i,j} \cong \bigoplus_a H_j(\mathrm{Row}_n, \ {}_{q^{a_n}}k_{q^{-a}}) \otimes \mathrm{CH}_i^{(a)}(M_q(n-1,n), \ {}_\alpha k), $$

where $\mathrm{CH}_*^{(a)}$ denotes the subcomplex of terms whose total degree in x_{in}, for $i = 1, \ldots, n-1$, are precisely $a \in \mathbb{Z}$. Let us remark also that since these terms act by η, the graded subspace $\mathrm{CH}_*^{(a)}(M_q(n-1,n), \ {}_\alpha k)$ of $\mathrm{CH}_*(M_q(n-1,n), \ {}_\alpha k)$ is indeed a subcomplex.

For the homology of the row algebra this time, we use the lattice of extensions $\mathrm{Row}_n(a,b) \subseteq \mathrm{Row}_n$, where $\mathrm{Row}_n(a,b)$ is the subalgebra of Row_n generated by x_{na}, $\ldots, x_{nb}$. Thus, we may express

$$ H_j(\mathrm{Row}_n, {}_{q^{a_n}}k_{q^{-a}}) = \bigoplus_c H_c(\mathrm{Row}_n(b+1,n), {}_{q^{a_n}}k_{q^{-a+c-j}}) \otimes \Lambda^{j-c}(x_{n1}, \ldots, x_{nb}) $$

for every $1 \leqslant b \leqslant n-1$. In particular,

$$ H_j(\mathrm{Row}_n, {}_{q^{a_n}}k_{q^{-a}}) = \bigoplus_c H_c(\mathrm{Row}_n(n,n), {}_{q^{a_n}}k_{q^{-a+c-j}}) \otimes \Lambda^{j-c}(x_{n1}, \ldots, x_{nn-1}). $$

Since $\mathrm{Row}_n(n,n) = k[x_{nn}]$, the direct sum above has only two non-zero terms: those with $c = 0$ and $c = 1$. Therefore,

$$ \begin{aligned} H_j(\mathrm{Row}_n, {}_{q^{a_n}}k_{q^{-a}}) = & \Big(H_0(k[x_{nn}], {}_{q^{a_n}}k_{q^{-j-a}}) \otimes \Lambda^j(x_{n1}, \ldots, x_{nn-1}) \Big) \\ & \oplus \Big(H_1(k[x_{nn}], {}_{q^{a_n}}k_{q^{-j-a+1}}) \otimes \Lambda^{j-1}(x_{n1}, \ldots, x_{nn-1}) \Big). \end{aligned} $$

On the other hand, we observe that the homology is zero unless ${}_{q^{a_n}}k_{q^{-a+c-j}}$ is symmetric. Thus,

$$H_j(\mathrm{Row}_n, {}_{q^{a_n}}k_{q^{-a}}) = \begin{cases} \Lambda_q^j(x_{n1}, \ldots, x_{nn-1}) & \text{if } a = -a_n - j \geqslant 0, \\ \Lambda^{j-1}(x_{n1}, \ldots, x_{nn-1})x_{nn} & \text{if } a = -a_n - j + 1 \geqslant 0, \\ 0 & \text{otherwise.} \end{cases}$$

As a result, the E^1-page of the spectral sequence reduces to

$$\begin{aligned} H_\ell(M_q(n), {}_\alpha k) \Leftarrow E^1_{i,j} &\cong \mathrm{CH}_i^{(-a_n-j)}(M_q(n-1,n), {}_\alpha k \otimes \Lambda^j(x_{n1}, \ldots, x_{n,n-1})) \\ &\oplus \mathrm{CH}_i^{(-a_n-j+1)}(M_q(n-1,n), {}_\alpha k \otimes \Lambda^{j-1}(x_{n1}, \ldots, x_{n,n-1})x_{nn}). \end{aligned}$$

Now, let $\mathrm{Col}_n(a,b)$ be the subalgebra generated by $x_{an}, \ldots, x_{bn}$ in $M_q(n-1,n)$. Then, in view of Proposition 1.6 we have

$$\begin{aligned} &H_i^{(a)}(M_q(n-1,n), {}_\alpha k \otimes \Lambda^j(x_{1n}, \ldots, x_{nn})) \\ &\qquad \Leftarrow E^1_{r,s} = H_s^{(a)}(\mathrm{Col}_n(1,n-1), \mathrm{CH}_r(M_q(n-1), {}_\alpha k \otimes \Lambda^j(x_{1n}, \ldots, x_{nn}))). \end{aligned}$$

Since $\mathrm{Col}_n(1,n-1)$ acts on the coefficient complex via η, the E^1-page splits, and we arrive at

$$\begin{aligned} &H_i^{(a)}(M_q(n-1,n), {}_\alpha k \otimes \Lambda^j(x_{1n}, \ldots, x_{nn})) \\ &\qquad \cong H_{i-a}(M_q(n-1), {}_\alpha k \otimes \Lambda^j(x_{1n}, \ldots, x_{nn}) \otimes \Lambda^a(x_{1n}, \ldots, x_{n-1,n})). \end{aligned}$$

From these we get the E^2-page

$$\begin{aligned} E^2_{i,j} \cong\, & H_{i+j+a_n}(M_q(n-1), {}_\alpha k \otimes \Lambda^j(x_{n1}, \ldots, x_{n,n-1}) \otimes \Lambda^{-a_n-j}(x_{1n}, \ldots, x_{n-1,n})) \\ &\oplus H_{i+j+a_n-1}(M_q(n-1), {}_\alpha k \otimes \Lambda^{j-1}(x_{n1}, \ldots, x_{n,n-1})x_{nn} \\ &\qquad\qquad \otimes \Lambda^{-a_n-j+1}(x_{1n}, \ldots, x_{n-1,n})). \end{aligned}$$

Therefore,

$$\begin{aligned} H_\ell(M_q(n), {}_\alpha k) \cong \bigoplus_{j,s} H_{\ell+a_n-s}(M_q(n-1), &{}_\alpha k \otimes \Lambda^j(x_{n1}, \ldots, x_{n,n-1}) \\ &\otimes \Lambda^{-a_n-j}(x_{1n}, \ldots, x_{n-1,n})) \otimes \Lambda^s(x_{nn}). \end{aligned}$$

Now, consider the subspace S_n^* of $\Lambda^*(x_{ij} \mid 1 \leqslant i,j \leqslant n)$ generated by x_{in} and x_{nj} with $1 \leqslant i \leqslant n-1$ and $1 \leqslant j \leqslant n-1$. Also, we use $S_n^*(b)$ to denote the homogeneous vector subspace of S_n^* of terms whose total degree over terms of type x_{in} and x_{ni} is b.

We observe that

$$\begin{aligned} H_\ell(M_q(n), {}_\alpha k) &\cong \bigoplus_s H_{\ell+a_n-s}(M_q(n-1), {}_\alpha k \otimes S^*_{n-1}(-a_n)) \otimes \Lambda^s(x_{nn}) \\ &\cong \bigoplus_{\beta,s} H_{\ell+a_n-s}(M_q(n-1), {}_\alpha k_{\beta^{-1}}) \otimes S^*_{n-1}(b_1, \ldots, b_{n-1}, -a_n) \otimes \Lambda^s(x_{nn}) \\ &\cong \bigoplus_{\beta,s} H_{\ell+a_n-s}(M_q(n-1), {}_{\alpha\beta}k) \otimes S^*_{n-1}(b_1, \ldots, b_{n-1}, -a_n) \otimes \Lambda^s(x_{nn}). \end{aligned}$$

The sum is taken over all $\beta = (q^{b_1}, \ldots, q^{b_n})$, where the multi-degree $(b_1, \ldots, b_n)$ indicates that we consider the k-vector space spanned by monomials Γ that has the total degree $\deg_i(\Gamma) = b_i$ in terms of x_{si} and x_{it} for all $s, t = 1, \dots, n$ and $i = 1, \dots, n$, and we set

$$\deg_i(x_{ij}) = \deg_i(x_{ji}) = \begin{cases} -1 & \text{if } j < i, \\ 0 & \text{if } j = i, \\ 1 & \text{if } j > i. \end{cases} \tag{3.2}$$

It follows at once that $b_n = -a_n$, since there are no indices $j > n$. Proceeding the computation recursively, we arrive at the following result.

Theorem 3.5. *Fix a sequence of non-zero scalars $\alpha = (q^{a_1}, \dots, q^{a_n})$, and let us define*

$$\Lambda^*_{(\alpha)}(x_{ij} \mid 1 \leqslant i, j \leqslant n)$$

as the subspace of differential forms with multi-degree $(a_1, \dots, a_n)$ where the a_i is the total degree – in the sense of (3.2) – of terms involving the indeterminates x_{si} and x_{it} for $s, t = 1, \dots, n$, and $i = 1, \dots, n$. Then,

$$H_{\ell+|\alpha|}(M_q(n), {}_\alpha k) \cong \bigoplus_s \Lambda^{\ell-s}_{(\alpha)}(x_{ij} \mid 1 \leqslant i \neq j \leqslant n) \otimes \Lambda^s(x_{11}, \dots, x_{nn})$$

as vector spaces for every $n \geqslant 1$, $\ell \geqslant 0$.

3.3. Homologies of $GL_q(n)$ and $SL_q(n)$ with coefficients in ${}_{f_{q,n}^{-1}}k$

We are going to derive the homology of $GL_q(n)$ and $SL_q(n)$ from that of $M_q(n)$ by using the localization of the Hochschild homology.

Let us first recall the localization of the homology, [**20**, Prop. 1.1.17].

Proposition 3.6. *Given an algebra A, and a multiplicative subset $S \subseteq Z(A)$ so that $1 \in S$ and $0 \notin S$, and an A-bimodule M, there are the following canonical isomorphisms:*

$$H_*(A, M)_S \cong H_*(A, M_S) \cong H_*(A_S, M_S),$$

where

$$M_S := Z(A)_S \otimes_{Z(A)} M,$$

$Z(A)$ denotes the center of A, and $Z(A)_S$ stands for the localization of $Z(A)$ at S.

Now, in view of the fact that $GL_q(n)$ is the localization of $M_q(n)$ at $S = \{\mathcal{D}_q^n \mid n \geqslant 0\}$, we obtain the Hochschild homology of $GL_q(n)$ readily from the above localization result.

Theorem 3.7. *For every $\ell \geqslant 0$, and every $n \geqslant 1$,*

$$H_\ell(M_q(n), {}_{f_{q,n}^{-1}}k) = H_\ell(GL_q(n), {}_{f_{q,n}^{-1}}k) \cong H_\ell(SL_q(n), {}_{f_{q,n}^{-1}}k) \oplus H_{\ell-1}(SL_q(n), {}_{f_{q,n}^{-1}}k).$$

Proof. Let us recall from [**23**, Thm. 1.6], see also [**25**], that

$$Z(M_q(n)) = k[\mathcal{D}_q],$$

and that $GL_q(n) = M_q(n)_S$ for the multiplicative system $S = \{\mathcal{D}_q^n \mid n \geqslant 0\}$ generated by the quantum determinant. Accordingly, we have

$$Z(M_q(n))_S = k[\mathcal{D}_q, \mathcal{D}_q^{-1}],$$

and

$${}_{f_{q,n}^{-1}}k_S = k[\mathcal{D}_q^{-1}] \otimes {}_{f_{q,n}^{-1}}k$$

is the $GL_q(n)$-bimodule so that the $M_q(n)$-bimodule structure concentrated on ${}_{f_{q,n}^{-1}}k$, and the $k[\mathcal{D}_q^{-1}]$-bimodule structure is on $k[\mathcal{D}_q^{-1}]$. Then we have

$$H_*(GL_q(n), {}_{f_{q,n}^{-1}}k_S) \cong k[\mathcal{D}_q^{-1}] \otimes H_*(GL_q(n), {}_{f_{q,n}^{-1}}k)$$

so that the $GL_q(n)$-bimodule structure on ${}_{f_{q,n}^{-1}}k$ is determined by the trivial action of $\mathcal{D}_q^{-1}$. On the other hand,

$$H_*(M_q(n), {}_{f_{q,n}^{-1}}k)_S \cong k[\mathcal{D}_q^{-1}] \otimes H_*(M_q(n), {}_{f_{q,n}^{-1}}k),$$

where the $GL_q(n)$-bimodule structure is given in such a way that the $M_q(n)$-bimodule structure is on $H_*(M_q(n), {}_{f_{q,n}^{-1}}k)$, and the $\mathcal{D}_q^{-1}$-action structure is concentrated on $k[\mathcal{D}_q^{-1}]$. Proposition 3.6 then yields the first claim. The second part the computation follows from [**19**, Proposition] and [**14**, Theorem 2.7]. □

3.4. Homologies of $GL_q(n)$ and $SL_q(n)$ with twisted coefficients in themselves

Let H be a Hopf algebra and let $\alpha\colon H \to k$ be a character. Using α we can define an algebra automorphism $a \mapsto \alpha(a_{(1)})a_{(2)}$ whose inverse is given by $a \mapsto \alpha(S(a_{(1)}))a_{(2)}$. Then one can twist the regular representation of H on itself as

$$h \blacktriangleright x = \alpha(h_{(1)})h_{(2)}x$$

for every $h, x \in H$. Let us denote this module by ${}_\alpha H$, and the resulting Hochschild homology $H_*(H, {}_\alpha H)$ by $H_*^\alpha(H)$.

Now, for the Hopf algebras $GL_q(n)$ and $SL_q(n)$, one can use the character $f_{q,n}^{-1}$ of Proposition 2.2, to write a map from the Hochschild homology of H with coefficients in ${}_{f_{q,n}^{-1}}k$ to its twisted Hochschild homology. The proof of the following Proposition is by direct verification, and therefore, is omitted. See also [**16**, Prop. 3.2].

Proposition 3.8. *Let H be the Hopf algebra $GL_q(n)$ or $SL_q(n)$. Then there is a characteristic homomorphism $\widetilde{\theta}\colon H_*(H, {}_{f_{q,n}^{-1}}k) \to H_*^{f_{q,n}^{-1}}(H)$, given by*

$$\widetilde{\theta}_\ell(h^1 \otimes \cdots \otimes h^\ell) := S(h^1_{(1)} \cdots h^\ell_{(1)}) \otimes h^1_{(2)} \otimes \cdots \otimes h^\ell_{(2)} \tag{3.3}$$

for every $\ell \geqslant 1$.

A more conceptual way of thinking of the morphism (3.3) is the following. Consider the adjoint action of H on the twisted regular representation given by

$${}^h x = f_{q,n}^{-1}(h_{(1)})h_{(2)}xS(h_{(3)}).$$

The adjoint module $ad({}_{f_{q,n}^{-1}}H)$ splits as a direct sum of the submodule ${}_{f_{q,n}^{-1}}k$ and its complement which is the kernel of the counit. Thus we get that $H_*(H, {}_{f_{q,n}^{-1}}k) =$

$\mathrm{Tor}^H_*(k, {}_{f_{q,n}^{-1}}k)$ is a direct summand of $\mathrm{Tor}^H_*(k, ad({}_{f_{q,n}^{-1}}H))$ which is isomorphic to $H_*^{f_{q,n}^{-1}}(H)$.

3.5. Transfer of classes

Following [**13**] along the lines of [**16**], let H be a Hopf algebra with an MPI (δ, σ), and let A be an H-comodule algebra equipped with a δ-trace; that is, a functional $\tau\colon A \to k$ satisfying

$$\tau(xy) = \tau(yx_{(1)})\delta(x_{(2)})$$

for any $x, y \in A$. Then, there is a characteristic map

$$\gamma\colon H_*(A) \to H_*(H, {}_\delta k) \tag{3.4}$$

which is, on the chain level, given by

$$\gamma(a^0 \otimes \cdots \otimes a^n) := \tau(a^0 a^1_{\langle 0\rangle} \cdots a^n_{\langle 0\rangle}) a^1_{\langle 1\rangle} \otimes \cdots \otimes a^n_{\langle 1\rangle},$$

where $a \mapsto a_{\langle 0\rangle} \otimes a_{\langle 1\rangle}$ denotes the right H-coaction on A.

However, the unique Haar functional $h\colon GL_q(n) \to k$ (resp. $h\colon SL_q(n) \to k$) has the modularity

$$h(xy) = h(y(f_{q,n}^{-1} \triangleright x \triangleleft f_{q,n}^{-1})) = f_{q,n}^{-1}(x_{(1)})h(yx_{(2)})f_{q,n}^{-1}(x_{(3)}),$$

for any $x, y \in GL_q(n)$ (resp. $SL_q(n)$), see for instance [**17**, Prop. 11.3.34] and [**24**, Thm. 5.4.2].

Accordingly, (3.4) may be modified into

$$\widetilde{\gamma}\colon H_*^{f_{q,n}}(M_q(n)) \to H_*(GL_q(n), {}_{f_{q,n}^{-1}}k) \tag{3.5}$$

given by

$$\widetilde{\gamma}_\ell(a^0 \otimes \cdots \otimes a^\ell) := h(a^0 a^1_{(1)} \cdots a^\ell_{(1)}) a^1_{(2)} \otimes \cdots \otimes a^\ell_{(2)}, \qquad \ell \geqslant 0.$$

More generally, we have the following twisted analogue of [**16**, Prop. 3.1].

Proposition 3.9. *Let (A, H) be any one of the pairs $(M_q(n), SL_q(n))$, $(M_q(n), GL_q(n))$, $(GL_q(n), GL_q(n))$, $(GL_q(n), SL_q(n))$, or $(SL_q(n), SL_q(n))$. Then there is a characteristic map in homology of the form*

$$H_*^{f_{q,n}}(A) \to H_*^{f_{q,n}^{-1}}(H). \tag{3.6}$$

Proof. A direct computation reveals that (3.5) works just as well in the form of

$$\widetilde{\gamma}\colon H_*^{f_{q,n}}(A) \to H_*(H, {}_{f_{q,n}^{-1}}k).$$

The claim, then, follows from the composition with (3.3). □

3.6. Untwisting the homology

As for the untwisting, we have two options. The first comes from [**18**] and the cap product of twisted homology and cohomology.

Let us recall from [18, Sect. 3] that if A is an unital associative algebra with two automorphisms σ and η, there is a cap product of the form

$$\cap\colon H_n^{\sigma}(A)\otimes H_{\eta}^{m}(A)\to H_{n-m}^{\eta\circ\sigma}(A), \qquad m\leqslant n,$$

given explicitly by

$$(a^0\otimes\cdots\otimes a^n)\cap\varphi := \eta(a^0)\varphi(a^1\otimes\cdots\otimes a^m)\otimes a^{m+1}\otimes\cdots\otimes a^n.$$

Here the left regular action of A on itself is twisted by the automorphisms via $a\blacktriangleright x=\eta(a)x$.

Proposition 3.10. *For $A=H=GL_q(n)$ or $A=H=SL_q(n)$, assume we have a twisted $f_{q,n}$-trace $\varphi\in H^0_{f_{q,n}}(A)=H^0(A,{}_{f_{q,n}}A)$. Then the characteristic map (3.6) can be extended to a map of the form*

$$H_*(H,{}_{f_{q,n}^{-1}}k)\to H_*(H).$$

The second option for untwisting comes from Hopf-cyclic cohomology:

Proposition 3.11. *For $A=H=GL_q(n)$ or $A=H=SL_q(n)$, the characteristic map (3.6) can be extended to a map of the form*

$$H_m(H,{}_{f_{q,n}^{-1}}k)\to H_{m+1}(H),$$

for every $m\geqslant 0$.

Proof. The Hopf algebra H is a module algebra over the Laurent polynomial Hopf algebra $k[\alpha,\alpha^{-1}]$ where α is the automorphism defined as $\alpha(h)=f_{q,n}(h_{(1)})h_{(2)}$. The rest follows from an untwist as in [15, Section 2.9] but for Hochschild homology. □

4. Explicit calculations

4.1. $M_q(2)$, $GL_q(2)$ and $SL_q(2)$

The character $f_{q,2}^{-1}$ is given by the sequence (q,q^{-1}). Then, $|\alpha|=0$ and

$$H_\ell(M_q(2),{}_{(q,q^{-1})}k)=\bigoplus_s \Lambda^{\ell-s}_{(q,q^{-1})}(x_{12},x_{21})\otimes\Lambda^s(x_{11},x_{22}).$$

One can write 4 different exterior products between x_{12} and x_{21}, and

$$\begin{aligned}\deg((1))&=(0,0),\\ \deg((x_{21}))=\deg((x_{12}))&=(1,-1),\\ \deg((x_{12},x_{21}))&=(2,-2)\end{aligned}$$

from which we only take the degree $(1,-1)$-terms of exterior degree 1. Thus

$$H_\ell(M_q(2),{}_{f_{q,2}^{-1}}k)=\begin{cases}0 & \text{if } \ell=0 \text{ or } \ell\geqslant 4,\\ Span_k((x_{12}),(x_{21})) & \text{if } \ell=1,\\ Span_k((x_{11},x_{12}),(x_{11},x_{21}),(x_{12},x_{22}),(x_{21},x_{22})) & \text{if } \ell=2,\\ Span_k((x_{11},x_{12},x_{22}),(x_{11},x_{21},x_{22})) & \text{if } \ell=3.\end{cases} \tag{4.1}$$

The Betti numbers of the homology are given in Figure 1.

m	1	2	3
$\dim_k H_m(M_q(2), {}_{f_{q,2}^{-1}}k)$	2	4	2
$\dim_k H_m(GL_q(2), {}_{f_{q,2}^{-1}}k)$	2	4	2
$\dim_k H_m(SL_q(2), {}_{f_{q,2}^{-1}}k)$	2	2	

Figure 1: The Betti numbers for $M_q(2)$, $GL_q(2)$ and $SL_q(2)$

Let us note also that (4.1) simply illustrates what each representative of $H_*(M_q(2), {}_{f_{q,2}^{-1}}k) \cong H_*(GL_q(2), {}_{f_{q,2}^{-1}}k)$ is made of. They can be converted into actual representatives quite straightforwardly as

$$(x_{ki}, x_{kj}) = \mathbf{1} \otimes \Big(x_{ki} \otimes x_{kj} - q^{-1}\, x_{kj} \otimes x_{ki}\Big),$$

for any $1 \leqslant k \leqslant n$, and any $1 \leqslant i < j \leqslant n$. The very same pattern holds for (x_{ik}, x_{jk}). As for the 3-classes, we have

$$\begin{aligned}(x_{11}, x_{ij}, x_{22}) = \mathbf{1} \otimes \Big(& -q^2\, x_{11} \otimes x_{ij} \otimes x_{22} + q\, x_{ij} \otimes x_{11} \otimes x_{22} + q\, x_{11} \otimes x_{22} \otimes x_{ij} \\ & + x_{22} \otimes x_{ij} \otimes x_{11} - q\, x_{22} \otimes x_{11} \otimes x_{ij} - q\, x_{ij} \otimes x_{22} \otimes x_{11} \\ & + (1-q^2)\, x_{ij} \otimes x_{ji} \otimes x_{ij}\Big)\end{aligned}$$

both for $(i,j) = (1,2)$ and $(j,i) = (1,2)$.

Let us next compare our results with those of [5]. To this end we begin, in view of Theorem 3.7, with

$$H_\ell(SL_q(2), {}_{f_{q,2}^{-1}}k) = \begin{cases} 0 & \text{if } \ell = 0 \text{ or } \ell \geqslant 3, \\ Span_k((x_{12}), (x_{21})) & \text{if } \ell = 1, \\ Span_k((x_{11}, x_{12}), (x_{11}, x_{21})) & \text{if } \ell = 2. \end{cases}$$

Next, we use (3.3), that is,

$$\widetilde{\theta}\colon H_1(SL_q(2), {}_{f_{q,2}^{-1}}k) \to H_1^{f_{q,2}^{-1}}(SL_q(2)),$$

to obtain the classes

$$\widetilde{\theta}(x_{12}) = x_{22} \otimes x_{12} - q\, x_{12} \otimes x_{22} \in H_1^{f_{q,2}^{-1}}(SL_q(2))$$

and

$$\widetilde{\theta}(x_{21}) = -q^{-1}\, x_{21} \otimes x_{11} + x_{11} \otimes x_{21} \in H_1^{f_{q,2}^{-1}}(SL_q(2)).$$

Moreover, it is immediate to observe that

$$x_{22} \otimes x_{12},\ x_{12} \otimes x_{22},\ x_{21} \otimes x_{11},\ x_{11} \otimes x_{21} \in H_1^{f_{q,2}^{-1}}(SL_q(2)),$$

which are the classes obtained in Case 4. of [5, Subsect. 4.5]. We do note that there is a $q \leftrightarrow q^{-1}$ difference between [5] and the present note, and that the twisting automorphism $\sigma_{f_{q,2}^{-1}}\colon SL_q(2) \to SL_q(2)$ given by $x \mapsto f_{q,2}^{-1}(x_{(1)})x_{(2)}$ corresponds to $\sigma_{q^{-1},q^{-1}}$ in [5, Prop. 3.1].

We proceed to

$$\widetilde{\theta}\colon H_2(SL_q(2), {}_{f_{q,2}^{-1}}k) \to H_2^{f_{q,2}^{-1}}(SL_q(2)).$$

We have,

$$(x_{11}, x_{12}) := x_{11} \otimes x_{12} - q^{-1}\, x_{12} \otimes x_{11} \in H_2(SL_q(2), {}_{f_{q,2}^{-1}}k)$$

and

$$(x_{11}, x_{21}) := x_{11} \otimes x_{21} - q^{-1}\, x_{21} \otimes x_{11} \in H_2(SL_q(2), {}_{f_{q,2}^{-1}}k).$$

Accordingly, we obtain

$$\widetilde{\theta}((x_{11}, x_{21})) = \widetilde{\theta}(x_{11} \otimes x_{21} - q^{-1}\, x_{21} \otimes x_{11}) = 1 \otimes \Big(x_{11} \otimes x_{21} - q^{-1}\, x_{21} \otimes x_{11}\Big),$$

and

$$\begin{aligned}
&\widetilde{\theta}((x_{11}, x_{12})) = \widetilde{\theta}(x_{11} \otimes x_{12} - q^{-1}\, x_{12} \otimes x_{11}) = \\
&\quad x_{22}^2 \otimes \Big(x_{11} \otimes x_{12} - q^{-1}\, x_{12} \otimes x_{11}\Big) - q\, x_{12}x_{22} \otimes \Big(x_{11} \otimes x_{22} - q^{-1}\, x_{12} \otimes x_{21}\Big) \\
&\qquad + q\, x_{12}x_{22} \otimes \Big(x_{22} \otimes x_{11} - q\, x_{21} \otimes x_{12}\Big) + q^2\, x_{12}^2 \otimes \Big(x_{21} \otimes x_{22} - q^{-1}\, x_{22} \otimes x_{21}\Big),
\end{aligned}$$

while we refer the reader to Case 4. of [**5**, Subsect. 4.6] for a comparison of the classes obtained there.

Finally, we turn our attention to the noncommutative covolume form $\mathrm{d}\mathcal{A} \in H_3^{\sigma_{\mathrm{mod}}}(SL_q(2))$, where $\sigma_{\mathrm{mod}} = \sigma_{q^2,1}$ is the modular automorphism given by

$$\sigma_{\mathrm{mod}}(x) := f_{q,2}^{-1}(x_{(1)})x_{(2)}f_{q,2}^{-1}(x_{(3)}),$$

for any $x \in SL_q(2)$. Accordingly, it follows from (3.3) that there is

$$\widetilde{\theta}\colon H_*(SL_q(2), {}_{\delta_{\mathrm{mod}}}k) \to H_*^{\sigma_{\mathrm{mod}}}(SL_q(2)),$$

where the character $\delta_{\mathrm{mod}}\colon SL_q(2) \to k$ is given by $\delta_{\mathrm{mod}} = f_{q,2}^{-1} * f_{q,2}^{-1}$, more explicitly,

$$\delta_{\mathrm{mod}}(x_{11}) = q^2\, x_{11}, \qquad \delta_{\mathrm{mod}}(x_{12}) = 0 = \delta_{\mathrm{mod}}(x_{21}), \qquad \delta_{\mathrm{mod}}(x_{22}) = q^{-2}\, x_{22}.$$

Then, we deduce from Theorem 3.5 and Theorem 3.7 that

$$H_\ell(GL_q(2), {}_{\delta_{\mathrm{mod}}}k) = H_\ell(M_q(2), {}_{\delta_{\mathrm{mod}}}k) = \begin{cases} 0 & \text{if } \ell \leqslant 1 \text{ or } \ell \geqslant 5, \\ Span_k((x_{12}, x_{21})) & \text{if } \ell = 2, \\ Span_k((x_{12}, x_{21}, x_{22}), & \\ \qquad\qquad (x_{12}, x_{11}, x_{21})) & \text{if } \ell = 3, \\ Span_k((x_{12}, x_{11}, x_{21}, x_{22})) & \text{if } \ell = 4, \end{cases}$$

and that

$$H_\ell(SL_q(2), {}_{\delta_{\mathrm{mod}}}k) = \begin{cases} 0 & \text{if } \ell \leqslant 1 \text{ or } \ell \geqslant 4, \\ Span_k((x_{12}, x_{21})) & \text{if } \ell = 2, \\ Span_k((x_{12}, x_{11}, x_{21})) & \text{if } \ell = 3, \end{cases}$$

where

$$(x_{12},x_{11},x_{21}) := q^2\, x_{11}\otimes x_{12}\otimes x_{21} - q\, x_{12}\otimes x_{11}\otimes x_{21} - q^2\, x_{11}\otimes x_{21}\otimes x_{12} \\ + x_{12}\otimes x_{21}\otimes x_{11} - x_{21}\otimes x_{12}\otimes x_{11} + q\, x_{21}\otimes x_{11}\otimes x_{12} \in H_3(SL_q(2), {}_{\delta_{\rm mod}}k).$$

Accordingly,

$$\begin{aligned}
&\mathrm{d}\mathcal{A} = \widetilde{\theta}((x_{12},x_{11},x_{21}))\\
&= -q\, x_{12}^2 x_{21}\otimes \Big(x_{21}\otimes x_{22}\otimes x_{11} + (1-q)\, x_{11}\otimes x_{21}\otimes x_{21} + (q^2-1)\, x_{21}\otimes x_{11}\otimes x_{21}\\
&\qquad + x_{22}\otimes x_{11}\otimes x_{21} - q\, x_{22}\otimes x_{21}\otimes x_{11} + (q-q^2)\, x_{21}\otimes x_{21}\otimes x_{11}\\
&\qquad + q\, x_{11}\otimes x_{21}\otimes x_{22} - x_{11}\otimes x_{22}\otimes x_{21} - q^2\, x_{21}\otimes x_{11}\otimes x_{22}\Big)\\
&+ x_{22}\otimes \Big(q^2\, x_{11}\otimes x_{12}\otimes x_{21} - q\, x_{12}\otimes x_{11}\otimes x_{21} + x_{12}\otimes x_{21}\otimes x_{11} - x_{21}\otimes x_{12}\otimes x_{11}\\
&\qquad + q\, x_{21}\otimes x_{11}\otimes x_{12} - q^2\, x_{11}\otimes x_{21}\otimes x_{12}\Big)\\
&- q\, x_{12}\otimes \Big((q-q^2)\, x_{11}\otimes x_{21}\otimes x_{21} + (1-q)\, x_{21}\otimes x_{21}\otimes x_{11} - x_{21}\otimes x_{12}\otimes x_{21}\Big).
\end{aligned}$$

4.2. $M_q(3)$, $GL_q(3)$ and $SL_q(3)$

The character $f_{q,3}^{-1}$ is given by $(q^2,1,q^{-2})$ and $|\alpha|=0$. The exterior degree 1 terms are

$$\begin{aligned}
\deg((x_{12})) = \deg((x_{21})) &= (1,-1,0),\\
\deg((x_{13})) = \deg((x_{31})) &= (1,0,-1),\\
\deg((x_{23})) = \deg((x_{32})) &= (0,1,-1)
\end{aligned}$$

and we need terms of degree signature $(2,0,-2)$. We must solve a system of $\mathbb{Z}$-linear equations

$$(\alpha_1,\alpha_2,\alpha_3)\begin{bmatrix}1 & -1 & 0\\ 1 & 0 & -1\\ 0 & 1 & -1\end{bmatrix} = (2,0,-2),$$

where $\alpha_i\in\{0,1,2\}$. The only solutions are

$$(\alpha_1,\alpha_2,\alpha_3) = (2,0,2) \quad\text{or}\quad (\alpha_1,\alpha_2,\alpha_3) = (1,1,1).$$

For the first solution, there is only one term of exterior degree 4: $(x_{12},x_{21},x_{23},x_{32})$. On the other hand, for the second solution there are 8 such terms of exterior degree 3. Then we use the exterior algebra on x_{11}, x_{22} and x_{33} to promote these terms to higher degrees. In short, we have:

$$\begin{aligned}
&H_{3+\ell}(M_q(3), {}_{(q^2,1,q^{-2})}k) =\\
&\quad Span_k\Big((x_{12},x_{13},x_{23}), (x_{12},x_{13},x_{32}), (x_{12},x_{31},x_{23}), (x_{12},x_{31},x_{32}), (x_{21},x_{13},x_{23}),\\
&\qquad (x_{21},x_{13},x_{32}), (x_{21},x_{31},x_{23}), (x_{21},x_{31},x_{32})\Big)\otimes\Lambda^{\ell}(x_{11},x_{22},x_{33})\\
&\quad \oplus Span_k((x_{12},x_{21},x_{23},x_{32}))\otimes\Lambda^{\ell-1}(x_{11},x_{22},x_{33}).
\end{aligned}$$

The Betti numbers of the homology are given in Figure 2.

m	3	4	5	6	7
$\dim_k H_m(M_q(3), {}_{f_{q,3}^{-1}}k)$	8	25	27	11	1
$\dim_k H_m(GL_q(3), {}_{f_{q,3}^{-1}}k)$	8	25	27	11	1
$\dim_k H_m(SL_q(3), {}_{f_{q,3}^{-1}}k)$	8	17	10	1	

Figure 2: The Betti numbers for $M_q(3)$, $GL_q(3)$ and $SL_q(3)$

4.3. $M_q(4)$**,** $GL_q(4)$ **and** $SL_q(4)$

The character $f_{q,4}^{-1}$ is now given by the sequence $\alpha = (q^3, q, q^{-1}, q^{-3})$ with $|\alpha| = 0$. The exterior degree 1 terms are

$$\begin{aligned} \deg((x_{12})) = \deg((x_{21})) =&(1,-1,0,0), & \deg((x_{23})) = \deg((x_{32})) =&(0,1,-1,0), \\ \deg((x_{13})) = \deg((x_{31})) =&(1,0,-1,0), & \deg((x_{24})) = \deg((x_{42})) =&(0,1,0,-1), \\ \deg((x_{14})) = \deg((x_{41})) =&(1,0,0,-1), & \deg((x_{34})) = \deg((x_{43})) =&(0,0,1,-1) \end{aligned} \tag{4.2}$$

and we need the total multi-degree $(3, 1, -1, -3)$. Thus we solve

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} = (3, 1, -1, -3)$$

again with the restriction that $\alpha_i \in \{0, 1, 2\}$. The Betti numbers of this case are given in Figure 3.

m	2	3	4	5	6	7	8	9	10	11	12	13	14
$\dim_k H_m(M_q(4), {}_{f_{q,4}^{-1}}k)$	8	40	80	96	176	408	560	408	176	96	80	40	8
$\dim_k H_m(GL_q(4), {}_{f_{q,4}^{-1}}k)$	8	40	80	96	176	408	560	408	176	96	80	40	8
$\dim_k H_m(SL_q(4), {}_{f_{q,4}^{-1}}k)$	8	32	48	48	128	280	280	128	48	48	32	8	

Figure 3: The Betti numbers for $M_q(4)$, $GL_q(4)$ and $SL_q(4)$

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A. Kaygun kaygun@itu.edu.tr

Istanbul Technical University, Istanbul, Turkey

S. Sütlü serkan.sutlu@isikun.edu.tr

Işık University, Istanbul, Turkey