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THE FULTON–MACPHERSON OPERAD AND THE W-CONSTRUCTION

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Abstract

We construct an $O(n)$-equivariant isomorphism of topological operads $F_n \cong WF_n$ where F_n is the Fulton–MacPherson operad and W is the Boardman–Vogt construction. For $n = 2$ the isomorphism is explicit.

1. Introduction

The Fulton–MacPherson operad F_n is a geometric E_n-operad that was introduced by Getzler and Jones. It is an operad in the category of $O(n)$-spaces, that is $O(n)$-equivariantly weakly equivalent to the operad of little n-discs. The weak equivalence is constructed in [7] by a zig-zag of $O(n)$-equivariant maps, although the equivariance is not explicitly mentioned there. In some sense the Fulton–MacPherson operad is the smallest topological E_n-operad. It is a crucial object in the Goodwillie–Weiss calculus studying spaces of embeddings. The Boardman–Vogt W-construction [3], introduced in the seventies, is a functor providing a standard resolution of topological operads. Our main result is that WF_n and F_n are isomorphic as operads in $O(n)$-spaces. We announced the non-equivariant version in [7]. An explicit isomorphism for $n = 1$ is described in [1]. The isomorphism for $n = 2$ can be constructed explicitly using the machinery of [8], as we explain later. An important application of the main theorem is the construction of an operadic cellular decomposition of the Fulton–MacPherson operad F_2 described in [8]. The key insight of the proof is that $F_n(k)$ is a manifold with corners, and $WF_n(k)$ can be identified to a standard fattening of $F_n(k)$, that is homeomorphic to it. For example each connected component of $F_1(4)$ is a Stasheff pentagon. The corresponding component of $WF_1(4)$ is the union of the pentagon with a rectangle for each edge, and with a square for each vertex.

2. Operads and trees

We can define a topological operad P as a functor from the category of finite sets and bijections to the category of topological spaces, together with composition maps of the form $\circ_i \colon P_I \times P_J \to P_{I-\{i\} \coprod J}$ satisfying appropriate conditions. We

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write $P(k) = P_{\{1,\dots,k\}}$. We say that P is an operad in G-spaces if the functor P takes values in G-spaces, and the $\circ_i$ operations are G-equivariant maps. Each $\circ_i$ operation is represented by a tree with a single internal edge, such that the leaves sources of the non-root vertex are in bijective correspondence with J, and the remaining leaves are in bijective correspondence with $I - \{i\}$. Consider a finite rooted tree T, such that any vertex has at least two incoming edges, and exactly one outgoing edge. Suppose that the edges with no source, the *leaves*, are labelled by $1, \dots, k$, and there is a unique edge with no target, the *root*. All other edges are called *internal*, and have both source and target. We say that T is a *nested trees on k leaves*. We call the number of incoming edges of a vertex v its *valence*, and denote it by $|v|$. By iterating $\circ_i$-operations, any nested tree T on k leaves defines an operad composition for a topological operad P of the form $\circ_T \colon \prod_v P(|v|) \to P(k)$, where the product runs over all vertices v of T.

3. The Fulton–MacPherson operad

The space $F_n(k)$ is a compactification of the quotient of the ordered configuration space $Conf_k(\mathbb{R}^n)$ of k points in $\mathbb{R}^n$ modulo translations and positive dilations. It is defined [6] as the closure of the image of the map

$$\iota \colon Conf_k(\mathbb{R}^n) \to (S^{n-1})^{\binom{n}{2}} \times [0, +\infty]^{\binom{n}{3}},$$
$$\iota(x_1, \dots, x_k) = \left(((x_i - x_j)/||x_i - x_j||)_{i<j} \, , \, (||x_i - x_j||/||x_i - x_k||)_{i<j<k} \right).$$

The symmetric group Σ_k acts freely on $F_n(k)$, compatibly with the action permuting labels of configurations in $\mathbb{R}^n$. Since the map ι is $O(n)$-equivariant there is an induced $O(n)$-action on $F_n(k)$, that commutes with the action of the symmetric group. The operad structure on the collection F_n was defined by Getzler and Jones [4]. We consider the reduced version of this operad, in the sense that $F_n(0) = \emptyset$. We recall that $F_n(k)$ is a smooth manifold with faces, i.e., a manifold with corners such that any codimension l stratum is the transverse intersections of l strata of codimension 1. The strata correspond to ways of clustering points together recursively, and are parametrized by trees. The strata of codimension l of $F_n(k)$ are indexed by nested trees on k leaves and l internal edges. The following property is well known and crucial.

Fact 3.1. *A stratum (respectively its closure) indexed by a nested tree T is canonically diffeomorphic to the product $\prod_v \overset{\circ}{F}_n (|v|)$ (resp. to $\prod_v F_n(|v|)$) over all vertices v of T.*

The diffeomorphisms of Fact 3.1 follows from the fact that

$$\circ_T \colon \prod_v F_n(|v|) \to F_n(k)$$

is an embedding, and its image is the closure of the stratum of $F_n(k)$ indexed by T. See Section 6 of [8].

4. The W-construction

The W-construction by Boardman and Vogt is a functor sending a reduced topological operad P (i.e., such that $P(0) = \emptyset$ and $P(1) = \{1_P\}$) to another reduced topological operad WP defined as follows. The space $WP(I)$ is a quotient of the disjoint union

$$\coprod_T (\prod_v [0,1]^{E(T)} \times P(v)),$$

where T varies among all nested I-trees (those having I as set of leaves), v varies among the vertices of T, and $E(T)$ is the set of edges of the tree T, that is in bijective correspondence with the non-root vertices of T, by associating each vertex to the unique outgoing edge. We visualize an element $((x_v)_{v\in T}, (l_e)_{e\in E(T)})_T$ as a labelling of the tree T, so that each vertex v is labelled by x_v, and each edge e has a length l_e between 0 and 1.

The equivalence relation defining the quotient $WP(I)$ is the following: if $l_e = 0$ for some edge e, then we collapse such edge to a single vertex labelled by the $\circ_i$-composition of the labels of the source and the target of e. We also add formally the unit of WP in arity one.

There is an operad structure on WP, defined as follows. For $a \in WP(I), b \in WP(J)$ and $i \in I$ the operad composition $a \circ_i b \in WP(I - \{i\} \coprod J)$ is the labelled tree resulting by grafting together the leaf i of the labelled tree a with the root of the labelled tree b, and assigning to the new internal edge the length 1. There is an operad map $p\colon WP \to P$ that forgets the lengths of the edges and composes together all labels of the vertices of a labelled tree according to the underlying abstract tree. The projection $p_I\colon WP(I) \to P(I)$ is a Σ_I-equivariant homotopy equivalence for each finite set I. If P is an operad in the category of G-spaces, then WP is also an operad in the category of G-spaces, by the diagonal G-action on the labels of the vertices, and the trivial action on the lengths of the edges. The projection $p\colon WP \to P$ is an operad morphism of G-spaces, that is levelwise a G-equivariant homotopy equivalence. In particular in our case F_n is an $O(n)$-operad, and so we have a levelwise $O(n)$-equivariant weak equivalence of operads $WF_n \to F_n$. Our main result replaces this equivalence by an isomorphism.

Theorem 4.1. *There is an $O(n)$-equivariant isomorphism of topological operads*

$$\beta : F_n \cong WF_n.$$

The geometric idea of the proof is that $WF_n(k)$ is a fattening of the manifold with corners $F_n(k)$, since $WF_n(k)$ decomposes as union

$$WF_n(k) = \bigcup_T [0,1]^{l(T)} \times F_n(T),$$

where $l(T)$ is the number of internal edges of T, and also the codimension of the corresponding closed stratum $F_n(T)$. This shows that

$$WF_n(k) \cong F_n(k) \cup ([0,1] \times \partial F_n(k)),$$

but the right hand side is Σ_k-equivariantly homeomorphic to $F_n(k)$ by the equivariant collar neighbourhood theorem for topological manifolds with boundary, that we consider next.

5. Equivariant collaring

We introduce the notion of locally linear action [5].

Definition 5.1. A Lie group G acts locally linearly on a topological manifold M with boundary ∂M if

- For each $x \in M - \partial M$ there is a representation V_x of the isotropy subgroup G_x, and a G-invariant neighbourhood of x that is G-equivariantly homeomorphic to $G \times_{G_x} V_x$;
- for each $x \in \partial M$ there is a G_x-representation V_x and a G-invariant neighbourhood of x that is G-equivariantly homeomorphic to $G \times_{G_x} V_x \times [0,1)$.

If M is a smooth manifold and G acts smoothly on M then the action is locally linear by choosing, via the exponential map, $V_x \cong T_x(M)/T_x(G \cdot x)$ if x is in the interior of M and $V_x \cong T_x(\partial M)/T_x(G \cdot x)$ if x is in the boundary of M, so that the homeomorphisms of Definition 5.1 are actually diffeomorphisms.

Lemma 5.2. *The action of $G = O(n) \times \Sigma_k$ on $F_n(k)$ is locally linear.*

Proof. If x is in the interior of $F_n(k)$, that is a smooth manifold, local linearity at x is immediate. If x is in the topological boundary of $F_n(k)$, then it belongs to a stratum T of codimension $l > 0$, and $x = \circ_T((x_v)_{v \in T})$, with x_v in the interior of $F_n(|v|)$ for each vertex v. Lambrechts and Volic construct an explicit chart in 5.9.2 of [6] for a neighbourhood of x of the form $\Phi_x \colon [0,1)^l \times U' \to F_n(k)$ where $U' = \prod_v U'_v$, and U'_v is a small open disc centered at x_v in the interior of $F_n(|v|)$. Here we are picking representatives of configurations modulo translations and positive dilations that have barycenter in the origin, and norm 1, and consider the euclidean distance between them. The charts are compatible with the G-action since $\Phi_{gx}(gu) = g\Phi_x(u)$ for $g \in G$ and $u \in [0,1)^l \times U'$. Then, setting $U := Im(\Phi_x)$, we have that $G \cdot U$ is a G-invariant open neighbourhood of x G-homeomorphic to $[0,1)^l \times G \cdot U'$. The local linearity of the product of configuration spaces, that is smooth, gives a G_x-representation V'_x such that $G \cdot U' \cong G \times_{G_x} V'_x$. We choose then $V_x = V'_x \times \mathbb{R}^{l-1}$, and a homeomorphism $[0,1)^l \cong \mathbb{R}^{l-1} \times [0,1)$ to deduce that $G \cdot U \cong G \times_{G_x} V_x \times [0,1)$. □

Lemma 5.3 (Equivariant collaring theorem (Bredon) [5])**.** *Let M be a compact topological manifold on which a compact Lie group G acts locally linearly. Then there is a G-equivariant collar of the boundary of M, i.e., a G-equivariant embedding*

$$c \colon [0,2] \times \partial M \to M$$

such that $c(2,x) = x$.

As a consequence of the previous two lemmas we have:

Proposition 5.4. *There is an $O(n) \times \Sigma_k$-equivariant collar*

$$c_k \colon [0,2] \times \partial F_n(k) \to F_n(k)$$

of the boundary of $F_n(k)$.

6. Proof of the main theorem

Let us write $c_k = c$. From now on we suppress the index n from the notation and write $F = F_n$.

We build inductively on the arity k the homeomorphism $\beta_k : F(k) \cong WF(k)$.

In arity $k = 2$, $\beta_2 : F(2) = WF(2)$ is the canonical identification. We recall that $F(2)$ is Σ_2-equivariantly homeomorphic to S^{n-1} with the antipodal action.

At the next stage $F(3)$ is a manifold with boundary equipped with a free action of Σ_3. The boundary $\partial F(3)$ is the union of three copies of $F(2) \times F(2)$, corresponding to the 3 nested trees on 3 leaves with an internal edge. The space $WF(3)$ is obtained as $WF(3) = F(3) \cup_{\{0\}\times\partial F(3)} ([0,1] \times \partial F(3))$.

Definition 6.1. The map $\beta_3 \colon F(3) \to WF(3)$ is given by

- $\beta_3(y) = y \in F(3) \subset WF(3)$ if $y \notin Im(c)$.
- $\beta_3(c(t,x)) = c(2t,x) \in F(3) \subset WF(3)$ for $t \in [0,1]$ and $x \in \partial F(3)$.
- $\beta_3(c(t,x)) = (t-1, x_1, x_2)_T \in [0,1] \times \partial F(3) \subset WF(3)$ for $t \in [1,2]$ and $x = x_1 \circ_T x_2 \in \partial F(3)$.

In the last expression $(t-1, x_1, x_2)_T$ indicates the labelled tree T with internal edge of length $t-1$, running from a valence 2 vertex labelled x_2 to a valence 2 vertex labelled x_1. It is easy to see that β_3 is a $O(n) \times \Sigma_3$-equivariant homeomorphism. It also respects the operad composition, since for $x_1, x_2 \in F(2)$, the composition in WF of $\beta_2(x_1) = x_1$ and $\beta_2(x_2) = x_2$ along a nested tree T on 3 leaves with two vertices and an internal edge is the labelled tree with vertices labelled x_1, x_2 and the internal edge of length 1. But this is

$$\beta_3(x_1 \circ_T x_2) = \beta(c(2, x_1 \circ_T x_2)) = (1, x_1, x_2)_T.$$

We construct inductively β_k for $k > 3$. We first extend the collar embedding c_{k-1} to an embedding

$$c'_{k-1} \colon [0,3] \times \partial F(k-1) \to WF(k-1)$$

defined by

$$c'_{k-1}(t,x) := \begin{cases} \beta_{k-1}(c_{k-1}(t-1,x)) & \text{for } 2 \leqslant t \leqslant 3, \\ c_{k-1}(t,x) & \text{for } 0 \leqslant t \leqslant 2. \end{cases}$$

The extension is well defined because for $t = 2$ the expressions coincide.

Now let us define $\beta_k \colon F(k) \to WF(k)$.

- If $y \notin \overset{\circ}{Im}(c_k)$ then $\beta_k(y) = y$.
- If $y = c(t,w)$ with $0 \leqslant t \leqslant 1$ then $\beta_k(c(t,w)) = c(2t,w) \in F(k) \subset WF(k)$.
- If $y = c(t,w)$ with $1 \leqslant t \leqslant 2$ and $w = x \circ_T \bar{x}$, then $\beta_k(c(t,w))$ is described by a labelled tree in $WF(k)$ that is obtained by grafting two labelled trees along T, a "lower" tree related to x and an "upper" tree related to $\bar{x}$, with the new internal edge of length $t-1$.

There are three subcases for each tree, and so $3 \cdot 3 = 9$ possible cases in total. We consider the lower tree:

(1) If $x \notin \overset{\circ}{Im}(c)$ then the lower tree is a single vertex with label x.

(2) If $x = c(s, z)$, with $0 \leqslant s \leqslant 1$ then the lower tree is a single vertex with label $c(st, z)$.

(3) If $x = c(s, z)$ with $1 \leqslant s \leqslant 2$ then the lower tree is $c'(s + t - 1, z)$.

A similar description holds for the upper tree (with the replacement $x \mapsto \bar{x}, s \mapsto \bar{s}, z \mapsto \bar{z}$). We suppressed from the notation the index of c that is the arity of x (resp. of $\bar{x}$).

The induction process continues defining c'_{k-1} and β_k for all $k > 3$.

Proposition 6.2. *the map β_k is well defined, $O(n) \times \Sigma_k$-equivariant, and continuous.*

Proof. The function β_k is defined as a piecewise continuous function on some closed sets and so we need to check that the definitions are compatible for $t = 0, t = 1, s = 0, s = 1, \bar{s} = 0, \bar{s} = 1$. Now the equality $c(0, w) = c(2 \cdot 0, w)$ settles the case $t = 0$. For $t = 1$ observe that $c(2 \cdot 1, w) = w = x \circ_T \bar{x}$ is equivalent to the labelled tree obtained by grafting x and $\bar{x}$ together, with a new internal edge of length $t - 1 = 0$. Notice that x and $\bar{x}$ do not change since in (2) $c(st, z) = c(s, z)$ and in (3) $c'(s + t - 1) = c'(s) = c(s)$. Let us consider the lower tree. For $s = 0$ the element $x = c(0, z)$ is sent to $c(0 \cdot t, z) = x$ in both (1) and (2). For $s = 1$ the element $x = c(1, z)$ is sent to $c'(1 + t - 1, z) = c(t, z)$ in both (2) and (3). A similar compatibility holds for the upper tree. We also have to check that Definition 6.1 does not depend on the operadic decomposition of w. But by iterated applications of the definition it turns out that if w is the operadic composition of elements x_i along a tree T, then for $1 \leqslant t \leqslant 2$ $\beta_k(c(t, w))$ is the labelled tree obtained by grafting with edges of length $t - 1$ the trees associated to x_i by (1), (2), (3), (with x, s, z replaced by appropriate x_i, s_i, z_i), and so the result does not depend on the order of the composition operations producing w.

The equivariance follows from the construction. □

Proposition 6.3. *The map β_k respects the operad composition.*

Proof. For two arbitrary elements $x, \bar{x}$ of the operad F we have that

$$\beta_k(x \circ_T \bar{x}) = \beta_k(c(2, x \circ_T \bar{x}))$$

is the labelled tree connecting a lower tree and an upper tree by an internal edge of length $t - 1 = 2 - 1 = 1$. The upper tree is x if $x \notin Im(c)$; it is $c(2s, z)$ if $x = c(s, z)$ and $0 \leqslant s \leqslant 1$; and it is

$$c'(s + 2 - 1, z) = c'(s + 1, z) = \beta(c(s, z)) = \beta(x)$$

if $x = c(s, z)$ and $1 \leqslant s \leqslant 2$. In all cases the upper tree is $\beta(x)$. Similarly the lower tree is $\beta(x')$, therefore

$$\beta_k(x \circ_T \bar{x}) = \beta(x) \circ_T \beta(\bar{x}),$$

where the latter composition takes place in WF. □

Proposition 6.4. *The map β_k is a homeomorphism.*

Proof. We prove this by induction. It is clear that β restricts to a homeomorphism from $(F(k) - Im(c_k)) \cup c_k([0, 1] \times \partial F(k))$ to $F(k) \subset WF(k)$ and we need only to verify that it restricts to a bijection from $c_k([1,2]\times\partial F(k))$ to $\overline{WF(k)-F(k)}$. We know that the proposition is true for $k=3$. If it is true for k then

$$c'_k\colon [0,3]\times\partial F(k)\to WF(k)$$

is an embedding. We prove simultaneously by induction that

$$c'(\{1+t\}\times\partial F(k))\subset WF(k)$$

contains exactly the labelled trees in $\overline{WF(k)-F(k)}$ with maximum edge length equal to $t-1$, for $1\leqslant t\leqslant 2$. Namely by definition $c'(1+t,w)=\beta_k(c(t,w))$, when $w=x\circ_T\bar{x}$, is given by a labelled tree with an edge of length $t-1$, and an upper tree is either a vertex, or a tree of type $c'(s+t-1,z)$ that by inductive hypothesis has a maximum edge length $\leqslant s+t-3\leqslant t-1$, and a lower tree that behaves similarly. Now given a labelled tree in $x\in WF(k)$ we can decompose it by cutting it along all edges of maximum length $t-1$, obtaining some subtrees T_i, and then write it as $x=c'(1+t,w)$, where w is the composition of appropriate indecomposable elements $x_i\in F(k_i)-\partial F(k_i)$ such that the operations (1), (2), (3) on x_i produce the trees T_i. This decomposition exists and is unique by inductive hypothesis. □

Propositions 6.2, 6.3 and 6.4 together prove Theorem 4.1.

7. The case $n=2$

In Theorem 6.12 of [8], using the complex structure, we construct a Σ_k-equivariant homeomorphism $\overline{\Xi}: F_2(k)\cong\bar{N}_k(\mathcal{C})$ (the homeomorphism is equivariant for the choice of parameters $a_1=\cdots=a_k=1/k$ in the theorem). The space $\bar{N}_k(\mathcal{C})$ is defined in terms of nested trees with k leaves and the cacti complexes $\mathcal{C}_m$, $m\geqslant 2$, introduced in Section 4 of [8]. Namely

$$\bar{N}_k=\coprod_T(\prod_v C_{|v|}\times[0,1]^{E(T)})/\sim,$$

where T runs among the nested trees on k leaves and v among the vertices of T. We might regard the parameters in $[0,1]$ as edge lengths, and the cactus elements as labels of the vertices. The equivalence relation composes cacti when the edge length is 1, and removes that edge. The boundary of $F_2(k)$ is sent by the homeomorphism $\overline{\Xi}$ to the subspace of $\bar{N}_k(\mathcal{C})$ of labelled trees with at least one edge length is 0. Clearly this definition looks like a W-construction on the cacti complexes, modulo the change of parameters $t\leftrightarrow 1-t$. However this is not precise because the cacti complexes do not form a strict operad [2]. Using this description it is straightforward to find an explicit $O(2)\times\Sigma_k$-equivariant collar $c\colon[0,2]\times\partial F_2(k)\to F_2(k)$ such that $c(2,x)=x$, just by rescaling edge lengths. It is sufficient to convert any edge length s in x into an edge length $\frac{1}{2}(s+1)+\frac{t}{4}(s-1)$ in $c(t,x)$.

8. Concluding remarks

8.1. Semi-algebraic case

It is known that $F_n(k)$ is a semi-algebraic manifold [6], and it has a semi-algebraic action of $O(n)\times\Sigma_k$. Together with Michael Ching we proved a version of Proposition 5.4 with c_k semi-algebraic. This implies that Theorem 4.1 holds in the semi-algebraic category, yielding an operad isomorphism $F_n \cong WF_n$ that is levelwise a $O(n) \times \Sigma_k$ isomorphism $F_n(k) \cong WF_n(k)$ of semi-algebraic manifolds.

8.2. Smooth structure

In this paper we consider only homeomorphisms $F_n(k) \cong WF_n(k)$. However the construction of $WF_n(k)$ as a fattening of the manifold with corners $F_n(k)$ has a smooth structure such that $WF_n(k)$ is also $O(n) \times \Sigma_k$-equivariantly diffeomorphic to $F_n(k)$. There is a preferred isotopy class of such diffeomorphisms, that we call the collar class. We conjecture that there exists a $O(n)$-equivariant operad isomorphism $F_n \cong WF_n$ realized levelwise by diffeomorphisms in the collar class.

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