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MINIMAL MODELS FOR MONOMIAL ALGEBRAS

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Abstract

We give, for any monomial algebra A, an explicit description of its minimal model, which also provides us with formulas for a canonical A_∞-structure on the Ext-algebra of the trivial A-module. We do this by exploiting the combinatorics of chains going back to works of Anick, Green, Happel and Zacharia, and the algebraic discrete Morse theory of Jöllenbeck, Welker and Sköldberg. We then show how this result can be used to obtain models for algebras with a chosen Gröbner basis, and briefly outline how to compute some classical homological invariants with it.

1. Introduction

Understanding A_∞-structures associated to differential graded associative (dga) algebras is central to understanding in turn, the homotopy category of the category Alg of dga algebras. More precisely, one can, in principle, compute in the homotopy category of Alg by considering the category of quasi-free dga algebras or, equivalently, A_∞-coalgebras, modulo the usual relation of homotopy between morphisms in Alg: the quasi-free dga algebras are cofibrant in Alg, where the weak equivalences are the quasi-isomorphisms and the fibrations are the degree-wise epimorphisms; see [**14**] and Proposition 1.5 in [**28**].

In particular, we may use A_∞-coalgebras to understand usual (non-dg) associative algebras. For any augmented algebra A over a field $\Bbbk$ one can produce, from the bar construction BA of A, the class of minimal A_∞-coalgebra structures on $\operatorname{Tor}_A(\Bbbk,\Bbbk)$. Among other things, these determine A up to isomorphism, and may be used to compute its Hochschild cohomology or obtain the minimal model of A; see [**17**, **18**]. The explicit computation of such higher structures is therefore of interest. The machinery of Gröbner bases and homological perturbation theory suggest that a possible step towards solving this problem is to first obtain an answer for monomial algebras. In this paper we provide a complete description of a canonical minimal A_∞-coalgebra structure on $\operatorname{Tor}_A(\Bbbk,\Bbbk)$ for a monomial algebra A in terms of the combinatorics of its chains. Equivalently, we completely describe a minimal model of A as the ∞-cobar construction $\Omega_\infty \operatorname{Tor}_A(\Bbbk,\Bbbk)$. The results extend without modification to describe minimal models of monomial quiver algebras in terms of the combinatorics

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of their chains; see [**11**].

Concretely, let γ be a basis element of $\mathrm{Tor}_A^{r+1}(\Bbbk,\Bbbk)$, represented by an Anick chain of length $r\in\mathbb{N}$ and let us take $n\in\mathbb{N}_{\geqslant 2}$. A *decomposition* of γ is a tuple $(\gamma_1,\dots,\gamma_n)$ of chains of respective lengths $(r_1,\dots,r_n)$ satisfying $r_1+\cdots+r_n=r-1$ and whose concatenation, in this order, is γ. Our result is the following.

Theorem. *For each monomial algebra A there is a minimal model $B\to A$ where B is the ∞-cobar construction on $\mathrm{Tor}_A(\Bbbk,\Bbbk)$. The differential d is such that for a chain $\gamma\in\mathrm{Tor}_A(\Bbbk,\Bbbk)$,*

$$d\gamma=-\sum_{n\geqslant 2}(-1)^{\binom{n+1}{2}+|\gamma_1|}\gamma_1\cdots\gamma_n,$$

where the sum ranges through all possible decompositions of γ. $\square$

This recovers, in particular, the results in [**12**] describing cup products in $\mathrm{Ext}_A(\Bbbk,\Bbbk)$ for a monomial quiver algebra A using a multiplicative basis of chains, and the results in [**13**] describing the A_∞-algebra structure of Ext_A for monomial algebras which are p-Koszul.

The paper is organised as follows. In Section 2 we recall the relevant definitions and constructions from homological and homotopical algebra to be used throughout the paper. In particular, we recall the essentials from [**1**], the central results of algebraic discrete Morse theory presented in [**15**], and the dual version of the homotopy transfer theorem for A_∞-algebras from [**21**]. In Section 3 we use the results of [**15**] to produce a homotopy retract datum from the bar construction of A to $\mathrm{Tor}_A(\Bbbk,\Bbbk)$ and therefore a minimal A_∞-coalgebra structure on $\mathrm{Tor}_A(\Bbbk,\Bbbk)$, which we describe explicitly in Section 4 in terms of decompositions of Anick chains into concatenations of smaller chains, and we note that our results generalize directly to the case of algebras defined by quivers with monomial relations. Finally, in Section 5 we outline how to exploit the results of Section 4 to compute invariants of algebras and models of algebras with a chosen Gröbner basis, and briefly explain how our model has been successfully used to study the support variety theory of monomial algebras [**6**].

We fix once and for all a field $\Bbbk$. All unadorned hom and $\otimes$, which denote the usual bifunctors on graded vector spaces, will be taken with respect to $\Bbbk$. We let $\Bbbk s^{-1}$ be the graded vector space concentrated in degree -1, where it is one dimensional, and write s^{-1} for its generator. If V is a graded vector space, we write $s^{-1}V$ for $\Bbbk s^{-1}\otimes V$, and denote $s^{-1}\otimes v$ by $s^{-1}v$, and write $V^\vee$ for the graded dual of V.

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2. Recollections

We recall that a monomial algebra over $\Bbbk$ is a quotient of a free algebra TV on a finite dimensional $\Bbbk$-module V by an ideal generated by finitely many monomials. We will follow the conventions and definitions from [**20**], and we refer the reader to it for the essentials on weight graded differential graded algebras. In particular, we follow their convention regarding gradings: if a dga algebra has an additional grading by weight, refer to it as the *weight grading*, as opposed to the *homological grading*, to avoid any confusion. Hence, the grading of a graded algebra with a zero differential is homological: a weight grading is, for us, an *extra grading* coming from some additional structure so that, for example, our monomial algebras will be weight graded but concentrated in homological degree zero.

As explained in the introduction, we will completely describe, for a given monomial algebra A, a minimal model $B \to A$. Recall this is a quasi-isomorphism onto A from a quasi-free dga algebra B (that is, a dga algebra whose underlying weight graded algebra is free) whose differential satisfies the Sullivan condition [**20**, B.6.8]; although this condition is necessary to have a well-behaved model of A, it will not be central to our exposition, and the fact that our model satisfies this condition will be immediate to check.

Although this gives us, a priori, information about A in the homotopy category of Alg, there is a rich feedback loop between homotopical and homological algebra, already present in the original work of Quillen, and successfully pursued in [**14, 17**], among others. Without going into details, we will content ourselves with giving a few examples:

A model of A, that is, its homotopy type, can be computed entirely by homological and perturbative methods, starting with homological invariants of it.

From this one may compute the Hochschild homology and cohomology of A and, in particular, obtain information about the derived category of its representations, and the representations of its enveloping algebra.

In fact, the homotopy type of the dg Lie algebra of derivations of a model determines the deformation theory of A.

All results of this paper can be proved for quiver algebras defined by monomial relations; for readability, we present all arguments in the case of associative algebras (that is, for one-vertex quivers) and then merely state the corresponding generalization.

2.1. Bar resolution and Tor

Let A be a weight graded $\Bbbk$-algebra. Observe that if $\Omega_\infty C \to A$ is a minimal model of A, then the space of indecomposable elements C of $\Omega_\infty C$ can be identified with $\operatorname{Tor}_A(\Bbbk,\Bbbk)$ and is, in fact, the Quillen homology of $\Omega_\infty C$: it will become apparent in what follows that our choice of basis for $\operatorname{Tor}_A(\Bbbk,\Bbbk)$, that of Anick chains, will be central in describing our choice of minimal model of A.

Write ${}_A\mathsf{Mod}$ and Mod_A for the respective categories of left and right A-modules. The bifunctor $-\otimes_A -\colon \mathsf{Mod}_A \times {}_A\mathsf{Mod} \to {}_\Bbbk\mathsf{Mod}$ gives us, upon derivation, the classical bifunctor $\operatorname{Tor}_A(-,-)\colon \mathsf{Mod}_A \times {}_A\mathsf{Mod} \to {}_\Bbbk\mathsf{GMod}$ to the category of *graded* $\Bbbk$-modules, defined as follows. For $M \in \mathsf{Mod}_A$ and $N \in {}_A\mathsf{Mod}$, let us pick respective projective resolutions $P \to M$ and $Q \to N$ in Mod_A and ${}_A\mathsf{Mod}$. The diagram

$$P \otimes_A N \longleftarrow P \otimes_A Q \longrightarrow M \otimes_A Q$$

connects the above three complexes by natural quasi-isomorphisms, up to our choice of resolutions, and their homology is the graded $\Bbbk$-module $\operatorname{Tor}_A(M,N)$. Let us remark that $\operatorname{Tor}_A(M,N)$ is usually denoted by $\operatorname{Tor}^A(M,N)$ but that for typographical purposes we will instead write it $\operatorname{Tor}_A(M,N)$. When A is connected or, more generally, augmented, we will write Tor_A for $\operatorname{Tor}_A(\Bbbk,\Bbbk)$, where $\Bbbk$ is made into an A-module via the augmentation $A \to \Bbbk$.

There is a particularly useful way we can construct such bifunctor following the definition above. Concretely, if $R \to A$ is any projective resolution of the A-bimodule A, then the homology of the complex $M \otimes_A R \otimes_A N$ is $\operatorname{Tor}_A(M,N)$. The advantage of this is we need only choose one resolution, namely that of A as an A-bimodule, to obtain resolutions for every left or right A-module, and we now fix this choice. Define $B(A,A,A)$, the *double-sided bar resolution of* A, to be the chain complex such that for each $n \in \mathbb{N}_0$, we have $B_n(A,A,A) = A \otimes \overline{A}^{\otimes n} \otimes A$, the free A-bimodule with basis $\overline{A}^{\otimes n}$, where $\overline{A}$ is the kernel of the augmentation $A \to \Bbbk$.

For each such integer, denote a generic bimodule basis element in degree n by $[a_1|\cdots|a_n]$. Its differential is then given by

$$-a_1[a_2|\cdots|a_n] + \sum_{i=1}^{n-1}(-1)^{i-1}[a_1|\cdots|a_ia_{i+1}|\cdots|a_n] + (-1)^{n-1}[a_1|\cdots|a_{n-1}]a_n$$

and is extended A-bilinearly. In particular, if $n=0$ we have $B_0(A,A,A) = A \otimes A$ and there is an augmentation $B_0(A,A,A) \to A$ given by multiplication which renders the augmented complex $B(A,A,A) \to A$ contractible both as a complex of left and as a complex of right A-modules. From this it follows that if M is right A-module and N a left A-module, the complex $B(M,A,N) := M \otimes_A B(A,A,A) \otimes_A N$ computes $\operatorname{Tor}_A(M,N)$.

From now on we assume that A is connected, which makes it naturally augmented, and endows $\Bbbk$ with a trivial A-module structure on both sides, in which elements of positive degree act by zero. From the previous remarks it follows that the complex $B(\Bbbk,A,A)$ is a resolution of the right A-module $\Bbbk$ by free right A-modules, which we will denote by $B(A,A)$ and call the *bar resolution of* $\Bbbk$, so that $\operatorname{Tor}_A(\Bbbk,\Bbbk)$ may be computed as the homology of the complex $B(\Bbbk,A,\Bbbk)$, which we simply denote by BA and call the *bar construction of* A. Concretely, we have for each $n \in \mathbb{N}_0$ a natural isomorphism $(BA)_n \to \overline{A}^{\otimes n}$, which we consider an identification, with differential given on basis elements $[a_1|\cdots|a_n]$ by

$$d[a_1|\cdots|a_n] = \sum_{i=1}^{n-1}(-1)^{i-1}[a_1|\cdots|a_ia_{i+1}|\cdots|a_n].$$

The complex BA admits a diagonal $\Delta_2' \colon BA \to BA \otimes BA$ given by deconcatenation, that makes it into a non-unital dga coalgebra. Concretely, on basis elements $[a_1|\cdots|a_n]$ of degree $n \in \mathbb{N}_0$ we have that

$$\Delta_2'[a_1|\cdots|a_n] = \sum_{i=1}^{n-1}[a_1|\cdots|a_i] \otimes [a_{i+1}|\cdots|a_n].$$

2.2. Anick's resolution

In his celebrated article [**1**], Anick constructs an A-free resolution of the trivial module for any augmented algebra A equipped with a Gröbner basis. This construction is generalized for quiver algebras in [**2**], where the authors use the notion of chains for such algebras from [**11**]. Since we will use the description of Tor_A by means of Anick's resolution, let us quickly recall his results.

Let us write S for a set of generators of A, the variables, and let $f\colon \Bbbk\langle S\rangle \to A$ be the quotient map by the ideal of relations of A, which is a map of augmented $\Bbbk$-algebras. We weight grade A by the length of a monomial, and give S a total order. This induces on the monoid of monomials M_S on S a well ordering in such a way that $m < m'$ if $|m| < |m'|$ (here $|m|$ is the length of a monomial), or if m and m' are of the same length but $m < m'$ in the lexicographical order, also known as the dictionary order, induced from the total order of the letters in S: if $m = x_1 \cdots x_t$ and $m' = y_1 \cdots y_t$ are monomials of the same length, then $m < m'$ if $x_i < y_i$ for the first index $j \in [t]$ such that $x_j \neq y_j$. Given monomials $u, v \in M_S$, say v *is a divisor of* u if $u = u'vu''$ for monomials u', u'', to obtain a partial ordering $\subseteq$ on M_S. A subset I of M_S is an *order ideal of monomials* if it is a lower set for $\subseteq$. Note that the reverse ordering associated to $\subseteq$ is sometimes used in the literature: for us, $v \subseteq u$ means that v is a divisor of u. It is readily checked by induction that the set $N = \{x \in M_S : f(x) \notin \langle f(y) : y < x\rangle\}$ is an order ideal of monomials, and that $f(N)$ is a basis of A as a $\Bbbk$-module.

From N Anick extracts the basic building blocks for his resolution, the *obstructions*. Concretely, let V consist of those $x \in M_S$ that are not in N, but all $y \subsetneq x$ are in N. These are simply the maximal elements of the order ideal N, and thus form an anti-chain. Its elements are the obstructions. From the definitions it follows that an element is in $M_S \smallsetminus N$ precisely when it contains as a divisor an obstruction. In case A is monomial, N consists of those monomials that contain no monomial relation as a divisor, and the obstructions are the minimal relations of A. Now set $V^{-1} = \Bbbk$, $V^0 = \Bbbk S$ and $V^1 = \Bbbk V$, to begin to construct a right A-free resolution

$$\cdots \longrightarrow V^2 \otimes A \xrightarrow{\delta_2} V^1 \otimes A \xrightarrow{\delta_1} V^0 \otimes A \xrightarrow{\delta_0} V^{-1} \otimes A \longrightarrow 0$$

of $\Bbbk$. For each $n \in \mathbb{N}$ we now obtain a vector space V^n with a basis of monomials, called the n-chains, in the following way. An *n-prechain* is a monomial $x_{i_1} \cdots x_{i_t}$ in B for which there exist strictly increasing sequences of integers $(a_1, \dots, a_n)$ and $(b_1, \dots, b_n)$ with $a_1 = 1$ and $b_n = t$ such that the sequences are interlaced, meaning that $a_{i+1} \leqslant b_i$ for each $i \in \{1, \dots, n-1\}$, and such that for each $j \in \{1, \dots, n\}$, the monomial $x_{i_{a_j}} \cdots x_{i_{b_j}}$ is an obstruction.

In particular, the collection of 1-prechains, which coincides with that of 1-chains, is a basis for V^1. We say an n-prechain is an *n-chain* if the two previous sequences may be chosen so that $x_{i_1} \cdots x_{i_s}$ is not an m-prechain for any $s < b_m$ and $m \in \{1, \dots, n\}$. Plainly, a chain is a prechain that satisfies a minimality condition regarding the overlappings between the obstructions that constitute it. It is readily verified that in this case these two sequences are uniquely determined, there is a unique $s = b_{n-1} < t$ such that $x_{i_1} \cdots x_{i_s}$ is an $(n-1)$-chain and the tail $x_{i_s+1} \cdots x_{i_t}$ contains no divisor that is an obstruction. This is the key observation to construct a sequence of boundary maps $(\delta_n\colon V^n \otimes A \to V^{n-1} \otimes A)_{n \geqslant 2}$ such that

$$\delta_n(x_{i_1} \cdots x_{i_t}) = x_{i_1} \cdots x_{i_s} \otimes x_{i_s+1} \cdots x_{i_t} + \text{lower terms}.$$

If A is monomial, there are no lower terms in the differential and this resolution is minimal, so that for each $n \in \mathbb{N}$, Tor_A^{n+1} is identified with the vector space V^n with basis consisting of the n-chains: this is the content of Lemma 3.3 in [**1**]. Let us make the important remark that, in what follows, we adhere to such identification strictly: our main result depends critically on using Anick chains to model Tor_A.

Since it will be useful to illustrate some of the rather technical constructions that will follow, let us consider the algebra $J = T(x,y)/(x^2, y^2x - xy^2 - xyx)$ and the monomial algebra $K = T(x,y,z)/(xy^2, y^2z)$. In the case of the algebra J with the order $y > x$, one can check that the specified generator for its ideal of relations constitute a Gröbner basis. In the case of K this is immediate for any order, since the relations are monomial and there are no redundant relations. For J, we obtain that for each $n \in \mathbb{N}$, the set of n-chains is $\{x^{n+1}, y^2x^n\}$, corresponding to n overlappings of the relation x^2 with itself, and of an overlapping of y^2x with $n-1$ copies of the relation x^n. In the case of K, we get finitely many Anick chains: the 0-chains are $\{x, y, z\}$, the 1-chains are the relations $\{xy^2, y^2z\}$, and the 2-chains are the overlappings $\{xy^2z, xy^3z\}$, and there are no other chains.

2.3. Algebraic discrete Morse theory

Let C be a non-negatively graded complex of $\Bbbk$-modules. Fix a basis $X = \{X_t\}_{t \geqslant 0}$ of homogeneous elements of C, so that for each $t \in \mathbb{N}_0$, the set X_t is a basis of C_t. Given $c \in X$ we introduce the notation

$$dc = \sum_{c' \in X} [c : c']c',$$

where $[c : c'] \in \Bbbk$. Let $G = G(C, X)$ be the directed weighted graph with vertices the set X and with an edge $c \to c'$ if c' appears in dc with non-zero coefficient $[c : c']$ which is, in that case, the weight of $c \to c'$. A finite subset M of edges of G is a *Morse matching* if it satisfies the following *Morse conditions*:

M1. Each vertex of G is in at most one edge of M.
M2. The weights of edges of M are invertible.
M3. The graph G_M obtained by inverting the edges of M in G has no directed cycles.

If $c' \to c$ is a edge in G_M with $c \to c' \in M$, we set its weight to be $-[c : c']^{-1}$. In our situation the coefficients $[c : c']$ will be either 1 or -1, which means **M2** is always satisfied. We write X^M for the collection of vertices not appearing in M, which we call *critical*. Write $P(c, c')$ for the set of paths in G_M from c to c', and assign a path the product of the weights of the edges it contains. Finally, write $\Gamma(c, c')$ for the sum of all the weights of paths from c to c' in G_M.

We define the *Morse complex of* C *with respect to* M, which we denote by C^M, as the complex with basis the critical vertices X^M and with differential given, on basis elements, by

$$dc = \sum_{c' \in X^M_{t-1}} \Gamma(c, c')c',$$

whenever $c \in X_t$. The result of main interest to us in [**15**] is the following theorem, which shows how to produce a homotopy retract datum from C to C^M given a Morse matching M on C relative to a basis X.

Theorem 2.1. *The complex C^M is homotopy equivalent to C. More precisely, there are maps $f\colon C \to C^M$ and $g\colon C^M \to C$ given on basis elements by*

$$f(c) = \sum_{c' \in X_t^M} \Gamma(c, c')c', \quad g(c) = \sum_{c' \in X_t} \Gamma(c, c')c'$$

for $c \in X_t$, respectively $c \in X_t^M$, which are inverse homotopy equivalences. In fact, $fg = 1$ and $gf - 1 = dh + hd$ where for a basis element $c \in X_t$,

$$h(c) = \sum_{c' \in X_{t+1}} \Gamma(c, c')c'.$$

Note that since for any two basis elements we have defined the coefficient $\Gamma(c, c')$ as a sum through paths, it is important that M is finite for the theorem above to hold. We can, however, consider matchings M of the complex C if C is the colimit of a finite sequence of finite subcomplexes $\{F^pC\}$ that is compatible with the matching, in the sense that $(F^pC)^M$ is a filtration by subcomplexes of C^M. This last condition means $\Gamma(c, c')$ is well defined and the last theorem extends in this situation. In particular, we will consider the situation of $\mathbb{N}$-multigraded complexes such that each homogeneous subcomplex is finite, and in this case the filtration by weight of tuples fulfills the condition above.

Let us note that in the homotopy h we can only have a path from an element in degree t to one in degree $t+1$ if it is given by a sequence of edges $e_0'e_1e_1' \cdots e_je_j'$ where e_i' is an inverted edge of the matching and e_i is a direct edge. Indeed, the first Morse condition forbids a concatenation of inverted edges, which means we also cannot have two consecutive non-inverted edges. Finally, let us observe that if $c \in C^M$ is a cycle then $g(c) = c$, that the last observation means that $h^2 = 0$, and that $hg = 0$ and $fh = 0$. Thus (f, g, h) is a homotopy datum that satisfies the side conditions, as defined in Section 2.5.

2.4. Anick's resolution via Morse theory

Let A be a weight graded $\mathbb{k}$-algebra presented by generators $\{x_1, \ldots, x_n\}$ and ideal of relations I, and assume that $\{f_1, \ldots, f_m\}$ is a reduced Gröbner basis with respect to a fixed monomial order $<$. Following [**15**], we show how to obtain the Anick resolution of A as the Morse complex of an acyclic matching on the normalized bar resolution $B(A, A)$ of $\mathbb{k}$, which we now denote more simply by B.

Let $\mathrm{in}(I)$ denote the ideal of leading terms of elements of I, which is generated as an ideal by the leading terms of the elements in $\{f_1, \ldots, f_m\}$. A monomial is *normal* if it is not divisible by a leading term of an element in $\{f_1, \ldots, f_m\}$, and we write SM for the collection of such monomials. A monomial is *reducible* if it is not normal, and we say that $uv = 0$ *minimally* if for every prefix v' of v, the monomial uv' is normal. The set SM is a basis of A as a $\mathbb{k}$-module. In particular, given two normal monomials u and v we can write $uv = \sum_{w \in \mathsf{SM}} \lambda_w w$ where $|w| \leqslant |uv|$ for any w with $\lambda_w \neq 0$. Observe that if A is a monomial algebra, that is, if the relations of A are given by monomials, the normal monomials are those that do not contain as a subword any monomial relation, and reducible monomials are zero in A.

We now define a Morse matching on B by induction. Recall that we denote a generic basis element of the bar resolution by $[a_1|\cdots|a_n]$. Define M_1 to be the collection of edges of the form $[x_i|w_1|w_2|\cdots|w_t] \longrightarrow [x_iw_1|w_2|\cdots|w_t]$. The critical vertices

$B^{(1)}$ with respect to M_1 are the variables $[x_1], \ldots, [x_n]$ in degree 1, and those words $[x_1|w_1|\cdots|w_t]$ of normal monomials such that $x_1 w_1$ can be reduced. We proceed inductively to define M_j for $j > 1$. Having defined M_{j-1}, let $B^{(j-1)}$ be the set of critical vertices with respect to $M_1 \cup \cdots \cup M_{j-1}$, and define E_j to be the set of edges that connect vertices of $B^{(j-1)}$. Suppose that

$$[x_{i_1}|w_2|\cdots|w_{j-1}|u_1|u_2|w_{j+1}|\cdots|w_t] \to [x_{i_1}|w_2|\cdots|w_{j-1}|w_j|w_{j+1}|\cdots|w_t]$$

is an edge in E_j. In particular, $w_j = u_1 u_2$. We say e satisfies the *matching condition* if

B1. the monomial u_1 is a prefix of w_j,

B2. the source of e is in $B^{(j-1)}$ and,

B3. for each prefix v_1 of u_1 and each v_2 such that $v_1 v_2 = w_j$, the vertex

$$[x_{i_1}|w_2|\cdots|w_{j-1}|v_1|v_2|w_{j+1}|\cdots|w_t]$$

is not in $B^{(j-1)}$.

We let M_j be the collection of edges in E_j that multiply monomials at the jth bar and satisfy the matching condition. Then the set $B^{(j)}$ is given by the variables $[x_i]$ in degree 1, the elements $[x_{i_1}|w]$ such that $x_{i_1}w$ is a minimal monomial generating the ideal of leading terms of I, and the elements of the form $[x_i|w_2|w_3|\cdots|w_t]$ such that for each prefix u of w_j the vertex $[x_i|w_2|\cdots|w_{j-1}|u|w_{j+1}|\cdots|w_t]$ is not in $B^{(j-1)}$ and the term $w_j w_{j+1}$ is reducible. We set $M = \bigcup_{j \geqslant 1} M_j$ to be the desired Morse matching, and let B^M be the collection of critical vertices with respect to M.

Lemma 2.2. *Assume that A is a weight graded monomial $\Bbbk$-algebra. Let $j \in \mathbb{N}$. The elements of M_j consist of those edges $[x_i|u_1|\cdots|u_{j-1}|u_j|\cdots] \to [x_i|u_1|\cdots|u_{j-1}u_j|\cdots]$ such that $x_i u_1 = u_1 u_2 = \cdots = u_{j-2}u_{j-1} = 0$ minimally and $u_{j-1}u_j \neq 0$. Moreover, the collection M is a Morse matching.*

Proof. This is a particular case of [**15**, Lemma 4.2]. ☐

We now describe the critical vertices B^M. Let $m_1, \ldots, m_l$ be minimal monomial generators of the ideal of leading monomials of I, such that for each $j \in \{1, \ldots, l\}$ we have $m_j = u_j v_j u_{j+1}$ where u_1 is a variable. We call the term $[u_1|v_1u_2|v_2u_3|\cdots|v_l u_{l+1}]$ *fully attached* if for all $j \in \{1, \ldots, l-1\}$ and each prefix u of $v_{j+1}u_{j+2}$ the monomial $v_j u_{j+1} u$ is normal. We denote by B_j the set of fully attached terms of degree $j \geqslant 2$ and let B_1 consist of the variables. We refer the reader to [**15**] for the proof of the following lemma, valid for any weight graded $\Bbbk$-algebra with a Gröbner basis, as in the beginning of this section.

Lemma 2.3. *The fully attached tuples are exactly the critical vertices, and the complex C^M is the Anick resolution of A. In case A is monomial, the critical vertices are the variables $[x_1], \ldots, [x_n]$ along with those terms $[x_i|u_1|\cdots|u_r]$ where if we set $x_i = u_0$, we have that $u_j u_{j+1} = 0$ minimally for $j \in \{0, \ldots, r-1\}$.* ☐

Let us briefly explain how one can go back and forth from an Anick chain γ to a cycle in Tor_A. If γ is a 1-chain then it is a monomial relation $x_1 \cdots x_t$, and the corresponding bar cycle (and critical vertex) is $[x_1|x_2\cdots x_t]$. Now suppose that γ is an n-chain. Then there exist a unique $(n-1)$-chain γ' and a unique monomial t such that $\gamma = \gamma' t$, and the bar term corresponding to γ is $[x_1|u_1|\cdots|u_l|t]$ where everything before t is the bar structure corresponding to γ'. In this way, for example, if γ is a 2-chain corresponding to an overlap of a relation $x_1 \cdots x_t$ with another relation $x_j \cdots x_t x_{t+1} \cdots x_s$, then the bar structure of the corresponding critical vertex is $[x_1|x_2 \cdots x_t|x_{t+1} \cdots x_s]$. To illustrate, in the case of J, for each $n \in \mathbb{N}$, the bar element corresponding to x^{n+1} is $[x|\cdots|x]$, where there are exactly n bars, and the bar element corresponding to y^2x^n is $[y|yx|x|\cdots|x]$ where, again, we have n bars. In the case of the algebra K, the Anick chains correspond to the bar elements

$$[x],\ [y],\ [z],\ [x|y^2],\ [y|yz],\ [x|y^2|z] \text{ and } [x|y^2|yz].$$

2.5. Homotopy transfer theorem and A_∞-coalgebras

Recall that an A_∞-coalgebra is a graded $\Bbbk$-module V along with sequence of locally finite maps $(\Delta_n \colon V \to V^{\otimes n})_{n \in \mathbb{N}}$, where for each $n \in \mathbb{N}$ we have $|\Delta_n| = n - 2$, that satisfy the following *Stasheff identities*

$$\mathrm{SI}(n): \qquad \sum_{r+s+t=n} (-1)^{r+st}(1^r \otimes \Delta_s \otimes 1^t)\Delta_u = 0.$$

That such sequence of maps be locally finite means that for each element $v \in V$ the set $\{\Delta_n(v) : n \in \mathbb{N}_0\}$ contains finitely many nonzero terms. We write (V, Δ) for an A_∞-coalgebra, and we call it *minimal* whenever Δ_1 vanishes. Observe that every graded vector space, every complex, and every dga coalgebra is, in an obvious way, an A_∞-coalgebra. Remark our A_∞-coalgebras are non-unital and positively graded. We warn the reader that conventions for the signs appearing in the Stasheff identities above vary in the literature. As explained in [**21**, Section 2], one can go from this convention to that of J. Stasheff by multiplying Δ_n by the sign $(-1)^{\binom{n}{2}}$.

We can associate to every A_∞-coalgebra (V, Δ) a dga algebra $(\Omega_\infty V, b)$, its ∞-*cobar construction*, as follows. The underlying algebra to $\Omega_\infty V$ is the free associative algebra on the suspension $s^{-1}V$. Define the family of maps

$$(b_n \colon s^{-1}V \longrightarrow (s^{-1}V)^{\otimes n})_{n \in \mathbb{N}}$$

by conjugation with the isomorphisms $s \colon s^{-1}V \to V$ and $(s^{-1})^{\otimes n} \colon V^{\otimes n} \to (s^{-1}V)^{\otimes n}$. This sequence gives a map $s^{-1}V \to \Omega_\infty V$, and we then have a unique derivation $b \colon \Omega_\infty V \to \Omega_\infty V$ that restricts to $s^{-1}V \to \Omega_\infty V$ on $s^{-1}V$. A straightforward computation shows that $b^2 = 0$ is equivalent to the Stasheff identities, so we have a dga algebra. Observe that since V is positively graded, $\Omega_\infty V$ is non-unital and non-negatively graded. If V has a weight grading, as it happens for Tor_A whenever A is a weight graded algebra, $\Omega_\infty V$ inherits a weight-grading from V.

The ∞-cobar construction allows us to define the category of A_∞-coalgebras, which we denote by Cog_∞, quite painlessly: its objects are the A_∞-coalgebras and the hom-sets are given by

$$\hom_{\mathsf{Cog}_\infty}(C, C') = \hom_{\mathsf{Alg}}(\Omega_\infty C, \Omega_\infty C').$$

Plainly, Cog_∞ is the full subcategory of Alg consisting of dga algebras that are quasi-free, that is, those which are free as graded algebras if we forget about their differential. Since in the category Alg we have defined the notion of homotopy between maps and weak equivalences, the quasi-isomorphisms, these notions are available to us in Cog_∞ by creating them with the functor Ω_∞; see [**18**]. Observe, moreover, that if $F \colon V \rightsquigarrow W$ is a map between A_∞-coalgebras, it is determined uniquely by a sequence of maps $(f_n \colon V \to W^{\otimes n})_{n \in \mathbb{N}}$ satisfying appropriate commutativity conditions with the coproducts of V and W. In view of this, we will identify such a map F with the sequence $f = (f_n)_{n\in\mathbb{N}}$, and write $\Omega_\infty(f)$ for F. Abusing notation a little, for a second sequence $(g_n)_{n\in\mathbb{N}}$, we write fg for the map corresponding to the composition $\Omega_\infty(f)\Omega_\infty(g)$.

Let C be a dga coalgebra, and assume that V is a complex of $\Bbbk$-modules which is a deformation retract of C, that is, there is a homotopy retract datum

$$\begin{tikzcd} V \arrow[r, shift left, "i"] & C \arrow[l, shift left, "p"] \arrow[loop right, "h"] \end{tikzcd}, \quad 1 - ip = dh + hd, \quad pi = 1,$$

which we denote by (i, p, h). We assume that such datum satisfies the side conditions, that is, all three maps h^2, hi and ph are zero, in which case we call it a *contraction*. The following result, which is a simplified form of Theorem 5 in [**21**], shows how to transfer on V an A_∞-coalgebra structure from the dga coalgebra structure of C and, further, how to produce from the homotopy retract datum another homotopy datum of A_∞-coalgebras.

Theorem 2.4. *Let (C, Δ_2') be a dga coalgebra and consider a homotopy retract as above. There exists an A_∞-coalgebra structure on V and a homotopy retract datum*

$$\begin{tikzcd} \Omega_\infty V \arrow[r, shift left, "j"] & \Omega_\infty C \arrow[l, shift left, "q"] \arrow[loop right, "k"] \end{tikzcd}, \quad 1 - jq = bk + kb, \quad qj = 1.$$

The A_∞-coalgebra structure on V is given by $\Delta_1 = d_V$ and, for $n \in \mathbb{N}_{\geqslant 2}$, by $\Delta_n = p^{\otimes n}\Delta_n' i$, where for $n \in \mathbb{N}_{\geqslant 3}$ the arrows $\Delta_n' \colon C \to C^{\otimes n}$ are defined by

$$\Delta_n' = \sum_{\substack{s+t=n \\ s,t>0}} (-1)^{s(t+1)} (\Delta_s' h \otimes \Delta_t' h)\Delta_2',$$

with the convention that $\Delta_1' h = 1$. □

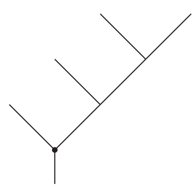


Figure 1: A right comb

There is a non-inductive definition of the maps $(\Delta_n)_{n\in\mathbb{N}}$ that will be useful to have in mind when we discuss A_∞-coalgebra structures on Tor_A, which can also be found in [**21**, Section 4]. Let T be a planar binary tree with n leaves, and let us assign to it a sign $\vartheta(T)$ as follows. For each vertex v of T, let r_1 be the number of paths from a leaf of T to the root that pass through the first (left) input of v, and let r_2 be the number of those that pass through the second (right). Set $\vartheta_T(v) = r_1(r_2 + 1)$ and $\vartheta(T) = \sum_{v\in T} \vartheta_T(v)$. It will be important later on to observe that if T is the right comb with n leaves then $\vartheta(T) = \binom{n+1}{2} - 1$. Let us write Δ_T for the cooperation of arity n obtained by decorating the leaves of T by p, the root of T by i, the inner vertices by Δ_2' and the inner edges by h; see Figure 1 in [**21**, Section 4]. We then have the following result.

Theorem 2.5. *Let $n \in \mathbb{N}$. Then Δ_n is given by the sum $\sum_T (-1)^{\vartheta(T)} \Delta_T$ as T ranges through all planar binary trees with n leaves.* □

3. The A_∞-coalgebra structure on Tor

3.1. The homotopy retract

Let A be an algebra with a reduced Gröbner as in the construction of Jöllenbeck–Welker. Using the contraction

$$\begin{tikzcd} \mathrm{Tor}_A \otimes A \arrow[r, shift left, "i"] & B(A,A) \arrow[l, shift left, "p"] \arrow[loop right, "h"] \end{tikzcd}, \quad 1 - ip = dh + hd, \quad pi = 1$$

obtained from Subsection 2.3 and from the Morse matching for $B(A,A)$ described in Subsection 2.4 we obtain, upon tensoring to the right with $\Bbbk$, a contraction

$$\begin{tikzcd} \mathrm{Tor}_A \arrow[r, shift left, "i"] & BA \arrow[l, shift left, "p"] \arrow[loop right, "h"] \end{tikzcd}, \quad 1 - ip = dh + hd, \quad pi = 1,$$

from the dga coalgebra BA to its homology, Tor_A. This and the Homotopy Transfer Theorem 2.4 provide us with a minimal A_∞-coalgebra structure on Tor_A, and which we will describe in detail by means of the combinatorics of Anick chains. It is worthwhile to note that one may obtain this retract directly, by applying the methods of [**15**] to the bar construction BA.

We recall that by construction $hi = 0$, that is, h vanishes on Tor_A. Suppose now that $\gamma = [x_{i_1}|u_1|\cdots|u_r]$ is a bar term representing an Anick chain in Tor_A. We then have that $\Delta_2'(\gamma) = \sum \gamma_{(i)} \otimes \gamma^{(i)}$ where each left term $\gamma_{(i)}$ is also a chain: $\gamma_{(i)}$ is the unique i-chain obtained from γ by removing a right divisor. Since Δ_n is obtained by projecting the map $\Delta_n' \colon BA \to BA^{\otimes n}$, defined recursively in Theorem 2.4, we obtain the following.

Proposition 3.1. *For $n \geqslant 3$ we have that $\Delta_n' = (-1)^n (1 \otimes \Delta_{n-1}' h)\Delta_2'$ on Tor_A.*

Proof. Let γ be a chain. Looking at the recursive definition of higher coproducts given by 2.4, any term which contains $\Delta_s' h$ on the left for some $s \geqslant 2$ will act by zero on $\Delta_2'(\gamma)$, since the all the terms to the left of the tensor appearing in this sum are also chains, and we already know $hi = 0$. □

3.2. Description of the homotopy

In this section, we consider only the case when A is monomial. From the last proposition of the previous section, it follows, in particular, that $\Delta_3' = -(1 \otimes \Delta_2' h)\Delta_2'$ on Tor_A, so the only tree that appears in Δ_3 is the right comb. Given $n \in \mathbb{N}_{\geqslant 3}$, we would like to show this is the case for the higher coproduct

$$\Delta_n \colon \mathrm{Tor}_A \longrightarrow \mathrm{Tor}_A^{\otimes n}$$

defined by $p^{\otimes n} \Delta_n' i$. We will need an explicit description of the homotopy h. Because it will be useful later on, we also give a description of the projection p: this is the content of the following lemma. Let us say a bar term $[u_0|\cdots|u_r]$ is *attached* if for $i \in \{1, \ldots, r\}$, we have $u_{i-1}u_i = 0$.

Suppose that $\gamma = [u_0|\cdots|u_r]$ is attached but is not a chain. Then there is a largest i_1 such that $u_i = u_i' u_i''$ and such that $\eta^1 = [u_0|\cdots|u_i']$ is a chain. Remark that by this we mean the *bar structure* is also the correct one; for example, $[t|t^2]$ and $[t^2|t]$ both have underlying monomial the chain t^3 in $\Bbbk[t]/(t^3)$ but only the first is a 1-chain. It may happen that $i=0$, in which case u_0' is simply the first variable in u_0, as it does for $[t^2|t]$. We define

$$\gamma^1 = (-1)^{i_1+1}[\eta^1|u_{i_1}''|u_{i_1+1}|\cdots|u_r], \quad \Gamma^1 = [\eta^1|u_{i_1}''u_{i_1+1}|\cdots|u_r].$$

If Γ^1 is a chain or zero, stop. Else, there is some largest $i_2 > i_1$ such that, keeping in with the notation above, $\eta^2 = [u_0|\cdots|u_{i_1}'|\cdots|u_{i_2}'']$ is a chain. In which case, set

$$\gamma^2 = (-1)^{i_2+1}[\eta^2|u_{i_2}''|u_{i_2+1}|\cdots|u_r], \quad \Gamma^2 = [\eta^2|u_{i_2}''u_{i_2+1}|\cdots|u_r].$$

Continuing in this way, we obtain terms $\gamma = \Gamma^0, \ldots, \Gamma^n$ and $\gamma^1, \ldots, \gamma^n$, where Γ^n is either zero or a chain. For convenience, we will agree that $\gamma^m = 0$ for $m > n$, and note that the sign accompanying γ^a is $(-1)^{i_a+1}$, where i_a is simply the length of the largest chain η^a contained in γ^a, starting from the left. If γ is a bar term in degree $r+1$ whose underlying monomial is an r-chain, we will write Γ for the r-chain obtained from γ at the end of the algorithm above, which we observe has no signs. Observe that by construction, the sequence $(i_a)_{a\geqslant 1}$ is strictly increasing, until it stabilizes.

Lemma 3.2. *With the notation above, we have that*

$$h(\gamma) = \sum_{i=1}^{n} \gamma^i, \quad p(\gamma) = \Gamma.$$

Proof. Suppose that $\gamma = [u_0|\cdots|u_r]$ is attached. From the description of the Morse graph in Lemma 2.2, we see that there is a unique inverted edge from γ to the element γ^1 in the previous paragraph. The face maps of γ^1 are all zero or γ except possibly for Γ^1, up to sign. If Γ^1 is critical, there is no inverted edge leaving Γ^1, and so h is what we claim. Else, we can repeat the argument above. The claim for p follows, since there is a unique path from γ to Γ in the Morse graph, which is obtained by following the terms $\gamma^1, \Gamma^1, \gamma^2, \Gamma^2, \ldots$ that we described above. Finally, the signs can be read off from the definition of the Morse graph and the differential of the bar construction. Indeed, the signs in the bar construction are such that when a bar is removed and two terms are multiplied, there is a sign $(-1)^N$ where N is the number of terms preceding the first factor that was multiplied, and the Morse graph inverts and negates every sign, effectively changing every -1 to a 1, and vice-versa. A close look shows that these are precisely the signs we have incorporated in our description of the elements $\gamma^1, \gamma^2, \ldots$ and $\Gamma^1, \Gamma^2, \ldots$, which completes the proof the Lemma. ☐

It is useful to observe that uniqueness follows precisely because our algebra is monomial, and hence one can either "undo" a differential in the Morse graph in a unique way, or fail to do so. If we had more complex rewriting rules, it would be a priori possible to obtain a term in multiple ways from reduced monomials, causing our paths to branch and making the description of the action of h and p above much more complicated. In the language of the Morse graph of M, we have the following corollary.

Corollary 3.3. *Let c be a vertex in G^M of degree t that is not critical. There is a unique element c' of degree $t+1$ and a unique element c'' of degree t, which is either zero or critical, a unique path in G^M from c to c' and, if c'' is nonzero, a unique edge* *from c' to c''. Thus, the coefficients in the homotopy of Theorem 2.1 are all* 1 *or* -1 *and $p(c)$ coincides with c''.*

Proof. The proof of Lemma 3.2 shows there is a unique path to follow when computing the action of the homotopy h on a non-critical vertex, and is given by the successive terms $\gamma^1, \gamma^2, \dots$ and $\Gamma^1, \Gamma^2, \dots$ that we wrote down explicitly before its proof. The conclusion that the coefficients in the homotopy are all -1 or 1 follow also from the careful description of the terms above, since we gave the signs explicitly. The unique element c'' corresponds to Γ, while the unique element c' corresponds to the last non-zero term in the sequence $(\gamma^1, \gamma^2, \dots)$. $\square$

To illustrate, let us consider a third algebra $L = T(t)/(t^N)$ where $N > 2$. We then have

$$h[\overbrace{t^{N-1}|t}^{m}] = -\sum_{i=0}^{m} [\overbrace{t|t^{N-1}}^{i}|t|t^{N-2}|\overbrace{t|t^{N-1}}^{m-i-1}|t],$$

where the brackets mean the terms are repeated the indicated amount of times. Note that, since in every summand the homotopy extracted a chain of odd homological degree, all the signs are the same. Using the results of Section 4 the reader may recover the A_∞-coalgebra structure on Tor_A for p-Koszul monomial algebras, dual to the A_∞-algebra structure on Ext_A obtained in [**13**].

3.3. The exchange rule and the right comb

We now prove the desired result that when computing the higher coproducts in Tor_A obtained from the homotopy retraction datum of Section 2.4, the only contributing tree is the right comb. The following exchange rule for h and Δ_2' will easily imply this result.

Lemma 3.4. *If γ is attached then $\Delta_2'(h(\gamma)) = (h \otimes 1)\Delta_2'(\gamma)$ modulo* $\operatorname{Tor}_A \otimes BA$.

Proof. This is a direct computation, albeit a bit cumbersome. We will use the notation of Section 3.2. Let $\gamma = [u_0|\cdots|u_r]$ and write $\Delta_2'(\gamma) = \sum_{i=1}^r \gamma_{(i)} \otimes \gamma^{(i)}$. From the definitions it follows that if $j \in \{1, \dots, r\}$ then:

1. $(\gamma_{(j)})^a = 0$ if $j < i_a$.
2. $(\gamma^a)_{(j)}$ is a chain for $j \leqslant i_a + 1$.
3. $(\gamma^a)_{(j)} = (\gamma_{(j-1)})^a$ for $j \geqslant i_a + 1$.
4. $(\gamma^a)^{(j)} = \gamma^{(j-1)}$ for $j > i_a + 1$.

That (1) holds follows, for if the chain we want to extract from γ appears after the jth bar, then $\gamma_{(j)}$ will be too short to contain it. It is clear that (2) holds, since γ^a has been "straightened" to chain structure up to the i_ath bar, and we already observed any initial bar term of a chain is again a chain. Note that (3) says that we can either straighten γ to a chain up to step a and then truncate far from where this chain ends, or we can truncate γ at worst at the boundary and then straighten it to a chain: the result is the same. Finally, (4) says that the tail of γ is not affected if cut beyond the straightening bar. This means that we can write

$$\begin{aligned}
\Delta_2'(h(\gamma)) &= \sum_{a\geqslant 1}\sum_{j\leqslant r+1} (\gamma^a)_{(j)} \otimes (\gamma^a)^{(j)} \\
&= \sum_{a\geqslant 1}\sum_{j\leqslant i_a+1} (\gamma^a)_{(j)} \otimes (\gamma^a)^{(j)} + \sum_{a\geqslant 1}\sum_{i_a<j-1\leqslant r} (\gamma^a)_{(j)} \otimes (\gamma^a)^{(j)} \\
&= \sum_{a\geqslant 1}\sum_{j\leqslant i_a+1} (\gamma^a)_{(j)} \otimes (\gamma^a)^{(j)} + \sum_{a\geqslant 1}\sum_{i_a<j-1\leqslant r} (\gamma_{(j-1)})^a \otimes \gamma^{(j-1)} \\
&= \sum_{a\geqslant 1}\sum_{j\leqslant i_a+1} (\gamma^a)_{(j)} \otimes (\gamma^a)^{(j)} + \sum_{a\geqslant 1}\sum_{i_a\leqslant j\leqslant r} (\gamma_{(j)})^a \otimes \gamma^{(j)},
\end{aligned}$$

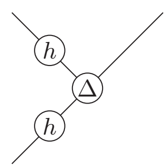
where the third equality uses (iii) and (iv), and from (ii) it follows the first summand is in $\mathrm{Tor}_A \otimes BA$. Finally, from (i) it follows that the second sum is, in fact, equal to $(h\otimes 1)(\Delta_2'(\gamma))$, which completes the proof of the lemma. □

Corollary 3.5. *We have $(h\otimes 1)\Delta_2' h = 0$ on attached bar terms.*

Proof. This now follows from our exchange rule and the fact h has square zero and vanishes on Tor_A. □

Theorem 3.6. *Let $n\in\mathbb{N}_{\geqslant 3}$ and let $\gamma\in\mathrm{Tor}_A$ be an element represented by an Anick chain. The only tree that contributes to $\Delta_n'(\gamma)$, and hence to $\Delta_n(\gamma)$, is the right comb.*

Proof. The fact that h vanishes on Tor_A means that, at the root, the left edge must be a leaf. Knowing this, the exchange rule means that if T is planar and contains any subtree of the form



which corresponds to $(h\otimes 1)(\Delta_2' h)$, the operator Δ_T will vanish identically. This means that the only tree that may possibly give a nonzero contribution to Δ_n is the right comb. □

Let us also record here the following easy proposition, which means, plainly, that the computation of the A_∞-structure of Tor_A depends only on the local information on a given chain. Thus, there seems to be no upshot from looking at induced maps when relations are added.

Proposition 3.7. *Suppose A is a monomial algebra and B is obtained by adjoining to A a non-redundant monomial relation. Let $\varphi\colon A\to B$ be the quotient map. Then the map $\mathrm{Tor}_\varphi\colon \mathrm{Tor}_A\to\mathrm{Tor}_B$ identifies Tor_A as a sub-A_∞-coalgebra of Tor_B in such a way that the coproducts of Tor_A are the restriction of those of Tor_B through Tor_φ.*

Proof. Since B is obtained from A by adjoining a non-redundant monomial relation, the collection of Anick chains for B can be computed from those of A by adding a (possibly infinite) new collection of chains, and the map Tor_φ is injective since it is induced by the inclusion map on Anick chains. To see that this a strict map of A_∞-coalgebras, meaning that it induces on Tor_A the correct higher coproducts, we note that we can arrange it so that the contraction on the bar complex of B onto Tor_A restricts to a contraction for the bar complex of A: according to the work of Jollenbeck–Welker, this datum can be produced exclusively from the Anick chains from B, and their procedure does not alter the underlying monomial of a chain in the monomial case, and hence restricts to the bar complex of A. Finally, the higher coproducts are built from the coproduct of the bar construction of B and the contraction, and for each Anick chain, this computation depends only on the underlying chain, and not on the inclusion of A into B. $\square$

4. Description of the minimal model

We now aim to give a more refined description of the terms appearing in a higher coproduct of a fixed chain γ, as stated in the following theorem. It will follow immediately from Theorem 4.6 and its proof. Unless stated otherwise, we are working exclusively with monomial algebras in what follows.

Theorem 4.1. *Let γ be a chain and $n \in \mathbb{N}_{\geqslant 2}$. The terms that appear in $\Delta_n(\gamma)$ are exactly those of the form $\gamma_1 \otimes \cdots \otimes \gamma_n$ with $(\gamma_1, \ldots, \gamma_n)$ a decomposition of γ. Moreover, if γ_i is of length r_i for each $i \in \{1, \ldots, n\}$, the coefficient of $\gamma_1 \otimes \cdots \gamma_n$ is $(-1)^N$ where*

$$N = \binom{n+1}{2} + r_1 + \sum_{i=1}^{n-1} (n-i)(r_i + 1).$$

4.1. Combinatorics of chains and tails

Suppose that $\gamma = x_{i_1} \cdots x_{i_s}$ is an Anick chain, with associated interlaced sequences $\{(a_j), (b_j)\}$. We will say a variable x_{i_s} is an overlapping variable if $s \in [a_{j+1}, b_j)$, and we will say that a bar is *inserted at* x_{i_s} if it is inserted immediately after it. A bar term obtained from γ is *regular* if it is obtained by inserting bars at non-overlapping variables, and it is *coregular* if it is obtained by inserting bars at overlapping variables. It may happen that $a_{j+1} = b_j$, in which case we agree that $x_{i_{a_{j+1}}}$ is both overlapping and non-overlapping. This always happens, for example, if A is quadratic. The following figure illustrates our definitions for the 4-chain $[t|t^3|t|t^3|t]$ in $\Bbbk\langle t|t^4\rangle$, where white circles represent overlapping variables, black ones represent non-overlapping variables, the cross represents the only variable that is both overlapping and non-overlapping, and bars mark the obstructions that constitute the chain.

Lemma 4.2. *Let γ be a monomial which is an r-chain. Any (co)regular bar term obtained by inserting*

1. *exactly r bars into γ is either attached and nonzero or is zero,*
2. *less than r bars into γ is zero, and*
3. *more than r bars into γ is not attached and nonzero or is zero.*

Proof. We prove this by induction on r. If $r = 1$, then γ is simply a monomial relation. Certainly inserting no bars gives a bar term of degree one which is zero and, since there are no overlapping variables to keep track of, inserting any bar gives a regular bar term, which is certainly nonzero, and inserting one more bar gives a non-attached term. Assume then $r \geqslant 1$ and that our claim holds for r-chains, and that we have an $(r+1)$-chain. We consider the three cases above separately:

1. *We have inserted* $r+1$ *bars regularly:* if the bar term is zero, we are done. Else the bar term obtained in nonzero, and there must be at least one bar inserted in a non-overlapping variable of the last chain. Moreover, there must be exactly one, else, by removing the tail of the $r+1$ chain, we would obtain a regular bar term from an r-chain which is nonzero but has $r-1$ bars, which cannot happen. Having settled this, we now remove the tail and proceed by induction.
2. *We have inserted less than* $r+1$ *bars regularly:* if no bar has been inserted on non-overlapping variables of the last monomial relation, we are done. Else, there is one variable inserted there. Removing the tail now gives a regular bar term obtained from an r-chain were less than r bars have been inserted, and induction does the rest.
3. *We have inserted more than* $r+1$ *bars regularly:* if two or more bars have been inserted in non-overlapping variables of the last monomial relation, we get a zero term, since removing the tail gives a bar term where at most $r-1$ bars have been inserted regularly into an r-chain. If there is exactly one bar in the tail, we may remove it and proceed inductively.

Analogous considerations apply to coregular terms. □

We now note that the homotopy h, which introduces and shifts bars in bar terms, produces bar terms whose subchains, starting from the left, have bars introduced regularly.

Lemma 4.3. *If* γ *is an element of* $\operatorname{Tor}_A^{r+1}$ *corresponding to an* r*-chain, it has its* r *bars inserted regularly. In particular, if* γ *is an attached term, and if* γ^a *is a nonzero summand in* $h(\gamma)$*, following the notation of Lemma 3.2, then for* $j \leqslant i_a$*, the* j*-chain* $(\gamma^a)_{(j+1)}$ *has its* j *bars inserted regularly.*

Proof. The insertion of bars follows Anick's interlaced sequence associated to a chain in such a way that we insert bars at variables $x_{i_1}, x_{i_{b_1}}, \ldots, x_{i_{b_{r-1}}}$ which are not overlapping, since the overlapping variables are precisely at the half-open intervals $[a_j, b_{j-1})$ for $j \in \{2, \ldots, r-1\}$. □

Let us now introduce the definitions that will be central to our proof of Theorems 4.1 and its equivalent formulation, Theorem 4.6, which we already stated the Introduction. Let γ be an r-chain and $j \in \mathbb{N}$. We will say a bar term Γ is a j*-tail of* γ if there is a term of the form $\gamma_1 \otimes \cdots \otimes \gamma_j \otimes \Gamma$ in $\Delta'_{j+1}(\gamma)$ appearing with nonzero coefficient, where the first j tensors are chains, and, moreover, Γ is a concatenation of at least two chains $\gamma_{j+1}, \ldots, \gamma_n$, in this order. Moreover, if for $i \in [n]$ we have that γ_i is an r_i chain, we require that $r_1 + \cdots + r_n = r-1$. The *length* of Γ is $n-j$. Let us call the n-tuple $(\gamma_1, \ldots, \gamma_n)$ a *decomposition* of γ. Remark that there is the notion of "tail" of a chain given in [**1**], but that this is not a special case of our definition, and that Γ may be a tail for several choices of the tuple $(\gamma_{j+1}, \ldots, \gamma_n)$.

We continue by observing that j-tails are obtained by cutting a chain in the form of a bar term either at a bar or at some place between bars.

Lemma 4.4. *Fix* $j \in \mathbb{N}$ *and suppose that* $\gamma = [u_0|u_1|\cdots|u_r]$ *is an* r*-chain, and that* Γ

is a j-tail of γ, with first chain γ_{j+1}. Then there exists $i \in \{1,\ldots,r\}$ and a decomposition $u_i = u_i' u_i''$ such that $u_i'' \neq 1$, $u_i'' u_{i+1} = 0$ minimally and $\Gamma = [u_i''|\cdots|u_r]$. Moreover:

1. *This decomposition is nontrivial whenever $j > 1$.*
2. *The tail Γ contains exactly $r_{j+1} + \cdots + r_n$ bars.*
3. *There is a unique $(j-1)$-tail Γ' and a unique term in $\Delta_2' h(\Gamma')$ of the form $\gamma_j \otimes \Gamma$ that gives rise to Γ, and it appears with a sign as a coefficient.*

Proof. The case that $j = 1$ and n is arbitrary is obvious, so let us assume $j > 1$, our claim true for $(j-1)$-tails, and analyse the claim for j.

Observe that by Theorem 3.6, if Γ is a j-tail of γ, it must come from a $(j-1)$-tail Γ' of γ by applying the operator $\Delta_2' h$ on the last factor. We will prove that Γ has the desired form and, moreover, that there is a unique way to obtain Γ from Γ', so that if the term corresponding to Γ' appears with coefficient 1 or -1, then so does the term corresponding to Γ. The description of h from Lemma 3.2 and the inductive hypothesis applied to Γ' means that $\Gamma' = [u_i''|u_{i+1}|\cdots|u_r]$ with $u_i'' u_{i+1} = 0$ minimally, or Γ' has no bars. In the latter case the chain γ_{j+2} is a variable and then Γ is obtained by removing this variable: we have that $h(\Gamma') = [\gamma_{j+1} \mid \cdots]$ and we obtain Γ uniquely from Γ'.

Let us then consider the case Γ' has bars, so that it contains $r - 1 - (r_1 + \cdots + r_j)$ bars by induction, and its first terms overlap minimally. We can certainly find some $k > i$ and a decomposition $u_k = u_k' u_k''$ in such a way that the underlying monomial of the bar term $[u_i''|u_{i+1}|\cdots|u_{k-1}|u_k']$ is precisely γ_j. Observe that the bar structure of γ_j is coregular, for it interlaces with that of γ. By Lemma 4.2 there are exactly r_j bars in such term, so that $k = i + r_j$. We now analyse two cases.

Case 1: u_i'' is a variable. In such case, it follows that $[u_i''|u_{i+1}|\cdots|u_{i+r_j}']$ is a honest chain belonging to Tor_A. We claim that the decomposition $u_k = u_k' u_k''$ is non-trivial, that $u_k'' u_{k+1} = 0$ minimally and that $h(\Gamma')$ is, up to signs, equal to the bar term

$$[u_i''|u_{i+1}|\cdots|u_{i+r_j}'|u_k''|u_{k+1}|\cdots|u_r]$$

which means, of course, that the description of Γ is the correct one. Indeed, note that if the decomposition were trivial, we would have a sequence of chains $\gamma_{j+1}\cdots\gamma_n$ underlying a $(j+1)$-tail with less than $r_{j+1} + \cdots + r_n$ bars. As before, the starting chain γ_{j+1} appears with bars inserted coregularly, so we may remove it along with exactly r_{j+1} bars. Repeating this argument, we end up with a coregular bar term underlying an r_n-chain with less than r_n bars, which contradicts Lemma 4.2. To see that $u_k'' u_{k+1} = 0$, note that otherwise we again would have a bar term $[u_k'' u_{k+1}|\cdots|u_r]$ whose underlying monomial has $r_{j+1} + \cdots + r_n - 1$ bars. The fact that the overlap $u_k'' u_{k+1}$ is minimal follows from the fact that overlap $u_k u_{k+1}$ is minimal. The description of $h(\Gamma')$ shows that Γ is obtained uniquely from Γ', possibly with a sign.

Case 2: u_i'' is not a variable. Arguing as before, we see that the overlap $u_k = u_k' u_k''$ is not trivial, and that $u_k'' u_{k+1} = 0$ minimally. We can write $u_i'' = xv$ were x is a variable and v a monomial, and we have that $h(\Gamma')$ has first term $[x|v|u_{i+1}|\cdots|u_r]$. Set j_* to be the last $k \geqslant j$ for which γ_k is a 0-chain. Our bar counting argument then shows that the concatenation $\gamma_j \cdots \gamma_{j_*}$ must be contained in the monomial u_i'', and then using our description of h it is clear we may extract the term $\gamma_j \otimes \cdots \otimes \gamma_{j_*}$ uniquely by iteration of $\Delta_2 h$. Let us assume then that γ_j is not a 0-chain. In such case,

$vu_{i+1} \neq 0$, since γ_j begins with a minimal monomial relation which, by minimality, must, in fact, be xvu_{i+1}. Since $u_i''u_{i+1}$ is a minimal monomial relation of A, it follows that $[x|vu_{i+1}|\cdots|u_k']$ begins with the initial 1-chain from γ_j, so that if γ_j is a 1-chain, we are done: this term is of the form $[x|vu_k']$. Else, we can find the initial 2-chain of γ in the form $[x|vu_{i+1}|u_{i+2}']$: since $u_{i+1}u_{i+2}$ is a minimal monomial relation of A, the second monomial relation of γ_j must be contained in a monomial of the form $vu_{i+1}u_{i+1}'$ where u_{i+1}' is a proper initial divisor of u_{i+1}. Continuing this way, we see the Anick structure of γ_j is interlaced inside that of γ, and that the last term in $h(\Gamma')$ is $[x|vu_{i+1}|\cdots|u_k'|u_k''|u_{k+1}|\cdots|u_r]$, proving the description of Γ is the correct one.

We also observe that the summands of $h(\Gamma')$ different from this one cannot create a summand corresponding to Γ so that again Γ is obtained uniquely from Γ'. Indeed, the only way to produce a bar term in the left factor with the same underlying monomial as γ_j, we would have to use Δ_2' to break such a term of $h(\Gamma')$ precisely at the bar dividing u_k' and u_k'', presently only on the last term. If we do it at a bar before or after this one, the resulting term has either its left factor or its right factor non-attached, since it contains $[\cdots|u_k'|u_k''|\cdots]$. This same argument shows that the previous summands of $\Delta_2'h(\Gamma')$ cannot contribute to Δ_{j+1}: the only place where we may break them is at the last opened bar, say $[\cdots|u_t'|u_t''|\cdots]$, but the fact we can continue the algorithm of Section 3.2 means that u_t'' has nonzero product with u_{t+1}, and hence this term does not contribute to Δ_{j+1}.

The final claim regarding the number of bars in Γ is immediate from the above. □

The following proposition is the central result about tails and chains we were after.

Proposition 4.5. *Let γ be a chain, $n \in \mathbb{N}_{\geqslant 2}$ and let $(\gamma_1, \ldots, \gamma_n)$ be a decomposition of γ. For each $j \in [n-1]$ there is a unique j-tail Γ of γ with underlying monomial $\gamma_{j+1}\cdots\gamma_n$ and a unique term $\gamma_1 \otimes \cdots \otimes \gamma_j \otimes \Gamma$ in $\Delta_{j+1}'(\gamma)$, and it appears with coefficient 1 or -1.*

Proof. Let $(\gamma_1, \ldots, \gamma_n)$ be a decomposition of γ and let Γ be a j-tail as in the statement of the Theorem. The claim is obvious for $j = 1$. Moreover, Lemma 4.4 shows that once we know that a $(j-1)$-tail Γ' corresponding to this decomposition of appears in $\Delta_j'(\gamma)$, there is a unique summand in $\Delta_2'h(\Gamma')$, with coefficient 1 or -1, that produces the term corresponding to Γ, which is what we wanted. □

Remark that the operators $(\Delta_j')_{j\geqslant 2}$ produce other terms than the ones described in the last proposition. However, the proof of Lemma 4.4 shows these terms have zero projection to tensor powers of Tor_A, since they contain factors that are not attached.

4.2. Main theorem

We now recall the promised description of the minimal model of a monomial algebra A. It follows immediately from Proposition 4.5 and Lemma 3.2, which in particular, describes the signs appearing in the homotopy h.

Theorem 4.6. *For each monomial algebra A there is a minimal model $B \to A$ where $B = \Omega_\infty \mathrm{Tor}_A$, and for a chain $\gamma \in \mathrm{Tor}_A$ the differential d acts by*

$$d\gamma = -\sum_{n\geqslant 2} (-1)^{\binom{n+1}{2}+|\gamma_1|}\gamma_1\cdots\gamma_n,$$

where the sum ranges through all possible decompositions of γ.

Proof. We need only address the claim about signs and the differential d. We already know that whenever $\Delta_2 h$ extracts an r-chain, it produces a sign $(-1)^{r+1}$. Moreover, whenever h goes through an r-chain γ it produces a sign $(-1)^{r+1}$. Thus when creating the term $\gamma_1 \otimes \cdots \otimes \gamma_n$ by extracting γ_{n-1}, we have a sign $(-1)^L$ where $L = \sum_{i=1}^{n-1}(r_i+1)$. Inductively accounting for the signs created by $\Delta_3, \ldots, \Delta_{n-1}$, for the missing sign r_1+1 that is *not* created by Δ_2 and for the sign given by 2.5, we obtain a sign congruent to

$$\binom{n+1}{2} + r_1 + \sum_{i=1}^{n-1}(n-i)(r_i+1) \pmod 2,$$

which is the integer N in Theorem 4.1. To see the claim about the minimal model, we observe that $(s^{-1})^{\otimes n}(\gamma_1 \otimes \cdots \otimes \gamma_n) = (-1)^M s^{-1}\gamma_1 \otimes \cdots \otimes s^{-1}\gamma_n$ where M is the integer $\sum_{i=1}^{n-1}(n-i)(r_i+1)$, giving the final result. □

The canonical identification of $\operatorname{Ext}_A := \operatorname{Ext}_A(\Bbbk, \Bbbk)$ as $\operatorname{Tor}_A^\vee$ gives us a result dual to Theorem 4.1 about the A_∞-algebra structure on Ext_A. Remark that it is quite crucial to have done all the work with A_∞-coalgebras and then dualizing to A_∞-algebras, and not otherwise, since not every A_∞-algebra is dualizable; see [**3**, 2.2]. It is important, however, to pay attention to the Koszul signs arising from the natural maps $D^n\colon \operatorname{Ext}_A^{\otimes n} \to (\operatorname{Tor}_A^{\otimes n})^\vee$ for $n \in \mathbb{N}$: if $f_1 \otimes \cdots \otimes f_n$ is an element in the domain, and if we pick $c_1 \otimes \cdots \otimes c_n \in \operatorname{Tor}_A^{\otimes n}$, then

$$D^n(f_1 \otimes \cdots \otimes f_n)(c_1 \otimes \cdots \otimes c_n) = (-1)^N f_1(c_1) \otimes \cdots \otimes f_n(c_n),$$

where $N = \sum_{i=2}^{n}(|c_1| + \cdots + |c_{i-1}|)|f_i|$. Observe that if $f\colon V \to W$ is a map between complexes, then $f^\vee(\varphi) = (-1)^{|f||\varphi|}\varphi f$, which explains the introduction of signs in the higher products of the graded dual Ext_A of Tor_A. Concretely, for each $n \in \mathbb{N}_{\geqslant 2}$, define $\mu_n\colon \operatorname{Ext}_A^{\otimes n} \to \operatorname{Ext}_A$ by

$$\mu_n(\varphi_1 \otimes \cdots \otimes \varphi_n) = (-1)^{n(|\varphi_1| + \cdots + |\varphi_n|)} D^n(\varphi_1 \otimes \cdots \otimes \varphi_n)\Delta_n.$$

Let us say an A_∞-algebra structure on Ext_A is *canonical* if it is A_∞-quasi-isomorphic to the dga algebra $BA^\vee$. We have the following result.

Theorem 4.7. *There is a canonical A_∞-algebra structure on Ext_A given as follows. If $n \in \mathbb{N}_{\geqslant 2}$ and if $\gamma_1^\vee, \ldots, \gamma_n^\vee$ are chains in Ext_A of lengths $r_1, \ldots, r_n$, respectively, then $\mu_n(\gamma_1^\vee \otimes \cdots \otimes \gamma_n^\vee) = (-1)^M \gamma^\vee$ if the concatenation $\gamma = \gamma_1 \cdots \gamma_n$ is a chain of length $r = r_1 + \cdots + r_n + 1$ where M is the integer $\binom{n+1}{2} - 1 + \sum_{i<j} r_i(r_j+1) + r_1 + r$. Otherwise, this higher product is zero.* □

Let us make the model explicit for the monomial algebra K, we will address the algebra J later (but note that its associated monomial algebra has the same Anick chains as J). The dg model for K is a free algebra $T(x, y, z, \alpha, \beta, \Gamma, \Lambda)$ where the first three generators are in homological degree 0, the next two in homological degree 1 and the last two in homological degree 2. The differentials are as follows:

$$d\alpha = xy^2, \quad d\beta = y^2 x, \quad d\Gamma = x\beta - \alpha z, \quad d\Lambda = xy\beta - \alpha yz.$$

These can be read off the (unique) 3-decompositions of the relations into a concatenation of variables (0-chains), the 2-decompositions of xy^2z into $x \cdot y^2 z$ and $xy^2 \cdot z$, and the 3-decompositions of xy^3z into $x \cdot y \cdot y^2 z$ and $xy^2 \cdot y \cdot z$.

4.3. The extension to monomial quiver algebras

We now observe that the results of these notes extend without any non-trivial modification to the more general class of monomial quiver algebras.

Fix a quiver $Q = (Q_0, Q_1, s, t)$ and a set R of paths in Q of length at least two, none of which is a divisor of another. We call $A = \Bbbk Q/(R)$ a *monomial quiver algebra*. Let us write $\underline{\Bbbk}$ for the semi-simple $\Bbbk$-algebra $\Bbbk Q_0$, so that there is an augmentation $A \to \underline{\Bbbk}$. We set $\mathrm{Tor}_A = \mathrm{Tor}_A(\underline{\Bbbk}, \underline{\Bbbk})$, and write BA for the bar construction of A, where unadorned $\otimes$ are now taken over $\underline{\Bbbk}$. Thus, a generic basis element of BA in degree $n \in \mathbb{N}$ is of the form $[a_1|\cdots|a_n]$ where $t(a_i) = s(a_{i+1})$ for each $i \in \{1, \ldots, n-1\}$. Since $\underline{\Bbbk}$ is semi-simple over $\Bbbk$, we can consider this alternative bar construction instead.

The methods of Subsection 2.3 go through to produce a homotopy retract datum from BA to Tor_A, and select a basis of Tor_A of critical vertices given by chains: Tor_A^1 has basis $\{[a] : a \in Q_1\}$, and for $n \in \mathbb{N}$ a basis of Tor_A^{n+1} is given by bar terms $[u_0|\cdots|u_n]$ where $t(u_i) = s(u_{i+1})$ and $u_i u_{i+1} = 0$ minimally for each $i \in \{0, \ldots, n-1\}$. The description of the action of the homotopy on fully attached terms is unchanged, as is the exchange rule.

The notion of a decomposition of a chain carries through to this setting, as well as the technical work of Section 3. As an end result we obtain the following description of a minimal model for monomial quiver algebras: there is no change on the Morse matching, the action of the homotopy and the exchange rule, or any other detail: the fact that we can do things relative to $\underline{\Bbbk}$ makes any pathology that can arise in $\Bbbk Q$ due to non-concatenable arrows disappear, since elements of the bar construction just look like those in the usual one over $\Bbbk$, but with the extra condition of concatenability. This last condition ensures that spurious cycles, say of the form $[x|y]$, arising from zero multiplication due x and y not being concatenable, disappear, so everything works like in the case Q is a bouquet. We refer the reader to [**5**, Lemma 2.1] where it is shown the relative double sided bar construction is a projective resolution of $A = \Bbbk Q/(R)$ as an A-bimodule (note the hypothesis that A be finite dimensional is not really needed there).

Let us remark that we also have, implicitly, obtained comparison maps between the bar resolution $B(A, A)$ of $\underline{\Bbbk}$ and the Green–Happel–Zacharia resolution $\mathrm{Tor}_A \otimes_\tau A$ of $\underline{\Bbbk}$ that are part of a homotopy retract datum; see [**11**] for details. Naturally, we have a dual result for the Yoneda algebra $\mathrm{Ext}_A(\underline{\Bbbk}, \underline{\Bbbk})$ of A, which we also record.

Theorem 4.8. *For each quiver monomial algebra A there is a minimal model $B \to A$ where $B = \Omega_\infty \mathrm{Tor}_A$, and for a chain $\gamma \in \mathrm{Tor}_A$ the differential d acts by*

$$d\gamma = -\sum_{n \geqslant 2} (-1)^{\binom{n+1}{2} + |\gamma_1|} \gamma_1 \cdots \gamma_n,$$

where the sum ranges through all possible decompositions of γ. □

Theorem 4.9. *There is a canonical A_∞-algebra structure on Ext_A given as follows. If $n \in \mathbb{N}_{\geqslant 2}$ and if $\gamma_1^\vee, \ldots, \gamma_n^\vee$ are chains in Ext_A of respective lengths $(r_1, \ldots, r_n)$, then*

$$\mu_n(\gamma_1^\vee \otimes \cdots \otimes \gamma_n^\vee) = (-1)^M \gamma^\vee$$

if the concatenation $\gamma = \gamma_1 \cdots \gamma_n$ is a chain of length $r = r_1 + \cdots + r_n + 1$ where M is the integer $\binom{n+1}{2} - 1 + \sum_{i<j} r_i(r_j + 1) + r_1 + r$. Otherwise, this higher product is zero. □

Let us remark that the theorem above is a common generalisation of the results in [**12**] and in [**13**], the latter in the case of monomial algebras. In the first the authors describe a multiplicative basis of Ext_A for A a monomial quiver algebra given in terms of Anick chains, and show if γ_1 and γ_2 are chains, then $\gamma_1 \smile \gamma_2$ is zero unless the concatenation $\gamma_1\gamma_2$ is a chain, in which case $\gamma_1 \smile \gamma_2 = \gamma_1\gamma_2$. In the second, the authors describe the higher products in Ext_A for monomial algebras that are p-Koszul, and show that the chains involved in a product $\mu_p(\gamma_1 \otimes \cdots \otimes \gamma_p)$ are all of odd homological degree. A calculation shows that the only term that contributes to a sign in the integer M of Theorem 4.7 is the binomial coefficient $\binom{p+1}{2}$. Switching to the sign convention for the Stasheff identities used in [**13**] removes this sign, and then our result coincides with their result exactly: the higher product of $\gamma_1 \otimes \cdots \otimes \gamma_p \in \mathrm{Ext}_A^{\otimes n}$ is zero unless the chains $\gamma_1, \ldots, \gamma_p$ concatenate, in this order, to a chain γ of the correct homological degree, in which case $\mu_p(\gamma_1 \otimes \cdots \otimes \gamma_p) = \gamma$.

5. Some applications

5.1. Computation of invariants and operations

We now use our description of the minimal model of a monomial algebra to obtain a model of its Hochschild cochain complex; we refer the reader to [**8**, 2.1] for the definition of this cohomology theory and a panorama of its relation to deformation theory, higher structures, and homotopy theory of algebras. If $f\colon B \to B'$ is a map of dga algebras, a map $\partial\colon B \to B'$ is an *f-derivation* if $\partial\mu = \mu(f \otimes \partial + \partial \otimes f)$, and we write $\mathrm{Der}_f(B, B')$ for the space of such f-derivations. When $B = B'$ and f is the identity of B, we write $\mathrm{Der}(B)$ for such space. For convenience, we will denote $\Omega_\infty \mathrm{Tor}_A$ by B in what follows. We write $\mathrm{HH}^*(A)$ for the Hochschild cohomology of an algebra A with coefficients in itself.

Having obtained a minimal model $\alpha\colon B \to A$ for A, we can produce a cochain complex to compute the Hochschild cohomology of A as follows. There is a map $\tau\colon \mathrm{Tor}_A \to A$ of degree -1 which extends uniquely to the map of algebras α, such that $\tau[x] = x$ for each variable of $x \in A$. This is a *twisting cochain* in the sense of [**22**]: it satisfies the *Maurer–Cartan equation*

$$\partial\tau + \sum_{n\geqslant 1} (-1)^{\binom{n}{2}} \tau^{[n]} = 0,$$

where $\tau^{[n]}\colon C \to A$ is defined by the composition $\mu^{(n)}\tau^{\otimes n}\Delta_n$. Indeed, $\partial\tau$ is zero since A has trivial differential, and for an Anick chain γ, $\tau^{[n]}(\gamma)$ is zero for trivial reasons unless γ is a 1-chain of length n, in which case $\tau^{[n]}(\gamma)$ is simply the image of γ in A, a relation, and is thus zero. Note the Maurer–Cartan equation is equivalent to the fact αd vanishes, where d is the map of Theorem 4.6.

From this we obtain the *twisted hom-complex associated to* τ, which we denote by $\hom_\tau(\mathrm{Tor}_A, A)$. Its underlying graded vector space is $\hom(\mathrm{Tor}_A, A)$, the space of graded $\Bbbk$-linear maps $\mathrm{Tor}_A \to A$, and its differential is obtained as follows. Let us write $\mathcal{D}_A$ for the space of α-derivations $\mathrm{Der}_\alpha(B, A)$ and $\mathcal{T}_A$ for the twisted chain complex $\hom_\tau(\mathrm{Tor}_A, A)$. Observe that if $f\colon \mathrm{Tor}_A^0 \to A$ is an element of $\mathcal{T}_A^0$, which amounts to an element $a \in A$, we have a map $d_f\colon \mathrm{Tor}_A^1 \to A$ given by $d_f[x] = [a, x]$, which extends uniquely to a derivation in $\mathcal{D}_A$, and gives us a map $j_A\colon A \to \mathcal{D}_A$. Moreover, if $F \in \mathcal{D}_A$ is a derivation, the fact that $\alpha d = 0$ means that $d^*(F) = (-1)^{|F|-1} Fd$ is an α-derivation, and $\mathcal{D}_A$ is then a cochain complex with differential d^*. We form the cone of j_A which we denote by $A \oplus \mathcal{D}_A[-1]$ and now record the following proposition and refer the reader to [**3**, 2.3] for details.

Proposition 5.1. *There is an isomorphism $A \oplus \mathcal{D}_A[-1] \to \mathcal{T}_A$ of graded vector spaces that sends a derivation in the domain to the suspension of its restriction to* Tor_A *and identifies A with* $\hom(\mathrm{Tor}_A^0, A)$*. The differential of $\mathcal{T}_A$ is induced from this isomorphism, so that if $f\colon \mathrm{Tor}_A \to A$ is a linear map of nonzero degree, df is the suspension of the restriction of $d^*(F)$ to* Tor_A*, where F is the unique derivation in $\mathcal{D}_A$ extending f. If $f\colon \mathrm{Tor}_A^0 \to A$ is linear, then $df\colon \mathrm{Tor}_A^1 \to A$ is the map given by $x \longmapsto [f[], x]$.* □

The usual Hochschild complex is the twisted complex $\hom_\pi(BA, A)$ where $\pi\colon BA \to A$ is the projection onto A from the bar construction of A, with twisted differential $\partial_{BA}^* + [\pi, -]$. The map $A \oplus \mathrm{Der}(B, A) \to A \oplus \mathrm{Der}(\Omega BA, A)$ induced by the homotopy equivalence $B \to \Omega BA$ from Theorem 2.4, induces, in turn, a morphism $\hom_\tau(\mathrm{Tor}_A, A) \to \hom_\pi(BA, A)$. Since $\Omega_\infty(q)$ is a homotopy equivalence, this map is a quasi-isomorphism, so the cohomology of $\mathcal{T}_A$ is precisely $\mathrm{HH}^*(A)$.

The next proposition addresses the computation of cup products in $\mathrm{HH}^*(A)$ using the complex $\mathcal{T}_A$ which computes it. We note that, in fact, this complex is an A_∞-algebra, and that its multiplication induces the cup product in Hochschild cohomology. We refer the reader to [**18**, Chapter 8, 1] for details.

Proposition 5.2. *For each $n \in \mathbb{N}_{\geqslant 2}$, define a higher product $\mu_n\colon \mathcal{T}_A^{\otimes n} \to \mathcal{T}_A$ so that for linear maps $f_1, \ldots, f_n \in \mathcal{T}_A$,*

$$\mu_n(f_1 \otimes \cdots \otimes f_n)(\gamma) = (-1)^N \mu_A^{(n)}(f_1 \otimes \cdots \otimes f_n)\Delta_n(\gamma),$$

where we set $N = n(|f_1| + \cdots + |f_n| + 1)$. These maps define on $\mathcal{T}_A$ an A_∞-algebra structure, and on cohomology the map μ_2 induces the cup product of $\mathrm{HH}^*(A)$. □

It is fair to observe that the construction of our minimal model requires the construction of a homotopy retract datum from BA to Tor_A, and thus of comparison morphisms, which are usually difficult to produce. However, the construction of this retraction is streamlined by the machinery of algebraic discrete Morse theory and, in fact, one may attempt to apply the methods outlined in [**15**] to any algebra admitting a Gröbner basis to produce a model of it. Let us also remark that one need not recourse to comparison maps to produce models of algebras. In the article [**7**], for example, the authors produce models for monomial operads, in particular, for monomial algebras, without doing this. As explained in that article, one may use this model to understand not necessarily monomial algebras admitting a Gröbner basis by the method of homological perturbation theory. Remark, too, that in [**23**] the authors produce chain comparison maps between the Bardzell resolution of a monomial quiver algebra and its usual bar resolution, and succeed in using them to compute the Gerstenhaber bracket on Hochschild cohomology of some examples. It may be the case that the maps of [**23**] are a part of a homotopy retract datum provided by algebraic discrete Morse theory [**15, 24**].

5.2. Computation of Tamarkin–Tsygan calculi

We noted that the twisted complex $\mathcal{T}_A$ is naturally isomorphic to the complex $A \oplus \mathcal{D}_A[-1]$. The morphism $\alpha\colon B \to A$ induces a map $\alpha \oplus \alpha_*\colon B \oplus \operatorname{Der}(B)[-1] \to A \oplus \mathcal{D}_A[-1]$ by post-composition, which one can check is a quasi-isomorphism. The domain of this map is, naturally, a dg Lie algebra, whose cohomology is $\mathrm{HH}^*(A)$, and it is not hard to prove its Lie bracket induces the Gerstenhaber bracket on $\mathrm{HH}^*(A)$, which gives a description of the Gerstenhaber bracket of A in terms of a model, without having recourse to the bar construction of A or comparison morphisms. It seems the first intrinsic definition of the Gerstenhaber bracket was given in [**25**] by Stasheff, where it is shown, among other things, that the Lie bracket in the complex $\operatorname{Coder}(BA)$ of coderivations of the bar construction of A induces the Gerstenhaber bracket on $\mathrm{HH}^*(A)$.

It is important to note that the computation of $\mathrm{HH}^*(A)$ through this dg Lie algebra is plausible, for example, if the model has finitely many generators; see [**9**] for two examples. In the case of monomial quiver algebras, it may very well happen that, although Tor_A is locally finitely dimensional, it is not finitely dimensional. There is, however, hope that computing Hochschild cohomology, and thus the Gerstenhaber bracket, using derivations of a minimal model is feasible. Let us mention, too, that one can also compute cyclic homology and non-commutative de Rham homology of A through a model following [**10**], using non-commutative differential forms. These are treated in detail, for example, in [**16**, Chapter 1] and [**19**, Chapter 2, 6].

One can, in fact, compute the Tamarkin–Tsygan calculus [**26**] of A through a model; we have pursued this in [**27**], where we use this minimal model to compute the Tamarkin–Tsygan calculus of some monomial algebras.

5.3. An application to support variety theory for Gorenstein monomial algebras

In joint work with Dotsenko and Gelinas [**6**], we used the higher structure on Tor_A obtained here and the notion of higher centres of Briggs–Gelinas to deduce that a monomial algebra satisfies the FG conditions of Snashall–Soldberg if and only if it is Gorenstein. We also showed that in this case, if the algebra is of Gorenstein dimension d, there is a periodicity operator on Hochschild cohomology whose cup product map induces isomorphism in degrees above d, and that its Tate–Hochschild cohomology is given by its periodic Hochschild cohomology: it is simply obtained by inverting this operator in Hochschild cohomology.

5.4. The case of algebras with a Gröbner basis

Let us put ourselves in the situation where A is a finitely generated algebra with generators V and ideal of relations (R). Pick a Gröbner basis with respect to a monomial order on TV, and let us write A' for the monomial algebra associated to A and $B' = (TW, d')$ for the minimal model of Theorem 4.6. Note that since W consists of monomials of TV, this graded space is partially ordered by looking at the support of a chain, and this order extends to monomials lexicographically.

We claim that there exists a model $B = (TW, d)$ of A such that for any $w \in W$, the terms appearing in $(d - d')(w)$ are smaller than w, and such that the associated graded morphism to $B \to A$ is the model $B' \to A'$ in the main theorem of these notes. As before, let (C, d) denote the complex obtained from the Anick resolution of A that computes Tor_A. Note that Proposition 3.1 is still valid if we replace Tor_A with C, since at no point we used A is monomial to prove it. We also observe that the differential on B' preserves the support of a chain.

Naturally, to prove our claim, it suffices we do it for each higher coproduct, including the possibly non-zero differential Δ_1' on C. The work of Anick shows this differential decreases the order of a chain, and the claim is obvious for Δ_2', so we may only worry about Δ_n' for $n \in \mathbb{N}_{\geqslant 3}$. In this case, the recursive formula of Proposition 3.1 means it suffices we do this for the homotopy h. But this follows from the fact it is built from the differential of BA, which, after rewriting possible non-zero products that appear, decreases the order of the underlying monomial of any bar term, independent of them being a cycle or not. From this we obtain the desired result:

Theorem 5.3. *Let A be a finitely generated algebra with a finite Gröbner basis, and let A' be its associated monomial algebra. There exists a (possibly non-minimal) model $(B, d) \to A$ such that the associated graded morphism $(B, d') \to A'$ is the model of Theorem 4.6. More precisely, we can arrange it so that $d - d'$ decreases the order of the underlying monomial of a chain in B.*

Proof. We have given some details in the discussion preceding the statement of the theorem to obtain a proof following the strategy used to prove our main theorem. Alternative, one can use a homological perturbation argument completely analogous to [**7**, Theorem 4.1], where instead of starting with the (usually non-minimal) model of the authors, one starts with the minimal model of our main theorem with the internal grading given by the underlying monomial of an Anick chain. ☐

We remark that this theorem is not too surprising, since it is the non-linear analog of the work of Chouhy in his PhD thesis [**4**], with A. Solotar. The lack of an explicit formula for the perturbed differential makes this theorem uninteresting for computations: in concrete examples, what we usually do is produce a perturbed differential which squares to zero, since it is usually possible to come up with a candidate of model and, through a filtration argument, show it is indeed acyclic. However, we would like to state the following

Conjecture 5.4. *Let A be as before, and let w be a chain in the generators of the model $(B, d) \to A$. Then the basis elements appearing in dw are obtained as follows:*

- **C1.** *Compute all possible decompositions of the chain w.*
- **C2.** *Starting from the left, rewrite the chain w once, and obtain all possible decompositions into chains of the terms that appear after this.*
- **C3.** *Repeat this procedure until all terms that appear are in normal form.*

As an example, let us consider the algebra J with two generators x and y subject to the relations $x^2 = 0$ and $y^2x = xy^2 + xyx$, and lexicographical order with respect to $y > x$. The associated monomial algebra J' has relations $x^2 = 0$ and $y^2x = 0$, and its model has generators $x_0, y_0, x_1, y_1 \ldots$ with differential

$$dy_{n+1} = y^2x_n + \sum_{\substack{s+t=n \\ s \geqslant 1}} (-1)^s y_s x_t, \quad dx_{n+1} = \sum_{s+t=n} (-1)^s x_s x_t.$$

Here, for $n \in \mathbb{N}$, the generator y_n has underlying ambiguity y^2x^n while x_n has underlying ambiguity x^{n+1}, which our differential preserves. The differential then codifies all possible 2-decompositions of y^2x^n into $y^2x^s \cdot x^t$ for $s+t=n$, and the unique 3-decomposition $y \cdot y \cdot x^n$. Similarly, x^{n+2} only admits 2-decompositions of the form $x^{s+1} \cdot x^{t+1}$ where $s+t=n$. The model corresponding to the original algebra J incorporates lower order terms as follows:

$$dy_{n+1} = [y^2, x_n] - \sum_{s+t=n} x_s y x_t - \sum_{\substack{s+t=n \\ t \geqslant 1}} (x_s y_t - (-1)^t y_t x_s), \qquad dx_{n+1} = \sum_{s+t=n} (-1)^s x_s x_t.$$

It is routine to check this perturbed differential squares to zero, so that we have obtained a model of J. To illustrate our conjecture, let us consider the term $y_2 = y^2x^2$. This can be decomposed into the chains $y_0^2 x_1$ and $y_1 x_0$ and no others. Rewriting, we obtain two terms, xy^2x and xyx^2. The first can be decomposed into $x_0 y_1$ only, and the second into xyx_1. We can only rewrite the first monomial, and we obtain x^2y^2 and x^2yx which rewrite to zero. We can decompose these into x_1y^2 and $x_1y_0x_0$, and no other terms. Summing up, the basis elements that appear are the following:

$$y^2x_1, \quad x_1y^2, \quad y_1x_0, \quad x_0y_1, \quad x_1yx_0, \quad x_1yx_0.$$

These are precisely those appearing in the formula for dy_2 above.

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