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LINEAR MOTION PLANNING WITH CONTROLLED COLLISIONS AND PURE PLANAR BRAIDS

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Abstract

We compute the Lusternik–Schnirelmann category (LS-cat) and higher topological complexity (TC_s, $s \geqslant 2$) of the "no-k-equal" configuration space $\mathrm{Conf}^{(k)}(\mathbb{R}, n)$. With $k = 3$, this yields the LS-cat and the higher topological complexity of Khovanov's group PP_n of pure planar braids on n strands, which is an $\mathbb{R}$-analogue of Artin's classical pure braid group on n strands. Our methods can be used to describe optimal motion planners for PP_n provided n is small.

1. Introduction

For a topological space X and a positive integer n, the configuration spaces $\mathrm{Conf}(X, n) = \{(x_1, \ldots, x_n) \in X^n : x_i \neq x_j \text{ for } i \neq j\}$, of n ordered points in X, and $\mathrm{UConf}(X, n)$, the orbit space of $\mathrm{Conf}(X, n)$ by the canonical action of the n-th permutation group, are central objects of study in pure and applied mathematics. The case $X = \mathbb{C}$ is historically and theoretically important: both $\mathrm{Conf}(\mathbb{C}, n)$ and $\mathrm{UConf}(\mathbb{C}, n)$ are Eilenberg–MacLane spaces of respective types $(\mathrm{P}_n, 1)$ and $(\mathrm{B}_n, 1)$. Here B_n stands for Artin's classical braid group on n strands, and P_n denotes the corresponding subgroup of pure braids.

Having contractible path components, $\mathrm{Conf}(\mathbb{R}, n)$ and $\mathrm{UConf}(\mathbb{R}, n)$ are topologically uninteresting. A meaningful and rich $\mathbb{R}$-analogue of $\mathbb{C}$-based configuration spaces arises when the actual definition of a configuration space is relaxed.

For X and n as above, and for an integer $k \geqslant 2$, the "no-k-equal" ordered configuration space $\mathrm{Conf}^{(k)}(X, n)$ is the subspace of the product X^n consisting of the n-tuples $(x_1, \ldots, x_n)$ for which no set $\{x_{i_1}, \ldots, x_{i_k}\}$, with $i_j \neq i_\ell$ for $j \neq \ell$, is a singleton. The corresponding unordered analogue $\mathrm{UConf}^{(k)}(X, n)$ is the orbit space of $\mathrm{Conf}^{(k)}(X, n)$ by the canonical action of the n-th permutation group. As shown

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in [**16**], the homotopy properties of $\mathrm{Conf}^{(k)}(X,n)$ (respectively, $\mathrm{UConf}^{(k)}(X,n)$) interpolate between those of the usual configuration space $\mathrm{Conf}(X,n) = \mathrm{Conf}^{(2)}(X,n)$ (respectively, $\mathrm{UConf}(X,n) = \mathrm{UConf}^{(2)}(X,n)$), and those of the cartesian (respectively, symmetric) n-th power $X^n = \mathrm{Conf}^{(k)}(X,n)$ (respectively, $\mathrm{SP}^n X = \mathrm{UConf}^{(k)}(X,n)$), for $k > n$. Moreover, as discussed in [**7**], no-k-equal configuration spaces play a subtle role in the study of the limit of Goodwillie's tower of a space of no k-self-intersecting immersions.

For the particular case $k = 3$, $\mathrm{Conf}^{(3)}(\mathbb{R},n)$ gives the desired $\mathbb{R}$-analogue of the classical Artin pure braid group. Indeed, Khovanov introduces in [**17**] the group PP_n which stands for the group of *planar* pure braids on n strands, also called the pure twin group, and proves that $\mathrm{Conf}^{(3)}(\mathbb{R},n)$ is an Eilenberg–MacLane space of type $(\mathrm{PP}_n, 1)$, i.e., an aspherical space which classifies PP_n-principal bundles—just as $\mathrm{Conf}^{(2)}(\mathbb{C},n)$ is an aspherical space classifying Artin pure braid group.

No-k-equal configuration spaces on the real line were first considered in [**4**], where methods for estimating the size and depth of decision trees are applied to the analysis of the complexity of the problem of determining whether, for given n real numbers, some k of them are equal.

A central goal of this paper is the computation of Farber's topological complexity, TC, of $\mathrm{Conf}^{(k)}(\mathbb{R},n)$ for $k \geqslant 3$. In the process, we compute the Lusternik–Schnirelmann category, cat, and all the higher topological complexities, TC_s, $s \geqslant 2$, of $\mathrm{Conf}^{(k)}(\mathbb{R},n)$.

Theorem 1.1. *Let $k \geqslant 3$. The Lusternik–Schnirelmann category and the topological complexity of $\mathrm{Conf}^{(k)}(\mathbb{R},n)$ are given by $\mathrm{cat}\left(\mathrm{Conf}^{(k)}(\mathbb{R},n)\right) = \lfloor n/k \rfloor$, the integral part of n/k, and*

$$\mathrm{TC}(\mathrm{Conf}^{(k)}(\mathbb{R},n)) = \begin{cases} 0, & n < k; \\ 1, & n = k \text{ with } k \text{ odd}; \\ 2, & n = k \text{ with } k \text{ even}; \\ 2\lfloor n/k \rfloor, & n > k. \end{cases} \tag{1}$$

See Corollary 4.1 for the corresponding description of all the higher topological complexities of $\mathrm{Conf}^{(k)}(\mathbb{R},n)$. Note that $\mathrm{TC}(\mathrm{Conf}^{(k)}(\mathbb{R},n)) = 2\lfloor n/k \rfloor$ unless $n = k = 2\ell + 1$ for some $\ell > 0$.

It is worth highlighting a couple of partial similarities between Theorem 1.1 and the topological complexity of the classical configuration spaces $\mathrm{Conf}(\mathbb{R}^d, n)$, for $d \geqslant 2$, described in [**12**] as

$$\mathrm{TC}(\mathrm{Conf}(\mathbb{R}^d,n)) = \begin{cases} 2n-3, & d \text{ even}; \\ 2n-2, & d \text{ odd}. \end{cases} \tag{2}$$

Firstly, both (1) and (2) are linear functions on n, of slope 2 in the case of (2), and slope roughly $2/k$ in the non-trivial instances of (1). The slope is precisely $1/k$ if $n = k = 2\ell + 1$. Further, just as in (2), (1) is at most one from maximal possible; (2) is precisely one less than maximal possible for d even, while (1) is so only for $n = k$, an odd number.

Since $\mathrm{TC}(X)$ is a homotopy invariant of X, the topological complexity of a discrete group G can be defined as that of any of its classifying spaces, just as in the case of the Lusternik–Schnirelmann category $\mathrm{cat}(G)$. In particular, when $G = \mathrm{PP}_n$, $\mathrm{Conf}^{(3)}(\mathbb{R}, n)$ is an Eilenberg–MacLane space of type $(\mathrm{PP}_n, 1)$, and Theorem 1.1 becomes:

Corollary 1.2. *Category and topological complexity of* PP_n *are* $\mathrm{cat}(\mathrm{PP}_n) = \lfloor n/3 \rfloor$ *and*

$$\mathrm{TC}(\mathrm{PP}_n) = \begin{cases} 0, & n < 3; \\ 1, & n = 3; \\ 2\lfloor n/3 \rfloor, & n > 3. \end{cases}$$

Remark 1.3. In the short but influential paper [**9**], Eilenberg and Ganea laid the grounds for establishing the fact that, for a discrete group G, $\mathrm{cat}(G)$ agrees with the projective dimension of the trivial $\mathbb{Z}[G]$-module $\mathbb{Z}$. On the other hand, a description of $\mathrm{TC}(G)$ depending solely on the algebraic properties of G is an open problem which has captured much of the current attention of the experts in the field.

Remark 1.4. J. Mostovoy pointed out to the authors that the cartesian product of $\lfloor n/3 \rfloor$ copies of PP_3 sits inside PP_n (by cabling sets of 3-strands). In particular, PP_n is hyperbolic only for $n = 3, 4, 5$. In this respect, it is relevant to observe that, while the main result in [**13**] asserts that the topological complexity of a hyperbolic group π must be $\mathrm{cdim}(\pi \times \pi) - \delta_\pi$ with $\delta_\pi \in \{0, 1\}$, $\mathrm{PP}_3 = \mathbb{Z}$ seems to be the only known hyperbolic group π with $\delta_\pi = 1$.

Cases with $n \leqslant 5$ in Corollary 1.2 are recovered in Section 5 with short proofs of the facts that PP_1 and PP_2 are trivial groups, whereas PP_3, PP_4 and PP_5 are free groups of respective ranks 1, 7, 31. The assertion for PP_5 appears as Conjecture 3.5 in [**1**]. Although PP_n is no longer free for $n > 5$, PP_6 decomposes as a free product $\mathrm{PP}_6 \cong H_6 * F$, where F is a free group of rank 71 and $H_6 = (\mathbb{Z} \oplus \mathbb{Z})^{*20}$ (see [**18**]). Here we remark that, in any decomposition $\mathrm{PP}_n \cong H_n * F$ with F free, the cat and TC values of H_n are forced to agree with those of PP_n.

Corollary 1.5. *Assume a group isomorphism* $\mathrm{PP}_n \cong H_n * F$ *holds for* $n \geqslant 6$ *with* F *a free group. Then* $\mathrm{cat}(H_n) = \lfloor n/3 \rfloor$ *and* $\mathrm{TC}(H_n) = 2\lfloor n/3 \rfloor$.

Proof. This is an immediate consequence of Corollary 1.2 and the formulae

$$\begin{aligned} \mathrm{cat}(G_1 * G_2) &= \max\{\mathrm{cat}(G_1), \mathrm{cat}(G_2)\}, \\ \mathrm{TC}(G_1 * G_s) &= \max\{\mathrm{TC}(G_1), \mathrm{TC}(G_2), \mathrm{cat}(G_1 \times G_2)\} \end{aligned}$$

for the free product $G_1 * G_2$ of arbitrary groups G_1 and G_2, with the TC-formula proved recently in [**8**]. □

Despite their motivation from motion planning in robotics, TC ideas have not yet found actual applications to current technological developments. We hope Theorem 1.1 would be closer to filling in this gap. For instance, $\mathrm{Conf}^{(k)}(\mathbb{R}, n)$ is the state space of a system consisting of n distinguishable points moving back and forth along an acyclic path subject to the restriction that k-multiple collisions are forbidden. For practical applications it is convenient to replace points by intervals of a suitably small fixed radius, changing the no-k-multiple-collision condition by the requirement that no k intervals have a common overlap. Indeed, it is known (see [**7**]) that the configuration space based on intervals is homotopic to that based on points. In this context, if the moving objects are equipped with communication sensors, and the radius of the intervals are thought of as the communication range of each of the moving objects, then the no-k-multiple-collision condition corresponds to the restriction that at most $k-1$ of the moving vehicles can communicate at any given time.

We close this introductory section by remarking that the methods in this paper are not enough to settle the topological complexity, or even the LS-category of the higher dimensional analogues $\mathrm{Conf}^{(k)}(\mathbb{R}^d, n)$. Difficulties seem to come from the fact that the latter spaces are no longer rationally formal for $d \geqslant 2$.

2. Preliminaries

2.1. LS category and topological complexity

For a space X, the Lusternik–Schnirelmann category, $\mathrm{cat}(X)$, and the topological complexity, $\mathrm{TC}(X)$, both homotopy invariants of X, are special cases of the notion of sectional category, a.k.a. Schwarz genus, of a fibration, which is taken in the reduced sense in this paper. Recall that the sectional category of a fibration $p\colon E \to B$, $\mathrm{secat}(p)$, is defined as the smallest non-negative integer k so that there exists an open covering of the base $B = U_0 \cup U_1 \cup \cdots \cup U_k$ such that the fibration p admits a continuous section on each U_i, see [**20**][1]. As a special case, we obtain the Lusternik–Schnirelmann category of a space X, $\mathrm{cat}(X)$, defined as the sectional category of the fibration $e_1\colon P_0(X) \to X$, where $P_0(X)$ is the space of based paths on X and e_1 is the evaluation map given by $e_1(\gamma) = \gamma(1)$. On the other hand, the topological complexity of a space X, $\mathrm{TC}(X)$, is defined as the sectional category of the fibration $e_{0,1}\colon P(X) \to X \times X$, where $P(X)$ is the space of free paths on X and $e_{0,1}$ is the double evaluation map given by $e_{0,1}(\gamma) = (\gamma(0), \gamma(1))$. The open[2] sets U_i covering $X \times X$ so that $e_{0,1}$ admits a continuous section on each U_i are called *local domains*, and the corresponding local sections are called *local rules*. The system of local domains and local rules is called a *motion planner* for X. A motion planner is said to be *optimal* if it has $\mathrm{TC}(X)$ local rules. As explained by Farber in his seminal work [**10, 11**], this concept gives a homotopical framework for studying the motion planning problem in robotics. Indeed, $\mathrm{TC}(X)$ gives a measure of the complexity of motion-planning an autonomous system with state-space X and which should perform robustly within a noisy environment.

Most of the existing methods to estimate the topological complexity of a given space are cohomological in nature and are based on some form of obstruction theory. One of the most simple and successful such methods is:

Proposition 2.1. *Let X be a c-connected space X having the homotopy type of a CW complex, then* $\mathrm{cl}(X) \leqslant \mathrm{cat}(X) \leqslant \mathrm{hdim}(X)/(c+1)$ *and* $\mathrm{zcl}(X) \leqslant \mathrm{TC}(X) \leqslant 2\,\mathrm{cat}(X)$.

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[1]Schwarz' original (unreduced) definition is recovered as $\mathrm{genus}(p) = \mathrm{secat}(p) + 1$.

[2]For practical purposes, the openness condition on local domains can be replaced, without altering the resulting numerical value of $\mathrm{TC}(X)$, by the requirement that local domains are pairwise disjoint Euclidean neighborhood retracts (ENR).

The notation $\mathrm{hdim}(X)$ stands for the cellular homotopy dimension of X, i.e. the minimal dimension of CW complexes having the homotopy type of X. On the other hand, the cup-length of X, $\mathrm{cl}(X)$, and the zero-divisor cup-length of X, $\mathrm{zcl}(X)$, are defined in purely cohomological terms. The former is the largest non-negative integer ℓ such that there are coefficients systems $A_1, \ldots, A_\ell$ over X and corresponding positive-dimensional classes $c_j \in H^*(X; A_j)$ so that the product $c_1 \cdots c_\ell \in H^*(X; \bigotimes_i A_i)$ is non-zero. Likewise, $\mathrm{zcl}(X)$ is the largest non-negative integer ℓ such that there are coefficients systems $A_1, \ldots, A_\ell$ over $X \times X$ and corresponding classes $z_j \in H^*(X \times X; A_j)$, each with trivial restriction under the diagonal inclusion $\Delta\colon X \hookrightarrow X \times X$, and so that the product $z_1 \cdots z_\ell \in H^*(X \times X; \bigotimes_i A_i)$ is non-zero. Each such class z_i is called a zero-divisor for X. Throughout this work, we will only be concerned with simple coefficients in $\mathbb{Z}_2$, and will omit reference of coefficients in writing a cohomology group $H^*(X)$. In these terms, $\Delta^*\colon H^*(X \times X) = H^*(X) \otimes H^*(X) \to H^*(X)$ is given by cup-multiplication, which explains the name "zero-divisors".

All results reviewed in this subsection have corresponding analogues for Rudyak's higher topological complexity of a space X, $\mathrm{TC}_s(X)$, defined as the sectional category of the s-fold evaluation map $e_{0,s-1}\colon P(X) \to X^s$ given by

$$e_{0,s-1}(\gamma) = \left(\gamma(0), \gamma\left(\frac{1}{s-1}\right), \ldots, \gamma\left(\frac{s-2}{s-1}\right), \gamma(1)\right).$$

For instance, the conclusion in Proposition 2.1 becomes $\mathrm{zcl}_s(X) \leqslant \mathrm{TC}_s(X) \leqslant s \cdot \mathrm{cat}(X)$, where the s-th zero-divisor cup-length of X, $\mathrm{zcl}_s(X)$, is the largest non-negative integer ℓ for which there is a non-trivial product of ℓ s-th zero-divisors, i.e., of cohomology classes $z \in H^*(X^s)$ with trivial restriction under the diagonal inclusion $X \hookrightarrow X^s$. See [**3, 19**] for details.

2.2. Preorders and the cohomology ring of $\mathrm{Conf}^{(k)}(\mathbb{R}, n)$

We recall the description of the cohomology ring[3] $H^*(\mathrm{Conf}^{(k)}(\mathbb{R}, n))$—see [**2, 7**]. A binary relation R on a set S is any subset of the cartesian product $S \times S$. As usual, we write xRy as a substitute for $(x, y) \in R$. A preorder is a binary relation $\preceq$ on S which is reflexive ($x \preceq x, \forall x \in S$) and transitive ($x \preceq y \preceq z \Rightarrow x \preceq z, \forall x, y, z \in S$). For instance, the diagonal $\Delta_S = \{(x, x)\colon x \in S\}$ and the entire cartesian product $S \times S$ are preorders which are called empty and full, respectively. Whenever the preorder is understood, a situation where $x \preceq y$ and $y \preceq x$ is denoted by $x \approx y$. Thus, a partial order is a preorder where $x \approx y$ occurs only with $x = y$. We write $x \prec y$ when both $x \preceq y$ and $x \neq y$ hold.

The transitive closure of a binary relation R on S is the smallest transitive binary relation on S containing R. In particular, the transitive closure of the union of two preorders is automatically a preorder. This yields a commutative and associative binary operation $\circ\colon \mathcal{P}(S) \times \mathcal{P}(S) \to \mathcal{P}(S)$ on the set $\mathcal{P}(S)$ of preorders on S having the empty preorder as a two-sided neutral element. We refer to this operation as the *transitive-closure product* or, simply, the *product* of preorders.

Fix positive integers n and k with $3 \leqslant k \leqslant n$. Baryshnikov describes the cohomology ring $H^*(\mathrm{Conf}^{(k)}(\mathbb{R}, n))$ in terms of what he calls "string preorders" and their transitive-closure products. We next recall the explicit details.

[3] Recall we only take mod 2 coefficients.

A preorder $\preceq$ on $[n]$ is *string* if there is a preorder-preserving map $h\colon [n] \to \mathbb{R}$, where $\mathbb{R}$ is equipped with the standard order, satisfying $x \prec y$ whenever $h(x) < h(y)$, and in such a way that the restriction of $\preceq$ to each non-empty "level" set $h^{-1}(r)$, with $r \in \mathbb{R}$, is either the empty preorder or the full preorder. Note that the "height" function h would recover the string preorder if the former would remember which level sets are empty and which are full. Non-empty level sets $h^{-1}(r)$ will be denoted by letters I, J, K, possibly with subindices; together they form a partition of $[n]$. Thus, a string preorder $\preceq$ can be spelled out through the ordered list (or string) of non-empty level sets of a corresponding height function for $\preceq$, where the list is ordered increasingly[4] from left to right according to the height values, and enclosing each level subset $I \subseteq [n]$ within either [ ]-brackets, if the restriction of $\preceq$ to I is full, or ( )-brackets, if the restriction of $\preceq$ to I is empty. By convention, a level set with a single element has to be enclosed within ( )-brackets.

A string preorder is said to be:

(a) *elementary*, if it has the form $(I)[J](K)$ with $\operatorname{card}(J) = k-1$.

(b) *admissible*, if it has the form $(I_0)[J_1](I_1)[J_2]\cdots[J_d](I_d)$ with $\operatorname{card}(J_i) = k-1$ for all $i = 1,\ldots,d$. In such a case, the admissible string preorder is said to have *dimension* $(k-2)d$. Elementary string preorders are thus admissible and have dimension $k-2$.

(c) *basic*, if it is specified by a string $(I_0)[J_1](I_1)[J_2]\cdots[J_d](I_d)$ satisfying $\operatorname{card}(J_i) = k-1$ and $\max(J_i \cup I_i) \in I_i$, for all $i = 1,\ldots,d$, where the maximal element of $J_i \cup I_i$ is taken with respect to the standard order of integers.

Remark 2.2. String preorders are used in [**2**] as a way to encode combinatorial information of cell decompositions of Euclidean spaces, where cells do not necessarily have compact closures. For instance, for $n = 5$ and $k = 4$, the level function $h\colon [5] \to \mathbb{R}$ given by $h(3) = 0$, $h(1) = h(2) = h(4) = 1$ and $h(5) = 2$ yields the elementary basic string preorder

$$(3)[124](5) = \{3 \prec 1 \approx 2 \approx 4 \prec 5\}$$

which, in turn, encodes the cell $C = \{(x_1,x_2,x_3,x_4,x_5) \in \mathbb{R}^5 \colon x_3 < x_1 = x_2 = x_4 < x_5\}$ in $\mathbb{R}^5$. Note that the boundary of C lies outside $\operatorname{Conf}^{(4)}(\mathbb{R},5)$. So, by thinking of C as a locally compact chain, we get a cohomology class in $\operatorname{Conf}^{(4)}(\mathbb{R},5)$. Cup-products of such classes can then be read off as intersection products of their locally-compact homology counterparts. See [**2**] for details.

Remark 2.3. An admissible (respectively, basic) preorder $(I_0)[J_1](I_1)[J_2]\cdots[J_d](I_d)$ of dimension $(k-2)d$ factors as $\varepsilon_1 \circ \cdots \circ \varepsilon_d$, where

$$\varepsilon_i = \Big(I_0 \cup J_1 \cup I_1 \cup \cdots \cup J_{i-1} \cup I_{i-1}\Big)\Big[J_i\Big]\Big(I_i \cup J_{i+1} \cup I_{i+1} \cup \cdots \cup J_d \cup I_d\Big)$$

is an elementary (respectively, basic) preorder of dimension $k-2$. As a partial converse, note that, for string preorders $(I)[J](K)$ and $(I')[J'](K')$ which may fail to be

[4]In [**2**], level sets are ordered decreasingly from left to right; this difference is immaterial.

elementary, the condition $I \cup J \subseteq I'$ implies the equality

$$\Big(I\Big)\Big[J\Big]\Big(K\Big) \circ \Big(I'\Big)\Big[J'\Big]\Big(K'\Big) = \Big(I\Big)\Big[J\Big]\Big(K \cap I'\Big)\Big[J'\Big]\Big(K'\Big), \tag{3}$$

where the ( )-level set $K \cap I'$ must be suppressed from the right term in (3) provided $K \cap I' = \varnothing$. Likewise, the condition $I' \cup J' \subseteq I$ implies the equality

$$\Big(I\Big)\Big[J\Big]\Big(K\Big) \circ \Big(I'\Big)\Big[J'\Big]\Big(K'\Big) = \Big(I'\Big)\Big[J'\Big]\Big(K' \cap I\Big)\Big[J\Big]\Big(K\Big), \tag{4}$$

where the ( )-level set $K' \cap I$ must be suppressed from the right term of (4) provided $K' \cap I = \varnothing$. The apparent symmetry on the right of (3) and (4) corresponds to the fact that the condition $I \cup J \subseteq I'$ (respectively, $I' \cup J' \subseteq I$) is equivalent, by complementing, to the condition $J' \cup K' \subseteq K$ (respectively, $J \cup K \subseteq K'$).

The fact that the products in (3) and (4) are string does not depend on the assumed inclusions $I \cup J \subseteq I'$ and $I' \cup J' \subseteq I$. Although not explicitly mentioned, such a property is used in the original works [**2, 7**]. We include proof details for completeness.

Lemma 2.4. *The product of two string preorders $(I)[J](K)$ and $(I')[J'](K')$ is string. In particular, if neither the inclusion $I \cup J \subseteq I'$ nor the inclusion $I' \cup J' \subseteq I$ hold, then*

$$\Big(I\Big)\Big[J\Big]\Big(K\Big) \circ \Big(I'\Big)\Big[J'\Big]\Big(K'\Big) = \Big(I \cap I'\Big)\Big[J \cup J' \cup (I \cap K') \cup (I' \cap K)\Big]\Big(K \cap K'\Big). \tag{5}$$

Remark 2.5. Baryshnikov's description of cup-products in $H^*(\mathrm{Conf}^{(k)}(\mathbb{R}, n))$ rests on the resulting restricted transitive-closure product $\circ\colon \mathcal{E}[n] \times \mathcal{E}[n] \to \mathcal{E}[n]$, where $\mathcal{E}[n]$ stands for the set of elementary preorders on $[n]$. See Remark 2.10 and Theorem 2.11, both at the end of the section.

Proof of Lemma 2.4. Let $\preceq$ stand for the product preorder $(I)[J](K) \circ (I')[J'](K')$, and for subsets A and B of $[n]$ write $A \preceq B$ (respectively, $A \approx B$) whenever $a \preceq b$ (respectively, $a \approx b$) for all $(a, b) \in A \times B$. For instance, $I \preceq J \preceq K$ as well as $I' \preceq J' \preceq K'$. Since

$$[n] = (I \cap I') \bigsqcup \Big(J \cup J' \cup (I \cap K') \cup (I' \cap K)\Big) \bigsqcup (K \cap K')$$

is clearly a partition, it suffices to show that $J \approx J' \approx (I \cap K') \approx (I' \cap K)$.

Pick $x \in (I \cup J) \setminus I'$ and $x' \in (I' \cup J') \setminus I$. For any $(j, j') \in J \times J'$, we have:

- $(x' \notin I \Rightarrow x' \in J \cup K \Rightarrow j \preceq x')$ and $(x' \in I' \cup J' \Rightarrow x' \preceq j')$, thus $j \preceq j'$;
- $(x \notin I' \Rightarrow x \in J' \cup K' \Rightarrow j' \preceq x)$ and $(x \in I \cup J \Rightarrow x \preceq j)$, thus $j' \preceq j$.

Therefore $J \approx J'$. The result follows from $(I \cap K' \preceq J \approx J' \preceq K' \Rightarrow I \cap K' \approx J \approx J')$ and $(I' \cap K \preceq J' \approx J \preceq K \Rightarrow I' \cap K \approx J' \approx J)$. ☐

Corollary 2.6. *Assume $(I)[J](K)$ and $(I')[J'](K')$ are elementary preorders with $I \cup J \nsubseteq I'$ and $I' \cup J' \nsubseteq I$. Then the product in (5):*

- *has the form $(I'')[J''](K'')$ with* $\mathrm{card}(J'') \geqslant k - 1$;
- *is elementary if and only if the preorders $(I)[J](K)$ and $(I')[J'](K')$ agree.*

Proof. (5) is elementary if and only if $J = J'$ and $I \cap K' = \varnothing = I' \cap K$. In such a case:

- $I \cap K' = \varnothing \Rightarrow I \subseteq I' \cup J' = I' \cup J \Rightarrow I \subseteq I'$;
- $I' \cap K = \varnothing \Rightarrow I' \subseteq I \cup J = I \cup J' \Rightarrow I' \subseteq I$;
- $I \cap K' = \varnothing \Rightarrow K' \subseteq J \cup K = J' \cup K \Rightarrow K' \subseteq K$;
- $I' \cap K = \varnothing \Rightarrow K \subseteq J' \cup K' = J \cup K' \Rightarrow K \subseteq K'$.

So, in fact, $I = I'$ and $K = K'$. □

Remark 2.7. It follows from equations (3)–(5) and Corollary 2.6 that a product of elementary preorders $(I)[J](K)$ and $(I')[J'](K')$ is admissible and has degree the sum of the degrees of the factors if and only if the following two conditions hold:

(a) $(I)[J](K) \neq (I')[J'](K')$;

(b) $I \cup J \subseteq I'$ or $I' \cup J' \subseteq I$.

This seemingly artificial combinatorial fact has a natural algebraic explanation in terms of Baryshnikov's description of the cohomology ring $H^*(\mathrm{Conf}^{(k)}(\mathbb{R}, n))$, stated in Theorems 2.8 and 2.11 below. For instance, condition (a) reflects the vanishing of all cup-squares in $H^*(\mathrm{Conf}^{(k)}(\mathbb{R}, n))$.

As illustrated in Remark 2.2, Poincaré duality allows us to think of elementary preorders as cohomology classes in $\mathrm{Conf}^{(k)}(\mathbb{R}, n)$. Recall we only take cohomology with $\mathbb{Z}_2$-coefficients.

Theorem 2.8 (Baryshnikov [**2**, Theorem 1], Dobrinskaya–Turchin [**7**, Section 4])**.** *For $k \geqslant 3$, the cohomology ring $H^*(\mathrm{Conf}^{(k)}(\mathbb{R}, n))$ is isomorphic to the (anti)commutative free exterior algebra generated in dimension $k-2$ by the elementary preorders subject to the following relations:*

1. $\sum_{a\in A} (A\setminus\{a\}) \left[\{a\}\cup B\right] (C) = \sum_{c\in C} (A) \left[B\cup\{c\}\right] (C\setminus\{c\})$, *whenever* $[n]$ *can be written as a disjoint union* $[n] = A \bigsqcup B \bigsqcup C$ *with* $\mathrm{card}(B) = k-2$.
2. $(I)[J](K) \cdot (I')[J'](K') = 0$, *for elementary preorders* $(I)[J](K)$ *and* $(I')[J'](K')$ *whose transitive closure* $(I)[J](K) \circ (I')[J'](K')$ *has a* $[\ ]$*-level set of cardinality larger than* $k-1$.

Remark 2.9. Since $H^*(\mathrm{Conf}^{(k)}(\mathbb{R}, n))$ is a quotient of an exterior algebra, Remark 2.7 implies that a cup-product $(I)[J](K) \cdot (I')[J'](K')$ of two elementary preorders of dimension $k-2$, $(I)[J](K)$ and $(I')[J'](K')$, can potentially be non-zero only when the transitive-closure product $(I)[J](K) \circ (I')[J'](K')$ is admissible and has dimension $2(k-2)$. Further, the latter condition holds precisely when one (and necessarily only one) of the inclusions $I \cup J \subseteq I'$ and $I' \cup J' \subseteq I$ holds, in which case the transitive closure product $(I)[J](K) \circ (I')[J'](K')$ is given by (3) and (4), respectively, and is therefore admissible.

Remark 2.10. Note that the first relation in Theorem 2.8 allows us to express any elementary string preorder, thought of as an element of $H^*(\mathrm{Conf}^{(k)}(\mathbb{R}, n))$, as a sum of basic elementary string preorders. We explain next how the latter fact can actually be extended to think of admissible string preorders as elements of $H^*(\mathrm{Conf}^{(k)}(\mathbb{R}, n))$ expressible as sums of basic string preorders, with the latter ones expressible in turn as cup-products of elementary basic preorders. In detail, start by noticing that Theorem 2.8 and Remark 2.9 imply that $H^{d(k-2)}(\mathrm{Conf}^{(k)}(\mathbb{R}, n))$ is additively generated by the set $\mathcal{G}_{d(k-2)}$ of cup-products

$$(I_1)[J_1](K_1)\cdot(I_2)[J_2](K_2)\cdots(I_d)[J_d](K_d) \tag{6}$$

of elementary preorders satisfying $I_\ell \cup J_\ell \subseteq I_{\ell+1}$ for $\ell = 1,\dots,d-1$. On the other hand, as indicated in Remark 2.3, if we replace cup-products in (6) by transitive-closure products, we get the admissible preorder

$$(I_1)[J_1](K_1\cap I_2)[J_2](K_2\cap I_3)\cdots(K_{d-1}\cap I_d)[J_d](K_d)$$

of dimension $d(k-2)$. The work in [**2**] then shows that the process of replacing cup-products by closure-transitive products leads to a one-to-one correspondence between $\mathcal{G}_{d(k-2)}$ and the set of admissible string preorders of dimension $d(k-2)$. This allows us to think of an admissible string preorder of dimension $d(k-2)$ as an element of $H^{d(k-2)}(\mathrm{Conf}^{(k)}(\mathbb{R},n))$ that factorizes as a cup-product of d elementary preorders. In particular, admissible non-basic preorders can be expressed in terms of basic ones by a systematic iterated use of the first relation in Theorem 2.8. We then have:

Theorem 2.11 (Baryshnikov [**2**, Theorem 2], Dobrinskaya–Turchin [**7**, Section 4])**.** *A graded basis, which we refer to as the* Baryshnikov basis*, for the graded $\mathbb{Z}_2$-vector space $H^*(\mathrm{Conf}^{(k)}(\mathbb{R},n))$ is given by the basic string preorders.*

Note that we can safely neglect the difference between cup-products and transitive-closure products of elementary preorders, as long as the resulting string preorders are admissible.

The process of writing admissible non-basic preorders as sums of basic ones is clarified in the next section, where we work extensively in terms of Baryshnikov basis elements in $H^*(\mathrm{Conf}^{(k)}(\mathbb{R},n))$, and the corresponding tensor basis elements in $H^*(\mathrm{Conf}^{(k)}(\mathbb{R},n))\otimes H^*(\mathrm{Conf}^{(k)}(\mathbb{R},n)) \cong H^*(\mathrm{Conf}^{(k)}(\mathbb{R},n)\times \mathrm{Conf}^{(k)}(\mathbb{R},n))$. For the time being we offer two examples that illustrate the additive (Example 2.12) and multiplicative (Example 2.13) structures.

Example 2.12. For $n=4$ and $k=3$, there are only 7 elementary basic string preorders, namely, $(1)[2,3](4)$, $(2)[1,3](4)$, $(3)[1,2](4)$, $[1,2](3,4)$, $[1,3](2,4)$, $[2,3](1,4)$ and $(4)[1,2](3)$, all of which have degree 1. Here and below we have omitted curly braces for level sets. Since there are no basic string preorders in larger degrees, we see no interesting cup-products. This is of course compatible with the fact that $\mathrm{Conf}^{(3)}(\mathbb{R},4)\simeq \bigvee_7 S^1$, see Section 5. On the other hand, the elementary non-basic preorder $(4)[1,3](2)$ can be written in terms of basic ones as follows. Take $A=\{4\}$, $B=\{1\}$ and $C=\{2,3\}$ in the first relation of Theorem 2.8 to get

$$[1,4](2,3) = (4)[1,2](3) + (4)[1,3](2),$$

while the combination $A=\varnothing$, $B=\{1\}$ and $C=\{2,3,4\}$ yields

$$0 = [1,2](3,4) + [1,3](2,4) + [1,4](2,3).$$

Therefore $(4)[1,3](2) = (4)[1,2](3) + [1,2](3,4) + [1,3](2,4)$. Recall we are ignoring signs.

Example 2.13. $\mathrm{Conf}^{(3)}(\mathbb{R},6)$ is the first instance where we actually see non-trivial cup-products. As noted in the paragraph following Remark 1.4, $\mathrm{Conf}^{(3)}(\mathbb{R},6)$ has the homotopy type of a wedge of 71 circles and 20 tori. In particular, there are 111 basic elementary preorders of degree one, which we do not attempt to write down. Here is the list of the $\binom{6}{3} = 20$ basic preorders of degree two:

$$\begin{array}{llll}
[1,2](3)[4,5](6), & [1,3](5)[2,4](6), & [2,3](4)[1,5](6), & [2,5](6)[1,3](4), \\
[1,2](4)[3,5](6), & [1,3](6)[2,4](5), & [2,3](5)[1,4](6), & [3,4](5)[1,2](6), \\
[1,2](5)[3,4](6), & [1,4](5)[2,3](6), & [2,3](6)[1,4](5), & [3,4](6)[1,2](5), \\
[1,2](6)[3,4](5), & [1,4](6)[2,3](5), & [2,4](5)[1,3](6), & [3,5](6)[1,2](4), \\
[1,3](4)[2,5](6), & [1,5](6)[2,3](4), & [2,4](6)[1,3](5). & [4,5](6)[1,2](3).
\end{array}$$

Each of these elements is a cup-product of two elementary preorders. For instance

$$[2,4](5)[1,3](6) = \Big([2,4](1,3,5,6)\Big) \cdot \Big((2,4,5)[1,3](6)\Big).$$

Note that in any such factorization, both factors are basic. Each of the 20 resulting pairs of factors can be thought of as giving a 1-dimensional basis for one of the 20 tori in the homotopy decomposition noted above.

3. $\mathrm{TC}(\mathrm{Conf}^{(k)}(\mathbb{R}, n))$

Theorem 1.1 is obvious for $n \leqslant k$. In fact, for $n < k$, $\mathrm{Conf}^{(k)}(\mathbb{R}, n) = \mathbb{R}^n$, which is contractible, so that $\mathrm{TC}(\mathrm{Conf}^{(k)}(\mathbb{R}, n)) = 0$. On the other hand, for $n = k$ and with $\Delta = \{(x, x, \ldots, x) \colon x \in \mathbb{R}\}$, $\mathrm{Conf}^{(k)}(\mathbb{R}, n) = \mathbb{R}^k - \Delta \simeq S^{k-2}$, whose topological complexity is well known to be 1 (respectively, 2) if k is odd (respectively, even). We thus assume $n > k$ in what follows. Recall we also assume $k \geqslant 3$.

The homotopy dimension, denoted by hdim, and the Lusternik-Schnirelmann category of $\mathrm{Conf}^{(k)}(\mathbb{R}, n)$ are easily established:

Proposition 3.1. $\mathrm{Conf}^{(k)}(\mathbb{R}, n)$ *is a* $(k-3)$*-connected space having LS-category* $\mathrm{cat}(\mathrm{Conf}^{(k)}(\mathbb{R}, n)) = \lfloor n/k \rfloor$ *and* $\mathrm{hdim}(\mathrm{Conf}^{(k)}(\mathbb{R}, n)) = (k-2)\lfloor n/k \rfloor$*. In particular, for* $k = 3$*, both the cohomological dimension and the geometric dimension, denoted respectively by* cdim *and* gdim*, of the group* PP_n *equal* $\lfloor n/k \rfloor$*.*

Proof. Let $q = \lfloor n/k \rfloor$. As noted in the first assertion of Remark 2.3, the Baryshnikov basis element

$$\Big[1, \ldots, k-1\Big]\Big(k\Big)\Big[k+1, \ldots, 2k-1\Big]\Big(2k\Big) \cdots \Big[(q-1)k+1, \ldots, qk-1\Big]\Big(qk\Big) \quad (7)$$

is a product of q factors, each being a basis element of dimension $k-2$. This implies $q \leqslant \mathrm{cat}(\mathrm{Conf}^{(k)}(\mathbb{R}, n))$. On the other hand, [**21**, Theorems 1.1 and 1.2] imply that $\mathrm{Conf}^{(k)}(\mathbb{R}, n)$ is $(k-3)$-connected, is not $(k-2)$-connected, and has the homotopy type of a cell complex of dimension $(k-2)q$. The first two assertions in the lemma then follow from the inequality $\mathrm{cat} \leqslant (\mathrm{hdim})/(\mathrm{conn}+1)$, which in turn follows from a standard obstruction-theory argument. When $k = 3$, so $\mathrm{hdim}(\mathrm{Conf}^{(k)}(\mathbb{R}, n)) = \mathrm{gdim}(\mathrm{PP}_n)$ by definition, the last assertion in the lemma follows from the relations $\mathrm{cat} = \mathrm{cdim} \leqslant \mathrm{gdim}$ in [**9**]. □

We have omitted the use of curly braces for level sets within the string preorder (7). This convention will be kept throughout the rest of the paper.

The standard inequality $\mathrm{TC}(X) \leqslant 2\,\mathrm{cat}(X)$ yields $\mathrm{TC}(\mathrm{Conf}^{(k)}(\mathbb{R},n)) \leqslant 2\lfloor n/k\rfloor$. Thus, in view of Proposition 2.1, the proof of Theorem 1.1 will be complete once we show

$$2\lfloor n/k\rfloor \leqslant \mathrm{zcl}(\mathrm{Conf}^{(k)}(\mathbb{R},n)), \quad \text{for } n > k \geqslant 3. \tag{8}$$

In order to establish (8), we introduce a few key elements in $H^*(\mathrm{Conf}^{(k)}(\mathbb{R},n))$ and in $H^*(\mathrm{Conf}^{(k)}(\mathbb{R},n))^{\otimes 2}$. Recall that all cohomology groups are taken with $\mathbb{Z}_2$-coefficients, a restriction that is not essential but allows us to simplify calculations.

Definition 3.2. For a positive integer m satisfying $m+k \leqslant n+2$, consider the elements $x_m, x'_m \in H^{k-2}(\mathrm{Conf}^{(k)}(\mathbb{R},n))$ given by

$$\begin{aligned} x_m &= \Big(1,\ldots,m-2,m-1\Big)\Big[m,m+1,\ldots,m+k-2\Big]\Big(m+k-1,\ldots,n\Big),\\ x'_m &= \Big(1,\ldots,m-2,m\Big)\Big[m-1,m+1,\ldots,m+k-2\Big]\Big(m+k-1,\ldots,n\Big),\end{aligned}$$

where x'_m is defined only for $m \geqslant 2$. Each of the corresponding zero-divisors $y_m = x_m \otimes 1 + 1 \otimes x_m$ for $\mathrm{Conf}^{(k)}(\mathbb{R},n)$ is pivotal in what follows, with the elements x'_m playing a subtle role.

Note that x_m and x'_m are Baryshnikov basis elements in $H^*(\mathrm{Conf}^{(k)}(\mathbb{R},n))$ provided $m+k \leqslant n+1$. In fact, as illustrated by the first assertion in Remark 2.3,

$$\begin{aligned}&\prod_{j=1}^{i} x_{(j-1)k+2} = x_2 x_{k+2}\cdots x_{(i-1)k+2} \qquad (9)\\ &= \Big(1\Big)\Big[2,\ldots,k\Big]\Big(k+1\Big)\Big[k+2,\ldots,2k\Big]\Big(2k+1\Big)\cdots\Big[(i-1)k+2,\ldots,ik\Big]\Big(ik+1,\ldots,n\Big)\end{aligned}$$

is a basis element in $H^*(\mathrm{Conf}^{(k)}(\mathbb{R},n))$ provided $ik+1 \leqslant n$. Likewise, if $\widetilde{x}_{(j-1)k+1}$ stands for either $x_{(j-1)k+1}$ or $x'_{(j-1)k+1}$, with the latter one being a possibility only for $j \geqslant 2$, then

$$\begin{aligned}&\prod_{j=1}^{i} \widetilde{x}_{(j-1)k+1} = \widetilde{x}_1 \widetilde{x}_{k+1}\cdots \widetilde{x}_{(i-1)k+1} \qquad (10)\\ &= \Big[1,\,\ldots,k-1\Big]\Big(k\Big)\Big[k+1,\,\ldots,2k-1\Big]\cdots\Big((i-1)k\Big)\Big[(i-1)k+1,\,\ldots,ik-1\Big]\Big(ik,\ldots,n\Big),\end{aligned}$$

is a basis element in $H^*(\mathrm{Conf}^{(k)}(\mathbb{R},n))$ provided $ik \leqslant n$. Here curved arrows indicate pairs of elements that might have to be switched, depending on the actual term $\widetilde{x}_{(j-1)k+1}$ under consideration.

Example 3.3. The condition $3 \leqslant k < n$ ensures that both x_1 and x_2 are Baryshnikov basis elements in $H^*(\mathrm{Conf}^{(k)}(\mathbb{R},n))$, and since $x_1 \neq x_2$, we obviously have

$$y_1 y_2 = (x_1\otimes 1 + 1\otimes x_1)(x_2\otimes 1 + 1 \otimes x_2) = \cdots + x_1\otimes x_2 + x_2 \otimes x_1 + \cdots \neq 0. \tag{11}$$

So $2 \leqslant \mathrm{zcl}(\mathrm{Conf}^{(k)}(\mathbb{R},n))$, which readily yields (8) for $2k > n > k \geqslant 3$.

The proof of (8) for $n \geqslant 2k$ and $k \geqslant 3$ requires a major generalization of the simple calculation in (11). The product indicated in (12) below will play the role of the product y_1y_2 on the left-hand side of (11). Most importantly, the tensor factors x_1 and x_2 in the two explicit summands after the second equality symbol in (11) will be replaced by products of the form (9), and by certain products of the form (10), some of which are made explicit as follows:

$$p_{i,1} = \begin{cases} x_1 \left(\prod_{j=1}^{a-1} x_{(2j-1)k+1} x'_{2jk+1}\right) x_{(2a-1)k+1}\,, & \text{if } i = 2a \geqslant 2; \\ x_1 \left(\prod_{j=1}^{a} x_{(2j-1)k+1} x'_{2jk+1}\right), & \text{if } i = 2a+1 \geqslant 3, \end{cases}$$

$$p_{i,2} = \begin{cases} x_1 \left(\prod_{j=1}^{a-1} x'_{(2j-1)k+1} x_{2jk+1}\right) x'_{(2a-1)k+1}\,, & \text{if } i = 2a \geqslant 2; \\ x_1 \left(\prod_{j=1}^{a} x'_{(2j-1)k+1} x_{2jk+1}\right), & \text{if } i = 2a+1 \geqslant 3. \end{cases}$$

Theorem 3.4. *If the integers i, k, n satisfy $2 \leqslant i$, $3 \leqslant k$, and $ik \leqslant n$, then the product*

$$\prod_{j=1}^{i} y_{(j-1)k+1} y_{(j-1)k+2} \in H^*(\mathrm{Conf}^{(k)}(\mathbb{R}, n))^{\otimes 2} \tag{12}$$

is non-zero. Explicitly:

1. *If $ik + 1 \leqslant n$, then the expression of (12) as a linear combination of Baryshnikov tensor basis elements for $H^*(\mathrm{Conf}^{(k)}(\mathbb{R}, n))^{\otimes 2}$ includes the Baryshnikov basis element*
$$\prod_{j=1}^{i} x_{(j-1)k+1} \otimes \prod_{j=1}^{i} x_{(j-1)k+2}.$$
2. *If $ki = n$, then the expression of (12) as a linear combination of Baryshnikov tensor basis elements for $H^*(\mathrm{Conf}^{(k)}(\mathbb{R}, n))^{\otimes 2}$ includes the Baryshnikov basis element $p_{i,1} \otimes p_{i,2}$.*

As distilled in Example 3.3, the hypothesis $i \geqslant 2$ is relevant only for the second half of Theorem 3.4. The actual exceptional case that has to be avoided is $n = k$, for which y_1y_2 is forced to vanish in view of the first paragraph of this section and the fact that we are using $\mathbb{Z}_2$-cohomology groups. If we had worked over the integers, rather than over $\mathbb{Z}_2$, the exceptional case would have been reduced to that where $n = k$ is odd.

The validity of (8) for $n \geqslant 2k$ and $k \geqslant 3$, i.e. the cases that remain to be considered, follows from Theorem 3.4 below by taking $i = \lfloor n/k \rfloor$. So, the rest of the section is devoted to the proof of Theorem 3.4.

Lemma 3.5. *The following relations hold in $H^*(\mathrm{Conf}^{(k)}(\mathbb{R}, n))$:*

1. $x_2x_{k+1} = x_1x_{k+1}$, *for $n \geqslant 2k - 1$.*
2. $x_{n-2k+4}x_{n-k+2} = x_{n-2k+3}x_{n-k+2} = 0$, *for $n \geqslant 2k - 2$.*
3. $x_{n-2k+2}x_{n-k+2} = x_{n-2k+2}x_{n-k+1}$, *for $n \geqslant 2k - 1$.*
4. $x_{n-2k+1}x_{n-k+2} = x_{n-2k+1}x_{n-k+1} + x_{n-2k+1}x'_{n-k+1}$, *for $n \geqslant 2k$.*
5. $x_rx_{r+k}x_{r+2k-1} = x_rx_{r+k-1}x_{r+2k-1}$, *for $n \geqslant r + 3k - 3$ and $r \geqslant 1$.*
6. $x_rx_{r+k+1}x_{r+2k} = x_rx_{r+k}x_{r+2k} + x_rx'_{r+k}x_{r+2k}$, *for $n \geqslant r + 3k - 2$ and $r \geqslant 1$.*

Remark 3.6. The numeric restrictions on k, n and r ensure that each of the factors x_m in the six items above is an element of $H^*(\mathrm{Conf}^{(k)}(\mathbb{R}, n))$.

Proof of Lemma 3.5. All these equalities follow from Theorem 2.8 and Remark 2.9. We give full details for completeness. Assume $n \geqslant 2k-2$. Take $A = \{1, \ldots, n-k+1\}$, $B = \{n-k+2, \ldots, n-1\}$ and $C = \{n\}$ in Theorem 2.8.1 to get

$$\begin{aligned} x_{n-k+2} &= (1, \ldots, n-k+1)[n-k+2, \ldots, n] \\ &= \sum_{i=1}^{n-k+1} (1, \ldots, \widehat{i}, \ldots, n-k+1)[i, n-k+2, \ldots, n-1](n). \end{aligned} \tag{13}$$

As explained in Remark 2.9, all terms in the summation in (13) vanish when multiplied by $x_{n-2k+3} = (1, \ldots, n-2k+2)[n-2k+3, \ldots, n-k+1](n-k+2, \cdots, n)$. This yields $x_{n-2k+3}x_{n-k+2} = 0$, while the equality $x_{n-2k+4}x_{n-k+2} = 0$ follows directly from the considerations in Remark 2.9. This proves item 2.

Assume $n \geqslant 2k-1$. Terms with $i \leqslant n-k$ in the summation in (13) vanish when multiplied by $x_{n-2k+2} = (1, \ldots, n-2k+1)[n-2k+2, \ldots, n-k](n-k+1, \ldots, n)$. This yields $x_{n-2k+2}x_{n-k+2} = x_{n-2k+2}x_{n-k+1}$ and proves item 3.

Assume $n \geqslant 2k$. Terms with $i < n-k$ in the summation in (13) vanish when multiplied by $x_{n-2k+1} = (1, \ldots, n-2k)[n-2k+1, \ldots, n-k-1](n-k, \ldots, n)$. This yields $x_{n-2k+1}x_{n-k+2} = x_{n-2k+1}x_{n-k+1} + x_{n-2k+1}x'_{n-k+1}$ and proves item 4.

Assume $n \geqslant r+3k-2$ and $r \geqslant 1$. Take $A = \{1, \ldots, r+k-1\}$, $B = \{r+k, \ldots, r+2k-3\}$ and $C = \{r+2k-2, \ldots, n\}$ in Theorem 2.8.1 to get

$$\begin{aligned} &\sum_{i=1}^{r+k-1} (1, \ldots, \widehat{i}, \ldots, r+k-1)[i, r+k, \ldots, r+2k-3](r+2k-2, \ldots, n) \\ &\quad = \sum_{i=r+2k-2}^{n} (1, \ldots, r+k-1)[r+k, \ldots, r+2k-3, i](r+2k-2, \ldots, \widehat{i}, \ldots, n). \end{aligned}$$

Terms with $i < r+k-1$ in the first summation vanish when multiplied by $x_r = (1, \ldots, r-1)[r, \ldots, r+k-2](r+k-1, \ldots, n)$, and terms with $i > r+2k-2$ in the second summation vanish when multiplied by $x_{r+2k-1} = (1, \ldots, r+2k-2)[r+2k-1, \ldots, r+3k-3](r+3k-2, \ldots, n)$. This yields the equality $x_r x_{r+k-1} x_{r+2k-1} = x_r x_{r+k} x_{r+2k-1}$ and proves item 5.

When $n \geqslant 2k-1$, the previous argument applies for $r = 2-k$—by vacuity in the case of the assertion about the first summation, whose only one term is x_1. This yields $x_1 x_{k+1} = x_2 x_{k+1}$ and proves item 1.

Assume $n \geqslant r+3k-2$ and $r \geqslant 1$. Take $A = \{1, \ldots, r+k\}$, $B = \{r+k+1, \ldots, r+2k-2\}$ and $C = \{r+2k-1, \ldots, n\}$ in Theorem 2.8.1 to get

$$\begin{aligned} &\sum_{i=1}^{r+k} (1, \ldots, \widehat{i}, \ldots, r+k)[i, r+k+1, \ldots, r+2k-2](r+2k-1, \ldots, n) \\ &\quad = \sum_{i=r+2k-1}^{n} (1, \ldots, r+k)[r+k+1, \ldots, r+2k-2, i](r+2k-1, \ldots, \widehat{i}, \ldots, n). \end{aligned}$$

Terms with $i < r+k-1$ in the first summation vanish when multiplied by $x_r = (1, \ldots, r-1)[r, \ldots, r+k-2](r+k-1, \ldots, n)$, while terms with $i > r+2k-1$ in the second summation vanish when multiplied by $x_{r+2k} = (1, \ldots, r+2k-1)$

$[r+2k,\ldots,r+3k-2](r+3k-1,\ldots,n)$. This yields the equality $x_r x'_{r+k} x_{r+2k} + x_r x_{r+k} x_{r+2k} = x_r x_{r+k+1} x_{r+2k}$ and proves item 6. □

Proof of part 1 in Theorem 3.4. By Remark 2.9,

$$\begin{aligned} y_{(j-1)k+1}y_{(j-1)k+2} &= (x_{(j-1)k+1}\otimes 1 + 1\otimes x_{(j-1)k+1})(x_{(j-1)k+2}\otimes 1 + 1\otimes x_{(j-1)k+2}) \\ &= x_{(j-1)k+1}\otimes x_{(j-1)k+2} + x_{(j-1)k+2}\otimes x_{(j-1)k+1}, \end{aligned}$$

so the product in (12) is

$$\begin{aligned} \prod_{j=1}^{i} y_{(j-1)k+1}y_{(j-1)k+2} &= (x_1\otimes x_2 + x_2\otimes x_1)(x_{k+1}\otimes x_{k+2} + x_{k+2}\otimes x_{k+1}) \;\cdots \\ &\qquad \cdots\; (x_{(i-1)k+1}\otimes x_{(i-1)k+2} + x_{(i-1)k+2}\otimes x_{(i-1)k+1}) \\ &= \sum_{\substack{\epsilon_j\,\in\,\{1,2\}\\ 1\leqslant j\leqslant i}} x_{3-\epsilon_1}x_{k+3-\epsilon_2}\cdots x_{(i-1)k+3-\epsilon_i}\otimes x_{\epsilon_1}x_{k+\epsilon_2}\cdots x_{(i-1)k+\epsilon_i}. \end{aligned} \tag{14}$$

The basis element we care about, namely

$$\prod_{j=1}^{i} x_{(j-1)k+1}\otimes\prod_{j=1}^{i} x_{(j-1)k+2}, \tag{15}$$

is the summand in (14) with $\epsilon_j = 2$ for all j. The proof task is to argue that, when we expand the other terms of (14) as sums of tensor of basis elements, the tensor (15) does not appear. This is obvious for the summand in (14) with $\epsilon_j = 1$ for all j. For all other summands, the assertion will be argued by focusing on the sequence of leaps associated to the subscripts of both tensor factors of each summand in (14). Explicitly, the first leap in the subscripts of $x_{3-\epsilon_1}x_{k+3-\epsilon_2}\cdots x_{(i-1)k+3-\epsilon_i}$ is $k+3-\epsilon_2-(3-\epsilon_1) = k+\epsilon_1-\epsilon_2$, and the full sequences of leaps associated to

$$x_{3-\epsilon_1}x_{k+3-\epsilon_2}\cdots x_{k(i-1)+3-\epsilon_i} \quad\text{and}\quad x_{\epsilon_1}x_{k+\epsilon_2}\cdots x_{k(i-1)+\epsilon_i} \tag{16}$$

are, respectively,

$$\begin{gathered} (k+\epsilon_1-\epsilon_2, k+\epsilon_2-\epsilon_3,\ldots,k+\epsilon_{i-1}-\epsilon_i) \quad\text{and} \\ (k-\epsilon_1+\epsilon_2, k-\epsilon_2+\epsilon_3,\ldots,k-\epsilon_{i-1}+\epsilon_i). \end{gathered} \tag{17}$$

Such sequences of leaps clearly satisfy:

(A) Leap values are either $k-1$, k, or $k+1$. Moreover, if all k-leaps are removed from either one of the sequences in (17), then the resulting sequence of leaps either is empty or, else, has leap values that alternate between $k-1$ and $k+1$: $(k-1,k+1,k-1,\ldots)$ or $(k+1,k-1,k+1,\ldots)$.

(B) Corresponding coordinates in the two sequences of leaps in (17) add up to $2k$.

(C) The first leap different from k, if any, in either of the sequences of leaps (17) is a $(k+1)$-leap (respectively, $(k-1)$-leap) provided the corresponding product in (16) starts with x_1 (respectively, x_2).

Since the right tensor factor in (15), i.e. $\prod_{j=1}^{i} x_{(j-1)k+2}$, is a basic string preorder starting as $(1)[2,\ldots,k]\cdots$, the proof is complete in view of Proposition 3.7 below. □

Proposition 3.7. *Any summand in (14) whose associated sequences of leaps (17) contain at least a $(k-1)$-leap, or equivalently a $(k+1)$-leap, is a linear combination of tensor basis elements $u \otimes v$ where* both *u and v are basic string preorders starting as*

$$[1, \ldots, k-1](I_1)\cdots(I_{i-1})[J_i](I_i).$$

Proof. Take a product $p = x_{k_1}x_{k_2}\cdots x_{k_i}$ in (16), so $k_1 \in \{1,2\}$, with associated sequence of leaps $(\ell_1, \ldots, \ell_{i-1})$ satisfying conditions (A)–(C) above, and so that not all leap values ℓ_j are k.

Case $k_1 = 1$: p has the form

$$x_1 \cdots \underbrace{x_{kr_1+1}x_{k(r_1+1)+2}}_{(k+1)\text{-leap}} \cdots \underbrace{x_{kr_2+2}x_{k(r_2+1)+1}}_{(k-1)\text{-leap}} \cdots \underbrace{x_{kr_3+1}x_{k(r_3+1)+2}}_{(k+1)\text{-leap}} \cdots \underbrace{x_{kr_4+2}x_{k(r_4+1)+1}}_{(k-1)\text{-leap}} \cdots, \tag{18}$$

where we only indicate $(k-1)$-leaps and $(k+1)$-leaps. Items 5 and 6 in Lemma 3.5 allow us to replace each portion $x_{kr_j+1}x_{k(r_j+1)+2}\cdots x_{kr_{j+1}+2}x_{k(r_{j+1}+1)+1}$, having an initial $(k+1)$-leap, a final $(k-1)$-leap, and perhaps some intermediate k-leaps, by

$$x_{kr_j+1}(x_{k(r_j+1)+1} + x'_{k(r_j+1)+1})x_{k(r_j+2)+1}\cdots x_{kr_{j+1}+1}x_{k(r_{j+1}+1)+1},$$

which only has k-leaps. The replacing process can be iterated since the initial and final terms in the replacing portion agree with those in the replaced portion. After all replacements are made, and sums are distributed, p becomes a sum of expressions each of which is similar to the original one (18), except that some of the initial x_{kj+1}'s get replaced by the corresponding x'_{kj+1}, and in such a way that no $(k-1)$-leaps show up, and at most one $(k+1)$-leap shows up. But any such expression is a basis element of the required form. The latter assertion uses the hypothesis $ik+1 \leqslant n$ in part 1 of Theorem 3.4 (see Remark 3.8 below).

Case $k_1 = 2$: p has the form

$$x_2 \cdots \underbrace{x_{kr_1+2}x_{k(r_1+1)+1}}_{(k-1)\text{-leap}} \cdots \underbrace{x_{kr_2+1}x_{k(r_2+1)+2}}_{(k+1)\text{-leap}} \cdots \underbrace{x_{kr_3+2}x_{k(r_3+1)+1}}_{(k-1)\text{-leap}} \cdots \underbrace{x_{kr_4+1}x_{k(r_4+1)+2}}_{(k+1)\text{-leap}} \cdots,$$

Items 1 and 5 in Lemma 3.5 allows to replace the initial portion $x_2\cdots x_{kr_1+2}x_{k(r_1+1)+1}$ by $x_1\cdots x_{kr_1+1}x_{k(r_1+1)+1}$. Then, the replacement process described in the previous case allows us to write p as a sum of basis elements of the required form. □

Remark 3.8. Part 2 in Theorem 3.4 will be proved using an argument similar to that in the previous proof, except that it will be necessary to deal first with an additional subtlety. Namely, note that when $ik = n$, we have

$$\begin{aligned} x_{(i-1)k+2} &= \Big(1, \ldots, (i-1)k+1\Big)\Big[(i-1)k+2, \ldots, ik\Big]\Big(ik+1, \ldots, n\Big) \\ &= \Big(1, \ldots, (i-1)k+1\Big)\Big[(i-1)k+2, \ldots, n\Big], \end{aligned}$$

which is an elementary *non-basic* element. So, when analyzing a typical tensor factor $x_{\epsilon_1}x_{k+\epsilon_2}\cdots x_{(i-1)k+\epsilon_i}$ in (14) with $\epsilon_i = 2$, the recursive process described in the previous proof will not end up producing sums of basis elements. This issue will be resolved using item 4 in Lemma 3.5.

Let us go back to the starting point for the proof of part 2 in Theorem 3.4, i.e., the expression in (14) for the product $\prod_{j=1}^{i} y_{(j-1)k+1}y_{(j-1)k+2}$. As noted in Remark 3.8, we no longer work with the basis element indicated in part 1 of Theorem 3.4. Instead, the basis element we now care about is $p_{i,1}\otimes p_{i,2}$, where $ki=n$, and which arises from one of the two summands in (14) for which the values of the indices ϵ_j alternate between 1 and 2.

In order to simplify the argument, it is convenient to note that all y_j, and therefore their product $\prod_{j=1}^{i} y_{(j-1)k+1}y_{(j-1)k+2}$, are invariant under the involution induced by the map that switches coordinates in $\mathrm{Conf}^{(k)}(\mathbb{R},n)\times\mathrm{Conf}^{(k)}(\mathbb{R},n)$. We show the following (equivalent, by the symmetry just noted, but slightly simpler-to-prove) version of part 2 in Theorem 3.4:

Lemma 3.9. *For $i\geqslant 2$, $k\geqslant 3$ and $n=ki$, both $p_{i,1}\otimes p_{i,2}$ and $p_{i,2}\otimes p_{i,1}$ are included in the expression of the product (12) as a linear combination of Baryshnikov tensor basis elements for $H^*(\mathrm{Conf}^{(k)}(\mathbb{R},n))^{\otimes 2}$.*

Proof. We provide full proof details when $i=2a$ is even; the parallel argument for i odd is left as an exercise for the reader. In order to simplify notation, we let $r_1\cdot r_2\cdots r_t$ and $r_1\cdot r_2\cdots r_t|s_1\cdot s_2\cdots s_t$ stand for $x_{r_1}x_{r_2}\cdots x_{r_t}$ and $x_{r_1}x_{r_2}\cdots x_{r_t}\otimes x_{s_1}x_{s_2}\cdots x_{s_t}$, respectively. With this notation, (14) becomes

$$\begin{aligned}&\Big(1|2+2|1\Big)\Big((k+1)|(k+2)+(k+2)|(k+1)\Big)\cdots\\&\qquad\cdots\Big(((2a-1)k+1)|((2a-1)k+2)+((2a-1)k+2)|((2a-1)k+1)\Big)\\&=\sum_{\substack{\epsilon_j\in\{1,2\}\\1\leqslant j\leqslant i}}(3-\epsilon_1)(k+3-\epsilon_2)\cdots((2a-1)k+3-\epsilon_{2a})\,\Big|\,(\epsilon_1)(k+\epsilon_2)\cdots((2a-1)k+\epsilon_{2a}).\end{aligned}\tag{19}$$

The summand with $(\epsilon_1,\epsilon_2,\ldots,\epsilon_{2a})=(1,2,1,\ldots,2)$ is

$$\begin{aligned}&2\cdot(k+1)\cdot(2k+2)\cdot(3k+1)\cdots((2a-2)k+2)\cdot((2a-1)k+1)\\&\qquad\Big|\,1\cdot(k+2)\cdot(2k+1)\cdot(3k+2)\cdots((2a-2)k+1)\cdot((2a-1)k+2),\end{aligned}\tag{20}$$

whose associated sequences of leaps are

$$(k-1,k+1,k-1,\ldots,k-1)\quad\text{and}\quad(k+1,k-1,k+1,\ldots,k+1).\tag{21}$$

Using the replacing process explained in the previous proof, it is clear that the expression of

$$2\cdot(k+1)\cdot(2k+2)\cdot(3k+1)\cdots((2a-2)k+2)\cdot((2a-1)k+1)$$

in terms of Baryshnikov basis elements includes $p_{2a,1}$, but not $p_{2a,2}$. Likewise, the replacing process and item 4 in Lemma 3.5 imply that the expression of

$$1\cdot(k+2)\cdot(2k+1)\cdot(3k+2)\cdots((2a-2)k+1)\cdot((2a-1)k+2)$$

in terms of Baryshnikov basis includes $p_{2a,2}$. Therefore the expression of (20) in terms of Baryshnikov tensor basis elements includes $p_{2a,1}\otimes p_{2a,2}$ but does not include $p_{2a,2}\otimes p_{2a,1}$. Further, the symmetry coming from the involution induced by the switching map on $\mathrm{Conf}^{(k)}(\mathbb{R},n)^{\times 2}$ implies that the expression in terms of Baryshnikov basis of the summand in (19) with $(\epsilon_1,\epsilon_2,\ldots,\epsilon_{2a})=(2,1,2,\ldots,1)$ includes $p_{2a,2}\otimes p_{2a,1}$ but does not include $p_{2a,1}\otimes p_{2a,2}$.

It remains to prove that neither $p_{2a,1} \otimes p_{2a,2}$ nor $p_{2a,2} \otimes p_{2a,1}$ are included in the expression in terms of basis elements of any summand in (19) whose associated sequences of leaps are different from those in (21). By symmetry, it suffices to consider the case of a summand

$$(3-\epsilon_1)(k+3-\epsilon_2)\cdots((2a-1)k+3-\epsilon_{2a}) \,\Big|\, (\epsilon_1)(k+\epsilon_2)\cdots((2a-1)k+\epsilon_{2a}) \tag{22}$$

with $\epsilon_1 = 1$. Let $\lambda \in \{k-1, k, k+1\}$ (respectively, $\rho \in \{k+1, k, k-1\}$) stand for the value of the last leap in the tensor factor on the left (respectively, right) of (22). Recall $\lambda + \rho = 2k$.

Case $\lambda = \rho = k$: The ending portion of one of the two tensor factors in (22) is forced to be

$$\cdots((2a-2)k+1)\cdot((2a-1)k+1).$$

The replacing process shows that such a factor cannot give rise to $p_{2a,1}$ or $p_{2a,2}$ in its expression in terms of Baryshnikov basis.

Case $(\lambda, \rho) = (k-1, k+1)$: The equalities $\epsilon_{2a-1} = 1$ and $\epsilon_{2a} = 2$ are now forced. Letting j' stand for x'_j, and ignoring Baryshnikov basis elements different from $p_{2a,1}$ and $p_{2a,2}$, the right factor in (22) then becomes

$$\begin{aligned} &1\cdot(k+\epsilon_2)\cdots((2a-2)k+1)((2a-1)k+2) \\ &\quad = 1\cdot(k+\epsilon_2)\cdots((2a-2)k+1)((2a-1)k+1)', \end{aligned}$$

in view of the replacing process and item 4 in Lemma 3.5. Further, the replacing process makes it clear that the expression of the latter element in terms of Baryshnikov basis elements does not include $p_{2a,1}$, and that it includes $p_{2a,2}$ only if the sequence of leaps associated to the right tensor factor in (22) is the second sequence in (21).

Case $(\lambda, \rho) = (k+1, k-1)$: The equalities $\epsilon_{2a-1} = 2$ and $\epsilon_{2a} = 1$ are now forced. Ignoring Baryshnikov basis elements different from $p_{2a,1}$ and $p_{2a,2}$, the left factor in (22) becomes

$$\begin{aligned} &2\cdot(k+3-\epsilon_2)\cdots((2a-2)k+1)((2a-1)k+2) \\ &\quad = 2\cdot(k+3-\epsilon_2)\cdots((2a-2)k+1)((2a-1)k+1)', \end{aligned}$$

where the latter expression further evolves under the replacing process, and still ignoring Baryshnikov basis elements different from $p_{2a,1}$ and $p_{2a,2}$, to either zero or to

$$2\cdot(k+1)\cdot(2k+1)\cdot(3k+1)'\cdots((2a-2)k+1)((2a-1)k+1)'. \tag{23}$$

Note the factor "$(k+1)$", rather than a primed "$(k+1)'$", due to the initial "2" in (23). In any case, a final application of item 1 in Lemma 3.5 shows that (23) vanishes modulo Baryshnikov basis elements different from $p_{2a,1}$ and $p_{2a,2}$. ☐

4. The higher topological complexity of $\mathrm{Conf}^{(k)}(\mathbb{R}, n)$

We now compute the higher topological complexity $\mathrm{TC}_s(\mathrm{Conf}^{(k)}(\mathbb{R}, n))$, for any $s > 2$.

Corollary 4.1. *For $s > 2$,*

$$\mathrm{TC}_s(\mathrm{Conf}^{(k)}(\mathbb{R}, n)) = \begin{cases} 0, & n < k; \\ s-1, & n = k \text{ with } k \text{ odd}; \\ s, & n = k \text{ with } k \text{ even}; \\ s\lfloor n/k \rfloor, & n > k. \end{cases}$$

Proof. The case $n \leqslant k$ is trivial. For $n > k$ and $s > 2$, Proposition 3.1 and [**3**, Theorem 3.9] imply the estimate $\mathrm{TC}_s(\mathrm{Conf}^{(k)}(\mathbb{R}, n)) \leqslant s\lfloor n/k \rfloor$. From [**3**, Definition 3.8

and Theorem 3.9], equality will follow once we exhibit a non-zero product of $s\lfloor n/k \rfloor$ "s-th zero-divisors" for $\mathrm{Conf}^{(k)}(\mathbb{R}, n)$, i.e., of elements in the kernel of the iterated cup-product $H^*(\mathrm{Conf}^{(k)}(\mathbb{R}, n))^{\otimes s} \to H^*(\mathrm{Conf}^{(k)}(\mathbb{R}, n))$.

Let $i = \lfloor n/k \rfloor$, $q \in \{1, \ldots, s-1\}$, and consider the s-th zero-divisors

$$z_{m,q} = 1 \otimes \cdots \otimes 1 \otimes \underbrace{x_m}_{q\text{-th}} \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes \cdots \otimes 1 \otimes x_m \in H^*(\mathrm{Conf}^{(k)}(\mathbb{R}, n))^{\otimes s},$$

whenever $m + k \leqslant n + 2$. For instance

$$\prod_{j=1}^{i} z_{(j-1)k+1,s-1} z_{(j-1)k+2,s-1} = 1 \otimes \cdots \otimes 1 \otimes \prod_{j=1}^{i} y_{(j-1)k+1} \cdot y_{(j-1)k+2}$$

and, for $q \leqslant s-2$,

$$\begin{aligned} z_{m,q} \prod_{j=1}^{i} & z_{(j-1)k+1,s-1} z_{(j-1)k+2,s-1} \\ &= 1 \otimes \cdots \otimes 1 \otimes \underbrace{x_m}_{q-\text{th}} \otimes 1 \otimes \cdots \otimes 1 \otimes \prod_{j=1}^{i} y_{(j-1)k+1} \cdot y_{(j-1)k+2} \\ &\quad + 1 \otimes \cdots \otimes 1 \otimes \left((1 \otimes x_m) \cdot \prod_{j=1}^{i} y_{(j-1)k+1} y_{(j-1)k+2} \right). \end{aligned}$$

The second summand in the latter expression vanishes in view of Proposition 3.1, and by dimensional considerations or, alternatively, by LS-category considerations. Consequently,

$$\begin{aligned} \prod_{j=1}^{i} z_{(j-1)k+1,1} \cdot \prod_{j=1}^{i} z_{(j-1)k+1,2} \cdots \prod_{j=1}^{i} z_{(j-1)k+1,s-2} \cdot \prod_{j=1}^{i} z_{(j-1)k+1,s-1} z_{(j-1)k+2,s-1} \\ = \left(\prod_{j=1}^{i} x_{(j-1)k+1} \right) \otimes \cdots \otimes \left(\prod_{j=1}^{i} x_{(j-1)k+1} \right) \otimes \prod_{j=1}^{i} y_{(j-1)k+1} y_{(j-1)k+2}, \end{aligned}$$

which is non-zero because the first $s-2$ tensor factors in the latter expression are Baryshnikov basis elements, whereas the last tensor factor is non-zero by Theorem 3.4. □

5. Planners for pure planar braids with few strands

In a recent work ([**1**]), Bardakov, Singh and Vesnin prove that (i) PP_n is free of rank $(1, 7)$ for $n = (3, 4)$, and that (ii) PP_n is not free for $n \geqslant 6$. They also conjecture that (iii) PP_5 is a free group of rank 31. The proof of (i) occupies a full section in [**1**], where two different proofs of the freeness of PP_4 are offered, one with a geometric flavor and another one with an algebraic flavor. The algebraic proof is technical, whereas the geometric proof is extensive. In this section we give short elementary arguments for both (i) and (ii), as well as a short argument proving the stronger form in Proposition 5.1 below of the conjectured (iii). In addition, we indicate a way to construct an explicit optimal motion planner for PP_n when n is small. Recall that the concept of an optimal motion planner is defined at the end of the first paragraph of Subsection 2.1.

Under this paper's perspective, the simplest case is that of (ii), which is an immediate consequence of Corollary 1.2 and the well-known fact that the topological complexity of a free group is at most 2. Even easier is the case $n = 3$ in (i). Indeed, as observed at the beginning of Section 3, $\mathrm{Conf}^{(k)}(\mathbb{R}, n)$ is either contractible or has the homotopy type of the sphere S^{k-2} for, respectively, $n < k$ or $n = k$. In particular, PP_1 and PP_2 are trivial, while PP_3 is an infinite cyclic group, which is relevant for (i).

Condition (iii) is a special case of:

Proposition 5.1. *For $3 \leqslant k < n < 2k$, $\mathrm{Conf}^{(k)}(\mathbb{R}, n)$ has the homotopy type of a wedge of $\beta(k, n)$ spheres of dimension $k-2$, where $\beta(k,n) = \Sigma_{i=k}^{n} \binom{n}{i}\binom{i-1}{k-1}$.*

Proof. Severs–White have shown in [**21**, Theorem 1.1] that $\mathrm{Conf}^{(k)}(\mathbb{R}, n)$ admits a minimal cellular model, i.e., $\mathrm{Conf}^{(k)}(\mathbb{R}, n)$ has the homotopy type of a cell complex having as many cells in each dimension d as the rank of the corresponding integral homology group $H_d(\mathrm{Conf}^{(k)}(\mathbb{R}, n); \mathbb{Z})$. The result then follows from [**5**, Theorem 1.1(c)], noticing that the hypothesis $k < n < 2k$ implies that the integral homology groups of $\mathrm{Conf}^{(k)}(\mathbb{R}, n)$ are free concentrated in dimensions 0 and $k-2$, in view of [**21**, Theorem 1.2]. □

We next give an elementary geometric argument leading to a proof of (i) and (iii), as well as to a description of explicit motion planners for the corresponding groups PP_n. Consider the subspace $X_n \subset \mathrm{Conf}^{(3)}(\mathbb{R}, n)$ consisting of the elements $x = (x_1, \ldots, x_n) \in \mathrm{Conf}^{(3)}(\mathbb{R}, n)$ with $x_n = 0$ and $|x| = 1$. For instance

$$X_3 = \big\{(x_1, x_2, 0) \in \mathbb{R}^3 \,:\, |(x_1, x_2)| = 1 \text{ and } (x_1, x_2) \neq (0,0)\big\} = S^1, \tag{24}$$

whereas

$$\begin{aligned} X_4 &= \left\{ (x_1, x_2, x_3, 0) \in \mathbb{R}^4 \,:\, \begin{array}{l} |(x_1,x_2,x_3)| = 1, \\ (0,0) \notin \{(x_1,x_2),(x_1,x_3),(x_2,x_3)\} \\ \text{not all } x_1,x_2,x_3 \text{ are equal} \end{array} \right\} \\ &= S^2 - \{\pm(0,0,1), \pm(0,1,0), \pm(1,0,0), \pm\tfrac{1}{\sqrt{3}}(1,1,1)\} \simeq \bigvee_7 S^1. \end{aligned} \tag{25}$$

Lemma 5.2 (cf. [**15**, Section III])**.** *Let $\mathbb{R}_+$ stand for the positive real numbers. For $n \geqslant 3$, the map $f\colon \mathrm{Conf}^{(3)}(\mathbb{R}, n) \times \mathbb{R}_+ \times \mathbb{R} \to \mathrm{Conf}^{(3)}(\mathbb{R}, n)$ sending the triple $((x_1, \ldots, x_n), r, a)$ into $(x_1 r + a, \ldots, x_n r + a)$ yields, by restriction, a homeomorphism $X_n \times \mathbb{R}_+ \times \mathbb{R} \cong \mathrm{Conf}^{(3)}(\mathbb{R}, n)$. Consequently, the subspace inclusion $X_n \hookrightarrow \mathrm{Conf}^{(3)}(\mathbb{R}, n)$ is a homotopy equivalence.*

Proof. For the first assertion, it is straightforward to check that the inverse of the restriction of f to $X_n \times \mathbb{R}_+ \times \mathbb{R}$ is given by the map $g\colon \mathrm{Conf}^{(3)}(\mathbb{R}, n) \to X_n \times \mathbb{R}_+ \times \mathbb{R}$ sending $(x_1, \ldots, x_n)$ into the triple $\left(\frac{1}{N}(x_1 - x_n, \ldots, x_{n-1} - x_n, 0), N, x_n\right)$, where N stands for the norm of $(x_1 - x_n, \ldots, x_{n-1} - x_n, 0)$. Note that $N \in \mathbb{R}_+$ since $(x_1, \ldots, x_n) \in \mathrm{Conf}^{(3)}(\mathbb{R}, n)$ and $n \geqslant 3$. For the second assertion, note that the composite $X_n \hookrightarrow \mathrm{Conf}^{(3)}(\mathbb{R}, n) \cong X_n \times \mathbb{R}_+ \times \mathbb{R}$ takes the form $x \mapsto (x, 1, 0)$. □

Note that (i) follows at once from (24), (25) and Lemma 5.2. A similar proof of (iii) is also possible; this involves the use of the stereographic projection from the pinched 3-sphere to the 3-dimensional euclidean space in order to express X_5 as the complement in $\mathbb{R}^3$ of ten unlinked and untangled curves. Details are omitted.[5]

Lemma 5.2 can be used to construct optimal motion planners on $\mathrm{Conf}^{(3)}(\mathbb{R}, n)$ for small values of n. Details are based on a couple of reductions using the following standard observation (see [**10**, Theorem 3]): Assume $\alpha\colon X \to Y$ is a homotopy equivalence with homotopy inverse $\beta\colon Y \to X$. Fix a homotopy $H\colon X \times [0,1] \to X$ between $H_0 = \beta \circ \alpha$ and the identity $H_1 = Id\colon X \to X$. Assume $s\colon U \to P(Y)$ is a local rule for Y, i.e. a section for the double evaluation map $e_{0,1}\colon P(Y) \to Y \times Y$ on the open set $U \subseteq Y \times Y$. Set $V = (\alpha \times \alpha)^{-1}(U)$. Then a local rule $\sigma\colon V \to P(X)$ for X is defined through the formula

$$\sigma(x_1, x_2)(t) = \begin{cases} H(x_1, 3t), & \text{for } 0 \leqslant t \leqslant 1/3; \\ \beta(s(\alpha(x_1), \alpha(x_2))(3t-1)), & \text{for } 1/3 \leqslant t \leqslant 2/3; \\ H(x_2, 3(1-t)), & \text{for } 2/3 \leqslant t \leqslant 1. \end{cases}$$

Applying the construction above to the homotopy equivalence

$$F = \left(\mathrm{Conf}^{(3)}(\mathbb{R}, n) \overset{\cong}{\to} X_n \times \mathbb{R}_+ \times \mathbb{R} \overset{\mathrm{proj}}{\longrightarrow} X_n\right),$$

we see that it suffices to describe an optimal motion planner on X_n. (Explicit formulae for F, the needed homotopy inverse G, and the needed homotopy between the identity and the composite $G \circ F$ are easily deduced from the proof of Lemma 5.2.) In turn, since X_n has the homotopy type of a wedge of circles for $n \leqslant 5$, and since explicit optimal motion planners for finite wedges of spheres have been described in [**6**] (see also [**14**]), it suffices to describe explicit homotopy equivalences $X_3 \simeq S^1$, $X_4 \simeq \vee_7 S^1$ and $X_5 \simeq \vee_{31} S^1$. The latter task has been accomplished in (24) and (25) for $n = 3$ and $n = 4$, where an obvious stereographic projection is needed in the latter case. The resulting motion planner in $\mathrm{Conf}^{(3)}(\mathbb{R}, 3)$ is spelled out next.

Example 5.3. Let D_0 be the subspace of $\mathrm{Conf}^{(3)}(\mathbb{R}, 3) \times \mathrm{Conf}^{(3)}(\mathbb{R}, 3)$ consisting of pairs (x, y) such that the line segment $[x, y]$ in $\mathbb{R}^3$ from x to y does not intersect the diagonal $\Delta = \{(z, z, z)\colon z \in \mathbb{R}\}$, and let D_1 be the complement of D_0 in $\mathrm{Conf}^{(3)}(\mathbb{R}, 3) \times \mathrm{Conf}^{(3)}(\mathbb{R}, 3)$. Both D_0 and D_1 are ENR's, so it suffices to describe a local rule on each. Motion planning in $\mathrm{Conf}^{(3)}(\mathbb{R}, 3)$ for points $(x, y) \in D_0$ can be done by following the segment $[x, y]$. On the other hand, for $(x, y) \in D_1$, let $p(x, y)$ be the point where the segment $[x, y]$ intersects Δ. Since the vectors $y - x$ and $(1, 1, 1)$ are linearly independent, their cross product $u(x, y)$ is nonzero. We then motion plan in $\mathrm{Conf}^{(3)}(\mathbb{R}, 3)$ from x to y (with $(x, y) \in D_1$) by following first the segment $[x, p(x, y) + u(x, y)]$, and then the segment $[p(x, y) + u(x, y), y]$.

[5]In private communications, Harshman and Knapp report having also carried out the reductions for X_5 analogous to (24) and (25), and which lead to a geometric verification of the fact that PP_5 is free of rank 31.

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