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TRUNCATED DERIVED FUNCTORS AND SPECTRAL SEQUENCES

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Abstract

The E_2-term of the Adams spectral sequence may be identified with certain derived functors, and this also holds for a number of other spectral sequences. Our goal is to show how the higher terms of such spectral sequences are determined by truncations of relative derived functors, defined in terms of certain simplicial functors called *mapping algebras*.

1. Introduction

The various types of Adams spectral sequences, which play a central role in algebraic topology (cf. [**A, BCM, BC, BK1, BK2, N, R**]), have a number of features in common:

(i) They are obtained from a space $\mathbf{Y}$ by constructing a (cosimplicial) resolution $\mathbf{Y} \to \mathbf{W}^\bullet$ with respect to a spectrum $\mathcal{A} = \{A_i\}_{i=-\infty}^{\infty}$, with its associated cohomology theory $\mathcal{A}^*$.

(ii) The spectral sequence in question is the homotopy spectral sequence for $\mathbf{TW}^\bullet$, for a suitable homotopy functor $\mathbf{T}$.

(iii) The E_2-term of the spectral sequence can be identified as the derived functors of an algebraic functor T associated to $\mathbf{T}$, applied to $\mathcal{A}^*\mathbf{Y}$.

The goal of this paper is to provide a description similar to (iii) for the E_{n+2}-term of the spectral sequence (for $n \geqslant 0$), as relative derived functors applied to the truncation $P^n\mathfrak{M}^{\mathcal{A}}\mathbf{Y}$ of a certain structure, called a *mapping algebra*, associated to $\mathbf{Y}$ (which reduces to $\mathcal{A}^*\mathbf{Y}$ when $n = 0$).

Just as for the E_2-term, this has two advantages:

(a) The truncated mapping algebra $P^n\mathfrak{M}^{\mathcal{A}}\mathbf{Y}$ has less information than $\mathbf{Y}$ itself, but still enough to determine the E_{n+2}-term.

(b) Relative derived functors may be calculated using *any* resolution of $P^n\mathfrak{M}^{\mathcal{A}}\mathbf{Y}$.

The first author carried out this program for the E_3-term of the stable Adams spectral sequence in [**Bau, BJ2**], showing that extended calculations may be made

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using such a construction. See [**BB3, CF**] for other general descriptions of the higher terms in the stable Adams spectral sequence, although those are not given in the form of truncated derived functors as defined here, and thus do not have the calculational flexibility in terms of varying resolutions (as shown in [**Bau**] for $n = 1$).

1.1. Mapping algebras and truncations

By (iii) above, the E_2-term of the Adams spectral sequence depends only on the sets $[\mathbf{Y}, A_i]_{i\in\mathbb{Z}}$ and operations on them induced by homotopy classes of maps between (products of) the spaces A_i. This suggests that for the higher terms, we should look at the function spaces $\mathrm{map}_*(\mathbf{Y}, A_i)$, with additional structure induced by maps between the representing spaces. This structure is encoded by the notion of a *mapping algebra*: that is, a simplicial functor $\mathfrak{X}\colon \mathbf{\Theta}^{\mathcal{A}} \to \mathcal{S}_*$ from the sub-simplicial category $\mathbf{\Theta}^{\mathcal{A}}$ of Top_0 whose objects are products of copies of the various spaces A_i, to the category $\mathcal{S}_*$ of pointed simplicial sets. For example, the *realizable* mapping algebra $\mathfrak{X} := \mathfrak{M}^{\mathcal{A}}\mathbf{Y}$ has the value $\mathrm{map}(\mathbf{Y}, \mathbf{A})$ at each $\mathbf{A} \in \mathbf{\Theta}^{\mathcal{A}}$.

Mapping algebras admit truncations, defined by applying the Postnikov section functor P^n to each mapping space. In particular, the 0-truncation contains the same information as the sets $[\mathbf{Y}, A_i]_{i\in\mathbb{Z}}$ of homotopy classes of maps, together with the operations on them induced by homotopy classes of maps between the spaces A_i: this is precisely what was needed to determine the E^2-term as suitable derived functors in (iii) above.

This leads to the idea that higher truncations of the mapping algebras may suffice to determine higher terms in the spectral sequence – depending, of course, on the homotopy functor $\mathbf{T}$ in question.

We may therefore summarize our program as follows:

(1) We need to show how a continuous functor $\mathbf{T}\colon \mathsf{Top}_* \to \mathsf{Top}_*$ factors through the category $\mathsf{Map}^{\mathcal{A}}$ of mapping algebras as $\mathfrak{T} \circ \mathfrak{M}^{\mathcal{A}}$, for a suitable homotopy functor $\mathfrak{T}\colon \mathsf{Map}^{\mathcal{A}} \to \mathsf{Top}_*$.

(2) We want $\mathfrak{W}_\bullet := \mathfrak{M}^{\mathcal{A}}\mathbf{W}^\bullet$ to be a resolution of $\mathfrak{M}^{\mathcal{A}}\mathbf{Y}$ in the resolution model category of simplicial mapping algebras, in order to guarantee that both the (functorial) cosimplicial resolution $\mathbf{W}^\bullet$ of $\mathbf{Y}$, and the resulting cosimplicial space $\mathbf{TW}^\bullet$, are homotopy functors of $\mathfrak{M}^{\mathcal{A}}\mathbf{Y}$. This will let us identify $\mathbf{TW}^\bullet$ as a certain relative left derived functor $(\mathbf{L}^{\mathrm{rel}}\mathfrak{T})\mathfrak{M}^{\mathcal{A}}\mathbf{Y} = \mathfrak{T}\mathfrak{W}_\bullet$ of $\mathfrak{T}$ applied to the mapping algebra $\mathfrak{M}^{\mathcal{A}}\mathbf{Y}$ (see §5.1).

(3) Finally, we must show that in the cases of interest to us, the E_r-term of the homotopy spectral sequence for $\mathbf{TW}^\bullet = (\mathbf{L}\mathfrak{T})\mathfrak{M}^{\mathcal{A}}\mathbf{Y}$ depends only on the n-truncation $P^{r+2}\mathfrak{W}_\bullet$, for each $r \geqslant 2$. Functors $\mathbf{T}$ with this property are called *level*.

Remark 1.1. There are also a number of less familiar spectral sequences obtained dually by constructing a simplicial resolution $\mathbf{X}_\bullet \to \mathbf{Y}$ with respect to $\mathcal{B} = \{\mathbf{S}^i\}_{i=1}^{\infty}$, applying a homotopy functor $\mathbf{T}\colon \mathcal{C} \to \mathcal{C}$, and then using the homotopy spectral sequence for the simplicial space $\mathbf{TX}_\bullet$ (see [**Sto, Bl1, DKSS**]). Here too, one can identify the E^2-term with the derived functors of an algebraic functor of $\pi_*\mathbf{Y}$ (the algebraic object corepresented by $\mathcal{B}$). We include these in the paper mainly in order to show that the formalism we describe here is not limited to the Adams spectral sequence, even though this is our most important example. Moreover, in a number of ways the simplicial-covariant version is cleaner than the cosimplicial-contravariant one.

However, since Eckmann–Hilton duality is not formal, we are forced to work carefully through the details in the two versions separately: for this reason, each section is divided into two parts, starting with the covariant case.

For reasons of space, we deal here only with the unstable spectral sequences. For the stable analogue, we must choose a simplicial model category of spectra (cf. [**BF, EKMM, HSS, L**]) and work there throughout; one can still take Postnikov n-sections of the mapping spaces $\mathrm{map}_*(\mathcal{B}, \mathbf{X}_\bullet)$.

1.2. Outline

In Section 2 we define enriched sketches and the associated mapping algebras (with their dual versions). It turns out that we have competing versions of mapping algebras: the category $\mathsf{sMap}^{\mathrm{St},R}_{\mathrm{re}}$, which allows us to factor $\mathbf{T}$ as $\mathfrak{T} \circ \mathfrak{M}^{\mathcal{A}}$ in §1.1(1), is not right proper, so we need a variant $\mathcal{S}_*^{\mathbf{\Theta}^{\mathcal{A}}}$ in which $\mathfrak{W}_\bullet := \mathfrak{M}^{\mathcal{A}}\mathbf{W}^\bullet$ is indeed a cofibrant replacement for $\mathfrak{M}^{\mathcal{A}}\mathbf{Y}$ in the resolution model category $\mathcal{S}_*^{\mathbf{\Theta}^{\mathcal{A}} \times \Delta^{\mathrm{op}}}$.

In Section 3 we describe a category $\mathsf{sMap}_{\mathrm{St}}$ of $\mathcal{B}$-mapping algebras (when $\mathcal{B}$ is the sphere spectrum – cf. Remark 1.1), with a realization functor $N\colon \mathsf{sMap}_{\mathrm{St}} \to \mathsf{Top}_*$ (see Theorem 3.10 and Corollary 3.11).

In Section 4 we construct the analogous category $\mathsf{sMap}^{\mathrm{St},R}_{\mathrm{re}}$ of dual mapping algebras, for $\mathcal{A}$ the Eilenberg–Mac Lane spectrum for a commutative ring R, and show:

Theorem A. *There is a* realization *functor* $N\colon (\mathsf{sMap}^{\mathrm{St},R}_{\mathrm{re}})^{\mathrm{op}} \to \mathcal{S}_*$*, equipped with a natural weak equivalence* $\mathfrak{M}^{\mathcal{A}} \circ N \to \mathrm{Id}$.

See Theorem 4.7 and Corollary 4.8 below.

Thus any homotopy functor $\mathbf{T}\colon \mathsf{Top}_* \to \mathsf{Top}_*$ which preserves R-equivalences, when restricted to R-good spaces, induces a functor $\mathfrak{T} := \mathbf{T} \circ N : (\mathsf{sMap}^{\mathrm{St},R}_{\mathrm{re}})^{\mathrm{op}} \to \mathsf{Top}_*$ equipped with a natural weak equivalence $\mathfrak{T} \circ \mathfrak{M}^{\mathrm{St},R} \to \mathbf{T}$.

In Section 5 we define the general notion of a relative derived functor (§5.1), and show how it applies to the functor $\mathfrak{T}\colon (\mathsf{sMap}^{\mathrm{St},R}_{\mathrm{re}})^{\mathrm{op}} \to \mathsf{Top}_*$ associated to the homotopy functor $\mathbf{T}\colon \mathsf{Top}_* \to \mathsf{Top}_*$. The dual notion is treated in Section 6.

In order to do so, we have to relate the two types of mapping algebras described in Section 2 – those that are used for resolutions, and those for which $\mathfrak{T}$ is defined – by means of Theorem 6.4, which implies:

Theorem B. *If* $\mathbf{Y}$ *is* R*-good, any simplicial resolution* $\mathfrak{V}_\bullet$ *of* $\mathfrak{M}^{\mathrm{St},R}\mathbf{Y}$ *in the resolution model category* $\mathcal{S}_*^{\mathbf{\Theta}^{\mathcal{A}} \times \Delta^{\mathrm{op}}}$ *is Reedy weakly equivalent (i.e., in each simplicial dimension) to a simplicial object* $\mathfrak{W}_\bullet$ *in* $(\mathsf{sMap}^{\mathrm{St},R}_{\mathrm{re}})^{\Delta^{\mathrm{op}}}$.

The dual version, for the sphere spectrum, is Theorem 5.8.

Finally, in Section 7 we deal with the truncated versions of our higher derived functors, and explain what data is needed to determine the E_r-term of the homotopy spectral sequence of a (co)simplicial space by formalizing the notion of a *level functor* (Definition 7.2), with the dual version described in Section 8. We then show

Theorem C. *For* $R = \mathbb{F}_p$ *or* $\mathbb{Q}$*,* $\mathbf{Z} \in \mathcal{S}_*$*, and* R*-good* $\mathbf{Y}$*, the unstable Adams spectral sequence for* $\mathrm{map}_*(\mathbf{Z}, \mathbf{Y})$ *is determined by a simplicial mapping algebra resolution* $\mathfrak{W}_\bullet$ *of* $\mathfrak{M}^{\mathrm{St},R}\mathbf{Y}$*, and for each* $r \geqslant 2$ *the* E_r*-term is determined by the corresponding* $(r-2)$*-truncated mapping algebras.*

See Theorem 8.2.

This implies that the mapping space functor $\mathrm{map}_*(\mathbf{Z}, -)$ is a level homotopy functor on R-good spaces. We also prove a number of similar results for functors related to the sphere spectrum (see Propositions 7.8, 7.9, and 7.10).

1.3. Notation

The category of finite ordered sets and order-preserving maps will be denoted by Δ (cf. [**Ma2**, §2]), so a simplicial object $G_\bullet$ in $\mathcal{C}$ is a functor $\Delta^{\mathrm{op}} \to \mathcal{C}$, and the category of such will be denoted by $\mathcal{C}^{\Delta^{\mathrm{op}}}$. Similarly, a cosimplicial object $G^\bullet$ in a category $\mathcal{C}$ is a functor $\Delta \to \mathcal{C}$, and the category of such will be denoted by $\mathcal{C}^{\Delta}$. There is a natural embedding $c(-)_\bullet : \mathcal{C} \to \mathcal{C}^{\Delta^{\mathrm{op}}}$ (the constant simplicial object), and similarly $c(-)^\bullet : \mathcal{C} \to \mathcal{C}^{\Delta}$.

Write Δ_+ for the subcategory of Δ with the same objects but only monic maps. A functor $G\colon \Delta_+^{\mathrm{op}} \to \mathcal{C}$ (respectively, $G\colon \Delta_+ \to \mathcal{C}$) is called a *restricted* (co)simplicial object in $\mathcal{C}$. The inclusion $i\colon \Delta_+ \hookrightarrow \Delta$ induces a forgetful functor $i^*\colon \mathcal{C}^{\Delta^{\mathrm{op}}} \to \mathcal{C}^{\Delta_+^{\mathrm{op}}}$, which has a left adjoint $\mathcal{L}\colon \mathcal{C}^{\Delta_+^{\mathrm{op}}} \to \mathcal{C}^{\Delta^{\mathrm{op}}}$ (for suitable $\mathcal{C}$).

The category of topological spaces will be denoted by Top, that of pointed spaces by Top_*, and that of pointed connected spaces by Top_0. The category of simplicial sets will be denoted by $\mathcal{S} = \mathsf{Set}^{\Delta^{\mathrm{op}}}$, that of pointed simplicial sets by $\mathcal{S}_* = \mathsf{Set}_*^{\Delta^{\mathrm{op}}}$, that of simplicial groups by $\mathcal{G} = \mathsf{Gp}^{\Delta^{\mathrm{op}}}$. Write $\mathrm{map}_*(\mathbf{X}, \mathbf{Y})$ for the standard function complex in $\mathcal{S}_*$, Top_0, or $\mathcal{G}$ (see [**GJ**, I, §1.5]). Note that both Top_0 and $\mathcal{S}_*$ are enriched over $(\mathcal{S}_*, \wedge)$, but if we forget the basepoints, the same mapping spaces $\mathrm{map}_{\mathcal{S}_*}(X, Y)$ or $\mathrm{map}_{\mathsf{Top}_0}(X, Y)$ also define an enrichment over $(\mathcal{S}, \times)$, which is the one we shall use (see [**H**, 9.1.14]).

We denote the category of pointed Kan complexes by $\mathcal{S}^{\mathrm{Kan}}$, that of *reduced* simplicial sets (with a single vertex) by $\mathcal{S}^{\mathrm{red}}$, and the full subcategory of n-types in $\mathcal{S}_*$ – i.e., spaces X with $\pi_i(X, x) = 0$ for $i > n$ and all $x \in X_0$ – by $\mathcal{S}_{[n]}$, with $P^n\colon \mathcal{S}_* \to \mathcal{S}_{[n]}$ the n-th *Postnikov section* functor.

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2. Enriched sketches and mapping algebras

The main technical tool in our approach is the notion of a mapping algebra, first introduced in [**BB2**, §8]. We shall need a number of variants of this notion.

Definition 2.1. Let $\mathcal{C}$ be a pointed simplicial model category, $\mathcal{B}$ a set of fibrant and cofibrant homotopy cogroup objects in $\mathcal{C}$, $\mathcal{F}$ a category of finite simplicial sets, and $\mathcal{E}$ a set of cocones in $\mathcal{C}$. The associated *enriched sketch*, or multi-sorted theory (cf. [**Bor**, §5.6]) $\mathbf{\Theta}_{\mathcal{B}} = \mathbf{\Theta}_{(\mathcal{B},\mathcal{F},\mathcal{E})}$ is the smallest full sub-simplicial category of $\mathcal{C}$ containing $\mathcal{B}$ and closed under the operations $-\otimes K$ for $K \in \mathcal{F}$ and taking colimits of the cocones in $\mathcal{E}$. In this setting:

(1) A $\mathcal{B}$-*presheaf* is a pointed simplicial functor $\mathfrak{X}\colon \mathbf{\Theta}_{\mathcal{B}}^{\mathrm{op}} \to \mathcal{S}_*$. The category of all $\mathcal{B}$-presheaves is denoted by $\mathcal{S}_*^{\mathbf{\Theta}_{\mathcal{B}}^{\mathrm{op}}}$, and the value of $\mathfrak{X}$ at $\mathbf{B} \in \mathbf{\Theta}_{\mathcal{B}}$ will be written $\mathfrak{X}\{\mathbf{B}\}$.
A map $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{Y}$ of $\mathcal{B}$-presheaves is called a *weak equivalence* if $\mathfrak{f}\{\mathbf{B}\}\colon \mathfrak{X}\{\mathbf{B}\} \to \mathfrak{Y}\{\mathbf{B}\}$ is a weak equivalence for each $\mathbf{B} \in \mathbf{\Theta}_{\mathcal{B}}$. Two $\mathcal{B}$-presheaves are said to be *weakly equivalent* if they are connected by a finite zigzag of weak equivalences.

(2) A *weak $\mathcal{B}$-mapping algebra* is a $\mathcal{B}$-presheaf $\mathfrak{X}$ for which the natural maps

$$\mathfrak{X}\{\mathbf{B} \otimes K\} \;\to\; \mathfrak{X}\{\mathbf{B}\}^K \qquad \text{and} \qquad \mathfrak{X}\{\mathrm{colim}_{i\in I}\, \mathbf{B}_i\} \;\to\; \lim_{i\in I}\, \mathfrak{X}\{\mathbf{B}_i\} \tag{2.2}$$

are isomorphisms for all $\mathbf{B} \in \mathbf{\Theta}_{\mathcal{B}}$, $K \in \mathcal{F}$, and diagrams I in $\mathcal{E}$. The full subcategory of strict $\mathcal{B}$-mapping algebras will be denoted by $\mathsf{sMap}_{\mathcal{B}}$.

(3) A *weak $\mathcal{B}$-mapping algebra* is a $\mathcal{B}$-presheaf $\mathfrak{X}$ which is weakly equivalent to a strict $\mathcal{B}$-mapping algebra. Thus in particular, the maps of (2.2) are weak equivalences. The full subcategory of weak $\mathcal{B}$-mapping algebras will be denoted by $\mathsf{wMap}_{\mathcal{B}}$.

Remark 2.3. In principle, we would like to identify a weak $\mathcal{B}$-mapping algebra more conceptually as a $\mathcal{B}$-presheaf for which not only the maps of (2.2) are weak equivalences, but also appropriate higher coherences hold. However, as we shall not, in fact, need to work explicitly with weak $\mathcal{B}$-mapping algebras, we can make do here with the above ad hoc definition.

Example 2.4. The main example of an enriched sketch we shall consider in this paper is the case where $\mathcal{C} = \mathsf{Top}_0$, $\mathcal{B} = \{\mathbf{S}^n\}_{n=1}^{\infty}$ and $\mathcal{F}$ consists of the inclusions $i_0, i_1\colon \Delta[0] \hookrightarrow \Delta[1]$. The cocone collection $\mathcal{E}$ contains all coproducts of cardinality $< \lambda$ for some fixed limit cardinal λ (e.g., $\aleph_0$), and the pushout squares

$$\begin{array}{ccc} \mathbf{B} & \hookrightarrow & \mathbf{B}\otimes\Delta[1] \\ \simeq\downarrow & \boxed{\mathrm{PO}} & \simeq\downarrow\mathrm{inc}_0 \\ * & \hookrightarrow & C\mathbf{B} \end{array} \qquad\qquad \begin{array}{ccc} \mathbf{B} & \hookrightarrow & C\mathbf{B} \\ \downarrow & \boxed{\mathrm{PO}} & \downarrow\mathrm{inc}_1 \\ * & \hookrightarrow & \Sigma\mathbf{A} \end{array} \tag{2.5}$$

for $\mathbf{B} \in \mathbf{\Theta}_{\mathcal{B}}$. (These will be our models for the cone CX and suspension ΣX of any $X \in \mathcal{C}$).

Thus a strict $\mathcal{B}$-mapping algebra $\mathfrak{X}$ will take the two squares of (2.5) to pullback squares:

$$\begin{array}{ccc} P\mathfrak{X}\{\mathbf{B}\} & \hookrightarrow & \mathfrak{X}\{\mathbf{B}\}^{\Delta[1]} \\ \simeq\downarrow & \boxed{\mathrm{PB}} & \mathrm{ev}_0\downarrow\simeq \\ * & \hookrightarrow & \mathfrak{X}\{\mathbf{B}\} \end{array} \qquad\qquad \begin{array}{ccc} \Omega\mathfrak{X}\{\mathbf{B}\} & \xhookrightarrow{\iota_{\mathbf{B}}} & P\mathfrak{X}\{\mathbf{B}\} \\ \downarrow & \boxed{\mathrm{PB}} & \mathrm{ev}_1\downarrow \\ * & \hookrightarrow & \mathfrak{X}\{\mathbf{B}\} \end{array} \tag{2.6}$$

One might also consider localized versions, where $\mathcal{B} = \{\mathbf{S}^n_R\}_{n=1}^{\infty}$ for some subring $R \subseteq \mathbb{Q}$ (cf. [**Bi**]). In particular, when $R = \mathbb{Q}$ we may replace $\mathcal{C} = \mathsf{Top}_0$ by a suitable algebraic model of rational homotopy types, such as the category of differential graded Lie algebras.

More generally, one could take any space $\mathbf{M} \in \mathsf{Top}_0$, and let $\mathcal{B} = \{\Sigma^n\mathbf{M}\}_{n=1}^{\infty}$. However, while the formal part of our program can be made to work in this case (see [**BBD**]), the application to the homotopy spectral sequence of a simplicial space is not available for $\mathbf{M}$ which is not essentially a sphere (see [**CDI**] and [**Bl2**, §4.6]).

Definition 2.7. For any enriched sketch $\mathbf{\Theta}_{\mathcal{B}}$ as above, the most important example of a $\mathcal{B}$-presheaf $\mathfrak{X}$ is a *realizable* one, associated to an object $\mathbf{Y} \in \mathcal{C}$, where $\mathfrak{X}\{\mathbf{B}\} := \mathrm{map}_{\mathcal{C}}(\mathbf{B}, \mathbf{Y})$ for any $\mathbf{B} \in \mathbf{\Theta}_{\mathcal{B}}$. Evidently, this will be a strict $\mathcal{B}$-mapping algebra, which we denote by $\mathfrak{M}_{\mathcal{B}}\mathbf{Y}$ (of course, it actually takes *all* colimits in $\mathbf{\Theta}_{\mathcal{B}}$ to the corresponding limits). When $\mathbf{Y} \in \mathrm{Obj}\,\mathbf{\Theta}_{\mathcal{B}}$, we say that $\mathfrak{M}_{\mathcal{B}}\mathbf{Y}$ is *free.*

The strong Yoneda Lemma for enriched categories (see [**K**, 2.4]) implies:

Lemma 2.8. *If $\mathfrak{Y}$ is a $\mathcal{B}$-presheaf and $\mathfrak{M}_{\mathcal{B}}\mathbf{B}$ is a free strict $\mathcal{B}$-mapping algebra (for $\mathbf{B} \in \mathbf{\Theta}_{\mathcal{B}}$), there is a natural isomorphism*

$$\Phi\colon\ \mathrm{map}_{\mathcal{S}_*^{\mathbf{\Theta}_{\mathcal{B}}^{\mathrm{op}}}}(\mathfrak{M}_{\mathcal{B}}\mathbf{B},\,\mathfrak{Y}) \ \xrightarrow{\cong}\ \mathfrak{Y}\{\mathbf{B}\},$$

with $\Phi(\mathfrak{f}) = \mathfrak{f}(\mathrm{Id}_{\mathbf{B}}) \in \mathfrak{Y}\{\mathbf{B}\}_0$ *for any* $\mathfrak{f} \in \mathrm{Hom}_{\mathcal{S}_*^{\mathbf{\Theta}_{\mathcal{B}}^{\mathrm{op}}}}(\mathfrak{M}_{\mathcal{B}}\mathbf{B},\,\mathfrak{Y}) = \mathrm{map}_{\mathcal{S}_*^{\mathbf{\Theta}_{\mathcal{B}}^{\mathrm{op}}}}(\mathfrak{M}_{\mathcal{B}}\mathbf{B},\,\mathfrak{Y})_0$.

Remark 2.9. It is sometimes convenient think of a $\mathcal{B}$-presheaf $\mathfrak{X}$ as a category $\mathcal{X}$ with object set $\mathcal{O} := \mathrm{Obj}(\mathbf{\Theta}_{\mathcal{B}}) \cup \{\star\}$, enriched in pointed simplicial sets as follows:

$$\mathrm{map}_{\mathcal{X}}(\mathbf{A},\mathbf{B}) = \begin{cases} \mathrm{map}_{\mathbf{\Theta}_{\mathcal{B}}}(\mathbf{A},\mathbf{B}) & \text{if } \mathbf{A},\mathbf{B} \in \mathrm{Obj}(\mathbf{\Theta}_{\mathcal{B}}),\\ \mathfrak{X}\{\mathbf{A}\} & \text{if } \mathbf{A} \in \mathrm{Obj}(\mathbf{\Theta}_{\mathcal{B}}) \text{ and } \mathbf{B} = \star,\\ c(\{*,\mathrm{Id}_{\star}\})_{\bullet} & \text{if } \mathbf{A} = \mathbf{B} = \star,\\ c(\{*\})_{\bullet} & \text{otherwise.} \end{cases} \tag{2.10}$$

Thus a realizable $\mathcal{B}$-presheaf $\mathfrak{X} = \mathfrak{M}_{\mathcal{B}}\mathbf{Y}$ corresponds to a sub-simplicial category $\mathcal{X}$ of $\mathcal{C}$ with object set $\mathrm{Obj}(\mathbf{\Theta}_{\mathcal{B}}) \cup \{\mathbf{Y}\}$ (compare [**BB2**, §8.1]).

Definition 2.11. An enriched sketch $\mathbf{\Theta}_{\mathcal{B}}$ in a model category $\mathcal{C}$ has an algebraic version, which is the (ordinary) sketch $\Theta_{\mathcal{B}} := \pi_0\mathbf{\Theta}_{\mathcal{B}}$ – that is, $\Theta_{\mathcal{B}}$ has the same objects as $\mathbf{\Theta}_{\mathcal{B}}$, and $\mathrm{Hom}_{\Theta_{\mathcal{B}}}(\mathbf{B},\mathbf{B}') := \pi_0\,\mathrm{map}_{\mathbf{\Theta}_{\mathcal{B}}}(\mathbf{B},\mathbf{B}')$. An *algebra* (or *model*) for $\Theta_{\mathcal{B}}$ is a functor $\Lambda\colon \Theta_{\mathcal{B}}^{\mathrm{op}} \to \mathsf{Set}$ which takes the coproduct of any discrete cocone in $\mathcal{E}$ to a product in Set (see [**Bor**, §5.6]).

These are called $\Pi_{\mathcal{B}}$*-algebras*, and the category of such is denoted by $\Pi_{\mathcal{B}}$-Alg: for $\mathcal{B} = \{\mathbf{S}^n\}_{n=1}^{\infty}$, these are simply the Π*-algebras* of [**DK2**]. Note that if $\mathfrak{X}$ is a (weak or strict) $\mathcal{B}$-mapping algebra, then $\pi_0\mathfrak{X}$ is a $\Pi_{\mathcal{B}}$-algebra; the same need not hold for an arbitrary $\mathcal{B}$-presheaf. We say that a $\Pi_{\mathcal{B}}$-algebra Λ is *realizable* if it is of the form $\pi_0\mathfrak{M}_{\mathcal{B}}\mathbf{Y}$ for some $\mathbf{Y} \in \mathcal{C}$. A coproduct of $\Pi_{\mathcal{B}}$-algebras of the form $\pi_0\mathfrak{M}_{\mathcal{B}}\mathbf{B}$ for $\mathbf{B} \in \mathrm{Obj}\,\mathbf{\Theta}_{\mathcal{B}}$ is called *free.*

There are dual versions of all three notions discussed in Section 2, defined as follows:

Definition 2.12. Let $\mathcal{C}$ be a pointed simplicial model category, $\mathcal{A}$ a set of fibrant and cofibrant homotopy group objects in $\mathcal{C}$, $\mathcal{F}$ a category of finite simplicial sets, and $\mathcal{L}$ a set of cones in $\mathcal{C}$. The associated *dual enriched sketch* $\mathbf{\Theta}^{\mathcal{A}} = \mathbf{\Theta}^{(\mathcal{A},\mathcal{K},\mathcal{L})}$ is the smallest full sub-simplicial category of $\mathcal{C}$ containing $\mathcal{A}$ and closed under the operations $(-)^K$ for $K \in \mathcal{F}$ and taking limits of the cones in $\mathcal{L}$. In this setting:

(1) An $\mathcal{A}$-*dual presheaf* is a pointed simplicial functor $\mathfrak{X}\colon \mathbf{\Theta}^{\mathcal{A}} \to \mathcal{S}_*$. The category of $\mathcal{A}$-dual presheaves is denoted by $\mathcal{S}_*^{\mathbf{\Theta}^{\mathcal{A}}}$, and the value of $\mathfrak{X}$ at $\mathbf{A} \in \mathbf{\Theta}^{\mathcal{A}}$ will again be written $\mathfrak{X}\{\mathbf{A}\}$.

(2) A *dual strict* $\mathcal{A}$*-mapping algebra* is a $\mathcal{A}$-dual presheaf $\mathfrak{X}$ for which the natural maps

$$\mathfrak{X}\{\mathbf{A}^K\} \to \mathfrak{X}\{\mathbf{A}\}^K \qquad \text{and} \qquad \mathfrak{X}\{\lim_{i\in I} \mathbf{A}_i\} \to \lim_{i\in I} \mathfrak{X}\{\mathbf{A}_i\} \tag{2.13}$$

are isomorphisms for all $\mathbf{A}, \mathbf{A}_i \in \mathbf{\Theta}^{\mathcal{A}}$, $K \in \mathcal{F}$, and diagrams I in $\mathcal{L}$. The subcategory of dual strict $\mathcal{A}$-mapping algebras will be denoted by $\mathsf{sMap}^{\mathcal{A}}$.

(3) A *dual weak* $\mathcal{A}$*-mapping algebra* is a $\mathcal{A}$-dual presheaf $\mathfrak{X}$ which is weakly equivalent to a dual strict $\mathcal{A}$-mapping algebra, so in particular, the maps of (2.13) are weak equivalences (see Remark 2.3 above). The subcategory of dual weak $\mathcal{A}$-mapping algebras will be denoted by $\mathsf{wMap}^{\mathcal{A}}$.

Example 2.14. The main example of an enriched dual sketch we consider here is the Ω-*spectrum* case, where $\mathcal{C} = \mathcal{S}_*$ and $\mathcal{A} = \{\mathbf{A}_n\}_{n=-\infty}^{\infty}$ are the spaces of an Ω-spectrum A (in the sense of [**BF**]). The category $\mathcal{F}$ then consists of the inclusions $i_0, i_1\colon \Delta[0] \hookrightarrow \Delta[1]$, and the cone collection $\mathcal{L}$ contains all products of cardinality $< \lambda$ for some fixed limit cardinal λ and the pullback squares

$$\begin{array}{ccc} P\mathbf{A} & \hookrightarrow & \mathbf{A}^{\Delta[1]} \\ {\scriptstyle\simeq}\downarrow & \boxed{\text{PB}} & \downarrow{\scriptstyle \mathrm{ev}_0}\,{\scriptstyle\simeq} \\ * & \hookrightarrow & \mathbf{A} \end{array} \qquad\qquad \begin{array}{ccc} \Omega\mathbf{A} & \overset{\iota_{\mathbf{A}}}{\hookrightarrow} & P\mathbf{A} \\ \downarrow & \boxed{\text{PB}} & \downarrow{\scriptstyle \mathrm{ev}_1} \\ * & \hookrightarrow & \mathbf{A} \end{array} \tag{2.15}$$

for any $\mathbf{A} \in \mathbf{\Theta}^{\mathcal{A}}$. Thus a dual strict $\mathcal{A}$-mapping algebra $\mathfrak{X}$ will take the two pullback squares of (2.15) to those of (2.6).

More generally, one might take any set of Ω-spectra − in particular, the set of all A-module spectra of bounded cardinality, for a fixed ring spectrum A.

Definition 2.16. For any dual enriched sketch $\mathbf{\Theta}^{\mathcal{A}}$, the *realizable* dual strict $\mathcal{A}$-mapping algebra $\mathfrak{X}$ associated to $\mathbf{Y} \in \mathcal{C}$ has $\mathfrak{X}\{\mathbf{A}\} := \mathrm{map}_{\mathcal{C}}(\mathbf{Y}, \mathbf{A})$ for each $\mathbf{A} \in \mathbf{\Theta}_{\mathcal{B}}$. We will denote it by $\mathfrak{M}^{\mathcal{A}}\mathbf{Y}$. When $\mathbf{Y} \in \mathrm{Obj}\,\mathbf{\Theta}^{\mathcal{A}}$, we again say that $\mathfrak{M}^{\mathcal{A}}\mathbf{Y}$ is *free.*

The analogue of Lemma 2.8 also holds:

Lemma 2.17 (cf. [**BS2**, Lemma 1.12])**.** *If* $\mathfrak{Y}$ *is an* $\mathcal{A}$*-dual presheaf and* $\mathfrak{M}^{\mathcal{A}}\mathbf{A}$ *is a free dual strict* $\mathcal{A}$*-mapping algebra (for* $\mathbf{A} \in \mathbf{\Theta}^{\mathcal{A}}$*), there is a natural isomorphism*

$$\Phi\colon\ \mathrm{map}_{\mathcal{S}_*^{\mathbf{\Theta}^{\mathcal{A}}}}(\mathfrak{M}^{\mathcal{A}}\mathbf{A}, \mathfrak{Y}) \xrightarrow{\cong} \mathfrak{Y}\{\mathbf{A}\},$$

with $\Phi(\mathfrak{f}) = \mathfrak{f}(\mathrm{Id}_{\mathbf{A}}) \in \mathfrak{Y}\{\mathbf{A}\}_0$ *for any* $\mathfrak{f} \in \mathrm{Hom}_{\mathcal{S}_*^{\mathbf{\Theta}^{\mathcal{A}}}}(\mathfrak{M}^{\mathcal{A}}\mathbf{A}, \mathfrak{Y})$*.*

Definition 2.18. As in Definition 2.11, given a dual enriched sketch $\mathbf{\Theta}^{\mathcal{A}}$, the corresponding "algebraic" sketch $\Theta^{\mathcal{A}} := \pi_0\mathbf{\Theta}^{\mathcal{A}}$, whose models are now functors $\Lambda\colon \Theta^{\mathcal{A}} \to \mathsf{Set}$ preserving all products among the cones listed in $\mathcal{E}$. These will be called $\Pi^{\mathcal{A}}$-*algebras*, and their category will be denoted by $\Pi^{\mathcal{A}}$-Alg. Again, if $\mathfrak{X}$ is a (weak or strict) mapping algebra, then $\pi_0\mathfrak{X}$ is a $\Pi^{\mathcal{A}}$-algebra. A $\Pi^{\mathcal{A}}$-algebrais *realizable* if it is isomorphic to $\pi_0\mathfrak{M}^{\mathcal{A}}\mathbf{Y}$ for some $\mathbf{Y} \in \mathcal{C}$, and it is *free* if it is of the form $\pi_0\mathfrak{M}^{\mathcal{A}}\mathbf{A}$ for $\mathbf{A} \in \mathrm{Obj}\,\mathbf{\Theta}^{\mathcal{A}}$.

Example 2.19. When $\mathcal{A} = \{K(\mathbb{F}_p, i)\}_{i=1}^{\infty}$ and $\lambda = \aleph_0$, $\mathbf{\Theta}^{\mathcal{A}}$ is the simplicial category of finite type $\mathbb{F}_p$-GEMs, and a $\Pi^{\mathcal{A}}$-algebrais simply an unstable algebra over the mod p Steenrod algebra (cf. [**Sc**]).

2.1. Model categories of mapping algebras

Like all categories of simplicial functors with small indexing category, the (dual) presheaf categories $\mathcal{S}_*^{\mathbf{\Theta}_{\mathcal{B}}^{\mathrm{op}}}$ and $\mathcal{S}_*^{\mathbf{\Theta}^{\mathcal{A}}}$ have proper simplicial model category structures (see [**H**, 13.1.14]), in which the fibrations and weak equivalences are defined objectwise (see [**DK1**, §1]). Thus a map $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{Y}$ of $\mathcal{B}$-presheaves is a weak equivalence if for every $\mathbf{B} \in \mathcal{B}$, $\mathfrak{f}_*\colon \mathfrak{X}\{\mathbf{B}\} \to \mathfrak{Y}\{\mathbf{B}\}$ is a weak equivalence in $\mathcal{C}$ (as in Definition 2.1).

By a suitable left Bousfield localization of $\mathcal{S}_*^{\mathbf{\Theta}_{\mathcal{B}}^{\mathrm{op}}}$ and $\mathcal{S}_*^{\mathbf{\Theta}^{\mathcal{A}}}$ we can obtain model categories for weak $\mathcal{B}$-mapping algebras and dual weak $\mathcal{A}$-mapping algebras (i.e., model structures on the (dual) presheaf category in which the latter are the fibrant objects). However, since we cannot guarantee that these localized model structures are right proper (cf. [**H**, 3.4.4]), they will not be used in this paper.

Remark 2.20. Note that since we assumed the objects of $\mathbf{\Theta}_{\mathcal{B}}$ are cofibrant, when $\mathbf{Y}$ is fibrant the realizable $\mathcal{B}$-presheaf $\mathfrak{M}_{\mathcal{B}}\mathbf{Y}$ will be fibrant (that is, $\mathfrak{M}_{\mathcal{B}}\mathbf{Y}\{\mathbf{B}\}$ is a Kan complex for each $\mathbf{B} \in \mathbf{\Theta}_{\mathcal{B}}$). Similarly, for $\mathcal{A}$-dual presheaves, $\mathfrak{M}^{\mathcal{A}}\mathbf{Y}$ is fibrant if $\mathbf{Y}$ is cofibrant in $\mathcal{C}$.

2.2. Model categories of simplicial Π-algebras

Because both $\Pi_{\mathcal{B}}$-algebras (Definition 2.11) and $\Pi^{\mathcal{A}}$-algebras (Definition 2.18) are universal algebras in the sense of [**Mc**, VI, §8] having an underlying graded group structure, there is a model category structure on both the category $\Pi_{\mathcal{B}}\text{-}\mathsf{Alg}^{\Delta^{\mathrm{op}}}$ of simplicial $\Pi_{\mathcal{B}}$-algebras and the category $\Pi^{\mathcal{A}}\text{-}\mathsf{Alg}^{\Delta^{\mathrm{op}}}$ of simplicial $\Pi^{\mathcal{A}}$-algebras. In both cases a map $f\colon U_\bullet \to V_\bullet$ of simplicial Π-algebras is a weak equivalence (respectively, fibration) if and only if the map $f_*\colon U_\bullet\{\mathbf{B}\} \to \mathbf{W}_\bullet\{\mathbf{B}\}$ is a weak equivalence (respectively, fibration) of simplicial groups for each $\mathbf{B} \in \operatorname{Obj}\mathbf{\Theta}$. The cofibrant objects are retracts of free simplicial objects.

2.3. Truncating mapping algebras

Fix $n \geqslant 0$. Given a $\mathcal{B}$-presheaf $\mathfrak{X}\colon \mathbf{\Theta}_{\mathcal{B}}^{\mathrm{op}} \to \mathcal{S}_*$, we may post-compose $\mathfrak{X}$ with the n-th Postnikov section functor $P^n\colon \mathcal{S}_* \to \mathcal{S}_{[n]}$ to obtain a new $\mathcal{B}$-presheaf $P^n\mathfrak{X}$, which we now think of as a continuous functor on $P^n\mathbf{\Theta}_{\mathcal{B}}$ – that is, the sketch enriched in $\mathcal{S}_{[n]}$ obtained from $\mathbf{\Theta}_{\mathcal{B}}$ by applying P^n to each mapping space.

This is simplest to describe when $\mathfrak{X}$ is fibrant (cf. Remark 2.20), since then we can use the $(n+1)$-coskeleton functor $\mathrm{csk}_{n+1}\colon \mathcal{S}_* \to \mathcal{S}_*$ (which strictly preserves products) as our model for P^n. Note that the mapping spaces of $\mathbf{\Theta}_{\mathcal{B}}$ are always fibrant, since we assumed that all its objects are both fibrant and cofibrant. In the general case, we must first apply a fibrant replacement functor to $\mathfrak{X}$ in the model category $\mathcal{S}_*^{\mathbf{\Theta}_{\mathcal{B}}^{\mathrm{op}}}$ of §2.1.

The category of *n-truncated $\mathcal{B}$-presheaves* will be denoted by $\mathcal{S}_{[n]}^{\mathbf{\Theta}_{\mathcal{B}}^{\mathrm{op}}} \subset \mathcal{S}_*^{\mathbf{\Theta}_{\mathcal{B}}^{\mathrm{op}}}$, with the truncation functor $\gamma_{[n]}\colon \mathcal{S}_*^{\mathbf{\Theta}_{\mathcal{B}}^{\mathrm{op}}} \to \mathcal{S}_{[n]}^{\mathbf{\Theta}_{\mathcal{B}}^{\mathrm{op}}}$.

If $\mathfrak{X}$ is a (strict or weak) $\mathcal{B}$-mapping algebra, this usually will not be true of $P^n\mathfrak{X}$, since in general

$$P^n \operatorname{map}(\Sigma\mathbf{B},\mathbf{Y}) \simeq P^n\Omega \operatorname{map}(\mathbf{B},\mathbf{Y}) \not\simeq P^{n-1}\Omega \operatorname{map}(\mathbf{B},\mathbf{Y}) \simeq \Omega P^n \operatorname{map}(\mathbf{B},\mathbf{Y}). \tag{2.21}$$

Thus we must modify Definition 2.1 as follows, assuming for simplicity that the category $\mathcal{F}$ consists as above of the inclusions $i_0, i_1 \colon \Delta[0] \hookrightarrow \Delta[1]$, and the cocone collection $\mathcal{E}$ contains all coproducts of cardinality $< \lambda$ for some fixed limit cardinal λ, and the pushout squares (2.5):

(1) An *n-truncated strict $\mathcal{B}$-mapping algebra* is an n-truncated $\mathcal{B}$-presheaf $\mathfrak{X}$ for which the natural maps of (2.2) are isomorphisms for all $\mathbf{B} \in \mathbf{\Theta}_{\mathcal{B}}$, $K \in \mathcal{F}$, and diagrams I in $\mathcal{E}$, except for the right hand square in (2.6), where we have instead:

$$\mathfrak{X}\{\Sigma\mathbf{B}\} \to P^{n-1}\mathfrak{X}\{\Sigma\mathbf{B}\} \xrightarrow{\cong} P^{n-1}\Omega\mathfrak{X}\{\mathbf{B}\} \xleftarrow{\cong} \Omega\mathfrak{X}\{\mathbf{B}\}, \tag{2.22}$$

where the first and last maps are the standard fibrations, the middle map is the natural map of (2.2), and $\Omega\mathfrak{X}\{\mathbf{B}\}$ is an $(n-1)$-type by assumption, with the last map an isomorphism.
The full subcategory of n-truncated strict $\mathcal{B}$-mapping algebras will be denoted by $\mathsf{sMap}^n_{\mathcal{B}}$.

(2) An *n-truncated weak $\mathcal{B}$-mapping algebra* is an n-truncated $\mathcal{B}$-presheaf $\mathfrak{X}$ weakly equivalent to an n-truncated strict $\mathcal{B}$-mapping algebra. This implies that the maps of (2.2), and the two right maps in (2.22), are weak equivalences (see Remark 2.3). The full subcategory of n-truncated weak $\mathcal{B}$-mapping algebras will be denoted by $\mathsf{wMap}^n_{\mathcal{B}}$.

In particular, for any $\mathbf{Y} \in \mathcal{C}$ we have the associated *realizable* n-truncated strict $\mathcal{B}$-mapping algebra $P^n\mathfrak{M}_{\mathcal{B}}\mathbf{Y}$, which is *free* if $\mathbf{Y} \in \mathbf{\Theta}_{\mathcal{B}}$, and the analogue of Lemma 2.8 still holds. We define the n-truncated versions of $\mathcal{A}$-dual presheaves and (strict or weak) dual $\mathcal{A}$-mapping algebras dually.

3. Factoring functors through mapping algebras

The first step in our program is to show that suitable homotopy functors $\mathbf{T} \colon \mathcal{C} \to \mathcal{D}$ factor up to weak equivalence through an appropriate category of mapping algebras: in other words, find an enriched sketch $\mathbf{\Theta}_{\mathcal{B}}$ and a functor $\mathfrak{T} \colon \mathcal{S}_*^{\mathbf{\Theta}_{\mathcal{B}}^{\mathrm{op}}} \to \mathcal{D}$, equipped with a natural weak equivalence $\mathfrak{T} \circ \mathfrak{M}_{\mathcal{B}} \to \mathbf{T}$. In fact, $\mathfrak{T}$ need not be defined on all of $\mathcal{S}_*^{\mathbf{\Theta}_{\mathcal{B}}^{\mathrm{op}}}$; it suffices if it is defined on the subcategory $\mathsf{sMap}_{\mathcal{B}}$ of strict $\mathcal{B}$-mapping algebras where $\mathfrak{M}_{\mathcal{B}}$ takes values.

Dually, we could try to find a dual enriched sketch $\mathbf{\Theta}^{\mathcal{A}}$ and a functor $\mathfrak{T}' \colon \mathsf{sMap}^{\mathcal{A}} \to \mathcal{D}$ with a natural weak equivalence $\mathbf{T} \to \mathfrak{T}' \circ \mathfrak{M}^{\mathcal{A}}$.

The simplest way to define such a functor $\mathfrak{T}$ is in the case where every strict $\mathcal{B}$-mapping algebra $\mathfrak{X}$ is (functorially) realizable. Essentially, the only case where this is known to be true is when $\mathcal{C} = \mathsf{Top}_0$ and $\mathcal{B} = \{\mathbf{S}^n\}_{n=1}^{\infty}$. We briefly summarize the construction of [**BB2**, §9] (based on that of [**Sto**, §2]):

3.1. The Stover construction

Recall that for a pointed Kan complex $K \in \mathcal{S}_*$, the path space PK is given by $(PK)_n := \{x \in K_{n+1} \ : \ d_1 \cdots d_{n+1} x = *\}$, with re-indexed face and degeneracy maps, and the universal fibration $p\colon PK \to K$ is induced by d_0 (cf. [**Mo**]). Thus when K is a simplicial group, the map on 0-simplices $p_0\colon (PK)_0 \to K_0$ suffices to compute $\pi_0 K$. We therefore choose the category $\mathcal{G} = \mathsf{Gp}^{\Delta^{\mathrm{op}}}$ of simplicial groups as our model $\mathcal{C}$ for the homotopy theory of pointed connected spaces, and set $\mathcal{B} := \{\mathcal{S}^n\}_{n=1}^{\infty}$ (where $\mathcal{S}^n := F\mathbf{S}^{n-1}$, as a free simplicial group, is a strict cogroup object modelling the n-sphere in $\mathcal{G}$). For any limit cardinal λ, the resulting enriched sketch $\mathbf{\Theta}_{\mathcal{B}} = \mathbf{\Theta}_{\mathcal{B}}^{\lambda}$ then has a strict mapping algebra functor $\mathfrak{M}_{\mathcal{B}}\colon \mathcal{G} \to \mathsf{sMap}_{\mathcal{B}}$ with each $\mathfrak{M}_{\mathcal{B}}\mathbf{Y}\{\mathbf{B}\}$ a simplicial group (though the structure maps are just maps of pointed simplicial sets, in general).

Definition 3.1. Let $\Gamma := \mathbf{1}^{\mathbb{N}}$ be the category consisting of a countable collection of arrows, indexed by the objects of $\mathcal{B}$, and Set_*^{Γ} the category of Γ-indexed diagrams $\Phi := (\phi_n\colon E_n \to F_n)_{n\in\mathbb{N}}$ in pointed sets, called *arrow sets*. We have a forgetful functor $\rho\colon \mathsf{sMap}_{\mathcal{B}} \to \mathsf{Set}_*^{\Gamma}$, with $(\rho\mathfrak{X})_n = (p_0\colon (P\mathfrak{X}\{\mathcal{S}^n\})_0 \to (\mathfrak{X}\{\mathcal{S}^n\})_0)$. In fact, $\rho\mathfrak{X}$ is defined for any presheaf $\mathfrak{X}\colon \mathbf{\Theta}_{\mathcal{B}}^{\mathrm{op}} \to \mathcal{S}_*$, but we are only interested in the composite $\mathcal{R}_{\mathcal{B}} := \rho\mathfrak{M}_{\mathcal{B}}\colon \mathcal{G} \to \mathsf{Set}_*^{\Gamma}$. This has a left adjoint $\mathcal{L}_{\mathcal{B}}\colon \mathsf{Set}_*^{\Gamma} \to \mathcal{G}$, which assigns to an arrow set $\Phi = (\phi_n\colon E_n \to F_n)_{n\in\mathbb{N}}$ the coproduct

$$\mathcal{L}_{\mathcal{B}}\Phi := \coprod_{n\in\mathbb{N}} \ \coprod_{f\in F_n} Q_{(f)}, \tag{3.2}$$

where we define $Q_{(f)}$ for $f \in F_n$ as follows:

(a) If $* \neq f \in \mathrm{Im}\ \phi_n$, then $Q_{(f)}$ is defined by the pushout square

$$\begin{array}{ccc} \coprod_{e\in\phi_n^{-1}(f)} \ \mathcal{S}^n_{(e)} & \xrightarrow{\quad \nabla \quad} & \mathcal{S}^n_{(f)} \\ \downarrow{\scriptstyle \coprod i_{(e)}} & & \downarrow \\ \coprod_{e\in\phi_n^{-1}(f)} \ C\mathcal{S}^n_{(e)} & \longrightarrow & Q_{(f)} \end{array} \tag{3.3}$$

in $\mathcal{G}$ (where $i\colon \mathcal{S}^n \to C\mathcal{S}^n$ is the inclusion into the cone, and ∇ is the fold map).

(b) If $f \notin \mathrm{Im}\ \phi_n$, we set $Q_{(f)} := \mathcal{S}^n$.

(c) If $f = *$, we set

$$Q_{(f)} := \coprod_{*\neq e\in\phi_n^{-1}(*)} \Sigma\mathcal{S}^n_{(e)}.$$

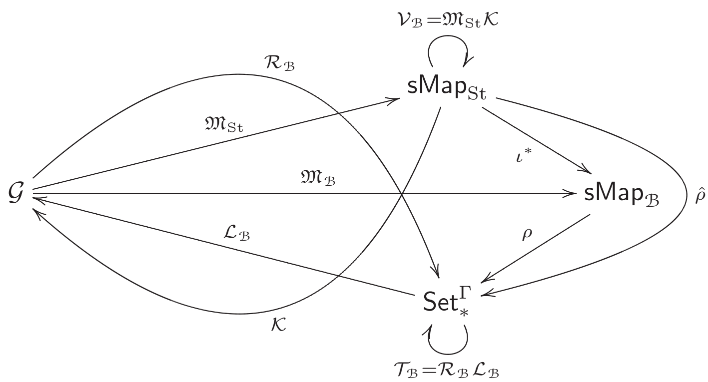
Compare [**BS2**, §2] and [**Sto**, §2], where the comonad $\mathcal{V}_{\mathcal{B}} = \mathcal{L}_{\mathcal{B}}\mathcal{R}_{\mathcal{B}}\colon \mathcal{G} \to \mathcal{G}$ (or rather, its analogue for Top_0) was used to construct functorial resolutions of pointed connected spaces by wedges of spheres.

Note that each $Q_{(f)}$, and thus $\mathcal{L}_{\mathcal{B}}\Phi$, is a strict cogroup object in $\mathcal{G}$ (fibrant and cofibrant) of the homotopy type of a wedge of spheres. If λ is any limit cardinal, we define a λ-*Stover space* to be any pushout of the form (3.3), with $\phi_n^{-1}(f)$ replaced by any set T of cardinality $< \lambda$. Let $\mathbf{\Theta}_{\mathrm{St}} = \mathbf{\Theta}_{\mathrm{St}}^{\lambda}$ denote a skeleton of the sub-simplicial category of $\mathcal{G}$ whose objects are coproducts of λ-Stover spaces over indexing sets of cardinality $< \lambda$. This is an enriched sketch, with $\mathcal{F}$ as in Example 2.4, and $\mathcal{E}$ consisting of the coproducts of cardinality $< \lambda$ in $\mathbf{\Theta}_{\mathrm{St}}$, together with the pushout squares of (2.5) and (3.3). The category of the corresponding strict mapping algebras, called *strict Stover mapping algebras,* will be denoted by $\mathsf{sMap}_{\mathrm{St}}$, with $\mathfrak{M}_{\mathrm{St}}\colon \mathcal{G} \to \mathsf{sMap}_{\mathrm{St}}$ the *strict Stover mapping algebra* functor.

3.2. The algebra structure

Since each sphere $\mathcal{S}^n \in \mathcal{G}$ is, in particular, a Stover space, $\mathbf{\Theta}_{\mathcal{B}} = \mathbf{\Theta}^{\lambda}_{\mathcal{B}}$ is a full simplicial subcategory of $\mathbf{\Theta}_{\mathrm{St}} = \mathbf{\Theta}^{\lambda}_{\mathrm{St}}$, with $\iota\colon \mathbf{\Theta}_{\mathcal{B}} \hookrightarrow \mathbf{\Theta}_{\mathrm{St}}$ the inclusion, inducing the restriction $\iota^*\colon \mathsf{sMap}_{\mathrm{St}} \to \mathsf{sMap}_{\mathcal{B}}$. Write $\hat{\rho}\colon \mathsf{sMap}_{\mathrm{St}} \to \mathsf{Set}^{\Gamma}_*$ for the composite $\rho \circ \iota^*$.

We claim that for every strict Stover mapping algebra $\mathfrak{X}$, the arrow set $\rho\mathfrak{X}$ has a natural $\mathcal{T}_{\mathcal{B}}$-algebra structure map $h\colon \mathcal{T}_{\mathcal{B}}\rho\mathfrak{X} \to \rho\mathfrak{X}$ for the monad $\mathcal{T}_{\mathcal{B}} = \mathcal{R}_{\mathcal{B}}\mathcal{L}_{\mathcal{B}}\colon \mathsf{Set}^{\Gamma}_* \to \mathsf{Set}^{\Gamma}_*$ (see [**Bor**, §4.1]). If we set $\mathcal{K} := \mathcal{L}_{\mathcal{B}} \circ \hat{\rho}\colon \mathsf{sMap}_{\mathrm{St}} \to \mathcal{G}$ and $\mathcal{V}_{\mathcal{B}} := \mathfrak{M}_{\mathrm{St}} \circ \mathcal{K}$, we may display the various functors defined in the following commuting diagram:



(3.4)

In this setting we have a stronger statement (cf. [**BB2**, 9.19]):

Lemma 3.5. *Every strict Stover mapping algebra $\mathfrak{X}$ has a natural map $\xi_{\mathfrak{X}}\colon \mathcal{V}_{\mathcal{B}}\mathfrak{X} \to \mathfrak{X}$ making the following diagram*

$$\begin{array}{ccc} \mathcal{V}_{\mathcal{B}}\mathcal{V}_{\mathcal{B}}\mathfrak{X} & \xrightarrow{\ \xi_{\mathcal{V}_{\mathcal{B}}\mathfrak{X}}\ } & \mathcal{V}_{\mathcal{B}}\mathfrak{X} \\ {\scriptstyle \mathcal{V}_{\mathcal{B}}\xi_{\mathfrak{X}}}\downarrow & & \downarrow{\scriptstyle \xi_{\mathfrak{X}}} \\ \mathcal{V}_{\mathcal{B}}\mathfrak{X} & \xrightarrow[\ \xi_{\mathfrak{X}}\ ]{} & \mathfrak{X} \end{array} \tag{3.6}$$

commute in $\mathsf{sMap}_{\mathrm{St}}$, where $\xi_{\mathcal{V}_{\mathcal{B}}\mathfrak{X}} = \mathfrak{M}_{\mathrm{St}}\varepsilon_{\mathcal{K}\mathfrak{X}}$ for $\varepsilon\colon \mathcal{K}\mathfrak{M}_{\mathrm{St}} \to \mathrm{Id}$ the counit of the comonad $\mathcal{L}_{\mathcal{B}}\mathcal{R}_{\mathcal{B}}$.

The structure map $h\colon \mathcal{T}_{\mathcal{B}}\hat{\rho}\mathfrak{X} \to \hat{\rho}\mathfrak{X}$ is then given by $\hat{\rho}(\xi_{\mathfrak{X}})$, since $\mathcal{T}_{\mathcal{B}} \circ \hat{\rho} = \hat{\rho} \circ \mathcal{V}_{\mathcal{B}}$ (see (3.4)).

Proof. Let $\mathcal{D}^i$ denote either $\mathcal{S}^{n_i}$ or $C\mathcal{S}^{n_i}$ in $\mathcal{G}$.

(a) Recall that $\mathcal{K}\mathfrak{X}$ is defined for any strict Stover mapping algebra $\mathfrak{X}$ by the colimit (3.3), which we may write as $\mathrm{colim}_i\, \mathcal{D}^i_{f_i}$, where $f_i \in \mathfrak{X}\{\mathcal{D}^i\}_0$. Since $\mathcal{K}\mathfrak{X} \in \mathbf{\Theta}_{\mathrm{St}}$, the strict Stover mapping algebra $\mathcal{V}_{\mathcal{B}}\mathfrak{X}$ is free, so to define the algebra structure map $\xi_{\mathfrak{X}}\colon \mathcal{V}_{\mathcal{B}}\mathfrak{X} \to \mathfrak{X}$ we need only specify $\xi_{\mathfrak{X}}(\mathrm{Id}_{\mathcal{K}\mathfrak{X}}) \in \mathfrak{X}\{\mathcal{K}\mathfrak{X}\}_0$. But $\mathfrak{X}$ takes the colimit of (3.3) to a limit, so $\xi_{\mathfrak{X}}(\mathrm{Id}_{\mathcal{K}\mathfrak{X}})$ is determined by the elements

$f_i \in \mathfrak{X}\{\mathcal{D}^i\}_0$. We therefore write $\xi_{\mathfrak{X}}(\mathrm{Id}_{\mathcal{K}\mathfrak{X}}) = \bigoplus_i f_i$, where $\oplus$ indicates that we are using the duality (2.2) between the colimits and the limits.

(b) Similarly, for any $Y \in \mathcal{G}$ we have $\mathcal{K}\mathfrak{M}_{\mathrm{St}}Y = \mathrm{colim}_j\, \mathcal{D}^j_{g_j}$. The counit $\varepsilon_Y : \mathcal{K}\mathfrak{M}_{\mathrm{St}}Y \to Y$ is again determined by the indexing maps as $\varepsilon_Y = \mathrm{colim}_j\, g_j$, with the induced map $\mathfrak{M}_{\mathrm{St}}\varepsilon_Y : \mathcal{V}\mathfrak{M}_{\mathrm{St}}Y \to \mathfrak{M}_{\mathrm{St}}Y$ sending $\mathrm{Id}_{\mathcal{K}\mathfrak{M}_{\mathrm{St}}Y}$ in $\mathfrak{M}_{\mathrm{St}}\mathcal{K}\mathfrak{M}_{\mathrm{St}}Y\{\mathcal{K}\mathfrak{M}_{\mathrm{St}}Y\}_0$ to $[\mathrm{colim}_j\, g_j]$ in $\mathfrak{M}_{\mathrm{St}}Y\{\mathcal{K}\mathfrak{M}_{\mathrm{St}}Y\}_0$.
Thus when $\mathfrak{X} = \mathfrak{M}_{\mathrm{St}}Y$, the map $\xi_{\mathfrak{X}}$ sends $\mathrm{Id}_{\mathcal{K}\mathfrak{M}_{\mathrm{St}}Y}$ to $\varepsilon_Y = \mathrm{colim}_j\, g_j$ in $\mathfrak{X}\{\mathcal{K}\mathfrak{M}_{\mathrm{St}}Y\}_0 = \mathrm{map}(\mathcal{K}\mathfrak{M}_{\mathrm{St}}Y,\, Y)_0$. This means that $\xi_{\mathfrak{M}_{\mathrm{St}}Y} = \mathfrak{M}_{\mathrm{St}}\varepsilon_Y$; in particular, the top horizontal map in (3.6) is $\xi_{\mathcal{V}_{\mathcal{B}}\mathfrak{X}}$.

(c) To evaluate the top right composite $\varphi := \xi_{\mathfrak{X}} \circ \mathfrak{M}_{\mathrm{St}}\varepsilon_{\mathcal{K}\mathfrak{X}} : \mathcal{V}_{\mathcal{B}}\mathcal{V}_{\mathcal{B}}\mathfrak{X} \to \mathfrak{X}$, note that $\mathcal{V}_{\mathcal{B}}\mathcal{V}_{\mathcal{B}}\mathfrak{X}$ is free on $\mathcal{K}\mathcal{V}_{\mathcal{B}}\mathfrak{X}$, so we need only specify $\varphi(\mathrm{Id}_{\mathcal{K}\mathcal{V}_{\mathcal{B}}\mathfrak{X}})$ in $\mathfrak{X}\{\mathcal{K}\mathcal{V}_{\mathcal{B}}\mathfrak{X}\}$. Since $\xi_{\mathfrak{X}}$ is a map of strict Stover mapping algebras, it sends $[\mathrm{colim}_j\, g_j] \in \mathcal{V}_{\mathcal{B}}\mathfrak{X}\{\mathcal{K}\mathcal{V}_{\mathcal{B}}\mathfrak{X}\}_0$ (for $Y := \mathcal{K}\mathfrak{X}$ in (b) above) to

$$[\perp_j g_j]^*(\xi_{\mathfrak{X}}(\mathrm{Id}_{\mathcal{K}\mathfrak{X}})) = \top_j\, g_j^*(\bigoplus_i f_i) \text{ in } \mathfrak{X}\{\mathcal{K}\mathcal{V}_{\mathcal{B}}\mathfrak{X}\}_0. \tag{3.7}$$

(d) Since $\mathcal{V}_{\mathcal{B}}\mathcal{V}_{\mathcal{B}}\mathfrak{X}$ is free, the map $\mathcal{V}_{\mathcal{B}}\xi_{\mathfrak{X}} : \mathcal{V}_{\mathcal{B}}\mathcal{V}_{\mathcal{B}}\mathfrak{X} \to \mathcal{V}_{\mathcal{B}}\mathfrak{X}$ is determined by where it sends $\mathrm{Id}_{\mathcal{K}\mathcal{V}_{\mathcal{B}}\mathfrak{X}}$ in $\mathcal{V}_{\mathcal{B}}\mathfrak{X}\{\mathcal{K}\mathcal{V}_{\mathcal{B}}\mathfrak{X}\}_0 = \mathrm{map}(\mathcal{K}\mathcal{V}_{\mathcal{B}}\mathfrak{X},\, \mathcal{K}\mathfrak{X})_0$, namely, to $\mathcal{K}\xi_{\mathfrak{X}} : \mathcal{K}\mathcal{V}_{\mathcal{B}}\mathfrak{X} \to \mathcal{K}\mathfrak{X}$. Since $\mathcal{K}\mathcal{V}_{\mathcal{B}}\mathfrak{X} = \mathrm{colim}_j\, \mathcal{D}^j_{g_j}$ where the colimit is over all maps $g_j : \mathcal{D}^j \to \mathcal{K}\mathfrak{X}$, we see from the description of $\xi_{\mathfrak{X}}$ above (and the construction of $\mathcal{K}$) that $\mathcal{K}\xi_{\mathfrak{X}}$ sends $\mathcal{D}^j_{g_j}$ to the copy of $\mathcal{D}^j$ in the colimit defining $\mathcal{K}\mathfrak{X}$ indexed by

$$\xi_{\mathfrak{X}}(g_j) = \xi_{\mathfrak{X}}(g_j^*(\mathrm{Id}_{\mathcal{K}\mathfrak{X}})) = g_j^*(\xi_{\mathfrak{X}}(\mathrm{Id}_{\mathcal{K}\mathfrak{X}})) = g_j^*(\bigoplus_i f_i) \tag{3.8}$$

in $\mathfrak{X}\{\mathcal{D}^j\}_0$, where $\xi_{\mathfrak{X}}(\mathrm{Id}_{\mathcal{K}\mathfrak{X}}) = \bigoplus_i f_i$ by (a).
Thus the element $\xi_{\mathfrak{X}}(\mathcal{V}_{\mathcal{B}}\xi_{\mathfrak{X}}(\mathrm{Id}_{\mathcal{K}\mathcal{V}_{\mathcal{B}}\mathfrak{X}}))$ in $\mathfrak{X}\{\mathcal{K}\mathcal{V}_{\mathcal{B}}\mathfrak{X}\}_0$ is determined by the fact that $\mathfrak{X}$ takes the colimit $\mathrm{colim}_j\, \mathcal{D}^j_{g_j}$ defining $\mathcal{K}\mathcal{V}_{\mathcal{B}}\mathfrak{X}$ to a limit, namely:

$$\xi_{\mathfrak{X}}(\mathcal{V}_{\mathcal{B}}\xi_{\mathfrak{X}}(\mathrm{Id}_{\mathcal{K}\mathcal{V}_{\mathcal{B}}\mathfrak{X}})) = \xi_{\mathfrak{X}}(\mathcal{K}\xi_{\mathfrak{X}}) = \xi_{\mathfrak{X}}(\perp_j g_j) = \top_j \xi_{\mathfrak{X}}(g_j) = \top_j g_j^*(\bigoplus_i f_i). \tag{3.9}$$

We see from (3.7) and (3.9) that the two composites agree on $\mathrm{Id}_{\mathcal{K}\mathcal{V}_{\mathcal{B}}\mathfrak{X}}$, so they are equal. □

3.3. The resolution model category of simplicial presheaves

For any set $\mathcal{B} \subset \mathcal{C}$ as in Definition 2.1, consider the category $(\mathcal{S}_*^{\mathbf{\Theta}_{\mathcal{B}}^{\mathrm{op}}})^{\Delta^{\mathrm{op}}} = \mathcal{S}_*^{\mathbf{\Theta}_{\mathcal{B}}^{\mathrm{op}} \times \Delta^{\mathrm{op}}}$ of *simplicial* $\mathcal{B}$*-presheaves* – that is, simplicial objects in the category of $\mathcal{B}$-presheaves. As noted in §2.1, the $\mathcal{B}$-presheaf category $\mathcal{S}_*^{\mathbf{\Theta}_{\mathcal{B}}^{\mathrm{op}}}$ has a proper simplicial model category structure. Moreover, the objects of $\mathcal{B}$ are homotopy cogroup objects in $\mathcal{C}$, as are their colimits under $\mathcal{E}$ as in Example 2.4. Therefore, as in [**J**, §2], there is a resolution model category structure on $\mathcal{S}_*^{\mathbf{\Theta}_{\mathcal{B}}^{\mathrm{op}} \times \Delta^{\mathrm{op}}}$, for which the projectives of $\mathcal{S}_*^{\mathbf{\Theta}_{\mathcal{B}}^{\mathrm{op}}}$ are the free strict $\mathcal{B}$-mapping algebras. A map $\mathfrak{f} : \mathfrak{V}_\bullet \to \mathfrak{W}_\bullet$ of simplicial $\mathcal{B}$-presheaves is a weak equivalence in this model category if and only if it is an E^2*-equivalence* – that is, if for each $\mathbf{B} \in \mathbf{\Theta}_{\mathcal{B}}$ and $t, s, \geqslant 0$, the map $\mathfrak{f}_* : \pi^h_t \pi^v_s \mathfrak{V}_\bullet\{\mathbf{B}\} \to \pi^h_t \pi^v_s \mathfrak{W}_\bullet\{\mathbf{B}\}$ is an isomorphism (the terminology comes from the E^2-term of the homotopy spectral sequence of a simplicial space – cf. [**DKS1**]).

Note that if a simplicial presheaf $\mathfrak{W}_\bullet$ is cofibrant, each $\mathfrak{W}_n$ is weakly equivalent to a coproduct of free strict $\mathcal{B}$-mapping algebras, so in particular, it is a weak $\mathcal{B}$-mapping algebra. Moreover, in order for $\mathfrak{W}_\bullet$ to be a resolution of a weak $\mathcal{B}$-mapping algebra $\mathfrak{X}$, in particular, $\pi_0\mathfrak{W}_\bullet$ must be a resolution of $\pi_0\mathfrak{X}$ in the model category of simplicial $\Pi_{\mathcal{B}}$-algebras (see §2.2), so that the augmented simplicial group $\pi_0\mathfrak{W}_\bullet\{\mathbf{B}\} \to \pi_0\mathfrak{X}\{\mathbf{B}\}$ is weakly contractible for any $\mathbf{B} \in \mathbf{\Theta}_{\mathcal{B}}$.

We observe also that $\mathcal{S}_*^{\mathbf{\Theta}_{\mathcal{B}}^{\mathrm{op}} \times \Delta^{\mathrm{op}}}$ has a Reedy model category structure, with weak equivalences and fibrations defined at each simplicial space $\mathfrak{V}_\bullet\{\mathbf{B}\}$ for every $\mathbf{B} \in \mathbf{\Theta}_{\mathcal{B}}$ (see [**H**, §15.3]).

Since P^n is a nullification, $\mathcal{S}_{[n]}$ is still right proper (see [**Bou3**, Theorem 9.9]), so we have an analogous resolution model category structure on the category $\mathcal{S}_{[n]}^{\mathbf{\Theta}_{\mathcal{B}}^{\mathrm{op}}}$ of n-truncated simplicial $\mathcal{B}$-presheaves (§2.3).

We deduce the following enhancement of [**BB2**, Proposition 9.23]:

Theorem 3.10. *There is a* realization functor $N\colon \mathsf{sMap}_{\mathrm{St}} \to \mathcal{G}$, *equipped with natural weak equivalences* $\theta\colon N \circ \mathfrak{M}_{\mathrm{St}} \to \mathrm{Id}_{\mathcal{G}}$ *and* $\zeta\colon \mathfrak{M}_{\mathrm{St}} \circ N \to \mathrm{Id}_{\mathsf{sMap}_{\mathrm{St}}}$.

Proof. Given a strict Stover mapping algebra $\mathfrak{X} \in \mathsf{sMap}_{\mathrm{St}}$, iterating the comonad $\mathcal{U} := \mathcal{L}_{\mathcal{B}}\mathcal{R}_{\mathcal{B}}\colon \mathcal{G} \to \mathcal{G}$ on $Y := \mathcal{K}\mathfrak{X} = \mathcal{L}_{\mathcal{B}}\hat{\rho}\mathfrak{X}$ yields an augmented simplicial space $\mathbf{Z}_\bullet \to Y$ with $\mathbf{Z}_n := \mathcal{U}^{n+1}Y$ and $d_i\colon \mathbf{Z}_n \to \mathbf{Z}_{n-1}$ given by as usual by $\mathcal{U}^i\varepsilon_{\mathcal{U}^{n-i}Y}$ (cf. [**W**, §8.6.4]).

Since by (3.4) $\mathcal{U} = \mathcal{L}_{\mathcal{B}}\mathcal{R}_{\mathcal{B}} = \mathcal{K}\mathfrak{M}_{\mathrm{St}}$ and $\mathcal{V}_{\mathcal{B}} = \mathfrak{M}_{\mathrm{St}}\mathcal{K}$, we have a simplicial strict Stover mapping algebra $\mathfrak{W}_\bullet = \mathfrak{M}_{\mathrm{St}}\mathbf{Z}_\bullet$, which augments to $\mathfrak{X}$ via $\xi_{\mathfrak{X}}\colon \mathfrak{M}_{\mathrm{St}}Y = \mathcal{V}_{\mathcal{B}}\mathfrak{X} \to \mathfrak{X}$, by Lemma 3.5. Applying $\mathcal{K}$ to $\mathfrak{W}_\bullet \to \mathfrak{X}$ recovers $\mathbf{Z}_\bullet \to Y$, but now with an extra degeneracy in each simplicial dimension coming from the unit $\eta\colon \mathrm{Id} \to \mathcal{T}_{\mathcal{B}} = \mathcal{R}_{\mathcal{B}}\mathcal{L}_{\mathcal{B}}$ of the corresponding monad, as well as an extra face map, obtained by iterating $\mathcal{U}$ on $\mathcal{K}\xi_{\mathfrak{X}}\colon \mathcal{K}\mathcal{V}_{\mathcal{B}}\mathfrak{X} = \mathbf{Z}_1 \to \mathcal{K}\mathfrak{X} = \mathbf{Z}_0$. By commutativity of (3.6), we see that $\mathbf{Z}_\bullet \to Y$ is, in fact, the décalage of a simplicial space $\mathbf{X}_\bullet$ (see [**I**]). Moreover, applying $\mathfrak{M}_{\mathrm{St}}$ to $\mathbf{X}_\bullet$ yields an augmented (free) simplicial strict Stover mapping algebra $\mathfrak{M}_{\mathrm{St}}\mathbf{X}_\bullet \to \mathfrak{X}$ which is a resolution of $\mathfrak{X}$ in the sense of §3.3.

This shows that the Quillen–Bousfield–Friedlander spectral sequence for $\mathbf{X}_\bullet$ (see [**Q1**] and [**BF**, Theorem B.5]) collapses, so that $N\mathfrak{X} := \|\mathbf{X}_\bullet\|$ realizes $\mathfrak{X}$ up to weak equivalence. Noting that $\mathbf{X}_\bullet$ is obtained by applying $\mathcal{K}$ to $\zeta_0\colon \mathfrak{M}_{\mathrm{St}}\mathbf{X}_\bullet \to \mathfrak{X}$, and that $\mathfrak{M}_{\mathrm{St}}\mathbf{X}_\bullet$ is constructed by iterating $\mathcal{V}_{\mathcal{B}}$ on $\mathfrak{X}$ (together with $\xi_{\mathfrak{X}}$), we have described a functorial procedure for realizing any strict Stover mapping algebra $\mathfrak{X}$. The natural weak equivalence ζ is induced by the augmentation ζ_0, while θ comes from the counit of the Stover comonad. □

Corollary 3.11. *Any homotopy functor* $\mathbf{T}\colon \mathcal{G} \to \mathcal{D}$ *to a model category* $\mathcal{D}$ *induces a functor* $\mathfrak{T} := \mathbf{T} \circ N\colon \mathsf{sMap}_{\mathrm{St}} \to \mathcal{D}$ *equipped with a natural weak equivalence* $\vartheta = \mathbf{T}\theta\colon \mathfrak{T} \circ \mathfrak{M}_{\mathrm{St}} \to \mathbf{T}$.

4. Realizing dual mapping algebras

To dualize the results of Section 3, we want a setting where every dual strict $\mathcal{A}$-mapping algebra $\mathfrak{X}$ is functorially realizable. Again we have only one case where this is known to be true, when $\mathcal{C} = \mathcal{S}^{\mathrm{red}}$ (or similar model categories for pointed connected spaces) and $\mathcal{A}$ consists of certain simplicial R-modules for some commutative ring R.

Definition 4.1. In general, we must include in the corresponding enriched sketch $\mathbf{\Theta}^{\mathcal{A}}$ all R-module GEMs up to a certain cardinality. In particular, when $\mathcal{C} = \mathcal{S}^{\mathrm{red}}$ we let $\mathbf{\Theta}^R = \mathbf{\Theta}^R_\lambda := s\mathcal{M}^R_\lambda$, be the full subsimplicial category of $\mathcal{C}$ consisting of all simplicial R-modules of cardinality $< \lambda$, for some limit cardinal λ (determined as in [**BS2**, §3.B]). The corresponding dual mapping algebras will be called *dual strict R-mapping algebras* (or *R-mapping algebras,* for short), and the category of such will be denoted by sMap^R, with $\mathfrak{M}^R\colon \mathcal{C}^{\mathrm{op}} \to \mathsf{sMap}^R$ the *realizable* R-mapping algebras.

4.1. The dual Stover construction

As in §3.1, we have a forgetful functor $\rho\colon \mathsf{sMap}^R \to (\mathsf{Set}^\Gamma_*)^{\mathrm{op}}$, with $(\rho\mathfrak{X})_n = (p_0\colon (P\mathfrak{X}\{K(R,n)\})_0 \to (\mathfrak{X}\{K(R,n)\})_0)$. The composite $\mathcal{L}^R := \rho\mathfrak{M}^R\colon \mathcal{C} \to (\mathsf{Set}^\Gamma_*)^{\mathrm{op}}$ has a right adjoint $\mathcal{R}^R\colon (\mathsf{Set}^\Gamma_*)^{\mathrm{op}} \to \mathcal{C}$, with $\mathcal{R}^R\Phi := \prod_{n\in\mathbb{N}} \prod_{f\in F_n} Q^{(f)}$ for any arrow set $\Phi = (\phi_n\colon E_n \to F_n)_{n\in\mathbb{N}}$.

When R is a field, we define $Q^{(f)}$ for $f \in F_n$ by the pullback square

$$\begin{array}{ccc} Q^{(f)} & \longrightarrow & \prod_{\phi_n^{-1}(f)} PK(R,n) \\ \downarrow & & \downarrow \prod p_{K(R,n)} \\ K(R,n) & \xrightarrow{\text{diag}} & \prod_{\phi_n^{-1}(f)} K(R,n) \end{array} \tag{4.2}$$

if $* \neq f \in \mathrm{Im}\ \phi_n$, while $Q^{(f)} := K(R,n)$ if $f \notin \mathrm{Im}\ \phi_n$. If $\phi = *$, we set $Q^{(f)} := \prod_{\phi_n^{-1}(*)\setminus\{*\}} \Omega K(R,n)$ (compare (3.3)).

Again, for any limit cardinal Λ we define a *λ-R-Stover space* to be any pullback of the form (4.2), with $\phi_n^{-1}(f)$ replaced by any set T of cardinality $< \lambda$. When R is not a field, we need to use the more complicated *modified Stover construction* of [**BS2**, §3.A] instead of the above.

We denote by $\mathbf{\Theta}^{\mathrm{St},R}_\lambda$ the corresponding dual enriched sketch, with $\mathcal{F}$ as in Example 2.14, and $\mathcal{L}$ consisting of products of cardinality $< \lambda$ in $\mathbf{\Theta}^{\mathrm{St},R}_\lambda$, together with the pullback squares of (2.15) and (4.2). The category of the corresponding dual strict mapping algebras, called *dual strict Stover mapping algebras,* will be denoted by $\mathsf{sMap}^{\mathrm{St},R}$, with $\mathfrak{M}^{\mathrm{St},R}\colon \mathcal{C} \to \mathsf{sMap}^{\mathrm{St},R}$ the *dual strict Stover mapping algebra* functor.

Since each $K(R,n)$ is, in particular, an R-Stover space, $\mathbf{\Theta}^R_\lambda$ is a full simplicial subcategory of $\mathbf{\Theta}^{\mathrm{St},R}_\lambda$, with $\iota\colon \mathbf{\Theta}^R_\lambda \hookrightarrow \mathbf{\Theta}^{\mathrm{St},R}_\lambda$ the inclusion, inducing the restriction $\iota^*\colon \mathsf{sMap}^{\mathrm{St},R} \to \mathsf{sMap}^R$ as in §3.2. Writing $\mathcal{V}^R := \mathfrak{M}^{\mathrm{St},R} \circ \mathcal{R}^R \circ \rho \circ \iota^*\colon \mathsf{sMap}^{\mathrm{St},R} \to \mathsf{sMap}^{\mathrm{St},R}$, we obtain the following categorical dual of Lemma 3.5 (compare [**BS2**, Proposition 2.19]):

Lemma 4.3. *Every dual strict Stover mapping algebra $\mathfrak{X}$ has a natural map $\zeta_{\mathfrak{X}}\colon \mathcal{V}^R\mathfrak{X} \to \mathfrak{X}$ making the following diagram commute in $\mathsf{sMap}^{\mathrm{St},R}$:*

$$\begin{array}{ccc} \mathcal{V}^R\mathcal{V}^R\mathfrak{X} & \xrightarrow{\zeta_{\mathcal{V}^R\mathfrak{X}}} & \mathcal{V}^R\mathfrak{X} \\ {\scriptstyle \mathcal{V}^R\zeta_{\mathfrak{X}}}\downarrow & & \downarrow{\scriptstyle \zeta_{\mathfrak{X}}} \\ \mathcal{V}^R\mathfrak{X} & \xrightarrow[\zeta_{\mathfrak{X}}]{} & \mathfrak{X} \end{array} \tag{4.4}$$

Definition 4.5. For any commutative ring R, we denote by $\mathcal{S}_R$ the full subcategory of R-good spaces in $\mathcal{S}_*$ (cf. [**BK1**, I, §5.1]), and by $\mathsf{sMap}^{\mathrm{St},R}_{\mathrm{re}}$ the full subcategory of $\mathsf{sMap}^{\mathrm{St},R}$ consisting of those dual strict Stover mapping algebras which are weakly equivalent to $\mathfrak{M}^{\mathrm{St},R}\mathbf{Y}$ for some $\mathbf{Y} \in \mathcal{S}_R$. These will be called *weakly R-good dual strict Stover mapping algebras.*

Remark 4.6. By [**H**, §15.3], $\mathcal{S}_*^{\mathbf{\Theta}^{\mathcal{A}} \times \Delta^{\mathrm{op}}}$ and $\mathcal{S}_{[n]}^{\mathbf{\Theta}^{\mathcal{A}} \times \Delta^{\mathrm{op}}}$ have Reedy model category structures, with weak equivalences and cofibrations defined at each simplicial space $\mathfrak{W}_\bullet\{\mathbf{B}\}$ for each $\mathbf{B} \in \mathbf{\Theta}^{\mathcal{A}}$.

As in §3.3, there is also a resolution model category structure on the category $(\mathcal{S}_*^{\mathbf{\Theta}^{\mathcal{A}}})^{\Delta^{\mathrm{op}}} = \mathcal{S}_*^{\mathbf{\Theta}^{\mathcal{A}} \times \Delta^{\mathrm{op}}}$ of simplicial dual $\mathcal{A}$-presheaves. Again, if a simplicial presheaf $\mathfrak{W}_\bullet$ is cofibrant, each $\mathfrak{W}_n$ is weakly equivalent to a coproduct of free dual strict $\mathcal{A}$-mapping algebras, so it is a dual weak $\mathcal{A}$-mapping algebra, and $\mathfrak{W}_\bullet \to \mathfrak{X}$ is a resolution of dual weak $\mathcal{A}$-mapping algebras only if $\pi_0\mathfrak{W}_\bullet \to \pi_0\mathfrak{X}$ is a resolution of $\Pi^{\mathcal{A}}$-algebras.

Since $\mathcal{S}_{[n]}$ is still proper, we also have a resolution model category structure on the category $\mathcal{S}_{[n]}^{\mathbf{\Theta}^{\mathcal{A}} \times \Delta^{\mathrm{op}}}$ of n-truncated simplicial dual $\mathcal{A}$-presheaves (§2.3).

The Eckmann–Hilton dual of Theorem 3.10 has the following more involved form:

Theorem 4.7. *Let R be any commutative ring, $\mathcal{C} = \mathcal{S}_*$, and $\mathfrak{X}$ a dual strict R-mapping algebra (for $\mathbf{\Theta}^R = s\mathcal{M}^R_\lambda$ as in Definition 4.1), which we assume to be a dual strict Stover mapping algebra.*

(a) *There is a functor associating to $\mathfrak{X}$ a cosimplicial object $\mathbf{W}^\bullet \in \mathcal{S}_*^{\Delta}$ with each W^n in $s\mathcal{M}^R_\lambda$, equipped with a natural augmentation of R-mapping algebras $\varepsilon\colon \mathfrak{M}^R\mathbf{W}^\bullet \to \mathfrak{X}$, such that $\pi_0\mathfrak{M}^R\mathbf{W}^\bullet \to \pi_0\mathfrak{X}\{M\}$ is a simplicial resolution of $\Pi^{\mathcal{A}}$-algebras.*

(b) *If $\mathfrak{X} \in \mathsf{sMap}^{\mathrm{St},R}_{\mathrm{re}}$ is weakly equivalent to $\mathfrak{M}^{\mathrm{St},R}\mathbf{Y}$ (for some R-good space $\mathbf{Y}$), then* $\operatorname{Tot}\mathbf{W}^\bullet$ *is homotopy equivalent to the R-completion of $\mathbf{Y}$ (so in particular,* $\operatorname{Tot}\mathbf{W}^\bullet$ *realizes $\mathfrak{X}$ up to weak equivalence).*

(c) *When R is a field, we can start with any dual strict $\mathcal{A}$-mapping algebra $\hat{\mathfrak{X}}$ (for $\mathcal{A} = \{K(R,n)\}_{n=1}^{\infty}$ in Example 2.14). If it extends to a dual strict Stover mapping algebra $\mathfrak{X}$ as defined in §4.1, and then (a) and (b) hold.*

(d) *When $R = \mathbb{F}_p$ or $\mathbb{Q}$, and $\mathfrak{X}$ is simply connected (that is, letting $\mathcal{A} = \{K(R,n)\}_{n=2}^{\infty}$ in Example 2.14),* any *R-mapping algebra (for a suitable limit cardinal λ) is weakly equivalent to $\mathfrak{M}^{\mathrm{St},R}\mathbf{Y}$ for some simply connected $\mathbf{Y}$, unique up to R-equivalence.*

Proof. This follows from various results in [**BS2**]:

(a) This is [**BS2**, Proposition 3.9].

(b) This is [**BS2**, Theorem 3.26].

(c) This combines [**BS2**, Proposition 2.23] and [**BS2**, Theorem 2.30], using the fact that a weak equivalence of dual strict Stover mapping algebras $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{Y}$ induces weak equivalence (in the model category of [**Bou2**, §3]) between the corresponding cosimplicial spaces (see [**Bou2**, §7.7]).

(d) This is [**BS2**, Theorem 4.23] (when $\lambda = \aleph_0$) or [**BS2**, Theorem 4.28] (otherwise).

□

Corollary 4.8. *If R is any commutative ring, then there is a* realization *functor* $N\colon (\mathsf{sMap}^{\mathrm{St},R}_{\mathrm{re}})^{\mathrm{op}} \to \mathcal{S}_*$ *with a natural weak equivalence* $\varepsilon\colon \mathrm{Id} \to N \circ \mathfrak{M}^{\mathrm{St},R}$. *Thus any functor* $\mathbf{T}\colon \mathcal{S}_R \to \mathcal{D}$ *(see Definition 4.5) to a model category* $\mathcal{D}$ *which preserves* R*-equivalences induces a functor* $\mathfrak{T} := \mathbf{T} \circ N\colon (\mathsf{sMap}^{\mathrm{St},R}_{\mathrm{re}})^{\mathrm{op}} \to \mathcal{D}$ *equipped with a natural weak equivalence* $\vartheta = \mathbf{T}\varepsilon\colon \mathfrak{T} \to \mathfrak{T} \circ \mathfrak{M}^{\mathrm{St},R}$.

Proof. We set $N := \mathrm{Tot}\,\mathbf{W}^\bullet$, where $\mathfrak{X} \mapsto \mathbf{W}^\bullet$ is the functor of Theorem 4.7. Once we know that $\mathfrak{X}$ is weakly R-good (see Definition 4.5), the natural augmentation $\varepsilon\colon \mathfrak{M}^{\mathrm{St},R}N \to \mathfrak{X}$ is a weak equivalence by Theorem 4.7(b) or (c). □

Example 4.9. For any $\mathbf{Z} \in \mathcal{S}_*$ with $\mathbf{T}\colon \mathcal{S}_R \to \mathcal{S}_*$ the functor $\mathrm{map}_*(\mathbf{Z}, -)$, the induced functor $\mathfrak{T} := \mathbf{T} \circ N\colon \mathsf{sMap}^{\mathrm{St},R}_{\mathrm{re}} \to \mathcal{S}_*$ has the property that if $\mathfrak{Z} := \mathfrak{M}^{\mathrm{St},R}\mathbf{Z}$ and $\mathfrak{X}$ is the realizable dual strict Stover mapping algebra $\mathfrak{M}^{\mathrm{St},R}\mathbf{Y}$ for some R-good space $\mathbf{Y}$, then $\mathfrak{T}(\mathfrak{X})$ is weakly equivalent to $\mathfrak{Z}\{\mathbf{Y}\}$.

Thus the n-truncation $P^n\mathfrak{T}$ (cf. §2.3) when evaluated at $\mathfrak{X} = \mathfrak{M}^{\mathrm{St},R}\mathbf{Y}$, is determined by $P^n\mathfrak{Z}$. Moreover, from the alternative description in Remark 2.9 we see that if $\mathbf{Y} \in \mathbf{\Theta}^{\mathrm{St},R}_{\lambda}$, then $\mathfrak{T}(\mathfrak{X})$ corresponds to the n-truncated simplicial category $P^n\mathcal{X}$, so that, in fact, $P^n\mathfrak{T}$, when evaluated at free dual strict Stover mapping algebras, factors through the n-truncation.

5. Relative left derived functors of mapping algebras

Let $\mathbf{T}\colon \mathcal{C} \to \mathcal{D}$ be a homotopy functor between model categories of spaces. We want to study the homotopy spectral sequence for the (co)simplicial object obtained by applying $\mathbf{T}$ to a (co)simplicial resolution of a space $\mathbf{Y} \in \mathcal{C}$, using a relative version of the total derived functor of the associated functor of mapping algebras $\mathfrak{T}$.

Definition 5.1. If $T\colon \mathcal{D} \to \mathcal{E}$ is a functor between model categories which preserves weak equivalences of cofibrant objects, recall that Quillen constructs the *total left derived functor* $\mathbf{L}T\colon\ \mathrm{ho}\,\mathcal{D} \to \mathrm{ho}\,\mathcal{E}$ on an object $x \in \mathcal{D}$ by applying T to any cofibrant replacement of x (see [**Q2**, I, §4]). In order for this to work, T need only be defined on the full subcategory $\mathcal{D}_{\mathrm{cof}}$ of all cofibrant objects in $\mathcal{D}$. In the spirit of the Eilenberg–Moore "relative homological algebra" (see [**EM**]), one could require only that T be defined on some full subcategory $\mathcal{P}$ of *special* cofibrant objects in $\mathcal{D}_{\mathrm{cof}}$ (e.g., free, rather than projective, resolutions) – as long as every object of $\mathcal{D}$ is weakly equivalent to an object of $\mathcal{P}$ (and T still takes weakly equivalent objects of $\mathcal{P}$ to weakly equivalent objects in $\mathcal{D}$). Moreover, if we are only given a full subcategory $\mathcal{D}_{\mathcal{P}}$ of $\mathcal{D}$, closed under weak equivalences, and every object of $\mathcal{D}_{\mathcal{P}}$ is weakly equivalent to one in $\mathcal{P}$, we still have $\mathbf{L}T\colon\ \mathrm{ho}\,\mathcal{D}_{\mathcal{P}} \to \mathrm{ho}\,\mathcal{E}$. Finally, $\mathcal{E}$ need not be a model category – all we need is the localization $\gamma\colon \mathcal{E} \to \mathrm{ho}\,\mathcal{E}$, with $\gamma \circ T$ taking weak equivalences to isomorphisms.

However, we shall be interested in a situation where we have two model category structures on $\mathcal{D}$ – or perhaps only a subcategory $\mathcal{W}'$ of the given weak equivalences $\mathcal{W}$. This commonly occurs when our model category $(\mathcal{D}, \mathcal{W}, \mathcal{D}_{\mathrm{cof}}, \mathcal{D}_{\mathrm{fib}})$ is obtained by localizing another.

In this case, we shall assume that $\mathcal{P}$ and $\mathcal{D}_{\mathcal{P}}$ satisfy the stronger requirement that for each $x \in \mathcal{D}_{\mathrm{cof}} \cap \mathcal{D}_{\mathcal{P}}$ there is a map $f\colon y \to x$ in $\mathcal{W}'$ with $y \in \mathcal{P}$. If $T\colon \mathcal{P} \to \mathcal{E}$ is then a functor which preserves $\mathcal{W}$-weak equivalences, the *relative left derived functor* of T (with respect to $\mathcal{P}$ and $\mathcal{W}'$) is the functor $\mathbf{L}^{\mathrm{rel}}T\colon \operatorname{ho}\mathcal{D} \to \operatorname{ho}\mathcal{E}$ defined on $z \in \mathcal{D}_{\mathcal{P}}$ by applying T to y, where $g\colon x \to z$ is a cofibrant replacement (with respect to $\mathcal{W}$) and $f\colon y \to x$ in $\mathcal{W}'$ is as above.

Dually, if we have full subcategories $\mathcal{F}$ of $\mathcal{D}_{\mathrm{fib}}$ (the fibrant objects) and $\mathcal{D}_{\mathcal{F}}$ of $\mathcal{D}$, both closed under weak equivalences, and $\mathcal{W}' \subseteq \mathcal{W}$, with the corresponding dual properties with respect to a homotopy functor $T\colon \mathcal{F} \to \mathcal{E}$, the *relative right derived functor* $\mathbf{R}^{\mathrm{rel}}T\colon \operatorname{ho}\mathcal{D}_{\mathcal{F}} \to \operatorname{ho}\mathcal{E}$ is defined analogously.

Remark 5.2. In the applications we have in mind, $\mathcal{D}$ will be a resolution model category of simplicial mapping algebras, so the weak equivalences $\mathcal{W}$ in $\mathcal{D}$ are E^2-equivalences. However, we also have a Reedy model structure on $\mathcal{D}$, and the special weak equivalences $\mathcal{W}'$ will be the E^1-equivalences. The ability to apply the functor T to a resolution which is $\mathcal{W}'$-equivalent to *any* cofibrant replacement (that is, simplicial resolution) y of an object $z \in \mathcal{D}_{\mathcal{F}}$ provides the flexibility we want in using particular resolutions – e.g., minimal – to calculate $(\mathbf{L}^{\mathrm{rel}}T)z$, and eventually, the appropriate terms of the spectral sequence.

5.1. CW resolutions

For $\mathcal{C} = \mathcal{G}$ and $\mathbf{\Theta}_{\mathcal{B}}$ as in Example 2.4, let $\mathbf{W}_\bullet$ be a resolution of $\mathbf{Y} \in \mathcal{G}$ in the resolution model category structure on $\mathcal{G}^{\Delta^{\mathrm{op}}}$. Given a homotopy functor $\mathbf{T}\colon \mathcal{G} \to \mathcal{D}$ for $\mathcal{D}$ a "category of spaces" such as Top_0, $\mathcal{S}_*$, or $\mathcal{G}$, we wish to study the homotopy spectral sequence for the simplicial space $\mathbf{TW}_\bullet \in \mathcal{D}^{\Delta^{\mathrm{op}}}$

By applying the functor $\mathfrak{M}_{\mathrm{St}}\colon \mathcal{C} \to \mathsf{sMap}_{\mathrm{St}}$ of §4.1 to $\mathbf{W}_\bullet$, we obtain a simplicial strict Stover mapping algebra $\mathfrak{W}_\bullet := \mathfrak{M}_{\mathrm{St}}\mathbf{W}_\bullet$ which is a cofibrant replacement for $\mathfrak{X} := \mathfrak{M}_{\mathrm{St}}\mathbf{Y}$ in the resolution model category structure on $\mathcal{S}_*^{\mathbf{\Theta}_{\mathcal{B}}^{\mathrm{op}} \times \Delta^{\mathrm{op}}}$ associated to the free dual strict Stover mapping algebras $\{\mathfrak{M}_{\mathrm{St}}\mathbf{S}^i\}_{i=1}^{\infty}$. By Corollary 3.11, there is functor $\mathfrak{T} = \mathbf{T}N\colon \mathsf{sMap}_{\mathrm{St}} \to \mathcal{D}$, with a natural Reedy (that is, levelwise) weak equivalence of simplicial spaces $\vartheta\colon \mathfrak{T}\mathfrak{W}_\bullet \to \mathbf{TW}_\bullet$.

We want to calculate the total left derived functor of $\mathfrak{T}$ evaluated at $\mathfrak{X}$ by applying $\mathfrak{T}$ to any resolution $\mathfrak{V}_\bullet \to \mathfrak{X}$. However, such an $\mathfrak{V}_\bullet$ is just a simplicial $\mathcal{B}$-presheaf, and the functor $\mathfrak{T}$ is only defined for *strict* Stover mapping algebras. As explained in §5.1, our solution to this difficulty is to show that any such $\mathfrak{V}_\bullet$ is, in fact, E^1-equivalent to a simplicial strict $\mathcal{B}$-mapping algebra $\mathfrak{W}_\bullet$. For this purpose we require some additional notions from [**BJT2**, §1]:

If $\mathcal{E}$ is any pointed complete category, the n-th *Moore chains* object of $G_\bullet \in \mathcal{E}^{\Delta^{\mathrm{op}}}$ is $C_nG_\bullet := \cap_{i=1}^{n} \operatorname{Ker}\{d_i\colon G_n \to G_{n-1}\}$. The differential is $\partial_n := d_0|_{C_nG_\bullet}\colon C_nG_\bullet \to C_{n-1}G_\bullet$ and the *cycles* objects is $Z_nG_\bullet := \operatorname{Ker}(\partial_n)$, with $v_n\colon Z_nG_\bullet \to C_nG_\bullet$ the inclusion. These are defined for any restricted simplicial object $G_\bullet \in \mathcal{E}^{\Delta_+^{\mathrm{op}}}$ (see §1.3).

The n-th *latching object* for $G_\bullet$ is the colimit

$$L_nG_\bullet := \operatorname*{colim}_{\theta^{\mathrm{op}}\colon [\mathbf{k}]\to[\mathbf{n}]} G_k, \tag{5.3}$$

where θ ranges over the surjective maps $[\mathbf{n}] \to [\mathbf{k}]$ in Δ for $k < n$.

A simplicial object $G_\bullet \in \mathcal{E}^{\Delta^{\mathrm{op}}}$ is called a *CW object* if it is equipped with a *CW basis* $(\overline{G}_n)_{n=0}^{\infty}$ in $\mathcal{E}$ such that $G_n = \overline{G}_n \amalg L_nG_\bullet$, and $d_i|_{\overline{G}_n} = 0$ for $1 \leqslant i \leqslant n$. The n-th *attaching map* for $G_\bullet$ is defined to be $\partial_n^G := d_0|_{\overline{G}_n}\colon \overline{G}_n \to C_{n-1}G_\bullet$ (which actually lands in $Z_{n-1}G_\bullet$).

When $\mathcal{E}$ is a suitable category of universal algebras, such as $\Pi_{\mathcal{B}}\text{-}\mathsf{Alg}$ (cf. Definition 2.11), a simplicial object $V_\bullet \in \mathcal{E}^{\Delta^{\mathrm{op}}}$ with an augmentation to $\Lambda \in \mathcal{C}$ is called a *CW resolution* if $V_\bullet \to \Lambda$ is acyclic, with a CW basis $(\overline{V}_n)_{n=0}^\infty$ having each $\overline{V}_n$ free. Moreover, in this case ∂_n^V surjects onto $Z_{n-1}V_\bullet$ (where $Z_{-1}V_\bullet := \Lambda$).

For $\mathcal{B} = \{\mathbf{S}^i\}_{i=1}^\infty$, by [**BJT2**, Lemma 1.38] every free simplicial $\Pi_{\mathcal{B}}$-algebra (Definition 2.11) has a free CW basis. Moreover, by [**BJT2**, Theorem 2.29], every CW resolution $V_\bullet$ of a realizable $\Pi_{\mathcal{B}}$-algebra $\Lambda = \pi_*\mathbf{Y} = \pi_0\mathfrak{M}_{\mathcal{B}}\mathbf{Y}$ can be realized by an augmented simplicial space $\mathbf{W}_\bullet \to \mathbf{Y}$. Therefore, every free simplicial $\Pi_{\mathcal{B}}$-algebra resolution $V_\bullet \to \pi_*\mathbf{Y}$ can be realized (non-canonically) by a simplicial resolution of strict $\mathcal{B}$-mapping algebras $\mathfrak{W}_\bullet \to \mathfrak{M}_{\mathcal{B}}\mathbf{Y}$, with $\pi_0\mathfrak{W}_\bullet \cong V_\bullet$. In order to apply the ideas of §5.1, we must show that any simplicial $\mathcal{B}$-presheaf resolution $\mathfrak{V}_\bullet$ of $\mathfrak{X} = \mathfrak{M}_{\mathcal{B}}\mathbf{Y}$ is Reedy weakly equivalent to a strict Stover mapping algebra resolution $\mathfrak{W}_\bullet$. To do so, we recall the following constructions from [**BJT2**]:

5.2. Sequential realizations

Assume given an enriched sketch $\mathbf{\Theta}_{\mathcal{B}} = \mathbf{\Theta}_{(\mathcal{B},\mathcal{F},\mathcal{E})}$ in a pointed simplicial model category $\mathcal{C}$, as in Definition 2.1, and a CW-resolution $V_\bullet$ of a realizable $\Pi_{\mathcal{B}}$-algebra $\Lambda = \pi_*^{\mathcal{B}}\mathbf{Y}$, with CW basis $\{\overline{V}_n\}_{n=0}^\infty$. We define a *sequential realization of* $V_\bullet$ (for $\mathbf{Y}$) to be a sequence $\mathcal{W}$ of maps

$$\mathbf{W}_\bullet^{[0]} \xrightarrow{\iota^{[0]}} \mathbf{W}_\bullet^{[1]} \xrightarrow{\iota^{[1]}} \mathbf{W}_\bullet^{[2]} \to \cdots \to \mathbf{W}_\bullet^{[n]} \xrightarrow{\iota^{[n]}} \mathbf{W}_\bullet^{[n+1]} \to \cdots \tag{5.4}$$

between Reedy fibrant and cofibrant objects in $C^{\Delta^{\mathrm{op}}}$, such that for each $n \geqslant 0$:

(i) $\overline{\mathbf{W}}_n \in \mathbf{\Theta}_{\mathcal{B}}$ realizes the given CW basis $\Pi^{\mathcal{A}}$-algebra$\overline{V}_n$.

(ii) There is an n-skeletal restricted simplicial object $\widetilde{\mathbf{W}}_\bullet^{[n]}$ with

$$\widetilde{\mathbf{W}}_k^{[n]} = \mathbf{W}_k^{[n-1]} \amalg C\Sigma^{n-k-1}\overline{\mathbf{W}}_n \quad \text{for } 0 \leqslant k \leqslant n, \tag{5.5}$$

where by convention $C\Sigma^0\overline{\mathbf{W}}_n := C\overline{\mathbf{W}}_n$, $C\Sigma^{-1}\overline{\mathbf{W}}_n := \overline{\mathbf{W}}_n$, and $\mathbf{W}_\bullet^{[-1]} = *$.

(iii) The face map $d_0|_{C\Sigma^{n-k-1}\overline{\mathbf{W}}_n}$ is the map F_k in the commuting diagram

$$\begin{array}{ccccc}
\Sigma^{n-k-1}\overline{\mathbf{W}}_n & \xhookrightarrow{i^k} & C\Sigma^{n-k-1}\overline{\mathbf{W}}_n & \xrightarrow{q^k} & \Sigma^{n-k}\overline{\mathbf{W}}_n \\
\downarrow{\scriptstyle a_k} & & \downarrow{\scriptstyle F_k} & & \downarrow{\scriptstyle a_{k-1}} \\
Z_{k-1}\mathbf{W}_\bullet^{[n-1]} & \xhookrightarrow{v_{k-1}} & C_{k-1}\mathbf{W}_\bullet^{[n-1]} & \xrightarrow{\partial_{k-1}} & Z_{k-2}\mathbf{W}_\bullet^{[n-1]}
\end{array} \tag{5.6}$$

in which the top row is a strict cofibration sequence and the bottom row a strict fibration sequence in $\mathcal{C}$. Thus F_k is a nullhomotopy for $v_{k-1} \circ a_k$, which in turn defines a_{k-1}, using (5.6). The first face map $d_1|_{C\Sigma^{n-k-1}\overline{\mathbf{W}}_n}$ is the composite $C\Sigma^{n-k-1}\overline{\mathbf{W}}_n \xrightarrow{q^k} \Sigma^{n-k}\overline{\mathbf{W}}_n \xrightarrow{i^{k-1}} C\Sigma^{n-k}\overline{\mathbf{W}}_n$, and $d_i|_{C\Sigma^{n-k-1}\overline{\mathbf{W}}_n} = 0$ for $i > 1$. We start with $F_n \colon \overline{\mathbf{W}}_n \to C_{n-1}\mathbf{W}_\bullet^{[n-1]}$ a realization of the n-th attaching map $\partial_n^V : \overline{V}_n \to C_{n-1}V_\bullet$ for the given CW resolution, and $a_{n-1} := \partial_{n-1} \circ F_n \colon \overline{\mathbf{W}}_n \to Z_{n-2}\mathbf{W}_\bullet^{[n-1]}$ (with $v_{n-2} \circ a_{n-1}$ indeed nullhomotopic).

(iv) Let $\widehat{\mathbf{W}}_\bullet^{[n]}$ be the pushout of the obvious maps

$$\mathbf{W}_\bullet^{[n-1]} \leftarrow \mathcal{L}i^*\mathbf{W}_\bullet^{[n-1]} \rightarrow \mathcal{L}\widetilde{\mathbf{W}}_\bullet^{[n]}, \tag{5.7}$$

where $\mathcal{L}\colon \mathcal{C}^{\Delta_+^{\mathrm{op}}} \to \mathcal{C}^{\Delta^{\mathrm{op}}}$ is the left adjoint of $i^*\colon \mathcal{C}^{\Delta^{\mathrm{op}}} \to \mathcal{C}^{\Delta_+^{\mathrm{op}}}$, as in §1.3. We then let $\mathbf{W}_\bullet^{[n]}$ be a Reedy fibrant and cofibrant replacement for $\widehat{\mathbf{W}}_\bullet^{[n]}$.

(v) There is an augmentation $\varepsilon^{[n]}\colon \mathbf{W}_\bullet^{[n]} \to \mathbf{Y}$ realizing $V_\bullet \to \Lambda$ through simplicial dimension n – that is, the n-truncation of the augmented simplicial $\Pi^{\mathcal{A}}$-algebra $\pi_*^{\mathcal{A}}\mathbf{W}_\bullet^{[n]} \to \pi_*^{\mathcal{A}}\mathbf{Y}$ is isomorphic to the n-truncation of $V_\bullet \to \Lambda$.

(vi) The maps $\iota^{[n]}$ restrict to a trivial cofibration $\iota_k^{[n]}\colon \mathbf{W}_k^{[n-1]} \xrightarrow{\simeq} \mathbf{W}_k^{[n]}$ for each $0 \leqslant k < n$.

It follows that $\mathbf{W}_\bullet := \operatorname{colim}_n \mathbf{W}_\bullet^{[n]} \xrightarrow{\varepsilon} \mathbf{Y}$ is a simplicial resolution in the resolution model category $\mathcal{C}^{\Delta^{\mathrm{op}}}$. See [**BJT2**, §2] for further details.

Theorem 5.8. *For an enriched sketch $\mathbf{\Theta}_{\mathcal{B}}$ as in Definition 2.1, $\mathbf{Y} \in \mathcal{C}$ fibrant, and $\mathfrak{X} := \mathfrak{M}_{\mathcal{B}}\mathbf{Y}$, let $\boldsymbol{\eta}\colon \mathfrak{V}_\bullet \to c(\mathfrak{X})_\bullet$ be a trivial fibration with $\mathfrak{V}_\bullet$ cofibrant in $\mathcal{S}_*^{\mathbf{\Theta}_{\mathcal{B}}^{\mathrm{op}} \times \Delta^{\mathrm{op}}}$. Then for any sequential realization $\mathcal{W}$ of the $\Pi_{\mathcal{B}}$-algebra resolution $\pi_0\mathfrak{V}_\bullet \to \pi_*^{\mathcal{B}}\mathbf{Y}$ as in §5.2, there is a Reedy weak equivalence of simplicial weak $\mathcal{B}$-mapping algebras $\mathfrak{f}\colon \mathfrak{M}_{\mathcal{B}}\mathbf{W}_\bullet \to \mathfrak{V}_\bullet$.*

Proof. By §3.3, the simplicial $\Pi_{\mathcal{B}}$-algebra $V_\bullet := \pi_o\mathfrak{V}_\bullet$ is a free resolution of the $\Pi_{\mathcal{B}}$-algebra $\Lambda := \pi_0\mathfrak{X}$, so it has a CW basis $\{\overline{V}_n\}_{n=0}^\infty$ by [**BJT2**, Lemma 1.38], with $\overline{V}_n = \pi_0\mathfrak{M}_{\mathcal{B}}\overline{\mathbf{W}}_n$ for some $\overline{\mathbf{W}}_n \in \operatorname{Obj}\mathbf{\Theta}_{\mathcal{B}}$. We may assume $\mathbf{\Theta}_{\mathcal{B}}$ contains all simplicial groups of the homotopy type of a (possibly trivial) wedge of objects of $\mathcal{B}$ of cardinality $< \lambda$. This will ensure that all objects $\mathbf{W}_k^{[n]}$, $\widetilde{\mathbf{W}}_k^{[n]}$, $\widehat{\mathbf{W}}_k^{[n]}$, and so on, in §5.2 are in $\mathbf{\Theta}_{\mathcal{B}}$.

We construct $\mathfrak{f}$ by a double induction: in the outer induction, we construct maps of simplicial weak $\mathcal{B}$-mapping algebras $\mathfrak{f}^{[n]}\colon \mathfrak{M}_{\mathcal{B}}\mathbf{W}_\bullet^{[n]} \to \mathfrak{V}_\bullet$. Assuming we have defined $\mathfrak{f}^{[n-1]}$, we need to extend it to a map of n-truncated restricted simplicial objects $\widetilde{\mathfrak{f}}^{[n]}\colon \mathfrak{M}_{\mathcal{B}}\widetilde{\mathbf{W}}_\bullet^{[n]} \to \mathfrak{V}_\bullet$, which we do by an inner downward induction on $0 \leqslant k \leqslant n$. Using Lemma 2.8, we see from (5.5) that $\widetilde{\mathfrak{f}}_k^{[n]}$ is determined by an element $\bar{f}_n^{(k)} \in \mathfrak{V}_k\{C\Sigma^{n-k-1}\overline{\mathbf{W}}_n\}_0$ with $d_i\bar{f}_n^{(k)} = 0$ for $i \geqslant 2$.

Step A. To start the outer induction, note that since $\mathbf{W}_0^{[0]} = \overline{\mathbf{W}}_0$, by Lemma 2.8 the augmentation $\varepsilon^{[0]}\colon \mathfrak{M}_{\mathcal{B}}\mathbf{W}_\bullet^{[0]} \to \mathfrak{X}$ is determined by an element $e \in \mathfrak{X}\{\overline{\mathbf{W}}_0\}_0 = \operatorname{Hom}(\overline{\mathbf{W}}_0, \mathbf{Y})$. Since $\boldsymbol{\eta}\colon \mathfrak{V}_\bullet \to c(\mathfrak{X})_\bullet$ is a Reedy fibration (see [**J**, §2]), $(\boldsymbol{\eta}_0)_*\colon \mathfrak{V}_0\{\overline{\mathbf{W}}_0\} \to \mathfrak{X}\{\overline{\mathbf{W}}_0\}$ is a fibration and, in particular, a surjection in $\mathcal{S}_*$. Moreover, $\pi_0\mathfrak{V}_0 \cong \pi_*^{\mathcal{B}}\overline{\mathbf{W}}_0$ is a free $\Pi_{\mathcal{B}}$-algebra, by our assumption on $\mathfrak{V}_\bullet$, so we have an element $\bar{f}_0^{(0)} \in \mathfrak{V}_0\{\overline{\mathbf{W}}_0\}_0$ representing $\operatorname{Id} \in \pi_0\mathfrak{V}_0\{\overline{\mathbf{W}}_0\}$ with $(\boldsymbol{\eta}_0)_*\bar{f}_0^{(0)} = e$ by [**BJT1**, Lemma 15.9], as required.

Step B. Given $\mathfrak{f}^{[n-1]}\colon \mathfrak{M}_{\mathcal{B}}\mathbf{W}_\bullet^{[n-1]} \to \mathfrak{V}_\bullet$, consider the augmented simplicial space $\mathbf{X}_\bullet := \mathfrak{V}_\bullet\{\overline{\mathbf{W}}_n\} \to \mathfrak{X}\{\overline{\mathbf{W}}_n\}$: we think of this as a bisimplicial set with vertical direction internal to each $\mathfrak{V}_k\{\overline{\mathbf{W}}_n\} \in \mathcal{S}_*$, and horizontal direction corresponding to the original simplicial direction of $\mathfrak{V}_\bullet$. The (split) inclusion $j_n\colon \overline{V}_n \hookrightarrow V_n$ for the CW basis $\Pi_{\mathcal{B}}$-algebra $\overline{V}_n = \pi_0\mathfrak{M}_{\mathcal{B}}\overline{\mathbf{W}}_n$ corresponds by the $\Pi_{\mathcal{B}}$-algebra analogue of Lemma 2.8 (the ordinary Yoneda embedding) to an element $\tilde{\jmath}_n \in V_n\{\overline{\mathbf{W}}_n\}$ – that is, a homotopy class $[\bar{f}_n^{(n)}] \in \pi_0\mathbf{X}_n = \pi_0\mathfrak{V}_n\{\overline{\mathbf{W}}_n\}$. Since $\mathbf{Y}$ is fibrant in $\mathcal{C}$, $\mathfrak{X} = \mathfrak{M}_{\mathcal{B}}\mathbf{Y}$ is fibrant in $\mathcal{S}_*^{\mathbf{\Theta}^{\mathcal{A}}}$, so $c(\mathfrak{X})_\bullet$ is Reedy fibrant. But $\mathfrak{V}_\bullet \to c(\mathfrak{X})_\bullet$ is a Reedy fibration, so $\mathfrak{V}_\bullet$ is Reedy fibrant, and therefore $\mathbf{X}_\bullet$ is, too. Thus by [**Sto**, Lemma 2.7] the inclusion of the (horizontal) Moore object $C_n\mathbf{X}_\bullet := C_n^h\mathbf{X}_\bullet \hookrightarrow \mathbf{X}_n$ induces an isomorphism $\pi_0 C_n\mathbf{X}_\bullet \to C_n\pi_0\mathbf{X}_\bullet = C_n V_\bullet\{\overline{\mathbf{W}}_n\}$ (see also [**BJT2**, Lemma 1.30]).

The functor $\mathfrak{M}_{\mathcal{B}}$ of Definition 2.7 takes any pointed limit in $\mathcal{C}$ to the corresponding limit of $\mathcal{B}$-presheaves, so $C_{n-1}\mathfrak{M}_{\mathcal{B}}\mathbf{W}_\bullet^{[n-1]} = \mathfrak{M}_{\mathcal{B}}C_{n-1}\mathbf{W}_\bullet^{[n-1]}$, and thus the attaching map $d_0 = \partial_n^W : \overline{\mathbf{W}}_n \to C_{n-1}\mathbf{W}_\bullet^{[n-1]}$ corresponds under Lemma 2.8 to an element $\gamma \in C_{n-1}\mathfrak{M}_{\mathcal{B}}\mathbf{W}_\bullet^{[n-1]}\{\overline{\mathbf{W}}_n\}$. Moreover, the given map of $\mathcal{B}$-presheaves $\mathfrak{f}^{[n-1]}$ induces $C_{n-1}\mathfrak{f}^{[n-1]} : C_{n-1}\mathfrak{M}_{\mathcal{B}}\mathbf{W}_\bullet^{[n-1]} \to C_{n-1}\mathfrak{V}_\bullet$, which takes γ to an element $\psi_n := \mathfrak{f}^{[n-1]}(\gamma)$ $\in (C_{n-1}^h\mathbf{X}_\bullet)_0$.

Since $\mathbf{X}_\bullet$ is Reedy fibrant, the matching structure map $\delta_n : \mathbf{X}_n \to M_n\mathbf{X}_\bullet$ is a fibration (cf. [**H**, §16.3]), and we have an inclusion $\iota : C_{n-1}\mathbf{X}_\bullet \hookrightarrow M_n\mathbf{X}_\bullet$, given by $x \mapsto (x, x, 0, \ldots, 0)$. Because $\mathbf{W}_\bullet$ realizes the CW resolution $V_\bullet \to \Lambda$ of $\Pi^{\mathcal{A}}$-algebras and $j_n : \overline{V}_n \hookrightarrow V_n$ factors through $C_n V_\bullet$, we have $(\delta_n)_*[\bar{f}_n^{(n)}] = [\iota \circ \psi_n]$. We may therefore change $\bar{f}_n^{(n)}$ within its homotopy class so that $\delta_n(\bar{f}_n^{(n)}) = \iota \circ \psi$ on the nose.

Lemma 2.8, together with (5.5) (and our assumption that $\mathbf{W}_k^{[n-1]}$ and $C\Sigma^{n-k-1}\overline{\mathbf{W}}_n$ are in $\mathbf{\Theta}_{\mathcal{B}}$), implies that $\mathfrak{M}_{\mathcal{B}}\widetilde{\mathbf{W}}_n^{[n]}$ is the coproduct of $\mathfrak{M}_{\mathcal{B}}\mathbf{W}_n^{[n-1]}$ and $\mathfrak{M}_{\mathcal{B}}\overline{\mathbf{W}}_n$. Therefore, this choice of $\bar{f}_n^{(n)}$ defines a map of $\mathcal{B}$-presheaves $\mathfrak{f}^{[n]} : \mathfrak{M}_{\mathcal{B}}\widetilde{\mathbf{W}}_n^{[n]} \to \mathfrak{V}_n$ (extending $\mathfrak{f}^{[n-1]}$). Since $F_{n-1}|_{\overline{\mathbf{W}}_n} = d_0^h(\gamma)$ (in the notation of §5.2(iii)), we have $d_0^h\bar{f}_n^{(n)} = d_1^h\bar{f}_n^{(n)}$.

Step C. In the k-th stage of the inner (downward) induction, with $k < n$, we assume that for each for $k < j \leqslant n$ we have chosen a map of weak $\mathcal{B}$-mapping algebras $\bar{f}_j^{(n)} : \mathfrak{M}_{\mathcal{B}}C\Sigma^{n-j-1}\overline{\mathbf{W}}_n \to \mathfrak{V}_j$, represented by an element $\psi_j \in \mathfrak{V}_j\{C\Sigma^{n-j-1}\overline{\mathbf{W}}_n\}_0$ with $d_i^h\psi_j = 0$ for $2 \leqslant i \leqslant j$. If $\iota_{n-j-1} : \Sigma^{n-j-1}\overline{\mathbf{W}}_n \hookrightarrow C\Sigma^{n-j-1}\overline{\mathbf{W}}_n$ is the inclusion, then $\varphi_j := \iota_{n-j-1}^*\psi_j$ lies in $C_{j-1}^h\mathfrak{V}_\bullet\{\Sigma^{n-j-1}\overline{\mathbf{W}}_n\}_0$, and by induction it represents

$$\mathfrak{M}_{\mathcal{B}}C\Sigma^{n-j-1}\overline{\mathbf{W}}_n \xrightarrow{(F_j)_*} C_{j-1}^h\mathfrak{M}_{\mathcal{B}}\mathbf{W}_\bullet^{[n-1]} \xrightarrow{C_{j-1}\mathfrak{f}^{[n-1]}} C_{j-1}^h\mathfrak{V}_\bullet \tag{5.9}$$

(in the notation of (5.6)). If $q_{n-j-2} : C\Sigma^{n-j-2}\overline{\mathbf{W}}_n \to \Sigma^{n-j-1}\overline{\mathbf{W}}_n$ is the quotient map, this implies that $q_{n-j-2}^*\varphi_j$ represents

$$\mathfrak{M}_{\mathcal{B}}\Sigma^{n-j-1}\overline{\mathbf{W}}_n \xrightarrow{(a_j)_*} Z_{j-1}^h\mathfrak{M}_{\mathcal{B}}\mathbf{W}_\bullet^{[n-1]} \xrightarrow{Z_{j-1}\mathfrak{f}^{[n-1]}} Z_{j-1}^h\mathfrak{V}_\bullet \tag{5.10}$$

(again using the notation of (5.6)), so $q_{n-j-2}^*\varphi_j$ is in $Z_{j-1}^h\mathfrak{V}_\bullet\{C\Sigma^{n-j-2}\overline{\mathbf{W}}_n\}_0$.

Similarly, $d_0^h\varphi_j$ actually lies in $Z_{j-2}^h\mathfrak{V}_\bullet\{C\Sigma^{n-j-2}\overline{\mathbf{W}}_n\}_0$, and represents

$$\mathfrak{M}_{\mathcal{B}}\Sigma^{n-j}\overline{\mathbf{W}}_n \xrightarrow{(a_{j-1})_*} Z_{j-2}^h\mathfrak{M}_{\mathcal{B}}\mathbf{W}_\bullet^{[n-1]} \xrightarrow{Z_{j-2}\mathfrak{f}^{[n-1]}} Z_{j-2}^h\mathfrak{V}_\bullet. \tag{5.11}$$

The nullhomotopy F_k for $v_{k-1} \circ a_k$ (cf. (5.6)) is represented by $\varphi_k \in C_{k-1}^h\mathfrak{V}_\bullet\{\Sigma^{n-k-1}\overline{\mathbf{W}}_n\}_0$, and as in Step B we use the embedding of $C_{k-1}^h\mathfrak{V}_\bullet\{\Sigma^{n-k-1}\overline{\mathbf{W}}_n\}$ in $M_k\mathfrak{V}_\bullet\{\Sigma^{n-k-1}\overline{\mathbf{W}}_n\}$ and the facts that $\delta_k : \mathfrak{V}_k\{\Sigma^{n-k-1}\overline{\mathbf{W}}_n\}$ $\to M_k\mathfrak{V}_\bullet\{\Sigma^{n-k-1}\overline{\mathbf{W}}_n\}$ is a fibration, and that φ_k lifts up to homotopy to $\mathfrak{V}_j\{C\Sigma^{n-k-1}\overline{\mathbf{W}}_n\}$ (since $C\Sigma^{n-k-1}\overline{\mathbf{W}}_n$ is contractible) to obtain an element ψ_k in $\mathfrak{V}_j\{C\Sigma^{n-k-1}\overline{\mathbf{W}}_n\}_0$ (with $d_i^h\psi_k = 0$ for $2 \leqslant i$), such that $\varphi_k := d_0^h\psi_k$.

Step D. The three conditions (5.9)–(5.10)–(5.11) on $\varphi_j := d_0^h \psi_j$ $(0 \leqslant j \leqslant n)$ are all that is needed in order for the elements ψ_j to fit together to define a map of restricted simplicial $\mathcal{B}$-presheaves $\widetilde{\mathfrak{f}}^{[n-1]} \colon \mathfrak{M}_{\mathcal{B}} \widetilde{\mathbf{W}}_\bullet^{[n]} \to i^*\mathfrak{V}_\bullet$ extending $i^*\mathfrak{f}^{[n-1]}$ (in the notation of §1.3), and so, using (5.7), an induced map of simplicial $\mathcal{B}$-presheaves $\widehat{\mathfrak{f}}^{[n-1]} \colon \mathfrak{M}_{\mathcal{B}} \widehat{\mathbf{W}}_\bullet^{[n]} \to \mathfrak{V}_\bullet$, which is a levelwise weak equivalence through dimension n.

Recall from [**BJT2**, 2.C] that $\mathbf{W}_\bullet^{[n]}$ is constructed by the following factorizations in the Reedy model category structure on $\mathcal{S}_*^{\mathbf{\Theta}_{\mathcal{B}}^{\mathrm{op}} \times \Delta^{\mathrm{op}}}$ (see §3.3):

$$\begin{array}{ccccc}
\mathbf{W}_\bullet^{[n-1]} & \longrightarrow & \widehat{\mathbf{W}}_\bullet^{[n]} & & \\
{\scriptstyle \iota^{[n-1]}} \big\downarrow & & \big\downarrow {\scriptstyle \simeq} & \searrow & \\
\mathbf{W}_\bullet^{[n]} & \underset{h}{\twoheadrightarrow} & {'\mathbf{W}}_\bullet^{[n]} & \twoheadrightarrow & *
\end{array} \tag{5.12}$$

where $\hookrightarrow$ indicates a cofibration and $\twoheadrightarrow$ a fibration, with the top horizontal map a levelwise weak equivalence in simplicial dimensions $\leqslant n-1$, so the same is true of the left vertical map.

Applying $\mathfrak{V}_\bullet\{-\}$ to (5.12) yields a diagram of bisimplicial spaces, and taking diagonals, a similar diagram in $\mathcal{S}_*^{\Delta^{\mathrm{op}}}$. Since by our initial assumption all objects of (5.12), in each simplicial dimension, are in $\mathbf{\Theta}_{\mathcal{B}}$, by Lemma 2.8 we obtain an analogous diagram of mapping spaces of $\mathcal{B}$-presheaves into $\mathfrak{V}_\bullet$. The sequence of elements in the simplicial set $\operatorname{diag} \mathfrak{V}_\bullet\{\mathbf{W}_\bullet^{[n-1]}\}_0$ in the upper left corner corresponding to $\mathfrak{f}^{[n-1]} \colon \mathfrak{M}_{\mathcal{B}} \mathbf{W}_\bullet^{[n-1]} \to \mathfrak{V}_\bullet$ map by construction to the sequence in $\operatorname{diag} \mathfrak{V}_\bullet\{\widehat{\mathbf{W}}_\bullet^{[n]}\}_0$ corresponding to $\widehat{\mathfrak{f}}^{[n-1]}$, mapping forward to a sequence β corresponding to ${'\mathfrak{f}}^{[n]} \colon \mathfrak{M}_{\mathcal{B}} {'\mathbf{W}}_\bullet^{[n]} \to \mathfrak{V}_\bullet$. Since the map h in (5.12) is a trivial fibration, and these are preserved by evaluation of $\mathfrak{V}_\bullet$ and diagonals, we see that the induced map of simplicial sets $h_* \colon \operatorname{diag} \mathfrak{V}_\bullet\{\mathbf{W}_\bullet^{[n]}\}_0 \to \operatorname{diag} \mathfrak{V}_\bullet\{{'\mathbf{W}}_\bullet^{[n]}\}_0$ is a trivial fibration. We can therefore lift β to a sequence representing the required map $\mathfrak{f}^{[n]} \colon \mathfrak{M}_{\mathcal{B}} \mathbf{W}_\bullet^{[n]} \to \mathfrak{V}_\bullet$, completing the outer induction step. □

Remark 5.13. The same result holds if we replace $\mathcal{B}$-presheaves by r-truncated $\mathcal{B}$-presheaves, since (as noted in §2.3), Lemma 2.8 still holds, and $\mathbf{W}_\bullet := P^r \mathfrak{M}_{\mathcal{B}} \mathbf{W}_\bullet$ is free in each simplicial dimension.

Summary 5.14. Assume given a homotopy functor $\mathbf{T} \colon \mathcal{G} \to \mathcal{M}$, inducing $\mathfrak{T} := \mathbf{T} \circ N \colon \mathsf{sMap}_{\mathrm{St}} \to \mathcal{M}$ as in Corollary 3.11. Let $\mathcal{D} := \mathcal{S}_*^{\mathbf{\Theta}_{\mathrm{St}}^{\mathrm{op}} \times \Delta^{\mathrm{op}}}$ and $\mathcal{E} := \mathcal{M}^{\Delta^{\mathrm{op}}}$, with the resolution model category structure on $\mathcal{D}$ determined by $\mathcal{B}$ for $\mathcal{G}$ as in §3.3, with respect to the structure of §2.1 for $\mathcal{S}_*^{\mathbf{\Theta}_{\mathrm{St}}^{\mathrm{op}}}$ (with E^s-weak equivalences on $\mathcal{E}$).

In the notation of §5.1, let $\mathcal{C}$ denote the category of simplicial strict $\mathcal{B}$-mapping algebras in $\mathcal{D}$ associated to sequential realizations as in §5.2, let $\mathcal{W}'$ be the Reedy weak equivalences in $\mathcal{D}$, and let $\mathcal{D}_{\mathcal{C}}$ be the full subcategory $\operatorname{ho} \mathsf{sMap}_{\mathrm{St}}$ of objects in $\operatorname{ho}(\mathcal{S}_*^{\mathbf{\Theta}_{\mathrm{St}}^{\mathrm{op}} \times \Delta^{\mathrm{op}}})$ weakly equivalent to a constant simplicial object on $\mathsf{sMap}_{\mathrm{St}}$. The *relative left derived functor* $\mathbf{L}^{\mathrm{rel}}\mathfrak{T} \colon \operatorname{ho} \mathsf{sMap}_{\mathrm{St}} \to \operatorname{ho} \mathcal{E}$ is then defined on a Stover mapping algebra $\mathfrak{X}$ (more formally, on $c(\mathfrak{X})_\bullet$) by

(a) Choosing a simplicial resolution $\boldsymbol{\eta} \colon \mathfrak{V}_\bullet \to \mathfrak{X}$ in $\mathcal{S}_*^{\mathbf{\Theta}_{\mathcal{B}}^{\mathrm{op}} \times \Delta^{\mathrm{op}}}$;

(b) Choosing a CW basis $\{\overline{V}_n\}_{n=0}^{\infty}$ for the $\Pi_{\mathcal{B}}$-algebra-resolution $V_\bullet := \pi_0 \mathfrak{V}_\bullet \to \pi_0 \mathfrak{X}$, a sequential realization $\mathcal{W}$ of $V_\bullet$ for $\mathbf{Y} := N\mathfrak{X}$, with an E^1-weak equivalence

$\mathfrak{M}_{\mathrm{St}}\mathbf{W}_\bullet \to \mathfrak{V}_\bullet$;

(c) Defining $(\mathbf{L}^{\mathrm{rel}}\mathfrak{T})\mathfrak{X}$ to be the simplicial object $\mathfrak{T}\mathfrak{M}_{\mathrm{St}}\mathbf{W}_\bullet$ in $\mathrm{ho}\,\mathcal{D}^{\Delta^{\mathrm{op}}}$ (uniquely determined up to E^1-weak equivalence).

6. Relative derived functors of dual mapping algebras

For a given commutative ring R, let $\mathbf{\Theta}^R = s\mathcal{M}^R_\lambda$ be the full subsimplicial category of $\mathcal{C} = \mathcal{S}_R$ consisting of all simplicial R-modules of cardinality $< \lambda$, as in Definition 4.1, and $\mathfrak{X} = \mathfrak{M}^R\mathbf{Y}$ for some $\mathbf{Y} \in \mathcal{C}$ (the cardinal λ we choose may depend on $\mathbf{Y}$). Essentially, we may dualize the results of Section 5 to this situation. Note that because $\mathbf{Y} \mapsto H^*(\mathbf{Y}; R)$ is contravariant the category $\Pi^{\mathcal{A}}$-Alg resembles $\Pi_{\mathcal{B}}$-Alg in being a category of graded universal algebras, so the resolutions we need for the $\Pi^{\mathcal{A}}$-algebra$\Lambda = H^*(\mathbf{Y}; R)$ will be simplicial, rather than cosimplicial, and we can use the notion of a CW resolution $V_\bullet \to \Lambda$ as in §5.1. However, only when R is a field do we know that any free simplicial resolution in $\Pi^{\mathcal{A}}\text{-Alg}^{\Delta^{\mathrm{op}}}$ has a CW basis $(\overline{V}_n)_{n=0}^\infty$ of free $\Pi^{\mathcal{A}}$-algebras (see [**Bl3**, Proposition 3.12]). For the cosimplicial resolutions of spaces, we need to dualize §5.1 as follows:

Definition 6.1. If $\mathcal{C}$ is cocomplete, the n-th *Moore cochain* object of a cosimplicial object $G^\bullet \in \mathcal{C}^\Delta$ is $C^nG^\bullet := \mathrm{Coker}\,(\coprod_{i=1}^{n-1} G^n \xrightarrow{\perp_i d^i} G^n)$, with differential $\delta^{n-1}\colon C^{n-1}G^\bullet \to C^nG^\bullet$ induced by d^0_{n-1}, and structure map $v^n : G^n \to C^nG^\bullet$. We denote the cofiber of δ^{n-1} by $Z^nG^\bullet$, with structure map $w^n\colon C^nG^\bullet \twoheadrightarrow Z^nG^\bullet$, and note that δ^{n-1} factors as $\overline{d}^0_{n-1} \circ w^{n-1}$.

6.1. Dual sequential realizations

Let R be a commutative ring and λ a limit cardinal, with $\Theta^{\mathcal{A}} := \pi_0\mathbf{\Theta}^R$ for $\mathbf{\Theta}^R = s\mathcal{M}^R_\lambda$. Assume given an R-good space $\mathbf{Y} \in \mathcal{S}_*$ and a CW resolution $V_\bullet$ of the $\Pi^{\mathcal{A}}$-algebra$\Lambda = \pi^{\mathcal{A}}_*\mathbf{Y}$, with CW basis $\{\overline{V}_n\}_{n=0}^\infty$, such that for each $n \geqslant 0$, $\overline{V}_n \cong \pi^{\mathcal{A}}_*\overline{\mathbf{W}}^n$ for some $\overline{\mathbf{W}}^n \in \mathbf{\Theta}^R$.

We define a (dual) *sequential realization of* $V_\bullet$ *for* $\mathbf{Y}$ to be a sequence $\mathcal{W}$ of maps

$$\cdots \mathbf{W}^\bullet_{[n+1]} \xrightarrow{p_{[n+1]}} \mathbf{W}^\bullet_{[n]} \xrightarrow{p_{[n]}} \mathbf{W}^\bullet_{[n-1]} \to \cdots \to \mathbf{W}^\bullet_{[1]} \xrightarrow{p_{[1]}} \mathbf{W}^\bullet_{[0]} \tag{6.2}$$

between Reedy fibrant and cofibrant objects in $\mathcal{S}^\Delta_*$, such that for each $n \geqslant 0$:

(i) There is an n-skeletal restricted cosimplicial object $\widetilde{\mathbf{W}}^\bullet_{[n]}$ with $\widetilde{\mathbf{W}}^k_{[n]} = \mathbf{W}^k_{[n-1]} \times P\Omega^{n-k-1}\overline{\mathbf{W}}^n$ for $0 \leqslant k \leqslant n$, where as before by convention $\Omega^0\overline{\mathbf{W}}^n = P\Omega^{-1}\overline{\mathbf{W}}^n = \overline{\mathbf{W}}^n$.

(ii) The coface map $d^0\colon C^k \to \widetilde{\mathbf{W}}^{k+1}_{[n]}$ into the factor $P\Omega^{n-k-2}\overline{\mathbf{W}}^n$ is the map F^k in the commuting diagram

$$\begin{array}{ccccc}
Z^{k-1}\mathbf{W}^\bullet_{[n-1]} & \xrightarrow{\overline{d}^0_{k-1}} & C^k\mathbf{W}^\bullet_{[n-1]} & \overset{w^k}{\twoheadrightarrow} & Z^k\mathbf{W}^\bullet_{[n-1]} \\
\downarrow{\scriptstyle a^{k-1}} & & \downarrow{\scriptstyle F^k} & & \downarrow{\scriptstyle a^k} \\
\Omega^{n-k-1}\overline{\mathbf{W}}^n & \overset{j^{n-k-1}}{\hookrightarrow} & P\Omega^{n-k-2}\overline{\mathbf{W}}^n & \overset{p^{n-k-2}}{\twoheadrightarrow} & \Omega^{n-k-2}\overline{\mathbf{W}}^n
\end{array} \tag{6.3}$$

(in the notation of Definition 6.1). The first coface map d^1 into $P\Omega^{n-k-2}\overline{\mathbf{W}}^n$ is

the composite of the projection onto $P\Omega^{n-k-1}\overline{\mathbf{W}}^n$ with $j^{n-k-1} \circ p^{n-k-1}$, and d^i into the factor $P\Omega^{n-k-2}\overline{\mathbf{W}}^n$ is zero for $i > 1$.
We start with a realization of the n-th attaching map $\partial^V_n : \overline{V}_n \to C_{n-1}V_\bullet$ for the given CW resolution as our choice for $F^{n-1} : C^{n-1}\mathbf{W}^\bullet_{[n-1]} \to \overline{\mathbf{W}}^n$.

(iii) Let $\widehat{\mathbf{W}}^{[n]}_\bullet$ be the pullback of $\mathbf{W}^\bullet_{[n-1]} \leftarrow \mathcal{F}i^*\mathbf{W}^\bullet_{[n-1]} \to \mathcal{F}\widetilde{\mathbf{W}}^{[n]}_\bullet$, where $\mathcal{F} : \mathcal{C}^\Delta \to \mathcal{C}^\Delta$ is the right adjoint of the forgetful functor $i^* : \mathcal{C}^\Delta \to \mathcal{C}^{\Delta_+}$ (see §1.3), with $\mathbf{W}^\bullet_{[n]}$ a Reedy fibrant and cofibrant replacement for $\widehat{\mathbf{W}}^{[n]}_\bullet$.

Again, $\mathbf{W}^\bullet := \lim_n \mathbf{W}^\bullet_{[n]}$ is a cosimplicial resolution of $\mathbf{Y}$ in the resolution model category $\mathcal{C}^\Delta$, and in fact, the sequential realization $\mathcal{W}$ can be constructed starting from any R-mapping algebra $\mathfrak{X}$. See [**BS1**, §2 & Appendix A] for further details.

The proof of Theorem 5.8 can be dualized to yield:

Theorem 6.4. *Given a commutative ring R with $\mathbf{\Theta}^{\mathcal{A}} = s\mathcal{M}^R_\lambda$ and an R-good space $\mathbf{Y}$, let $\boldsymbol{\eta} : \mathfrak{V}_\bullet \to \mathfrak{X} = \mathfrak{M}^R\mathbf{Y}$ be a simplicial resolution in $\mathcal{S}_*^{\mathbf{\Theta}^R \times \Delta^{\mathrm{op}}}$ with a CW basis $\{\overline{V}_n\}_{n=0}^\infty$ for the $\Pi^{\mathcal{A}}$-algebra-resolution $V_\bullet := \pi_0\mathfrak{V}_\bullet \to \Lambda = \pi^{\mathcal{A}}_*\mathbf{Y}$. Then for any sequential realization $\mathcal{W}$ of $V_\bullet$ for $\mathbf{Y}$, there is a Reedy weak equivalence of simplicial dual weak $\mathcal{A}$-mapping algebras $\mathfrak{f} : \mathfrak{W}_\bullet := \mathfrak{M}^{\mathcal{A}}\mathbf{W}^\bullet \to \mathfrak{V}_\bullet$.*

The dual of Remark 5.13, for the n-truncated case, also holds.

Summary 6.5. Given a functor $\mathfrak{T} : \mathsf{sMap}^{\mathrm{St},R}_{\mathrm{re}} \to \mathcal{D}$ as in Corollary 4.8, the *relative right derived functor* $\mathbf{R}^{\mathrm{rel}}\mathfrak{T} : \operatorname{ho}\mathsf{sMap}^{\mathrm{St},R}_{\mathrm{re}} \to \operatorname{ho}(\mathcal{D}^\Delta)$ applied to $\mathfrak{X} := \mathfrak{M}^{\mathrm{St},R}\mathbf{Y}$ for R-good $\mathbf{Y} \in \mathcal{S}_*$, is obtained by

(a) Choosing a simplicial resolution $\boldsymbol{\eta} : \mathfrak{V}_\bullet \to \mathfrak{X}$ in the model category $\mathcal{S}_*^{\mathbf{\Theta}^{\mathcal{A}} \times \Delta^{\mathrm{op}}}$;
(b) Assuming the $\Pi^{\mathcal{A}}$-algebra-resolution $V_\bullet := \pi_0\mathfrak{V}_\bullet \to \pi^{\mathcal{A}}_*\mathbf{Y}$ has a CW basis $\{\overline{V}_n\}_{n=0}^\infty$ (e.g., if R is a field), choosing a sequential realization $\mathcal{W}$ of $V_\bullet$;
(c) Defining $(\mathbf{R}^{\mathrm{rel}}\mathfrak{T})\mathfrak{X}$ to be the cosimplicial object $\mathfrak{T}\mathfrak{M}^{\mathrm{St},R}\mathbf{W}^\bullet$ in $\mathcal{D}^\Delta$.

Example 6.6. For $\mathbf{Z} \in \mathcal{S}_*$ and $\mathbf{T} := \operatorname{map}_*(\mathbf{Z}, -)$ as in Example 4.9, if $\mathfrak{Z} = \mathfrak{M}^{\mathrm{St},R}\mathbf{Z}$ and $\mathfrak{X} = \mathfrak{M}^{\mathrm{St},R}\mathbf{Y}$ for some R-good space $\mathbf{Y}$, and $\mathfrak{V}_\bullet = \mathfrak{M}^{\mathrm{St},R}\mathbf{W}^\bullet$ for some cosimplicial resolution $\mathbf{Y} \to \mathbf{W}^\bullet$, then $(\mathbf{R}^{\mathrm{rel}}\mathfrak{T})\mathfrak{X} := \mathfrak{T}\mathfrak{V}_\bullet$ is the cosimplicial space $\mathfrak{Z}\{\mathbf{W}^\bullet\}$ (up to E^2-equivalence).

7. Truncating derived functors of mapping algebras

So far we have shown only that the usual total derived functor $\mathbf{LT}$ of a continuous functor $\mathbf{T} : \mathcal{C} \to \mathcal{D}$ can be interpreted (under suitable assumptions) as derived functors of the corresponding mapping algebras. Although there are many technicalities involved, the result is hardly surprising, since, under these assumptions, mapping algebras carry the same homotopy information as objects in $\mathcal{C}$ (Theorems 3.10 and 4.7).

The point is that mapping algebras are the right framework for *truncating* the homotopy information (using Postnikov sections), while still retaining enough to compute the required term in the homotopy spectral sequences for $\mathbf{TW}_\bullet$ or $\mathbf{TW}^\bullet$.

Not every homotopy functor $\mathbf{T}$ (and the corresponding $\mathfrak{T}$) will behave as we want with respect to such truncation. We therefore require the following:

Definition 7.1. For any $2 \leqslant r \leqslant \infty$, let $\mathcal{E}^r$ denote the category of r-truncated homological spectral sequences $\{E^k_{**}\}^r_{k=1}$, equipped with a differential $d^r\colon E^r_{t,i} \to E^r_{t-r-1,i+r}$, which need not satisfy $d^r \circ d^r = 0$. A map in $\mathcal{E}^r$ is called a *weak equivalence* if it induces an isomorphism in E^2_{**} (and thus also for $r \leqslant k > 2$). This defines the corresponding localized category $\mathrm{ho}\,\mathcal{E}^r$. We have truncation functors $P^r\colon \mathcal{E}^n \to \mathcal{E}^r$ for each $r \leqslant n \leqslant \infty$. Note that the homotopy spectral sequence of a simplicial space defines a homotopy functor $\mathcal{S}^\infty\colon \mathcal{G}^{\Delta^{\mathrm{op}}} \to \mathcal{E}^\infty$ (with respect to E^2-equivalences in the source and target), and write $\mathcal{S}^r := P^r \circ \mathcal{S}^\infty$.

Definition 7.2. Any homotopy functor $\mathbf{T}\colon \mathcal{G} \to \mathcal{G}$, and the corresponding $\mathfrak{T}\colon \mathsf{sMap}_{\mathrm{St}} \to \mathcal{G}$, induce a functor $\mathcal{S}^r \circ \mathbf{L}^{\mathrm{rel}}\mathfrak{T}\colon\ \mathrm{ho}\,\mathsf{sMap}_{\mathrm{St}} \to \mathrm{ho}\,\mathcal{E}^r$ (see Summary 5.14) for each $r \geqslant 2$. We say that $\mathbf{T}$ (and $\mathfrak{T}$) are *level* if for every $r \geqslant 2$, this functor $\mathcal{S}^r \circ \mathbf{L}^{\mathrm{rel}}\mathfrak{T}$ factors through a functor $\mathbf{L}^{\mathrm{rel}}\mathfrak{T}^{r-2}\colon\ \mathrm{ho}\,\mathsf{sMap}^{r-2}_{\mathrm{St}} \to \mathrm{ho}\,\mathcal{E}^r$.

Here $\mathrm{ho}\,\mathsf{sMap}^n_{\mathrm{St}}$ is the subcategory of $\mathrm{ho}(\mathcal{S}_{[n]}^{\mathbf{\Theta}^{\mathrm{op}}_{\mathrm{St}}\times\Delta^{\mathrm{op}}})$ weakly equivalent to $c(\mathfrak{X})_\bullet$, for $\mathfrak{X}$ in the subcategory $\mathsf{sMap}^n_{\mathrm{St}}$ of n-truncated Stover mapping algebras (cf. §2.3).

In order to identify which homotopy functors are level, we shall need the following notion introduced in [**BB1**, §1] (see also [**BDG**]):

Definition 7.3. Let $\mathcal{C}$ be Top_0, $\mathcal{S}_*$, or $\mathcal{G}$: for any $n \geqslant 0$, an *n-stem* in $\mathcal{C}$ is a tower:

$$\mathcal{Q} := \Big(\cdots \to Q_{k+1} \xrightarrow{q_{k+1}} Q_k \xrightarrow{q_k} Q_{k-1} \cdots Q_1\Big) \tag{7.4}$$

in $\mathcal{C}^{(\mathbb{N},\leqslant)}$, in which $\pi_i(Q_k) = 0$ for $i < k$ or $i > n+k$, and $\pi_i q_k$ is an isomorphism for $k \leqslant i < n+k$. Here $(\mathbb{N},\leqslant)$ is the usual linearly ordered category of the natural numbers. The object $Q_k \in \mathcal{C}$ is called the k-th *n-window* of $\mathcal{Q}$.

We denote by $\mathsf{Stem}[n]$ the full subcategory of n-stems in the functor category $\mathcal{C}^{(\mathbb{N},\leqslant)}$, with the model category structure on the latter as in [**H**, 11.6]. The *Postnikov n-stem* functor $\mathcal{P}[n]\colon \mathcal{C} \to \mathsf{Stem}[n]$ is given by $\mathcal{P}[n]X := \{P^{n+k+1}X\langle k\rangle\}^\infty_{k=1}$.

To avoid the need to distinguish the cases $\mathcal{C} = \mathsf{Top}_0$ or $\mathcal{G}$, we everywhere use the Top-indexing for spheres, homotopy groups, Postnikov systems, and connected covers (as in §3.1).

By [**BB1**, Theorem 4.13 & Corollary 4.16] we have:

Theorem 7.5. *For each $r \geqslant 2$ there is a functor $\hat{\mathcal{S}}^r\colon \mathsf{Stem}[r-1]^{\Delta^{\mathrm{op}}} \to \mathcal{E}^r$ which associates to any simplicial $(r-1)$-stem $Q_\bullet$ an r-truncated spectral sequence. Moreover, $\hat{\mathcal{S}}^r \circ \mathcal{P}[r-1]\colon \mathcal{C}^{\Delta^{\mathrm{op}}} \to \mathcal{E}^r$ is naturally equivalent to $\mathcal{S}^r$, so when $Q_\bullet = \mathcal{P}[r-1]\mathbf{X}_\bullet$ this is the truncation of the usual homotopy spectral sequence for $\mathbf{X}_\bullet$. In this case we have $d^r \circ d^r = 0$, so in fact, the spectral sequence is determined through E^{r+1}_{**} (though without d_{r+1}).*

Corollary 7.6. *A functor $\mathfrak{T}\colon \mathsf{sMap}_{\mathrm{St}} \to \mathcal{G}$ associated to a homotopy functor $\mathbf{T}\colon \mathcal{G} \to \mathcal{G}$ is level if for each $r \geqslant 1$, the relative derived functor $\mathcal{S}^r \circ \mathbf{L}^{\mathrm{rel}}\mathfrak{T}\colon\ \mathrm{ho}\,\mathsf{sMap}_{\mathrm{St}} \to \mathcal{E}^r$ factors as $\hat{\mathcal{S}}^r \circ \mathbf{L}^{\mathrm{rel}}\mathfrak{T}^{r-1}$ for some functor $\mathbf{L}^{\mathrm{rel}}\mathfrak{T}^{r-1}\colon\ \mathrm{ho}\,\mathsf{sMap}^{r-1}_{\mathrm{St}} \to \mathrm{ho}(\mathsf{Stem}[r-1]^{\Delta^{\mathrm{op}}})$.*

In order for Corollary 7.6 to be of any use, we must identify level homotopy functors $\mathbf{T}$ for which the homotopy spectral sequence of $\mathbf{TX}_\bullet$ is of interest. We first note:

Lemma 7.7. *For $\mathcal{B}$ as in Example 2.4, any n-truncated weak $\mathcal{B}$-mapping algebra $\mathfrak{X} \in \mathsf{sMap}^n_{\mathrm{St}}$ is functorially realizable by an n-stem $\mathcal{Q} = \{Q_k\}_{k=1}^\infty$. Moreover, if $\mathfrak{X} = P^n\mathfrak{M}_{\mathcal{B}}\mathbf{Y}$ for some $\mathbf{Y} \in \mathcal{G}$, then $\mathcal{Q}$ is naturally weakly equivalent to the Postnikov n-stem $\mathcal{P}[n]\mathbf{Y}$.*

Proof. This result appears in [**BB2**, §10.5] for Stover mapping algebras, but in fact, we need only observe that for $n \geqslant 1$, the action of $P^n\mathbf{\Theta}_{\mathcal{B}}$ on $\mathfrak{X}$ includes *inter alia* an A_n-structure on $X_k := \mathfrak{X}\{\mathbf{S}^k\}$, so allowing $P^n X_k$ to be delooped to produce the window Q_k by [**Sta**, Corollary 11.12]. The weak equivalences (2.22), together with [**Ma1**, Theorem 12.7], yield the structure maps for the n-stem $\mathcal{Q}$. □

The simplest example is from [**Bl1**], where it is used to construct a spectral sequence for computing $H_*\mathbf{Y}$ from the Π-algebra $\pi_*\mathbf{Y}$:

Proposition 7.8. *The abelianization functor* $\mathrm{Ab}\colon \mathcal{G} \to \mathcal{G}$ *is level.*

Proof. Let $\mathcal{Q} = \{Q_k\}_{k=1}^\infty$ denote the Postnikov n-stem of a space $\mathbf{X}$, and $\mathcal{R} = \{R_k\}_{k=1}^\infty$ that of $\mathrm{Ab}\,\mathbf{X}$. Note that for each $k \geqslant 0$, the covering map $\rho\colon \mathbf{X}\langle k\rangle \to \mathbf{X}$ induces a map $\rho_*\colon \mathrm{Ab}(\mathbf{X}\langle k\rangle) \to \mathrm{Ab}\,\mathbf{X}$, which factors through $(\mathrm{Ab}\,\mathbf{X})\langle k\rangle$ by cellularity (uniquely, if we choose a $(k+1)$-reduced model for connected covers $-$ which is an inclusion of a sub-simplicial group, in $\mathcal{G}$). Furthermore, by the Hurewicz Theorem, for each $m \geqslant 0$ the structure map $p_m\colon \mathbf{X} \to P^m\mathbf{X}$ induces an isomorphism $H_i\mathbf{X} \to H_iP^m\mathbf{X}$ for $i \leqslant m$, and an epimorphism $H_{m+1}\mathbf{X} \to H_{m+1}P^m\mathbf{X}$, so the natural map $P^m(\mathrm{Ab}\,\mathbf{X}) \to P^m\,\mathrm{Ab}(P^m\mathbf{X})$ is a weak equivalence. Thus we have a natural weak equivalence $P^{n+k+1}(\mathrm{Ab}\,Q_k) \simeq R_k$ for each $k \geqslant 0$.

Thus a given a simplicial resolution $\mathfrak{V}_\bullet \to P^n\mathfrak{X} = P^n\mathfrak{M}_{\mathcal{B}}\mathbf{Y}$ of n-truncated $\mathcal{B}$-presheaves in the model category $\mathcal{S}_{[n]}^{\mathbf{\Theta}_{\mathcal{B}}^{\mathrm{op}}}$, by Lemma 7.7, we obtain a simplicial n-stem $\mathcal{Q}_\bullet$, which yields in turn the required simplicial n-stem $\mathcal{R}_\bullet := \mathcal{P}[n](\mathrm{Ab}\,\mathcal{Q}_\bullet)$. □

Here are two additional examples from [**Sto**]. The first is used to construct a spectral sequence for computing $\pi_*\Sigma\mathbf{Y}$ from $\pi_*\mathbf{Y}$:

Proposition 7.9. *The suspension functor* $\Sigma\colon \mathcal{G} \to \mathcal{G}$ *is level.*

Proof. For each $n \geqslant 1$, any n-truncated weak $\mathcal{B}$-mapping algebra has a corresponding n-stem $\mathcal{Q}$ by Lemma 7.7, and the Π-algebra $\Lambda := \pi_0\mathfrak{X}$ determines the Π-algebra structure on π_*Q_k for each $k \geqslant 0$. If $\mathfrak{X} \simeq P^n\mathfrak{M}_{\mathcal{B}}\mathbf{X}$ for some space $\mathbf{X}$, then Λ is isomorphic to $\pi_*\mathbf{X}$ and $Q_k \simeq P^{n+k+1}\mathbf{X}\langle k\rangle$. To understand $\mathbf{L}\mathfrak{T}$, we need only consider the case when Λ is a free Π-algebra.

Now let $\mathcal{R} = \{R_k\}_{k=1}^\infty$, denote the Postnikov n-stem of $\Sigma\mathbf{X}$. As in the proof of Proposition 7.8, the covering map $\rho\colon \mathbf{X}\langle k\rangle \to \mathbf{X}$ induces a map $\rho_*\colon \Sigma(\mathbf{X}\langle k\rangle) \to (\Sigma\mathbf{X})\langle k+1\rangle$. Taking Postnikov sections yields natural maps $p_k\colon P^{n+k+2}(\Sigma Q_k) \to R_{k+1}$. In particular, $p_0\colon P^{n+2}(\Sigma Q_0) \to R_1 = P^{n+1}(\Sigma\mathbf{X})$ is a weak equivalence by the Hurewicz Theorem, with $P^1R_1 \simeq \mathbf{X}_1$ (a wedge of 1-spheres, and thus aspherical).

However, for $k > 1$ there is no functorial description of R_k in terms of $\mathcal{Q}$. Thus if $\mathfrak{T} := \Sigma \circ N\colon \mathsf{sMap}_{\mathrm{St}} \to \mathcal{G}$ is induced by $\Sigma\colon \mathcal{G} \to \mathcal{G}$ as in Corollary 3.11, in order to define $\mathbf{L}\mathfrak{T}^n\colon \mathrm{ho}\,\mathsf{sMap}^n_{\mathrm{St}} \to \mathsf{Stem}[n]$ we must proceed as follows:

By Lemma 7.7 a simplicial resolution $\mathfrak{V}_\bullet \to P^n\mathfrak{X} = P^n\mathfrak{M}_{\mathcal{B}}\mathbf{Y}$ of n-truncated weak $\mathcal{B}$-mapping algebras yields a simplicial n-stem $\mathcal{Q}_\bullet$. Since the simplicial Π-algebra

$V_\bullet := \pi_o \mathfrak{V}_\bullet$ is a free resolution of $\Lambda := \pi_0 \mathfrak{X}$, it has a (non-canonical) CW basis $\{\overline{V}_n\}_{n=0}^\infty$ for it, which in turn has a sequential realization $\mathcal{W}$ (see §5.2). By Remark 5.13, there is a Reedy weak equivalence of simplicial n-truncated weak $\mathcal{B}$-mapping algebras $\mathfrak{f}\colon P^n\mathfrak{W}_\bullet \to \mathfrak{V}_\bullet$, where $\mathfrak{W}_\bullet$ is realizable as $\mathfrak{M}_\mathcal{B}\mathbf{W}_\bullet$. We can realize $P^n\mathfrak{W}_\bullet$ by the simplicial n-stem $\widehat{\mathcal{Q}}_\bullet \simeq \mathcal{P}[n]\mathbf{W}_\bullet$, and let $\Sigma\widehat{\mathcal{Q}}_\bullet$ denote the simplicial n-stem obtained by applying Σ to each window of $\widehat{\mathcal{Q}}_\bullet$ (and taking appropriate Postnikov sections). If $\widehat{\mathcal{R}}_\bullet$ denotes the simplicial Postnikov n-stem $\mathcal{P}[n]\Sigma\mathbf{W}_\bullet$, we have a map of simplicial n-stems $\widehat{p}\colon \Sigma\widehat{\mathcal{Q}}_\bullet \to \widehat{\mathcal{R}}_\bullet$, as explained above.

Similarly, the simplicial n-truncated $\mathcal{B}$-presheaf $\mathfrak{V}_\bullet$ yields a simplicial n-stem $\mathcal{Q}_\bullet$, and $\mathfrak{f}\colon P^n\mathfrak{W}_\bullet \to \mathfrak{V}_\bullet$ induces a levelwise weak equivalence of simplicial n-stems $\widehat{f}\colon \Sigma\widehat{\mathcal{Q}}_\bullet \to \Sigma\mathcal{Q}_\bullet$ (in the Reedy model structure). We may assume that each window of all the simplicial n-stems described here are cofibrant in $\mathcal{G}$, so they are Reedy cofibrant. Thus if we let $\mathcal{R}_\bullet$ denote the homotopy pushout of $\widehat{f}$ and $\widehat{p}$ (in the Reedy model category of simplicial $\mathcal{B}$-presheaves), we have a Reedy weak equivalence $\widehat{\mathcal{R}}_\bullet \to \mathcal{R}_\bullet$ (cf. [**H**, Proposition 13.1.2]), as well as a structure map of simplicial n-stems $p\colon \mathcal{Q}_\bullet \to \mathcal{R}_\bullet$.

We define $(\mathbf{L}\mathfrak{T}^n)P^n\mathfrak{X}$ to be the simplicial n-stem $\mathcal{R}_\bullet$ To see that $\mathbf{L}\mathfrak{T}^n$ is well-defined, replace $\mathfrak{V}_\bullet$ by some other simplicial resolution $\mathfrak{U}_\bullet \to P^n\mathfrak{X}$ of n-truncated $\mathcal{B}$-presheaves, with $\mathcal{Z}$ a sequential realization of $\pi_0\mathfrak{U}_\bullet$ for $\mathbf{Y}$. Let $\mathcal{R}_\bullet$ and $\mathcal{S}_\bullet$ denote the simplicial n-stems associated as above to $\mathfrak{V}_\bullet$ and $\mathfrak{U}_\bullet$ respectively. We then have a weak equivalence of simplicial spaces $g\colon \mathbf{W}_\bullet \to \mathbf{Z}_\bullet$ in the resolution model category structure with respect to $\mathbf{\Theta}_\mathcal{B}$ (since both are cofibrant replacements for $c(\mathbf{Y})_\bullet$), and this will induce a weak equivalence $\mathfrak{V}_\bullet \to \mathfrak{U}_\bullet$ in the resolution model structure of §3.3, and thus the same holds for the simplicial n-stems $\mathcal{R}_\bullet$ and $\mathcal{S}_\bullet$ (cf. [**Sto**, Theorem 1.9]). □

The next example is used to construct a van Kampen spectral sequence to compute $\pi_*(\mathbf{Y} \vee \mathbf{Z})$ from $\pi_*\mathbf{Y}$ and $\pi_*\mathbf{Z}$:

Proposition 7.10. *The wedge bifunctor* $\vee\colon \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ *is level.*

Proof. The proof is entirely analogous to that of Proposition 7.9: given two Stover mapping algebras $\mathfrak{X}$ and $\mathfrak{Y}$, realizable by $\mathbf{Y}$ and $\mathbf{Z}$, respectively, their n-truncations are realizable by n-stems $\mathcal{Q}$ and $\mathcal{S}$, weakly equivalent to the Postnikov n-stem $\mathcal{P}[n]\mathbf{Y}$ and $\mathcal{P}[n]\mathbf{Z}$, respectively. Once again we cannot reconstruct the Postnikov n-stem for $\mathbf{Y} \vee \mathbf{Z}$ directly from the window-wise wedge of $\mathcal{Q}$ and $\mathcal{S}$ (except for the bottom window), but must have recourse to sequential realizations of the full simplicial resolutions. □

Remark 7.11. Stover set up spectral sequences for arbitrary homotopy colimits in Top_0 (see [**Sto**, Theorem 1.2]), and one can obtain similar results for the left derived functors appearing as the E^2-terms of these spectral sequences.

8. Truncating derived functors of dual mapping algebras

We may dualize Definitions 7.1 and 7.2 of Section 7 as follows:

Definition 8.1. For any $2 \leqslant r \leqslant \infty$ we let $\mathcal{E}_r$ denote the category of r-truncated cohomological spectral sequences $\{E_k^{**}\}_{k=1}^r$ (again, the last differential need not satisfy $d_r \circ d_r = 0$). A *weak equivalence* in $\mathcal{E}_r$ is a map inducing an isomorphism in E_2^{**}. Again we have truncation functors $P^r \colon \mathcal{E}_n \to \mathcal{E}_r$. The homotopy spectral sequence of a cosimplicial space defines a homotopy functor $\mathcal{S}_\infty \colon \mathcal{S}_*^\Delta \to \mathcal{E}_\infty$, and we write $\mathcal{S}_r := P^r \circ \mathcal{S}_\infty$.

If $\mathbf{T} \colon \mathcal{S}_R \to \mathcal{D}$ is a homotopy functor preserving R-equivalences, we say that $\mathbf{T}$, and the corresponding $\mathfrak{T} \colon \mathsf{sMap}_{\mathrm{re}}^{\mathrm{St},R} \to \mathcal{D}$ of Corollary 4.8, are *level* if for any $r \geqslant 2$ and weakly R-good dual strict Stover mapping algebra $\mathfrak{X} = \mathfrak{M}^{\mathrm{St},R}\mathbf{Y}$ (see Definition 4.5), $\mathcal{S}_r \mathbf{R}^{\mathrm{rel}}\mathfrak{T}\mathfrak{X}$ factors up to isomorphism through the $(r-2)$-truncated simplicial dual strict Stover mapping algebra $P^{r-2}\mathfrak{M}^{\mathrm{St},R}(\mathbf{R}^{\mathrm{rel}}\mathfrak{T}\mathfrak{X})$, up to weak equivalence in $\mathcal{S}_{[n]}^{\mathbf{\Theta}^{\mathcal{A}} \times \Delta^{\mathrm{op}}}$ (see Remark 4.6).

Although the analogue of Theorem 7.5 was also shown in [**BB1**] to hold for the homotopy spectral sequence of a cosimplicial space, this does not appear to be helpful in showing that functors of R-mapping algebras are level − mainly because there is no simple connection between maps into Eilenberg–Mac Lane spaces and maps out of spheres. Thus a more direct approach is needed here.

Our main result in this connection, which may be of independent interest, is the following reinterpretation of the results of [**BBS**]:

Theorem 8.2. *For any $\mathbf{Z} \in \mathcal{S}_*$ and $R = \mathbb{F}_p$ or $\mathbb{Q}$, the unstable R-Adams spectral sequence for $\mathbf{T} := \mathrm{map}_*(\mathbf{Z}, -)$ applied to $\mathbf{Y} \in \mathcal{S}_R$ (see [**BK1**, §7.2]) is determined by the simplicial R-mapping algebra $(\mathfrak{M}^R\mathbf{R}^{\mathrm{rel}}\mathfrak{T})\mathfrak{M}^{\mathrm{St},R}\mathbf{Y}$, and $\mathbf{T}$ is level.*

Proof. Let $\mathbf{Y} \to \mathbf{W}^\bullet$ be a cosimplicial resolution, which we may assume without loss of generality to be associated to a dual sequential realization $\mathcal{W}$ as in §6.1, by Theorem 6.4.

We know that the homotopy spectral sequence for the cosimplicial space $\mathbf{X}^\bullet := \mathrm{map}(\mathbf{Z}, \mathbf{W}^\bullet)$ is determined in principle by the simplicial dual strict $\mathcal{A}$-mapping algebra $\mathfrak{W}_\bullet := \mathfrak{M}^{\mathrm{St},R}\mathbf{W}^\bullet$. Following the description in [**BBS**] (and compare [**Bou1**]) we now explain how this can be made explicit:

By [**BBS**, Proposition 4.18] the unstable Adams spectral sequence for $\mathbf{Y}$ as above agrees from the E_2-term on with that associated to the fibration sequences

$$\Omega^n \overline{\mathbf{W}}^n \;\to\; \mathrm{Tot}_n \widehat{\mathbf{W}}_{[n]}^\bullet \;\to\; \mathrm{Tot}_{n-1} \mathbf{W}_{[n-1]}^\bullet, \tag{8.3}$$

in the notation of §6.1, so the same is true of the homotopy spectral sequence for $\mathbf{X}^\bullet := \mathrm{map}(\mathbf{Z}, \mathbf{W}^\bullet)$, if we apply $\mathrm{map}_*(\mathbf{Z}, -)$ before taking Tot.

An element $\gamma \in E_1^{n,k+n}$ is thus represented by a map $\Sigma^k\mathbf{Z} \to \mathrm{Tot}_n \mathbf{\Sigma} D_{[n]}^\bullet$, where $\mathbf{\Sigma} D_{[n]}^\bullet$ is the fiber of the Reedy fibration $\widehat{\mathbf{W}}_{[n]}^\bullet \to \mathbf{W}_{[n-1]}^\bullet$ and $\mathrm{Tot}_n \mathbf{\Sigma} D_{[n]}^\bullet \simeq \Omega^n \overline{\mathbf{W}}^n$ (see [**BBS**, Proposition 4.12]). This is represented in turn by a map of cosimplicial spaces $G^\bullet \colon \mathbf{\Delta}^\bullet \ltimes \Sigma^k\mathbf{Z} \to \mathbf{W}_{[n]}^\bullet$ (see §6.1(iii)) − that is, a sequence of maps $G_{[n]}^j \colon \mathbf{\Delta}^j \ltimes \Sigma^k\mathbf{Z} \to \mathbf{W}_{[n]}^j$ (where we may assume $G_{[n]}^j = 0$ for $j < n$ by [**BBS**, (3.6)]).

By [**BBS**, Theorem 5.9], for each $r \geqslant 2$ and $N := n + r - 1$, the differential $d_r \colon E_r^{n,k+n} \to E_r^{N+1,k+N}$ is defined on $\langle\gamma\rangle$ by the value $\phi \colon \Sigma^k\mathbf{Z} \to \Omega^N \overline{\mathbf{W}}^{N+1}$ of a certain r-th order R-cohomology operation. This operation is defined when the associated sequence of lower order operations vanish, so that there exists a chosen lift of $G^{\bullet}$ to $G_{[N]}\colon \mathbf{\Delta}^{\bullet} \ltimes \Sigma^{k}\mathbf{Z} \to \mathbf{W}^{\bullet}_{[N]}$.

The map ϕ is obtained by patching together the composite of the maps $G^{i}_{[N]}$ with the given maps $F^{j}_{[N+1]}\colon \mathbf{W}^{j}_{[N]} \to P\Omega^{N-j-1}\overline{\mathbf{W}}^{N+1}$ of (6.3), yielding a map from the boundary of a certain $(N+1)$-dimensional polyhedron $\mathcal{P}^{N+1}_{r}$, described in [**BBS**, §4.3] to $\mathrm{map}_{*}(\Sigma^{k}\mathbf{Z}, \overline{\mathbf{W}}^{N+1})$. This is adjoint to a map $\widetilde{\phi}\colon \Sigma^{k}\mathbf{Z} \to \Omega^{N}\overline{\mathbf{W}}^{N+1}$, and by [**BBS**, Theorem 5.10], the class

$$[\widetilde{\phi}] \in\ [\Sigma^{k}\mathbf{Z},\, \Omega^{N}\overline{\mathbf{W}}^{N+1}] \cong [\Sigma^{k-1}\mathbf{Z},\, \Omega^{N+1}\overline{\mathbf{W}}^{N+1}] \cong E_{1}^{N+1,k+N}$$

(using the usual Σ-Ω adjunction on the left) represents $d_{r}\langle\gamma\rangle \in E_{r}^{N+1,k+N}$. In particular, by [**BBS**, Lemma 5.7], $[\widetilde{\phi}]$ vanishes if and only if $G_{[N]}$ lifts to a map $G_{[N+1]}\colon \mathbf{\Delta}^{\bullet} \ltimes \Sigma^{k}\mathbf{Z} \to \mathbf{W}^{\bullet}_{[N+1]}$.

Because we assumed that each $\overline{\mathbf{W}}^{N}$ is in $\mathbf{\Theta}^{R}$ (see §6.1), the information used in defining this higher operation is encoded by $\mathfrak{W}_{\bullet} := \mathfrak{M}^{R}\mathbf{W}^{\bullet}$ and $\mathfrak{Z} := \mathfrak{M}^{R}\mathbf{Z}$ Furthermore, since $G^{j}_{[N]} = 0$ for $j < n$, and $\mathbf{W}^{\bullet}_{[N]}$ is $(n+r-1)$-skeletal by §6.1(i), from the description above we see that we only need $P^{r-1}\mathfrak{Z}\{\Omega^{k}\overline{\mathbf{W}}^{N}\}$ in order to calculate d_{r}, and thus E^{**}_{r+1}. Finally, by Example 4.9, $P^{r-1}\mathfrak{Z}$ is completely determined by the $(r-1)$-truncated R-mapping algebra $P^{r-1}\mathfrak{W}_{\bullet}$, and this in turn depends only on $P^{r-1}\mathfrak{M}^{\mathrm{St},R}\mathbf{Y}$, up to E_{2}-equivalence. □

Corollary 8.4. *For any* $\mathbf{Z} \in \mathcal{S}_{*}$ *and* $R = \mathbb{F}_{p}$ *or* $\mathbb{Q}$, *the mapping space functor* $\mathrm{map}_{*}(\mathbf{Z}, -)$ *is a level homotopy functor* $\mathcal{S}_{R} \to \mathcal{S}_{*}$.

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