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VECTOR BUNDLES AND COHOMOTOPIES OF SPIN 5-MANIFOLDS

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Abstract

The purpose of this paper is two-fold: On the one side we would like to fill a gap on the classification of vector bundles over 5-manifolds. Therefore it will be necessary to study quaternionic line bundles over 5-manifolds which are in 1-1 correspondence to elements in the cohomotopy group $\pi^4(M) = [M, S^4]$ of M. From results in [**22, 24**] this group fits into a short exact sequence, which splits into $H^4(M;\mathbb{Z}) \oplus \mathbb{Z}_2$ if M is spin. The second intent is to provide a bordism theoretic splitting map for this short exact sequence, which will lead to a $\mathbb{Z}_2$-invariant for quaternionic line bundles. This invariant is related to the generalized Kervaire semi-characteristic of [**23**].

1. Introduction

The classification of isomorphism classes of vector bundles over a fixed manifold (or more general over a CW-complex) in terms of computable invariants (e.g. by characteristic classes) is a classical and everlasting problem in topology. In particular, in low dimensions such classifications are feasible.

Knowingly every real and complex line bundle is completely determined by its first Stiefel–Whitney and its first Chern class respectively. One of the first classification results was acquired by *Dold* and *Whitney* in [**10**] on 4-complexes. And later *Woodward* classified oriented n-dimensional vector bundles over n-complexes for $n = 3, 4, 6, 7, 8$ in terms of characteristic classes by using elementary homotopy theoretic methods in [**28**].

The gap in Woodward's classification, namely the case $n = 5$, appears to be somewhat different as the example of S^5 shows: By the clutching construction the isomorphism classes of oriented rank 5 vector bundles over S^5 are enumerated by $\pi_4(\mathbf{SO}(5)) \cong \mathbb{Z}_2$ and are represented by the trivial and the tangent bundle of S^5. Of course both vector bundles have trivial characteristic classes and therefore these cannot be used to distinguish them. In [**5**] *Čadek* and *Vanžura* classify oriented rank 5 vector bundles over a 5-complex X provided the following assumptions are fulfilled (see [**5**, p. 755])

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(A) $H^4(X;\mathbb{Z})$ has no element of order 4,

(B) $\mathrm{Sq}^2 H^3(X;\mathbb{Z}_2) = H^5(X;\mathbb{Z}_2)$.

An important example of a 5-complex satisfying condition **(B)** is an oriented, closed 5-manifold M with $w_2(M) \neq 0$ (by orientability the Wu class of Sq^2 is just $w_2(M)$). The authors obtain

Theorem 1.1 (Theorem 1, [**5**]). *Let X be a CW-complex of dimension $\leqslant 5$ and suppose*

$$\gamma\colon [X, B\boldsymbol{SO}(5)] \to H^2(M;\mathbb{Z}_2) \oplus H^4(M;\mathbb{Z}_2) \oplus H^4(M;\mathbb{Z})$$

is defined by

$$\gamma(V) = (w_2(V), w_4(V), p_1(V)),$$

where the triple consists of the second and fourth Stiefel–Whitney, and the first Pontryagin class of V respectively. Then

- *(a) $\mathrm{im}\,\gamma = \{(a,b,c) : \rho_4(c) = \mathfrak{P}a + i_* b\}$ (where ρ_4 is the* mod 4 *reduction in cohomology, $\mathfrak{P}\colon H^2(M;\mathbb{Z}_2) \to H^4(M;\mathbb{Z}_4)$ is a Pontryagin square (cf. [**18**, Chapter 2]) and $i_*\colon H^*(M;\mathbb{Z}_2) \to H^*(M;\mathbb{Z}_4)$ the homomorphism induced by the map $\mathbb{Z}_2 \to \mathbb{Z}_4$).*
- *(b) If conditions **(A)** and **(B)** are fulfilled then γ is injective.*

Hereby we note that results in [**28**] are in similar fashion as Theorem 1.1, especially with conditions like **(A)**.

The paper in hand will fill the gap for $n = 5$ where M is a smooth, closed 5-manifold with $w_2(M) = 0$. We call a vector bundle $V \to M$ *spinnable* if $w_1(V) = w_2(V) = 0$. We prove in Proposition 2.3 that every spinnable vector bundle of rank 5 over M is decomposed by $E \oplus \varepsilon^1$ where E is a quaternionic line bundle.

Unlike the case of real and complex vector bundles the set of quaternionic line bundles do not possess in general a group structure. However if M is of dimension five, the quaternionic line bundles over M are in 1-1 correspondence with elements of the cohomotopy group $\pi^4(M) = [M, S^4]$ which has a natural group structure, see Remark 3.1. In section 3.1 we explain that, from previous works, $\pi^4(M)$ fits into a short exact sequence which splits such that $\pi^4(M) \cong H^4(M;\mathbb{Z}) \oplus \mathbb{Z}_2$ if M is spin. The *Pontryagin–Thom construction* provides an isomorphism between $\pi^4(M)$ and $\Omega^{\mathrm{fr}}_{1;M}$, the bordism classes of normally framed closed 1-submanifolds of M. In section 3.2 we assign to every quaternionic line bundle $E \to M$ a *framed divisor* $[L, \varphi_E] \in \Omega^{\mathrm{fr}}_{1;M}$ where L is the zero locus of a generic real section of E and φ_E a framing on the normal bundle of L induced by the $\mathbf{Sp}(1)$-structure of E. In section 3.3 we construct an invariant

$$\kappa\colon \pi^4(M) \longrightarrow \Omega^{\mathrm{fr}}_1,$$

where $\kappa(E)$ is the *stabilized framed divisor* of E and Ω^{fr}_1 the bordism group of stably framed closed 1-manifolds, see Definition 3.8. The definition of κ can depend on the choice of a spin structure on M. The main result in this section is Theorem 3.13 where we show that the $\mathbb{Z}_2$-part of $\pi^4(M)$ is isomorphic to $\pi_5(S^4)$ and furthermore that κ is a section for the short exact sequence of $\pi^4(M)$. We obtain

Corollary 1.2. *Let M be a closed and connected spin 5-manifold. Then for any $\boldsymbol{Sp}(1)$-structure on M the map*

$$\pi^4(M) \to H^4(M;\mathbb{Z}) \times \Omega_1^{fr}, \quad E \mapsto \Big(\frac{p_1}{2}(E), \kappa(E)\Big)$$

is an isomorphism of groups (where $\frac{p_1}{2}(E)$ is the spin characteristic class of E).

Cohomotopy groups of manifolds were also studied by *Kirby, Teichner* and *Melvin* in [**13**] where the authors compute by elementary geometric arguments the cohomotopy group π^3 of 4-manifolds. For X an odd 4-manifold the authors show in [**13**, Theorem 1] that the short exact sequence of $\pi^3(X)$ splits if and only if X is spin.

The classification of quaternionic line bundles over quaternionic projective spaces was studied in [**12, 11**]. Our results show similarities to the work of [**7**]. *Crowley* and *Goette* introduce an index theoretic t-invariant (cf. [**7**, Definition 1.4]) for quaternionic line bundles E on $(4k-1)$-manifolds N with $H^3(N;\mathbb{Q}) = 0$ and such that the spin characteristic class of E is torsion. If $\mathrm{Bun}(N)$ denotes the set of isomorphism classes of quaternionic line bundles, then the t-invariant is a map $t\colon \mathrm{Bun}(N) \to \mathbb{Q}/\mathbb{Z}$ such that

$$\mathrm{Bun}(N) \longrightarrow H^4(N;\mathbb{Z}) \times \mathbb{Q}/\mathbb{Z}, \quad E \mapsto \big(\tfrac{p_1}{2}(E), t(E)\big)$$

is injective provided N is a smooth, 2-connected oriented rational homology 7-sphere, see [**7**, Theorem 0.1] and cf. Corollary 1.2 (note that $\frac{p_1}{2}(E) = -c_2(E)$, if we consider E as a vector bundle with $\mathbf{SU}(2)$-structure, see section 3.2). Moreover, the concept of a *divisor* for quaternionic line bundles was also used in [**7**] to show that the t-invariant localizes near its divisor, cf. [**7**, Proposition 1.10]. Finally, the t-invariant and the κ-invariant indicate more resemblances as [**7**, Proposition 1.11] shows: For a stably framed manifold N one obtains $t(E) = -e(Y)$, where Y is a divisor of E and e Adam's e-invariant, cf. [**1**, Section 7]. Thus t is, as κ, related to the theory of stable homotopy groups and the J-homomorphism.

In section 4 we will use the developed theory of quaternionic line bundles to classify spinnable vector bundles of rank 5 over spin 5-manifolds in terms of κ and the spin characteristic class $\frac{p_1}{2}$:

Theorem 1.3. *Let M be a closed and connected spin 5-manifold and consider the sets*

$$W_1 := \{V \in [M, B\boldsymbol{SO}(5)] : w_2(V) = w_4(V) = 0\},$$
$$W_2 := \{V \in [M, B\boldsymbol{SO}(5)] : w_2(V) = 0,\ w_4(V) \neq 0\}.$$

Then W_1 is a group such that the map

$$W_1 \to \ker \rho_2 \oplus \Omega_1^{fr}, \quad V \mapsto \big(\tfrac{p_1}{2}(V), \kappa(V)\big)$$

is an isomorphism of groups, where $\rho_2\colon H^4(M;\mathbb{Z}) \to H^4(M;\mathbb{Z}_2)$ is the induced homomorphism by the mod 2 reduction $\mathbb{Z} \to \mathbb{Z}_2$.

Furthermore if $\dim H^4(M;\mathbb{Z}_2) > 0$ then every element in W_2 is uniquely determined by its spin characteristic class $\frac{p_1}{2}$ which are in one-to-one correspondence to $\rho_2^{-1}(H^4(M,\mathbb{Z}_2) \setminus \{0\})$. In particular, every vector bundle in W_2 is determined by its stable class.

Moreover, for $V \in W_1$ we define $\kappa(V)$ to be $\kappa(E)$, where E is the unique quaternionic line bundle such that $V \cong E \oplus \varepsilon^1$. Proposition 4.3 shows that $\kappa(V)$ is the *generalized Kervaire semi-characteristic* of V, cf. [**23**]. Thus we provide a bordism theoretic definition of the generalized Kervaire semi-characteristic.

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2. Spinnable vector bundles

From now on, if not otherwise stated, M should always denote a *connected* and *oriented* manifold of dimension 5.

Let $V \to M$ be an oriented, spinnable (i.e. $w_2(V) = 0$) vector bundle over M of rank 5. Before we prove a structure result for V we have to discuss subgroups of $\mathbf{SO}(n)$ and $\mathbf{Sp}(n)$ and their relations in low dimensions.

Let $i\colon \mathbf{SU}(2) \to \mathbf{SO}(5)$ denote the canonical inclusion $\mathbf{SU}(2) \subset \mathbf{SO}(4) \subset \mathbf{SO}(5)$. If we identify $\mathbf{Sp}(1)$ with $\mathbf{SU}(2)$ we may consider $\mathbf{Sp}(1)$ as a subgroup of $\mathbf{SO}(5)$ by i. Furthermore we consider one of the possible two diagonal embeddings of $\mathbf{Sp}(1)$ into $\mathbf{Sp}(2)$ which we call a *standard embedding*. We remark, that we will not distinguish between an $\mathbf{Sp}(1)$ and an $\mathbf{SU}(2)$-structure.

Lemma 2.1 (see Section 2 of [**19**])**.** *Let $i\colon \boldsymbol{Sp}(1) \to \boldsymbol{SO}(5)$ be the inclusion described above. Then i factors as the standard embedding of $\boldsymbol{Sp}(1)$ into $\boldsymbol{Sp}(2)$ through the universal cover map $\pi\colon \boldsymbol{Sp}(2) \to \boldsymbol{SO}(5)$.*

We define a quaternionic line bundle by means of reductions of structure groups.

Definition 2.2. Let $E \to M$ be an oriented real vector bundle of rank 4 such that its structure group can be reduced to $\mathbf{Sp}(1)$. We say E is a *quaternionic line bundle* if a choice of a reduction is made.

Now the important structure result for V is

Proposition 2.3. *There is a quaternionic line bundle $E \to M$ such that $V \cong E \oplus \varepsilon^1$.*

Proof. Let B_k denote the classifying space $B\mathbf{SO}(k)$ and let $B_{\mathbb{H}} := B\mathbf{SU}(2) \cong B\mathbf{Sp}(1)$. Then consider the fibration

$$\mathbf{SO}(5)/\mathbf{SU}(2) \longrightarrow B_{\mathbb{H}} \xrightarrow{Bi} B_5,$$

where Bi is the map induced from $i\colon \mathbf{Sp}(1) \to \mathbf{SO}(5)$ on the classifying spaces. Denote with the same letter $V\colon M \to B_5$ the classifying map of V. Since $w_2(V) = 0$ there is lift of V across $B\mathbf{Sp}(2) \to B_5$, which can be reduced to $B_{\mathbb{H}}$, since $B_{\mathbb{H}} \to B\mathbf{Sp}(2)$ is 7-connected (note that $\mathbf{Sp}(2)/\mathbf{Sp}(1) \cong S^7$). The proposition follows now from Lemma 2.1. □

Thus to understand the spinnable rank 5 vector bundles one has to understand first the set of quaternionic line bundles over M.

Remark 2.4. The geometric properties of $\mathbf{SU}(2)$-structures on 5-manifolds were studied intensively in [**6, 9**].

3. Quaternionic line bundles

The classifying space $B\mathbf{Sp}(1) \cong B\mathbf{SU}(2) \cong B\mathbf{Spin}(3)$ for quaternionic line bundles is given by the infinite quaternionic projective space $\mathbb{HP}^\infty$ and the inclusion $S^4 \to \mathbb{HP}^\infty$ is an 7-equivalence. Thus if M is a 5-dimensional manifold the set of isomorphism classes of quaternionic line bundles are in 1-1 correspondence to

$$[M, S^4] = \pi^4(M),$$

see also [**21**]. Hence, the set of quaternionic line bundles over M possesses naturally the structure of group, which is in general false in higher dimensions except the cases of real and complex line bundles. We conclude that every quaternionic vector bundle $E \to M$ is the pull-back of the tautological quaternionic line bundle H over $S^4 = \mathbb{HP}^1$. We will mix notations and denote a quaternionic line bundle over a 5-manifold also by a homotopy class of a continuous map $M \to S^4$.

Remark 3.1. (a) Let us describe the group structure of $\pi^4(M)$: Consider the inclusion $j\colon S^4 \vee S^4 \to S^4 \times S^4$ of the 7-skeleton of $S^4 \times S^4$ (endowed with the standard CW-structure). Since M is 5-dimensional, the induced map $j_\#\colon [M, S^4 \vee S^4] \to [M, S^4 \times S^4]$ is bijective. For $f, g \in \pi^4(M)$ the group structure is defined by

$$f + g := (\mathrm{id}_{S^4} \vee \mathrm{id}_{S^4})_\# \circ (j_\#)^{-1}(f, g).$$

This makes $\pi^4(M)$ into an abelian group, cf. [**21**, Theorem 6.3].

(b) A more sophisticated view to the group structure can be found in [**24**, 6]: Let SE_4 be the first two stages of the Postnikov decomposition of S^4 then the map $S^4 \to SE_4$ induces a bijection $[M, S^4] \to [M, SE_4]$ (cf. [**20**]). But we have $SE_4 \cong \Omega SE_5$, thus SE_4 is a homotopy H-space and therefore $[M, S^4]$ carries a natural group structure.

3.1. Steenrod's enumeration of $\pi^4(M)$

Steenrod investigated $\pi^n(X)$ in [**22**, Theorem 28.1, p. 318] for a CW complex X of dimension $\dim X = n + 1$. If $\sigma \in H^n(S^n; \mathbb{Z})$ is a generator then the *Hurewicz-homomorphism*

$$\Phi\colon \pi^n(X) \to H^n(X; \mathbb{Z}), \quad f \mapsto f^*(\sigma)$$

is a surjective group homomorphism and the kernel is isomorphic to

$$H^{n+1}(X; \mathbb{Z}_2)/\mathrm{Sq}^2\,\mu(H^{n-1}(X; \mathbb{Z}))$$

where $\mu\colon H^i(X; \mathbb{Z}) \to H^i(X; \mathbb{Z}_2)$ is the map induced by the coefficient homomorphism $\mathbb{Z} \to \mathbb{Z}_2$. Thus, if M is spin, the *Wu class* of Sq^2 vanishes and therefore we obtain a central extension of $H^4(M; \mathbb{Z})$ by $\mathbb{Z}_2$

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \pi^4(M) \longrightarrow H^4(M; \mathbb{Z}) \longrightarrow 0 \tag{1}$$

Moreover, in [**24**, 6] Taylor uses methods of Larmore and Thomas [**15**] to study this extension of $H^4(M, \mathbb{Z})$ by $\mathbb{Z}_2$. A purely homotopy theoretical argument shows that the above short exact sequence splits if $\mathrm{Sq}^2\colon H^3(M; \mathbb{Z}) \to H^5(M; \mathbb{Z}_2)$ and $\mathrm{Sq}^2\colon H^3(M; \mathbb{Z}_2) \to H^5(M; \mathbb{Z}_2)$ have the same image, cf. [**24**, Example 6.3]. This is obviously the case when M is spin. We would like to give a geometrical meaning to splitting $\pi^4(M) = H^4(M;\mathbb{Z}) \oplus \mathbb{Z}_2$.

Before doing this we would like to explain that Φ is equal to the map which assigns to every quaternionic line bundle its spin characteristic class. Let $\mathbf{SU} = \bigcup_n \mathbf{SU}(n)$ and $\mathbf{Spin} = \bigcup_n \mathbf{Spin}(n)$. The canonical inclusion $\mathbf{SU} \to \mathbf{Spin}$ induces isomorphisms $\pi_i\,\mathbf{SU} \to \pi_i\,\mathbf{Spin}$ for $i \leqslant 5$, see [**4**, Lemma 2.4]. Define the *(universal) spin characteristic class* $\frac{p_1}{2} \in H^4(B\mathbf{Spin};\mathbb{Z})$ to be the preimage of the universal second Chern class $-c_2 \in H^4(B\mathbf{SU};\mathbb{Z})$ under the map $B\mathbf{SU} \to B\mathbf{Spin}$ induced by the canonical inclusion.

Let $W \to X$ be a spinnable vector bundle over a finite CW complex X. A choice of spin structure on W defines a map $g\colon X \to B\mathbf{Spin}$ which is a lift of the classifying map $X \to \mathbf{SO} = \bigcup_n \mathbf{SO}(n)$. Define $\frac{p_1}{2}(W)$ to be $g^*(\frac{p_1}{2}) \in H^4(X;\mathbb{Z})$, which is independent of the choice of spin structure, cf. [**4**, p. 170]. Moreover, we have

$$\tfrac{p_1}{2}(W) \equiv w_4(W) \mod 2, \quad 2 \cdot \tfrac{p_1}{2}(W) = p_1(W),$$

see again [**4**, p. 170].

The classifying map

$$\frac{p_1}{2}\colon B\mathbf{Spin} \longrightarrow K(Z,4)$$

is an 8-equivalence, thus two spinnable vector bundles over a CW-complex of dimension $\leqslant 7$ are stably isomorphic if and only if their spin characteristic class $\frac{p_1}{2}$ are equal, cf. [**28**] and [**8**, p. 5].

The inclusion $i\colon S^4 = \mathbb{HP}^1 \hookrightarrow \mathbb{HP}^\infty$ induces an isomorphism on integer cohomology in dimension 4. By construction $\frac{p_1}{2}(H) \in H^4(S^4;\mathbb{Z})$ is a generator where H can be described as the pull back under i of the tautological quaternionic line bundle over $\mathbb{HP}^\infty$. Thus the map Φ is given as

$$\Phi\colon \pi^4(X) \to H^4(X;\mathbb{Z}), \quad f \mapsto \tfrac{p_1}{2}(f^*(H)) = f^*\left(\tfrac{p_1}{2}(H)\right).$$

3.2. Geometric interpretation of $\pi^4(M) \cong H^4(M;\mathbb{Z}) \oplus \mathbb{Z}_2$.

Suppose now $E \to M$ is a quaternionic line bundle over a spin 5-manifold M. We would like to show how to obtain an element of $\pi^4(M)$ out of the bundle data of E. Therefore we need to make a detour over $\Omega^{\mathrm{fr}}_{1;M}$, the bordism group of normally framed closed 1-manifolds of M (see [**17**, ▮7] for a definition). The groups $\pi^4(M)$ and $\Omega^{\mathrm{fr}}_{1;M}$ are isomorphic by the *Pontryagin–Thom construction*, see again [**17**, ▮7].

Choose a (real) section $\sigma \in \Gamma(E)$ transverse to the zero section of E. The zero set of σ, call it L, is a closed 1-dimensional submanifold of M. The bundle E defines a trivialization of the real vector bundle $\nu_L := \nu(L \hookrightarrow M)$ as follows. The restricted vector bundle $E|_L$ is the trivial quaternionic line bundle and any non vanishing quaternionic section of $E|_L$ gives a trivialization $E|_L \cong L \times \mathbb{H}$ which is unique up to homotopy (since $\pi_1(\mathbf{Sp}(1)) = 1$). The quaternionic structure gives now, up to homotopy, a unique trivialization of the underlying real vector bundle $E|_L$. The derivative of σ along L gives a bundle isomorphism between ν_L and $E|_L$. Let us denote this induced framing of ν_L with φ_E. Thus we obtain a class $[L, \varphi_E] \in \Omega^{\mathrm{fr}}_{1;M}$.

Lemma 3.2. *The class $[L, \varphi_E]$ does not depend on the choice of the transverse section σ.*

Proof. Let σ' be another section of E transverse to the zero section and denote the set of zeros by L'. These data induce, like above, a framing φ'_E on $\nu_{L'}$. Then there is a section $\tau \in \Gamma(\tilde{E})$ such that $\tau|_{M\times 0} = \sigma$ and $\tau|_{M\times 1} = \sigma'$ where $\tilde{E} = \mathrm{pr}^*(E)$ and $\mathrm{pr}\colon M \times [0,1] \to M$ is the projection onto the first factor. We may also assume that τ is transverse to the zero section of $\mathrm{pr}^*(E)$ and $\tau(p,t) = \sigma(p)$, $\tau(p, 1-t) = \tilde{\sigma}(p)$ for $t \in [0,\varepsilon)$ where $\varepsilon > 0$ is small. The zero set Σ of τ is a two-dimensional submanifold with boundary $(L \times 0) \cup (L' \times 1)$. We may assume that the possible closed components of Σ do not touch the boundary of $M \times [0,1]$ and therefore we may ignore them and we will still have a bordism between $L \times 0 =$ and $L' \times 1$. This means that we may assume that Σ is a CW-complex of dimension at most 1. The section τ provides an isomorphism of $\nu(\Sigma \hookrightarrow M \times [0,1])$ and $\tilde{E}|_\Sigma$. And as before $\tilde{E}|_\Sigma$ is the trivial bundle with a canonical framing induced by the quaternionic structure.

Thus Σ has a normal framing which restricts on the boundary to the given ones (by construction), which means that (L, φ_E) and (L', φ'_E) are normally framed bordant, hence they represent the same element in $\Omega^{\mathrm{fr}}_{1;M}$. □

Definition 3.3. For a quaternionic line bundle $E \to M$ we call the class $[L, \varphi_E]$ the *framed divisor of E.*

Thus we obtain a well-defined map from $\mathrm{Bun}(M) \to \Omega^{\mathrm{fr}}_{1,M}$, $E \mapsto [L, \varphi_E]$, where $\mathrm{Bun}(M)$ denotes the set of isomorphism classes of quaternionic line bundles over M.

Lemma 3.4. *Let $f\colon M \to S^4$ be a classifying map for E. Under the Pontryagin–Thom isomorphism $\Omega^{fr}_{1;M} \to \pi^4(M)$ the framed divisor $[L, \varphi_E]$ is mapped to f. Moreover, if $L \subset M$ is a closed submanifold representing the Poincaré dual of $\frac{p_1}{2}(E)$ then there is a section of E with zero locus L*

Proof. Suppose $x_0 \in S^4$ is a regular value of f and denote by $(f^{-1}(x_0), \varphi_f)$ the Pontryagin-manifold to f. Then there is a section $\sigma_0\colon S^4 \to H$ with only one zero exactly at x_0 and transverse to the zero section. Observe now, that the pull back $\sigma := f^*(\sigma_0)$ in $E = f^*(H)$ is transverse to the zero section and has $f^{-1}(x_0)$ as its set of zeros. Choosing a non-zero element in H_{x_0} corresponds to a quaternionic section of $E|_{f^{-1}(x_0)}$ and the framings φ_E and φ_f coincide.

Suppose now $L \subset M$ is representing the Poincaré dual of $\frac{p_1}{2}(E)$. We may choose a framing φ of $\nu(L)$ such that the bordism class $[L, \varphi] \in \Omega^{\mathrm{fr}}_{1;M}$ corresponds to $E \in \mathrm{Bun}(M) \cong \pi^4(M)$ under the Pontryagin–Thom isomorphism. Thus there is a map $f\colon M \to S^4$ such that (L, φ) is a Pontryagin manifold to f. Suppose $x_0 \in S^4$ is a regular value for f such that $L = f^{-1}(x_0)$. As above, we may choose a section σ_0 of $H \to S^4$ such that L is the zero locus of $\sigma = f^*(\sigma_0)$. □

Remark 3.5. Note that we may always assume that the zero locus is connected: If the number of connected components of the zero locus L is bigger than one, then we may use a *pair of pants* to reduce the connected components to one less but maintain the homology class. This leads to an embedded circle L' representing the Poincaré dual $\frac{p_1}{2}(E)$. By Lemma 3.4 we may assume that L' is the zero locus of a transverse section of E.

Remark 3.6. Thus we showed that the following diagram commutes

$$\begin{array}{ccc} & \mathrm{Bun}(M) & \\ {\scriptstyle f\mapsto f^*(H)}\nearrow & & \searrow{\scriptstyle E\mapsto[L,\varphi_E]} \\ \pi^4(M) & \xleftarrow{\text{Pontryagin–Thom}} & \Omega^{\mathrm{fr}}_{1;M}. \end{array}$$

We define on $\mathrm{Bun}(M)$ a group structure such that the above diagram commutes as abelian groups.

3.3. Splitting map

We will construct in this paragraph a splitting map for the short exact sequence (1) Therefore we choose first a $\mathbf{Sp}(1)$-structure on TM (which is equivalent to choose a spin structure for M, cf. Proposition 2.3.). This is possible since we assume $w_1(M) = w_2(M) = 0$. If L is a closed 1-dimensional submanifold of M then $TM|_L$ admits a unique trivialization up to homotopy compatible with the $\mathbf{Sp}(1)$-structure of TM.

Recall that if $E \in \pi^4(M)$ and $\sigma \in \Gamma(E)$ is a transverse section then the set of zeros L is a 1-dimensional closed submanifold. The section σ induces on ν_L a framing as described above. Altogether this produces a stable framing of TL since

$$\varepsilon^5 \cong TM|_L \cong TL \oplus \nu_L \cong TL \oplus \varepsilon^4.$$

We denote this stable framing by $[L, \varphi_E^S]$ which is an element in $\Omega_1^{\mathrm{fr}} \cong \mathbb{Z}_2$. We show that $[L, \varphi_E^S]$ is well-defined.

Lemma 3.7. *Let $E \to M$ be a quaternionic line bundle over M and let $\sigma, \sigma' \in \Gamma(E)$ be two transverse sections. Denote by L and L' the corresponding zero loci. Then*

$$[L, \varphi_E^S] = [L', (\varphi'_E)^S] \in \Omega_1^{fr}.$$

Proof. Let $\mathrm{pr}\colon M \times [0,1] \to M$ be the obvious projection. We have seen above, that $[L, \varphi_E]$ and $[L, \varphi'_E]$ are normally framed bordant. Denote by $\Sigma \subset M \times [0,1]$ the normally framed bordism. Then Σ is also stably framed

$$\varepsilon^6 \cong T(M \times [0,1])|_\Sigma \cong T\Sigma \oplus \nu(\Sigma \hookrightarrow M \times [0,1]) \cong T\Sigma \oplus \mathrm{pr}^*(E)|_\Sigma \cong T\Sigma \oplus \varepsilon^4$$

and clearly this shows that $[L, \varphi_E^S]$ and $[L', (\varphi'_E)^S]$ are stably framed bordant. □

Definition 3.8. For a quaternionic line bundle $E \to M$ we call $[L, \varphi_E^S]$ the *stabilized framed divisor* of E and we define

$$\kappa(E) := [L, \varphi_E^S] \in \Omega_1^{\mathrm{fr}}.$$

We call κ the *generalized Kervaire semi-characteristic.*

Example 3.9. We consider $S^5 \subset \mathbb{C}^3$ and note that the unique $\mathbf{SU}(2)$ structure is represented by the $\mathbf{SU}(2)$-principal bundle $\mathbf{SU}(3) \to S^5$, where a matrix $A \in \mathbf{SU}(3)$ is mapped to its first column. The map

$$V(z) = iz$$

defines a nowhere vanishing vector field on S^5. For $z \in S^5$ define E_0 to be the orthogonal complement to z with respect to the standard hermitian inner product on $\mathbb{C}^3$. Thus $(E_0)_z$ is a complex vector space of dimension 2. Then $TS^5 \cong (\mathbb{R} \cdot V) \oplus E_0$ and E_0 has an $\mathbf{SU}(2)$ structure: The group $\mathbf{SU}(3)$ is the (special) unitary frame bundle E_0, since the first column maps to the basepoint of E_0 and the second as well as the third column span the complex vector space E_0 over that basepoint. This gives E_0 an $\mathbf{SU}(2)$ structure (which also represents the $\mathbf{SU}(2)$-structure of S^5).

The section $\sigma\colon S^5 \to E_0$,

$$\sigma(z_1, z_2, z_3) = (0, -\overline{z_3}, \overline{z_2})$$

is transverse to the zero section of E_0 and its zero locus is given by

$$L = \{(z, 0, 0) \in S^5 : |z| = 1\}.$$

To determine the framing of E_0 induced by its $\mathbf{SU}(2)$-structure we have to find a section of $\mathbf{SU}(3)|_L \to L$, e.g.

$$\tau\colon L \to \mathbf{SU}(3)|_L, \quad (z, 0, 0) \mapsto \begin{pmatrix} z & 0 & 0 \\ 0 & \bar{z} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where $z = x_1 + ix_2$. The homotopy class of that framing is equal to the Lie group framing of S^1, which can be verified after identifying $\mathbb{C}^3$ with $\mathbb{R}^6$. Thus $\kappa(E_0) \neq 0$.

Note that E_0 is the pullback of the tangent bundle of $\mathbb{CP}^2$ by the Hopf fibration $S^5 \to \mathbb{CP}^2$. We will explain that E_0 is isomorphic to $\nu^*(H)$, where $\nu\colon S^5 \to S^4$ represents the non-trivial element of $\pi_5(S^4)$. To see this consider the following: The zero locus L_0 of $\nu^*(H)$ is the preimage of ν by a regular value $x_0 \in S^4$. The framing φ_0 induced by the quaternionic structure of $\nu^*(H)$ can be constructed by choosing a basis of H_{x_0}. Thus (L_0, φ_0) is just the Pontryagin manifold to ν and therefore we may conclude that $\kappa(\nu^*(H)) \neq 0$. This implies $\nu^*(H) \cong E_0$, since there are at most $\pi_5(S^4) \cong \mathbb{Z}_2$ quaternionic line bundles and both bundles are not trivial since their generalized Kervaire semi-characteristic does not vanish.

There is a subtle point in the definition of κ with the choice of the $\mathbf{Sp}(1)$-structure on M. In general the definition of this invariant depends on the $\mathbf{Sp}(1)$-structure but for some quaternionic line bundles the invariant is independent from that choice.

Fix a $\mathbf{Sp}(1)$-structure of M, then, by obstruction theory, any other $\mathbf{Sp}(1)$-structure is determined by an element of $H^1(M; \pi_1(\mathbf{SO}(5)/\mathbf{Sp}(1))) = H^1(M; \mathbb{Z}_2)$ which is the obstruction that any other structure is homotopic to the fixed one.

Proposition 3.10. *Fix a $\mathbf{Sp}(1)$-structure on M and choose another one represented by $\alpha \in H^1(M; \mathbb{Z}_2)$. Denote by κ^α the generalized Kervaire semi-characteristic induced by the $\mathbf{Sp}(1)$-structure α. Then we have*

$$\kappa(E) = \kappa^\alpha(E) + \delta(\alpha \smile w_4(E)),$$

where $\delta\colon H^5(M; \mathbb{Z}_2) \to \Omega_1^{fr}$ is the unique isomorphism. In particular, if $\alpha \smile w_4(E) = 0$ then $\kappa(E) = \kappa^\alpha(E)$.

Proof. Let $\sigma \in \Gamma(E)$ be a real transverse section then the zero locus L of σ is the Poincaré dual to the Euler class $e(E)$ of the underlying vector bundle with structure group $\mathbf{SO}(4)$, cf. [**3**, Proposition 12.8]. Furthermore we have the well-known relation $w_4(E) \equiv e(E_{\mathbb{R}}) \mod 2$.

Suppose first that L is connected and denote by $i\colon L \to M$ the embedding of L into M. Then the spin structure induced by α on $TM|_L$ is represented by $i^*(\alpha) \in H^1(L; \mathbb{Z}_2)$. Of course $TM|_L$ can have at most two different spin structures. By construction we must have

$$[L, \varphi_E^S] = [L, \varphi_E^{S,\alpha}] + \delta(i^*(\alpha))$$

where $\varphi_E^{S,\alpha}$ is the stabilized framed circle with respect to the spin structure defined by α and $\delta\colon H^1(L;\mathbb{Z}_2) \to \Omega_1^{\mathrm{fr}}$ the unique isomorphism.

The element $i^*(\alpha) \frown [L] \in H_0(L;\mathbb{Z}_2)$ is mapped to $\alpha \frown i_*([L]) \in H_0(M;\mathbb{Z}_2)$ under i_* where $[L]$ is the $\mathbb{Z}_2$ fundamental class of L. Let $[M] \in H_5(M;\mathbb{Z}_2)$ denote the $\mathbb{Z}_2$ fundamental class of M, then we infer

$$\alpha \frown i_*([L]) = \alpha \frown (\mathrm{PD}[L] \frown [M]) = (\alpha \smile \mathrm{PD}[L]) \frown [M],$$

where $\mathrm{PD}[L]$ refers to the Poincaré dual of $i_*([L])$. Since the cap products with $[L]$ and $[M]$ are isomorphisms by Poincaré duality and $i_*\colon H_0(L;\mathbb{Z}_2) \to H_0(M;\mathbb{Z}_2)$ is also an isomorphism (since L and M are connected) we obtain

$$[L, \varphi_E^S] = [L, \varphi_E^{S,\alpha}] + \delta(\alpha \smile \mathrm{PD}[L])$$

where $\delta\colon H^5(M;\mathbb{Z}_2) \to \Omega_1^{\mathrm{fr}}$ is again the unique isomorphism.

Now suppose $L = L_1 \cup \ldots \cup L_k$ are the connected components of L. Put $\varphi_{E,j} := \varphi_E|_{L_j}$ and $\varphi_{E,j}^\alpha := \varphi_E^\alpha|_{L_j}$. With the previous computations and by the group structure of Ω_1^{fr} we have

$$\begin{aligned}\kappa(E) = [L, \varphi_E^S] = \sum_{j=1}^k [L_j, \varphi_{E,j}^S] &= \sum_{j=1}^k \left([L_j, \varphi_{E,j}^{S,\alpha}] + \delta(\alpha \smile \mathrm{PD}[L_j])\right) \\ &= \kappa^\alpha(E) + \delta(\alpha \smile w_4(E))\end{aligned}$$

since $\sum_{j=1}^k \mathrm{PD}[L_j] \equiv e(E_{\mathbb{R}}) \mod 2$. □

Corollary 3.11. *If $w_4(E) = 0$ then $\kappa(E)$ is independent of the chosen $\boldsymbol{Sp}(1)$-structure on M. In particular, this is true if $H^1(M;\mathbb{Z}_2) = 0$.*

Lemma 3.12. *Let M and N be oriented, closed spin 5-manifolds. Assume that (L_M, φ_M) and (L_N, φ_N) are two normally framed circles such that*

- *(a) There are open neighborhoods U_M and U_N of L_M and L_N respectively such that there is an orientation-preserving diffeomorphism $\Phi\colon U_M \to U_N$,*
- *(b) $\Phi(L_M) = L_N$ and $d\Phi(\varphi_M) = \varphi_N$,*
- *(c) the induced spin structure on U_M is equal to the pull-back spin structure by Φ from U_N.*

Then we have

$$[L_M, \varphi_M^S] = [L_N, \varphi_N^S].$$

Proof. A stably framed bordism between (L_M, φ_M) and (L_N, φ_N) is obtain by glueing the two stably framed bordisms $L_M \times I$ and $L_N \times I$ together using the diffeomorphism Φ. □

Corollary 1.2 is a consequence of

Theorem 3.13. *Let M be a connected, oriented and closed spin 5-manifold. Then for any $\boldsymbol{Sp}(1)$-structure on M we have*

- *(a) the generator of $\mathbb{Z}_2 \cong \ker \frac{p_1}{2} \subset \pi^4(M)$ is given by the homotopy class of $\nu \circ p\colon M \to S^4$ where $p\colon M \to S^5$ is a map of odd degree and ν represents the generator of $\pi_5(S^4)$.*

(b) *if* Ω_1^{fr} *is identified with* $\pi_1^S \cong \pi_5(S^4) \cong \mathbb{Z}_2 \cong \ker \frac{p_1}{2} \subset \pi^4(M)$, *then* $\kappa\colon \pi^4(M) \to \Omega_1^{fr}$ *is a splitting map for the short exact sequence* (1).

Proof. Clearly the quaternionic line bundle $E := (\nu \circ p)^*(H)$ lies in the kernel of Φ as well as the trivial bundle $\underline{\mathbb{H}}$. Then part (a) follows if we show that E is not the trivial bundle. Assume first $p\colon M \to S^5$ is a map of degree 1. If necessary we may homotope p to a map, such that the preimage of a regular value of p consists of a single point. Thus there are contractible open sets $U \subset M$ and $V \subset S^5$ such that $p|_U : U \to V$ is a diffeomorphism with $p^{-1}(V) = U$. Since over a contractible set there is only one spin structure, the spin structures over U and V are compatible by means of $p|_U$. Let $[L_0, \varphi_0] \in \pi^4(S^5)$ be the generator and we may assume that $L_0 \subset V$. Set $L := p^{-1}(L_0)$ and pull back the framing $\varphi := p^*(\varphi_0)$. The element $[L_0, \varphi_0]$ is the framed divisor of $E_0 := \nu^*(H)$ and $[L, \varphi]$ that of $E := (\nu \circ p)^*(H)$. It follows that $\kappa(E_0) = \kappa(E)$, hence $\kappa(E)$ is a generator of Ω_1^{fr} since $\kappa(E_0)$ is a generator, see Example 3.9. The same argument shows that $\kappa(\underline{\mathbb{H}})$ is zero.

If $\deg p$ is an odd number different from 1, then there is an odd number of points in the preimage of a regular value for which p is a local diffeomorphism around these points. Hence for $\kappa(E)$ we would sum - by definition - an odd number of $\kappa(E_0)$, which again gives the generator of Ω_1^{fr}. Same holds for $\kappa(\underline{\mathbb{H}})$. Thus E and $\underline{\mathbb{H}}$ cannot be isomorphic, which proves part (a).

We explain next that $\kappa(E)$ is a homomorphism. For $E, F \in \pi^4(M)$ the corresponding framed divisor for $E + F$ is given by disjoint union, cf. Remark 3.6. The same holds for the group structure of Ω_1^{fr}. Thus by definition we have for the stabilized framed divisors

$$\kappa(E + F) = [L_E \cup L_F, (\varphi_E \cup \varphi_F)^S] = [L_E, \varphi_E^S] + [L_F, \varphi_F^S] = \kappa(E) + \kappa(F).$$

Now it follows that κ splits the short exact sequence (1), since κ is a homomorphism and κ restricted to $\ker \frac{p_1}{2}$ is the identity, which follows from part (a). □

Moreover, from the proof of Theorem 3.13 we conclude

Corollary 3.14. *Suppose M and N are two connected, closed and oriented spin 5-manifolds and $\Phi\colon M \to N$ is a local diffeomorphism. Suppose furthermore that the* ***Sp***(1)*-structure of M is the same to the pullback of the* ***Sp***(1)*-structure of N by Φ. If $E \to N$ is a quaternionic line bundle, then we have*

$$\kappa(\Phi^*(E)) = \deg_2 \Phi \cdot \kappa(E),$$

where $\deg_2 \Phi$ *is the mod* 2 *degree of* Φ.

4. Classification of spin vector bundles of rank 5

Let $V \to M$ be a spinnable vector bundles of rank 5 and recall that M is assumed to be a connected and closed spin 5-manifold.

Lemma 4.1. *If $w_4(V) = 0$ then there is a unique quaternionic vector bundle E such that $V \cong E \oplus \varepsilon^1$.*

Proof. First fix an $\mathbf{Sp}(1)$-structure on M and choose a quaternionic line bundle E such that $V \cong E \oplus \varepsilon^1$. Any other quaternionic line bundle E' with this property is stably isomorphic to E, hence they have the same spin characteristic class $\frac{p_1}{2}$. We claim also that we have $\kappa(E) = \kappa(E')$ which proves this lemma using Corollary 1.2.

From Lemma 3.4 and Remark 3.5 we infer that we may assume that the zero loci of E and E' are connected and that they coincide. If we fix a $\mathbf{Sp}(1)$-structure of V (such that the associated vector bundle to the $\mathbf{Sp}(1)$-principal bundle is E) then any other $\mathbf{Sp}(1)$-structure (corresponding to E') is represented by an $\alpha \in H^1(M; \mathbb{Z}_2)$, cf. the paragraph before Proposition 3.10. The framings φ of $E|_L$ and φ' of $E'|_L$ as quaternionic line bundles correspond to the choice of the $\mathbf{Sp}(1)$-structures of V and they agree if $i^*(\alpha) = 0$, where $i\colon L \to M$ the inclusion map. Thus by definition we have

$$\kappa(E) = [L, \varphi^S] \quad \text{and} \quad \kappa(E') = [L, (\varphi')^S].$$

If we shift the $\mathbf{Sp}(1)$-structure of M by α then we obtain $[L, (\varphi')^S] = [L, \varphi^{\alpha,S}]$ in the notation of the proof of Proposition 3.10. Therefore

$$\kappa(E') = \kappa^\alpha(E).$$

Since $w_4(E) = w_4(V) = 0$ we use again Proposition 3.10 to deduce $\kappa^\alpha(E) = \kappa(E)$ which finally proves the lemma. □

From Lemma 4.1 we may define for V with $w_4(V) = 0$ also a $\mathbb{Z}_2$-invariant by $\kappa(V) := \kappa(E)$, where E is the unique quaternionic line bundle of V. We will explain in the next lines that $\kappa(V)$ is the same invariant as the *generalized Kervaire semi-characteristic* $k(V)$ defined in [**23**]. We give a brief description of $k(V)$ (cf. [**26, 23**]; note that the following lines can be generalized to higher dimensions): Since $w_4(V) = 0$ there are two cross sections σ_1, σ_2 of V which are linearly independent on the complement of a finite set of points $\{p_1, \dots, p_k\}$. We may assume that p_i lies in the interior of a closed simplex s. The vector bundle V restricted to s is the trivial bundle $s \times \mathbb{R}^5$ and thus $(\sigma_1(q), \sigma_2(q))$ can be regarded as an orthonormal 2-frame in $\mathbb{R}^5$ on $s \setminus \{p_i\}$, thus a point in the Stiefel manifold $V_{5,2}$. The boundary ∂s of s is a 4-sphere and the restriction of (σ_1, σ_2) on ∂s produces a map $\partial S \to V_{5,2}$. Its homotopy class, call it I_{p_i}, is therefore an element of $\pi_4(V_{5,2})$. One defines now

$$k(V) := \sum_{i=1}^{k} I_{p_i} \in \pi_4(V_{5,2}) \cong \mathbb{Z}_2.$$

The authors show

Lemma 4.2 (From Corollary 2.2 in [**23**])**.** *$k(V)$ is independent of the choices made above and moreover V admits two linearly independent cross sections if and only if $k(V)$ vanishes.*

This invariant coincides with the invariant k of Thomas defined in [**26**, p. 108], which is the k-invariant of the Moore–Postnikov tower of a certain fibration. In case $V = TM$ the invariant is well known as the *Kervaire semi-characteristic of M* [**26, 2**] and can be easily computed by the formula

$$\begin{aligned} k(M) &\equiv \sum_i \dim H^{2i}(M; \mathbb{Z}_2) \mod 2 \\ &\equiv \sum_i \dim H^{2i}(M; \mathbb{R}) \mod 2, \end{aligned}$$

where the last equality is a consequence of the *Lusztig–Milnor–Peterson formula* [**16**].

Proposition 4.3. *Let V be an oriented, spinnable vector bundle over M of rank 5 and such that $w_4(V) = 0$. Then $\kappa(V) = k(V)$.*

Proof. We would like to show first that $k(V) = 0$ implies $\kappa(V) = 0$. Note that from [**23**, Corollary 2.2] we have $k(V) = 0$ if and only if there exist two linearly independent sections of V. We will use in the following lines the *singularity approach* of [**14**].

Let $\psi \in \Gamma(V)$ be a nowhere vanishing section such that $V \cong E \oplus \mathbb{R} \cdot \psi$, where E is the unique quaternionic line bundle to V. Let σ be a transverse section of E. This data induces a bundle morphism

$$u\colon \varepsilon^2 \to V, \quad u_p(x_1, x_2) := x_1\sigma(p) + x_2\psi(p), \quad p \in M,\ (x_1, x_2) \in \mathbb{R}^2,$$

which in turn defines a section, denoted by s_u, in the homomorphism bundle $\mathbf{Hom}(\varepsilon^2, V)$. Let $W^1, A^1 \subset \mathbf{Hom}(\varepsilon^2, V)$ be the subfibrations where the fibers consist of all linear maps of rank $\geqslant 1$ and rank equal to 1 respectively. Then W^1 is open in $\mathbf{Hom}(\varepsilon^2, V)$ and A^1 is a closed smooth submanifold of W^1, cf. [**14**, p. 17]. Of course we have $s_u \in \Gamma\left(W^1\right)$ and s_u is transverse to A^1. Moreover, it is evident that $s_u^{-1}(A^1) =: L$ is the zero locus of σ. Let $s_0 \in \Gamma(W^1)$ be the morphism induced by the two linearly independent sections of V, which exist, since $k(V) = 0$ is assumed.

Next, we look at $\widetilde{\mathbf{Hom}} := \mathbf{Hom}(\varepsilon^2, \mathrm{pr}^*(V)) = \mathrm{pr}^*\left(\mathbf{Hom}(\varepsilon^2, V)\right)$ (where $\mathrm{pr}\colon M \times I \to M$ is the projection) and consider $\widetilde{A}^1$ as well as $\widetilde{W}^1$ analogously to the definitions above. Then there exists a section $S \in \Gamma(\widetilde{W}^1)$ with the properties that S is transverse to $\widetilde{A}^1$, $S|_{M\times 0} = s_u$ and $S|_{M\times 1} = s_0$. The set $\Sigma := S^{-1}(\widetilde{A}^1)$ is a smooth compact surface with $\partial\Sigma = L$, hence Σ is a null-bordism for L. According to the definition of $\kappa(V)$ we have to show that the stable framing of TL in the definition of $\kappa(V)$ is induced by a stable framing of $T\Sigma$.

The normal bundle of $\widetilde{A}^1$ in $\widetilde{W^1}$ is isomorphic to $\mathbf{Hom}(\widetilde{\ker}, \widetilde{\operatorname{coker}})$ where

$$\widetilde{\ker} := \bigcup_{h\in\widetilde{A}^1} h \times \ker h, \quad \widetilde{\operatorname{coker}} := \bigcup_{h\in\widetilde{A}^1} h \times \operatorname{coker} h,$$

see [**14**, equation (1.1)]. This implies $\nu(\Sigma \hookrightarrow M \times I) \cong \mathbf{Hom}(\varepsilon^1, \mathrm{pr}^*(V)/\operatorname{im} S)|_\Sigma \cong (\mathrm{pr}^*(V)/\operatorname{im} S)|_\Sigma$ and therefore we have

$$\nu\left(\Sigma \hookrightarrow M \times I\right) \oplus \mathbb{R} \cdot S|_\Sigma \cong \mathrm{pr}^*(V)|_\Sigma.$$

Since $\mathrm{pr}^*(V)|_\Sigma$ has, up to homotopy, a unique framing determined by the quaternionic line bundle underlying V. We obtain, after a choice of an $\mathbf{Sp}(1)$-structure on M, a stable framing of $T\Sigma$ by

$$\varepsilon^7 \cong (T(M \times I) \oplus \mathbb{R} \cdot S)|_\Sigma \cong T\Sigma \oplus \nu(\Sigma \hookrightarrow M \times I) \oplus \mathbb{R} \cdot S|_\Sigma \cong T\Sigma \oplus \varepsilon^5.$$

One sees immediately that the restricted stable framing on $L \times 0$ is that which is used for the definition of $\kappa(V)$. This shows that $k(V) = 0$ implies $\kappa(V) = 0$.

It remains to show that $k(V) = 1$ implies $\kappa(V) = 1$. We will use the following statement which follows from the proof of [**27**, Lemma 3].

Lemma. *Let X be a closed spin manifold of dimension $4k + 1$ and V an orientable vector bundle of rank $4k + 1$ with $w_2(V) = 0$ over X. Let W be another orientable vector bundle of rank $4k + 1$ which is stably isomorphic to V. Then V is isomorphic to W if and only if $k(V) = k(W)$.*

Let $\overline{V}$ be the vector bundle with the same spin characteristic class as V but not isomorphic to it, thus $\kappa(\overline{V}) = \kappa(V) + 1$. Hence V and $\overline{V}$ are stably isomorphic but not isomorphic. From the Lemma above we must have $k(\overline{V}) = 0$ since $k(V) = 1$. Therefore from the first part we infer $\kappa(\overline{V}) = 0$ and therefore $\kappa(V) = 1$. □

Lemma 4.4. *Suppose $w_4(V) \neq 0$. Then V is uniquely determined by $\frac{p_1}{2}(V)$.*

Proof. Since $w_4(V) \neq 0$ there is a non-zero $\alpha \in H^1(M;\mathbb{Z}_2)$ such that $\alpha \cup w_4(V) \neq 0$. From the proof of Lemma 4.1 we deduce that there are exactly two quaternionic line bundles E and E' such that their spin characteristic classes coincide but $\kappa(E) \neq \kappa(E')$. This means there are two non-isomorphic quaternionic line bundles E and E' such that $E \oplus \varepsilon^1 \cong V \cong E' \oplus \varepsilon^1$. As mentioned above, there can be at most two such quaternionic line bundles. Hence V is completely determined by $\frac{p_1}{2}(V)$. □

We recall from [**25**] that the mod 2 reduction of the $\frac{p_1}{2}(V)$ is equal to $w_4(V)$. Then the first claim about W_1 of Theorem 1.3 follows from Lemma 4.1 and Corollary 1.2. The second claim on W_2 follows from Lemma 4.4 and again Corollary 1.2, since for every element of $H^4(M;\mathbb{Z})$ there are at most two quaternionic line bundles over M with this element as spin characteristic class and after adding a trivial real line bundle to it, they have to be isomorphic by Lemma 4.4 since $w_4(V)$ is not zero.

We may use the above results to prove

Proposition 4.5. *Suppose M is a closed, oriented spin 5-manifold with finite fundamental group and let $\pi\colon \widetilde{M} \to M$ be the universal covering. Denote by $\#\pi$ the cardinality of the fibers of π. Then*

(a) If $\#\pi \equiv 0 \mod 2$ then $\widetilde{M}$ has trivial tangent bundle. Thus $\widetilde{\beta}_2 \equiv \widetilde{b}_2 \equiv 1 \mod 2$, where $\widetilde{\beta}_2, \widetilde{b}_2$ are the second Betti numbers of $\widetilde{M}$ with $\mathbb{Z}_2$ and $\mathbb{R}$ coefficients respectively.

(b) If $\#\pi \equiv 1 \mod 2$ then $\widetilde{M}$ has trivial tangent bundle if and only if $\beta_2 + \beta_4 \equiv b_2 + b_4 \equiv 1 \mod 2$.

Proof. Since $T\widetilde{M} = \pi^*(TM)$ we have $\kappa(T\widetilde{M}) = \deg_2 \pi \cdot \kappa(TM) = \#\pi \cdot \kappa(TM)$, see Corollary 3.14. The proposition follows from Proposition 4.3 and the formula for the Kervaire semi-characteristic. □

We end the paper with a comparison of our results and the counting formula for the Kervaire semi-characteristic of [**29**]. The author considers a $(4k+1)$-dimensional, oriented closed manifold N. Since the dimension is odd we have a decomposition $TN = W \oplus \varepsilon^1$ and suppose $L_1, \ldots, L_k$ are the connected components of the zero locus of a transversal section of W. Zhang associates to every L_i a real line bundle $\gamma_i \to L_i$ and he proves using twisted Dirac operators and index theory that

$$k(TN) = \#\{L_i : w_1(\gamma_i) = 0\} \mod 2.$$

We may prove the analogous result for our setting. Let $L_1, \ldots, L_k$ be the connected components of the zero locus L of a transversal section of E where $V \cong E \oplus \varepsilon^1$. The group structure of Ω_1^{fr} permits us to write

$$\kappa(V) = [L, \varphi^{\mathrm{st}}] = \sum_{i=1}^{k} [L_i, \varphi_i^{\mathrm{st}}],$$

where φ_i is the framing induced by $E|_{L_i}$ and the transversal section of E. Now it is possible to associated to every element of Ω_1^{fr} a real line bundle as follows: For $x := [L, \varphi^S] \in \Omega_1^{\text{fr}}$ with L connected there is a map $A_x \colon L \to \mathbf{SO}(n)$ $(n > 2)$ such that its homotopy class $[A_x]$ in $\pi_1(\mathbf{SO}(n))$ represents $[L, \varphi^S]$ under the J-homomorphism. Denote by $B \colon L \to \mathbf{SO}(n)$ a map which generates $\pi_1(\mathbf{SO}(n))$, then we consider the map

$$\mu_x \colon L \to \mathbf{SO}(n), \quad \mu_x := A_x \cdot B$$

where the latter means pointwise multiplication of the matrices in $\mathbf{SO}(n)$. Thus $[\mu_x] \in \pi_1(\mathbf{SO}(n))$ is zero if x is non trivial and the other way around. Note that the homotopy class of μ_x does not depend on the choices made. Let $\gamma_x \to L$ be defined by $\mu_x^*(\gamma)$, where γ is the real line bundle associated to the generator of $H^1(\mathbf{SO}(n); \mathbb{Z}_2)$. From the prior discussion we infer that γ_x is trivial (or orientable) if and only if x is non-trivial. This proves

Proposition 4.6. *Let M be a connected, oriented and closed spin manifold of dimension 5 and $V \to M$ a spinnable vector bundle of rank 5 with $w_4(V) = 0$. Furthermore let $V = E \oplus \varepsilon^1$ where E is the unique quaternionic line bundle. Denote by $L_1, \ldots, L_k$ the connected components of the zero locus of a transverse section of E. Then, after identifying Ω_1^{fr} uniquely with $\mathbb{Z}_2$ we have*

$$\kappa(V) = \#\{L_i : w_1(\gamma_i) = 0\} \mod 2,$$

where γ_i is the real line bundle over L_i defined by $[L_i, (\varphi_i)^S]$.

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