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THE WEDGE FAMILY OF THE COHOMOLOGY OF THE $\mathbb{C}$-MOTIVIC STEENROD ALGEBRA

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Abstract

We describe some regular behavior in the motivic wedge, which is an infinite family in the cohomology $\mathrm{Ext}_{\mathbf{A}}(\mathbb{M}_2, \mathbb{M}_2)$ of the $\mathbb{C}$-motivic Steenrod algebra. The key tool is to compare motivic computations to classical computations, to $\mathrm{Ext}_{\mathbf{A}(2)}(\mathbb{M}_2, \mathbb{M}_2)$, or to h_1-localization of $\mathrm{Ext}_{\mathbf{A}}(\mathbb{M}_2, \mathbb{M}_2)$.

We also give two conjectures on the behavior of the families $e_0^t g^k$ and $\Delta h_1 e_0^t g^k$ in $\mathrm{Ext}_{\mathbf{A}}(\mathbb{M}_2, \mathbb{M}_2)$ which raise naturally from the study of the motivic wedge family.

1. Introduction

Computing the stable homotopy groups of the sphere spectrum is one of the most important problems of stable homotopy theory. Focusing on the 2-complete stable homotopy groups instead of the integral homotopy groups, the Adams spectral sequence appears to be one of the most effective tools to compute the homotopy groups. The spectral sequence has been studied by Adams [**1**, **2**], Mahowald [**3**, **14**], Tangora [**19**], May [**16**] and others [**4**].

In 1999, Morel and Voevodsky introduced motivic homotopy theory [**17**]. One of its consequences is the realization that almost any object studied in classical algebraic topology could be given a motivic analog. In particular, we can define the motivic Steenrod algebra $\mathbf{A}$ [**22**], the motivic stable homotopy groups of spheres [**17**] and the motivic Adams spectral sequence [**5**]. In the motivic perspective, there are many more non-zero classes in the motivic Adams spectral sequence, which allows the detection of otherwise elusive phenomena. Also, the additional motivic weight grading can eliminate possibilities which appear plausible in the classical perspective.

Let $\mathbb{M}_2$ denote the motivic cohomology of a point, which is isomorphic to $\mathbb{F}_2[\tau]$ where τ has bidegree $(0, 1)$ [**20**]. The motivic Steenrod algebra $\mathbf{A}$ is the $\mathbb{M}_2$-algebra generated by elements Sq^{2k} and Sq^{2k-1} for all $k \geqslant 1$, of bidegrees $(2k, k)$ and $(2k - 1, k - 1)$ respectively, subject to Adem relations [**21**, **22**]. Let $\mathrm{Ext}_{\mathbf{A}}(\mathbb{M}_2, \mathbb{M}_2)$ denote the cohomology of the motivic Steenrod algebra. To run the motivic Adams spectral sequence, one begins with $\mathrm{Ext}_{\mathbf{A}}(\mathbb{M}_2, \mathbb{M}_2)$. The cohomology $\mathrm{Ext}_{\mathbf{A}}(\mathbb{M}_2, \mathbb{M}_2)$ has an $\mathbb{M}_2$-algebra structure. Inverting τ in $\mathrm{Ext}_{\mathbf{A}}(\mathbb{M}_2, \mathbb{M}_2)$ gives the cohomology $\mathrm{Ext}_{\mathbf{A}_{cl}}(\mathbb{F}_2, \mathbb{F}_2)$ of the classical Steenrod algebra $\mathbf{A}_{cl}$ [**10**]. Given a classical element, there are many

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corresponding motivic elements. We typically want to find the corresponding element with the highest weight. For example, the classical element g corresponds to the motivic elements $\tau^k g$ for all $k \geqslant 1$. The element τg has weight 11, but there is no motivic element of weight 12 that corresponds to the classical element g.

The algebra $\mathrm{Ext}_{\mathbf{A}}(\mathbb{M}_2, \mathbb{M}_2)$ is infinitely generated and irregular. A natural approach is to look for systematic phenomena in $\mathrm{Ext}_{\mathbf{A}}(\mathbb{M}_2, \mathbb{M}_2)$. One potential candidate is the wedge family in $\mathrm{Ext}_{\mathbf{A}}(\mathbb{M}_2, \mathbb{M}_2)$.

The classical wedge family was studied by Mahowald and Tangora [**13**]. It is a subset of the cohomology $\mathrm{Ext}_{\mathbf{A}_{\mathrm{cl}}}(\mathbb{F}_2, \mathbb{F}_2)$ of the classical Steenrod algebra, consisting of non-zero elements $P^i g^j \lambda$ and $g^j t$ in which λ is in $\mathbf{\Lambda}$, t is in $\mathbf{T}$, $i \geqslant 0$ and $j \geqslant 0$. The sets $\mathbf{\Lambda}$ and $\mathbf{T}$ are specific subsets of $\mathrm{Ext}_{\mathbf{A}_{\mathrm{cl}}}(\mathbb{F}_2, \mathbb{F}_2)$. The wedge family gives an infinite wedge-shaped diagram inside the cohomology of the classical Steenrod algebra, which fills out an angle with vertex at g^2 in degree (40,8) (i.e., g^2 has stem 40 and Adams filtration 8), bounded above by the line $f = \frac{1}{2}s - 12$, parallel to the Adams edge [**1**], and bounded below by the line $s = 5f$, in which f is the Adams filtration and s is the stem. The wedge family is a large piece of $\mathrm{Ext}_{\mathbf{A}_{\mathrm{cl}}}(\mathbb{F}_2, \mathbb{F}_2)$ which is regular, of considerable size and easy to understand.

Using this idea we build the motivic version of the wedge. However, it appears to be more complicated than the classical one. The highest weights of the motivic wedge elements follow a somewhat irregular pattern. We will discuss this irregularity in more detail later.

Let $\mathbf{A}(2)$ denote the $\mathbb{M}_2$-subalgebra of $\mathbf{A}$ generated by $\mathrm{Sq}^1, \mathrm{Sq}^2$ and Sq^4. Let $\mathrm{Ext}_{\mathbf{A}(2)}(\mathbb{M}_2, \mathbb{M}_2)$ denote the cohomology of $\mathbf{A}(2)$. The finitely generated algebra $\mathrm{Ext}_{\mathbf{A}(2)}(\mathbb{M}_2, \mathbb{M}_2)$ is fully understood by [**9**]. We use a new technique of comparison to $\mathrm{Ext}_{\mathbf{A}(2)}(\mathbb{M}_2, \mathbb{M}_2)$ which makes the proof of the non-triviality of the wedge elements easy. We consider the ring homomorphism ϕ from $\mathrm{Ext}_{\mathbf{A}}(\mathbb{M}_2, \mathbb{M}_2)$ to $\mathrm{Ext}_{\mathbf{A}(2)}(\mathbb{M}_2, \mathbb{M}_2)$ induced by the inclusion from $\mathbf{A}(2)$ to $\mathbf{A}$. We use the map ϕ to detect structure in $\mathrm{Ext}_{\mathbf{A}}(\mathbb{M}_2, \mathbb{M}_2)$. Most of the elements studied in this article have non-zero images via ϕ [**9**]. Therefore, they are all non-trivial elements in $\mathrm{Ext}_{\mathbf{A}}(\mathbb{M}_2, \mathbb{M}_2)$.

We define set-valued operations $\mathbf{P}$ and $\mathbf{g}$ on $\mathrm{Ext}_{\mathbf{A}}(\mathbb{M}_2, \mathbb{M}_2)$. Classically, g is an element of the cohomology of the classical Steenrod algebra. However, this is not true motivically. Rather, τg is an element in $\mathrm{Ext}_{\mathbf{A}}(\mathbb{M}_2, \mathbb{M}_2)$, while g itself does not survive the motivic May spectral sequence. Consequently, multiplication by g does not make sense motivically. Also, P is not an element in $\mathrm{Ext}_{\mathbf{A}}(\mathbb{M}_2, \mathbb{M}_2)$ either. We instead consider the set-valued operations $\mathbf{P}$ and $\mathbf{g}$ whose actions can be seen as multiplications by P and g in $\mathrm{Ext}_{\mathbf{A}(2)}(\mathbb{M}_2, \mathbb{M}_2)$ respectively.

For any λ in $\mathrm{Ext}_{\mathbf{A}}(\mathbb{M}_2, \mathbb{M}_2)$, $i \geqslant 0$ and $j \geqslant 0$, let $\mathbf{P}^i \mathbf{g}^j \lambda$ be the set consisting of all elements x in $\mathrm{Ext}_{\mathbf{A}}(\mathbb{M}_2, \mathbb{M}_2)$ such that $\phi(x) = P^i g^j \phi(\lambda)$ in $\mathrm{Ext}_{\mathbf{A}(2)}(\mathbb{M}_2, \mathbb{M}_2)$.

We define the wedge family via the actions of $\mathbf{P}$ and $\mathbf{g}$. The wedge family is the set consisting of all elements in the set $\tau^k \mathbf{P}^i \mathbf{g}^j \lambda$ with $i \geqslant 0$ $j \geqslant 0$ and $k \geqslant 0$, where λ is an element in a specific 16-element subset $\mathbf{\Lambda}$ of $\mathrm{Ext}_{\mathbf{A}}(\mathbb{M}_2, \mathbb{M}_2)$ to be defined in Table 3.

The motivic wedge family takes the same position and same shape as the classical one (Figure 1). However the vertex of the motivic wedge is at τg^2 in degree $(40, 8, 23)$ having weight 23. Note that g^2 in degree $(40, 8, 24)$ does not survive the motivic May spectral sequence [**10**]. Our main result, Theorem 3.8, states that the subsets $\tau^k \mathbf{P}^i \mathbf{g}^j \lambda$ are non-empty and consist of non-zero elements for all λ in $\mathbf{\Lambda}$.

However, our main result is not optimal in the sense that there exist elements of weight greater than the weight of elements in $\mathbf{P}^i\mathbf{g}^j\lambda$ for some values of i, j, and λ. Some such elements are listed in Table 4.

We can not even conjecture the optimal result in general. However, we know a bit more about elements in the set $e_0^t\mathbf{g}^k$ for $t \geqslant 0$ and $k \geqslant 0$, which are part of the wedge. We will show that $\tau e_0^t\mathbf{g}^k$ is non-empty for all $t \geqslant 0$ and $k \geqslant 0$. We do not know whether $e_0^t\mathbf{g}^k$ is non-empty in general, but we make the following conjecture.

Conjecture 1.1. *The set $e_0^t\mathbf{g}^k$ is non-empty if and only if $k = (\sum_{i=1}^t 2^{n_i}) - t$ for some integer $n_i \geqslant 1$.*

The conjecture holds if and only if $e_0\mathbf{g}^{2^n-1}$ is non-empty for all $n \geqslant 1$, since

$$e_0^t\mathbf{g}^k \supseteq e_0\mathbf{g}^{2^{n_1}-1} \cdots e_0\mathbf{g}^{2^{n_t}-1}.$$

By explicit computations we know that $e_0, e_0\mathbf{g}$ and $e_0\mathbf{g}^3$ are non-empty and $e_0\mathbf{g}^2$ and $e_0\mathbf{g}^4$ are empty [**10**]. This means that the subsets $e_0\mathbf{g}^k$ are non-empty sometimes but empty other times. The analogous classical question is trivial, since $e_0^t g^k$ is a product of e_0^t and g^k.

1.1. Organization

The article contains four sections. In the second section, we recall techniques of comparison to classical computations, to $\mathrm{Ext}_{\mathbf{A}(2)}(\mathbb{M}_2, \mathbb{M}_2)$ and to h_1-localization of $\mathrm{Ext}_{\mathbf{A}}(\mathbb{M}_2, \mathbb{M}_2)$. In the third section, we introduce our main result on the motivic wedge. In the last section, we study the families $e_0^t\mathbf{g}^k$ and $\Delta h_1 e_0^t\mathbf{g}^k$ in $\mathrm{Ext}_{\mathbf{A}}(\mathbb{M}_2, \mathbb{M}_2)$ and give conjectures on their behavior.

1.2. Notation

We will use the following notation.

1. $\mathbb{M}_2 = \mathbb{F}_2[\tau]$ is the mod 2 motivic cohomology of a point, where τ has bidegree (0,1).
2. $\mathbf{A}$ is the mod 2 motivic Steenrod algebra over $\mathbb{C}$.
3. $\mathbf{A}(2)$ is the $\mathbb{M}_2$-subalgebra of $\mathbf{A}$ generated by $\mathrm{Sq}^1, \mathrm{Sq}^2$ and Sq^4.
4. $\mathbf{A}_{\mathrm{cl}}$ is the mod 2 classical Steenrod algebra.
5. Ext is the trigraded ring $\mathrm{Ext}_{\mathbf{A}}(\mathbb{M}_2, \mathbb{M}_2)$, the cohomology of $\mathbf{A}$.
6. $\mathrm{Ext}_{\mathbf{A}(2)}$ is the trigraded ring $\mathrm{Ext}_{\mathbf{A}(2)}(\mathbb{M}_2, \mathbb{M}_2)$, the cohomology of $\mathbf{A}(2)$.
7. $\mathrm{Ext}_{\mathrm{cl}}$ is the bigraded ring $\mathrm{Ext}_{\mathbf{A}_{\mathrm{cl}}}(\mathbb{F}_2, \mathbb{F}_2)$, the cohomology of $\mathbf{A}_{\mathrm{cl}}$.
8. We use the notation of [**11**] for elements in Ext.
9. We use the notation of [**9**] for elements in $\mathrm{Ext}_{\mathbf{A}(2)}$, except that we use a and n instead of α and ν respectively.
10. An element x in Ext has degree of the form (s, f, w) where:
    (a) f is the Adams filtration, i.e., the homological degree.
    (b) $s + f$ is the internal degree, i.e., corresponds to the first coordinate in the bidegrees of $\mathbf{A}$.
    (c) s is the stem, i.e., the internal degree minus the Adams filtration.
    (d) w is the motivic weight.

11. The Chow degree of an element of degree (s,f,w) is $s+f-2w$.
12. The coweight of an element of degree (s,f,w) is $s-w$.

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2. Comparison criteria

We know $\mathrm{Ext}_{\mathrm{cl}}$ [**16**, **19**] quite well. We know $\mathrm{Ext}_{\mathbf{A}(2)}$ completely [**9**, Theorem 4.13]. Computations in Ext can be studied via the relations with $\mathrm{Ext}_{\mathrm{cl}}$ and $\mathrm{Ext}_{\mathbf{A}(2)}$ in certain cases.

The following theorem plays a key role in comparing the motivic and the classical computations, saying that they become the same after inverting τ.

Theorem 2.1 ([**5**, Proposition 3.5])**.** *There is an isomorphism of rings*

$$\mathrm{Ext} \otimes_{\mathbb{M}_2} \mathbb{M}_2[\tau^{-1}] \cong \mathrm{Ext}_{\mathrm{cl}} \otimes_{\mathbb{F}_2} \mathbb{F}_2[\tau, \tau^{-1}].$$

Furthermore, the part of Ext at Chow degree 0 is isomorphic to $\mathrm{Ext}_{\mathrm{cl}}$.

Theorem 2.2 ([**10**, Theorem 2.1.12])**.** *There is an isomorphism from* $\mathrm{Ext}_{\mathrm{cl}}$ *to the subalgebra of* Ext *consisting of elements in degrees* (s,f,w) *such that* $s+f-2w=0$. *This isomorphism takes classical elements of degrees* (s,f) *to motivic elements of degrees* $(2s+f, f, s+f)$*, and it preserves all higher structure including products, squaring operations, and Massey products.*

In other words,

$$\mathrm{Ext}|_{s+f-2w=0} \cong \mathrm{Ext}_{\mathrm{cl}}.$$

The inclusion $\mathbf{A}(2) \hookrightarrow \mathbf{A}$ induces a homomorphism $\phi\colon \mathrm{Ext} \to \mathrm{Ext}_{\mathbf{A}(2)}$ which allows us to detect some structure in Ext via $\mathrm{Ext}_{\mathbf{A}(2)}$. We emphasize that $\mathrm{Ext}_{\mathbf{A}(2)}$ is described completely in [**9**, Theorem 4.13]. Table 1 gives some values of ϕ that we will need.

Remark 2.3. In some cases, $\phi(x)$ is decomposable in $\mathrm{Ext}_{\mathbf{A}(2)}$ but x is indecomposable in Ext. For example, the element $\Delta h_1 d_0$ in Ext is indecomposable but $\phi(\Delta h_1 d_0) = \Delta h_1 \cdot d_0$ is the product of Δh_1 and d_0 in $\mathrm{Ext}_{\mathbf{A}(2)}$.

Remark 2.4. We know values of ϕ by comparing the May spectral sequence computing Ext and the May spectral sequence computing $\mathrm{Ext}_{\mathbf{A}(2)}$.

If we invert h_1 on Ext, then we obtain $\mathrm{Ext}[h_1^{-1}]$ which is simpler than Ext. We can use $\mathrm{Ext}[h_1^{-1}]$ to detect some structure in Ext. The following theorems describe $\mathrm{Ext}[h_1^{-1}]$ and $\mathrm{Ext}_{\mathbf{A}(2)}[h_1^{-1}]$.

Theorem 2.5 ([**7**, Theorem 1.1])**.** *The h_1-localization* $\mathrm{Ext}[h_1^{-1}]$ *is a polynomial algebra over* $\mathbb{F}_2[h_1^{\pm 1}]$ *on generators* v_1^4 *and* v_n *for* $n \geqslant 2$*, where:*

Ext	$\mathrm{Ext}_{\mathbf{A}(2)}$	(s, f, w)
i	Pn	$(23, 7, 12)$
k	dn	$(29, 7, 16)$
r	n^2	$(30, 6, 16)$
m	ng	$(35, 7, 20)$
$\Delta h_1 d_0$	$\Delta h_1 \cdot d_0$	$(39, 9, 21)$
τg^2	$\tau \cdot g^2$	$(40, 8, 23)$
$\tau \Delta h_1 g$	$\tau \cdot \Delta h_1 \cdot g$	$(45, 9, 24)$
$h_2 g^j$	$h_2 \cdot g^j$	$(20j + 3, 4j + 1, 12j + 2)$
$P^i d_0$	$P^i \cdot d_0$	$(8i + 14, 4i + 4, 4i + 8)$
$P^i e_0$	$P^i \cdot e_0$	$(8i + 17, 4i + 4, 4i + 10)$

Table 1: Some values of the map $\phi\colon \mathrm{Ext} \longrightarrow \mathrm{Ext}_{\mathbf{A}(2)}$.

the element v_1^4 *has degree* $(8, 4, 4)$.

the element v_n *has degree* $(2^{n+1} - 2, 1, 2^n - 1)$.

Theorem 2.6 ([**7**, Proposition 3.7])**.** *The* h_1*-localization* $\mathrm{Ext}_{\mathbf{A}(2)}[h_1^{-1}]$ *is a polynomial algebra*

$$\mathrm{Ext}_{\mathbf{A}(2)}[h_1^{-1}] \cong \mathbb{F}_2[h_1^{\pm 1}, a_1, v_1^4, v_2]$$

in which a_1 *has degree* $(11, 3, 7)$*;* v_1^4 *has degree* $(8, 4, 4)$*; and* v_2 *has degree* $(6, 1, 3)$*.*

We can use h_1-localization to prove the non-existence of certain elements x in Ext. Guillou and Isaksen [**7**, 5] used the May spectral sequence analysis of $\mathrm{Ext}[h_1^{-1}]$ to determine the localization map

$$L\colon \mathrm{Ext} \longrightarrow \mathrm{Ext}[h_1^{-1}]$$

in a range. Some values of L are given in Table 2 [**7**, Table 13].

x	$L(x)$
$P^k h_1$	$h_1 v_1^{4k}$
$P^k d_0$	$h_1^2 v_1^{4k} v_2^2$
$P^k e_0$	$h_1^3 v_1^{4k} v_3$
$e_0 g$	$h_1^7 v_4$

Table 2: Some values of the localization map $L\colon \mathrm{Ext} \to \mathrm{Ext}[h_1^{-1}]$.

There is also a localization map $L\colon \mathrm{Ext}_{\mathbf{A}(2)} \longrightarrow \mathrm{Ext}_{\mathbf{A}(2)}[h_1^{-1}]$ [**7**, 5.1 and Table 14]. The following diagram is commutative.

$$\begin{array}{ccc} \mathrm{Ext} & \xrightarrow{\phi} & \mathrm{Ext}_{\mathbf{A}(2)} \\ \downarrow L & & \downarrow L \\ \mathrm{Ext}[h_1^{-1}] & \xrightarrow{\phi} & \mathrm{Ext}_{\mathbf{A}(2)}[h_1^{-1}] \end{array}$$

Remark 2.7. We know that the above diagram is commutative by analyzing the two localization maps as well as the four May spectral sequences computing Ext, $\mathrm{Ext}_{\mathbf{A}(2)}$, $\mathrm{Ext}[h_1^{-1}]$ and $\mathrm{Ext}_{\mathbf{A}(2)}[h_1^{-1}]$.

Definition 2.8. Let t be a non-negative integer. We define $\alpha(t)$ to be the number of 1's in the binary expansion of t.

Lemma 2.9. *Let t, k and s be non-negative integers, $s \geqslant 1$. We have:*

$\alpha(t) \leqslant t.$

$\alpha(t+k) \leqslant \alpha(t) + \alpha(k).$

$\alpha(2^s t) = \alpha(t).$

Proof. Suppose that $t = \sum_{i=1}^{n} 2^{m_i}$ in which $m_i \geqslant 0$ and $m_i \neq m_j$ if $i \neq j$. Consequently, $\alpha(t) = n$. Since $t = \sum_{i=1}^{n} 2^{m_i} \geqslant 1 \cdot n = n$, we obtain the first inequality.

With the above t we suppose further that $k = \sum_{i=1}^{p} 2^{q_i}$ in which $q_i \geqslant 0$ and $q_i \neq q_j$ if $i \neq j$. Consequently, $\alpha(k) = p$. We have

$$t + k = \sum_{i=1}^{n} 2^{m_i} + \sum_{j=1}^{p} 2^{q_j},$$

where the right hand side has $n + p$ powers of 2. If there is no pair (m_i, q_j) such that $m_i = q_j$, then $\alpha(t+k) = n + p = \alpha(t) + \alpha(k)$. If there exists at least one pair (m_i, q_j) such that $m_i = q_j = c$, since $2^{m_i} + 2^{q_j} = 2^{c+1}$ we have $\alpha(t+k) < n + p = \alpha(t) + \alpha(k)$. Therefore, $\alpha(t+k) \leqslant \alpha(t) + \alpha(k)$.

The last identity can be proven by the observation that if $t = \sum_{i=1}^{n} 2^{m_i}$, then $2^s t = \sum_{i=1}^{n} 2^{m_i + s}$. □

Lemma 2.10. *The map $\phi\colon \mathrm{Ext}[h_1^{-1}] \longrightarrow \mathrm{Ext}_{\mathbf{A}(2)}[h_1^{-1}]$ takes v_1^4 to v_1^4 and for all $n \geqslant 2$, ϕ maps v_n to $h_1^{-3(2^{n-2}-1)} a_1^{2^{n-2}-1} v_2$.*

Proof. This statement is stated as Conjecture 5.5 in [**7**] by Guillou and Isaksen. They also prove that if a "$\mathbb{C}$-motivic modular form" spectrum exists, then the conjecture holds [**7**, Proposition 6.4]. This spectrum has recently been constructed by Gheorghe, Isaksen, Krause and Ricka [**6**, 5], so we obtain the desired statement. □

Lemma 2.11. *The image of*

$$\phi\colon \mathrm{Ext}[h_1^{-1}] \longrightarrow \mathrm{Ext}_{\mathbf{A}(2)}[h_1^{-1}]$$

is spanned by the monomials $h_1^d v_1^{4a} v_2^b a_1^c$ where a, b and c are non-negative integers for which $\alpha(b+c) \leqslant b$ and d is an integer.

Proof. Denote by $\mathcal{G}$ the $\mathbb{M}_2$-submodule of $\mathrm{Ext}_{\mathbf{A}(2)}[h_1^{-1}]$ spanned by the monomials $h_1^d v_1^{4a} v_2^b a_1^c$ where a, b and c are non-negative integers for which $\alpha(b+c) \leqslant b$ and d is an integer.

Using Lemma 2.10 to get

$$\phi\colon v_1^{4a} \prod_{j \in J} v_j \longmapsto h_1^{-3\sum_{j\in J}(2^{j-2}-1)} v_1^{4a} v_2^m a_1^{\sum_{j\in J}(2^{j-2}-1)}$$

in which J is a sequence $(j_1, \ldots, j_m)$ of length m such that $j_k \geqslant 2$ (repeats are allowed).

Consequently, the image of ϕ equals the $\mathbb{M}_2$-submodule $\mathcal{H}$ of $\mathrm{Ext}_{\mathbf{A}(2)}[h_1^{-1}]$ spanned by the monomials of the form $h_1^d v_1^{4a} v_2^m a_1^{\sum_{j\in J}(2^{j-2}-1)}$ in which J is a sequence $(j_1, \dots, j_m)$ of length m such that $j_k \geqslant 2$ (repeats are allowed).

Since J has length m,

$$\alpha(m + \sum_{j\in J}(2^{j-2} - 1)) = \alpha(\sum_{j\in J} 2^{j-2}) \leqslant m.$$

As a result, $\mathcal{H}$ is contained in $\mathcal{G}$.

Conversely, for any monomial $h_1^d v_1^{4a} v_2^b a_1^c$ for which $\alpha(b+c) \leqslant b$, we can suppose that $b + c = \sum_{j\in J} 2^j$ where J is a sequence $(j_1, \dots, j_r)$ of length $r \leqslant b$ such that $j_k \geqslant 0$ for k in $\{1, \dots, r\}$. By replacing 2^j by $2^{j-1} + 2^{j-1}$ as necessary, we can rewrite $b + c$ as

$$b + c = \sum_{i\in I} 2^i,$$

where I is a sequence $(i_1, \dots, i_b)$ of length b such that $i_k \geqslant 0$ for k in $\{1, \dots, b\}$. Then

$$c = \sum_{i\in I} 2^i - b = \sum_{i\in I}(2^i - 1).$$

This shows that $\mathcal{G}$ is contained in $\mathcal{H}$. □

3. The wedge family

Recall that the inclusion $\mathbf{A}(2) \hookrightarrow \mathbf{A}$ induces a homomorphism of algebras $\phi\colon \mathrm{Ext} \to \mathrm{Ext}_{\mathbf{A}(2)}$.

Definition 3.1. For any λ in Ext, $i \geqslant 0$ and $j \geqslant 0$, $\mathbf{P}^i\mathbf{g}^j\lambda$ is the set which consists of all elements x in $\mathrm{Ext}^{s,f,w}$ such that $\phi(x) = P^i g^j \phi(\lambda)$ having degree (s, f, w) in $\mathrm{Ext}_{\mathbf{A}(2)}$.

Example 3.2. The set $\mathbf{g}^2 r$ contains m^2 because $\phi(m^2) = g^2 n^2 = g^2\phi(r)$.

Remark 3.3. We differentiate $\mathbf{P}$ and $\mathbf{g}$ with P and g. By the bold $\mathbf{P}$ and $\mathbf{g}$ we mean set-valued operations from Ext to Ext. Remember that $\mathbf{P}$ and $\mathbf{g}$ do not exist in Ext as elements. By P and g, we mean elements in $\mathrm{Ext}_{\mathbf{A}(2)}$.

Remark 3.4. We sometimes write the symbols $\mathbf{P}$ and $\mathbf{g}$ in a different order for consistency with standard notation. For example:

By $e_0\mathbf{g}^2$ we mean $\mathbf{g}^2(e_0)$. The set $e_0\mathbf{g}^2$ is empty (see Corollary 4.10).

By $\tau\Delta h_1\mathbf{g}^{j+1}$ we mean the set $\mathbf{g}^j(\tau\Delta h_1 g)$.

By $\tau\mathbf{P}^i\mathbf{g}^{j+1}$ we mean $\mathbf{P}^i\mathbf{g}^{j+1}(\tau)$.

The same convention is applied for $\tau\mathbf{g}^k$, $\tau e_0^t\mathbf{g}^k$ and many others.

Remark 3.5. From Definition 3.1 we have $\mathbf{P}^i\mathbf{g}^j x \cdot \mathbf{P}^a\mathbf{g}^b y \subseteq \mathbf{P}^{i+a}\mathbf{g}^{j+b}xy$. However, the inverse inclusion is not correct generally. For example, by low dimension calculation [**10**] we have

$$e_0 \cdot \tau^2\mathbf{g} = \{e_0\}\{\tau^2 g\} \subsetneq \tau^2 e_0\mathbf{g} = \{\tau^2 e_0 g, \tau^2 e_0 g + h_0^3 x\}.$$

Definition 3.6. We define $\mathbf{\Lambda}$ to be the following sixteen elements of Ext.

element	(s,f,w)	element	(s,f,w)
τg^2	$(40,8,23)$	$d_0 r$	$(44,10,24)$
$\tau\Delta h_1 g$	$(45,9,24)$	$d_0 m$	$(49,11,28)$
gr	$(50,10,28)$	$\tau e_0^2 g$	$(54,12,31)$
gm	$(55,11,32)$	$\tau\Delta h_1 e_0^2$	$(59,13,32)$
$\tau\Delta h_1 e_0$	$(42,9,22)$	$d_0 l$	$(46,11,26)$
$e_0 r$	$(47,10,26)$	τe_0^3	$(51,12,29)$
$e_0 m$	$(52,11,30)$	$\tau\Delta h_1 d_0 e_0$	$(56,13,30)$
$\tau e_0 g^2$	$(57,12,33)$	$d_0 e_0 r$	$(61,14,34)$

Table 3: Sixteen elements of the set $\mathbf{\Lambda}$.

Remark 3.7. The elements in the set $\mathbf{\Lambda}$ are not optimal in the sense that there may exist elements of weight greater than the weight of elements in $\mathbf{\Lambda}$ (see Table 4). For example, the element τe_0^3 in $\mathbf{\Lambda}$ has weight 29 but the element e_0^3 in Ext has weight 30. The reason for this choice is that our proof for Theorem 3.8 works for τe_0^3, $\tau\Delta h_1 e_0$, $\tau e_0^2 g$, $\tau\Delta h_1 e_0$ and $\tau\Delta h_1 d_0 e_0$ but does not work for e_0^3, $\Delta h_1 e_0$, $e_0^2 g$, $\Delta h_1 e_0$ and $\Delta h_1 d_0 e_0$.

The following theorem is our main result.

Theorem 3.8. *For any λ in $\mathbf{\Lambda}$, $i \geqslant 0$, $j \geqslant 0$ and $k \geqslant 0$, the set $\tau^k \mathbf{P}^i \mathbf{g}^j \lambda$ is non-empty and consists of non-zero elements.*

Remark 3.9. The set $\tau^k \mathbf{P}^i \mathbf{g}^j \lambda$ consists of all elements $\tau^k x$ with x in $\mathbf{P}^i \mathbf{g}^j \lambda$.

Combining all elements of $\tau^k \mathbf{P}^i \mathbf{g}^j \lambda$ with $i \geqslant 0$, $j \geqslant 0$, $k \geqslant 0$ and λ in $\mathbf{\Lambda}$, we obtain an infinite wedge-shaped diagram, filling out the angle with vertex at τg^2 in degree $(40,8,23)$, bounded above by the line $f = \frac{1}{2}s - 12$ parallel to the Adams edge [**8**], and bounded below by the line $s = 5f$ in Ext (Figure 1). We call this set the wedge. To be precise, we have the following definition.

Definition 3.10. For $i \geqslant 0, j \geqslant 0, k \geqslant 0$, and $\lambda \in \mathbf{\Lambda}$ the set

$$\{x \in \text{Ext} : x \in \tau^k \mathbf{P}^i \mathbf{g}^j \lambda\}$$

is called the wedge family of the cohomology of the motivic Steenrod algebra.

We need a couple of preliminary results before proving Theorem 3.8.

Lemma 3.11. *The sets $\mathbf{P}^i d_0$, $\mathbf{P}^i e_0$ and $\mathbf{P}^i \Delta h_1 e_0$ are non-empty for $i \geqslant 0$.*

Proof. Since d_0, e_0 and $\Delta h_1 e_0$ are generators in Ext, the statement is trivial when $i = 0$.

We now consider the case $i > 0$. The Adams periodicity operator P^i is an isomorphism on Ext in specified ranges [**12**, Theorem 1.4]. Since the element d_0 lies in these ranges, then $\mathbf{P}^i d_0$ contains the element $P^i d_0$. Therefore, the set $\mathbf{P}^i d_0$ is non-empty. The same argument is applied for $\mathbf{P}^i e_0$ and $\mathbf{P}^i \Delta h_1 e_0$. □

Lemma 3.12. *Let x be an element in* Ext *such that $h_1^3\phi(x) = 0$. Then $\mathbf{P}^{i+1}\mathbf{g}x$ contains the non-empty set $\mathbf{P}^i d_0^2 x$ for all $i \geqslant 0$. As a result, $\mathbf{P}^{i+1}\mathbf{g}x$ is non-empty.*

Proof. Since $\mathbf{P}^i d_0$ is non-empty by Lemma 3.11, the set $\mathbf{P}^i d_0^2 x$ is non-empty. Consider an element β in $\mathbf{P}^i d_0^2 x$. Since $\phi(d_0) = d$ and $d^2 = Pg + h_1^3 \cdot \Delta h_1$ in $\mathrm{Ext}_{\mathbf{A}(2)}$ [**9**, Table 8], we have

$$\begin{aligned}\phi(\beta) = P^i d^2 \phi(x) &= P^i(Pg + h_1^3 \cdot \Delta h_1)\phi(x) \\ &= P^{i+1} g\phi(x) + P^i \Delta h_1 \cdot h_1^3 \phi(x) = P^{i+1} g\phi(x).\end{aligned}$$

Consequently, $\mathbf{P}^{i+1}\mathbf{g}x$ contains the element β of $\mathbf{P}^i d_0^2 x$. □

Lemma 3.13. *Consider $j \geqslant 2$ and suppose that $j = 2^r(2k+1)$ for $r \geqslant 0$ and $k \geqslant 0$. We have the differential: $d_{2^{r+2}}(P^j) = h_0^5 x_j$ for some x_j in the classical May spectral sequence.*

Proof. Since $j \geqslant 2$, we do not consider $r = 0$ and $k = 0$. When $r = 0$ and $k \geqslant 1$ we have

$$d_4(P^{2k+1}) = d_4(P) \cdot P^{2k} + d_4(P^{2k}) \cdot P = d_4(P) \cdot P^{2k} = h_0^4 h_3 \cdot P^{2k}.$$

In the E_4 page of the classical May spectral sequence we have

$$P^2 \cdot h_3 = h_0^2 i + \tau P h_1 d_0.$$

Then

$$h_0^4 h_3 \cdot P^{2k} = h_0^4 \cdot P^{2k-2} \cdot P^2 h_3 = h_0^4 \cdot P^{2k-2} \cdot (h_0^2 i + \tau P h_1 d_0) = h_0^5 \cdot h_0 P^{2k-2} i.$$

When $r \geqslant 1$ we have

$$\begin{aligned}d_{2^{r+2}}(P^{2^r(2k+1)}) &= d_{2^{r+2}}(P^{2^r k \cdot 2} \cdot P^{2^r}) = d_{2^{r+2}}(P^{2^r}) \cdot P^{2^r k \cdot 2} + d_{2^{r+2}}(P^{2^r k \cdot 2}) \cdot P^{2^r} \\ &= d_{2^{r+2}}(P^{2^r}) \cdot P^{2^{r+1} k}.\end{aligned}$$

To obtain the last identity we note that P^{2^r} is an element on the $E_{2^{r+2}}$-term then we use Leibniz rule for $d_{2^{r+2}}(P^{2^r k} \cdot P^{2^r k})$ to obtain $d_{2^{r+2}}(P^{2^r k \cdot 2}) = 0$. Finally, we use Nakamura's formula [**18**, 4 page 14] to get

$$d_{2^{r+2}}(P^{2^r}) \cdot P^{2^{r+1} k} = h_0^{2^{r+2}} h_{r+3} \cdot P^{2^{r+1} k} = h_0^5 \cdot h_0^{2^{r+2}-5} h_{r+3} P^{2^{r+1} k}. \qquad □$$

We denote by $\tilde{x}_j$ the motivic element of Chow degree zero corresponding to x_j (defined in Lemma 3.13) via the Chow degree zero isomorphism in Theorem 2.2.

Remark 3.14. In the proof of Lemma 3.13, we actually show that $d_{2^{r+2}}(P^j) = h_0^6 y_j$ for some y_j in the classical May spectral sequence. However, in order to prepare for Lemma 3.15, we prefer the statement stated in Lemma 3.13.

Lemma 3.15. *In* Ext, *for $j \geqslant 2$, the Massey product $\langle h_2, h_1, h_1^4 \tilde{x}_j \rangle$ equals $h_2 g^j$.*

Proof. The motivic elements $h_2 g^j, h_2, h_1$ and $h_1^4 \tilde{x}_j$ all have Chow degree zero. They correspond to classical elements $P^j h_1$, h_1, h_0 and $h_0^4 x_j$ via the Chow degree zero isomorphism in Theorem 2.2.

Classically we have $P^j h_1 = \langle h_1, h_0, h_0^4 x_j \rangle$. We obtain the desired identity by the Chow degree zero isomorphism. □

Remark 3.16. The g in $h_2 g^j$ in the above argument is not the operator $\mathbf{g}$. We write $h_2 g^j$ for the element of Ext which corresponds to the classical element $P^j h_1$ via the Chow degree zero isomorphism in Theorem 2.2.

Lemma 3.17. *The sets* $\mathbf{g}^j m$*,* $\mathbf{g}^j l$ *and* $\mathbf{g}^j r$ *are non-empty for all* $j \geqslant 0$*.*

Proof. For $j = 0$ the set $\mathbf{g}^j m$ contains m, $\mathbf{g}^j l$ contains l and $\mathbf{g}^j r$ contains r. For $j = 1$ the set $\mathbf{g}^j m$ contains gm, $\mathbf{g}^j l$ contains $e_0 m$ and $\mathbf{g}^j r$ contains gr. For $j \geqslant 2$ we have

$$h_2\langle h_1, h_1^4\tilde{x}_j, m\rangle = \langle h_2, h_1, h_1^4\tilde{x}_j\rangle \cdot m = h_2 g^j \cdot m$$

in which the last identity is by Lemma 3.15. Consider an element β in $\langle h_1, h_1^4\tilde{x}_j, m\rangle$. We apply ϕ to get

$$h_2\phi(\beta) = h_2 g^j \phi(m) = h_2 n g^{j+1}.$$

By inspection of $\mathrm{Ext}_{\mathbf{A}(2)}$, we have

$$\phi(\beta) = n g^{j+1}.$$

Therefore $\mathbf{g}^j m$ contains β, and is non-empty.

The same argument is applied to $\mathbf{g}^j l$ and $\mathbf{g}^j r$. □

Lemma 3.18. *The set* $\tau\Delta h_1 \mathbf{g}^{j+1}$ *is non-empty for all* $j \geqslant 0$*.*

Proof. The set $\tau\Delta h_1 \mathbf{g}$ is non-empty since it contains $\tau\Delta h_1 g$.

When $j \geqslant 1$, the set $\tau\Delta h_1 \mathbf{g}^{j+1} \supseteq r \cdot \mathbf{g}^{j-1} m$ is non-empty since $\mathbf{g}^j m$ is non-empty by Lemma 3.17. Here we are using the identity $\tau\Delta h_1 g \cdot g = \phi(r) \cdot \phi(m)$ in $\mathrm{Ext}_{\mathbf{A}(2)}$. □

Lemma 3.19. *The set* $\tau\mathbf{g}^j$ *is non-empty for any* $j \geqslant 0$*.*

Proof. The claim for $j = 0$ and $j = 1$ is proven by explicit low dimension calculation [**5**, **10**].

When $j \geqslant 2$, by Lemma 3.15

$$\langle \tau, h_1^4\tilde{x}_j, h_1\rangle h_2 = \tau\langle h_1^4\tilde{x}_j, h_1, h_2\rangle = \tau h_2 g^j.$$

Consider an element γ in $\langle \tau, h_1^4\tilde{x}_j, h_1\rangle$, we apply ϕ to get

$$\phi(\gamma) h_2 = \tau h_2 g^j.$$

By inspection of $\mathrm{Ext}_{\mathbf{A}(2)}$ [**9**, Theorem 4.13],

$$\phi(\gamma) = \tau g^j.$$

Therefore $\tau\mathbf{g}^j$ contains γ, and is non-empty. □

Lemma 3.20. *The set* $\tau\mathbf{P}^i\mathbf{g}^{j+1}$ *is non-empty for* $i \geqslant 0$ *and* $j \geqslant 0$*.*

Proof. The case $i = 0$ is established in Lemma 3.19. Now we assume $i > 0$. When $j = 0$, by Lemma 3.11 the set $\mathbf{P}^{i-1}d_0$ is non-empty. Consequently, $\tau\mathbf{P}^{i-1}d_0^2$ is non-empty. We consider an element x in $\tau\mathbf{P}^{i-1}d_0^2$. The set $\mathbf{P}^i(\tau g)$ contains x because

$$\phi(x) = \tau P^{i-1}d_0^2 = \tau P^{i-1}(Pg + h_1^3\Delta h_1) = \tau P^i g.$$

When $j \geqslant 1$, consider x in $\tau\mathbf{g}^j$. Since $h_1^3\phi(x) = h_1^3 \cdot \tau g^j = 0$ in $\mathrm{Ext}_{\mathbf{A}(2)}$, $\mathbf{P}^i\mathbf{g}x$ is non-empty by Lemma 3.12. Therefore, $\tau\mathbf{P}^i\mathbf{g}^{j+1} = \mathbf{P}^i\mathbf{g}x$ is non-empty. □

Example 3.21. The set $\tau\mathbf{Pg}$ contains τd_0^2.

Now we can prove Theorem 3.8.

Proof. We will prove that $\mathbf{P}^i\mathbf{g}^j\lambda$ is non-empty. For all $i \geqslant 0$ and $j \geqslant 0$ we have the following inclusions of sets:

$$\mathbf{P}^i\mathbf{g}^j\tau\Delta h_1e_0 \supseteq \tau\mathbf{g}^j \cdot \mathbf{P}^i\Delta h_1e_0, \qquad \mathbf{P}^i\mathbf{g}^j e_0r \supseteq \mathbf{P}^ie_0 \cdot \mathbf{g}^jr,$$
$$\mathbf{P}^i\mathbf{g}^j e_0m \supseteq \mathbf{P}^ie_0 \cdot \mathbf{g}^jm, \qquad \mathbf{P}^i\mathbf{g}^j\tau e_0g^2 \supseteq \mathbf{P}^ie_0 \cdot \tau\mathbf{g}^{j+2},$$
$$\mathbf{P}^i\mathbf{g}^j d_0r \supseteq \mathbf{P}^id_0 \cdot \mathbf{g}^jr, \qquad \mathbf{P}^i\mathbf{g}^j d_0m \supseteq \mathbf{P}^id_0 \cdot \mathbf{g}^jm,$$
$$\mathbf{P}^i\mathbf{g}^j\tau e_0^2g \supseteq \mathbf{P}^ie_0^2 \cdot \tau\mathbf{g}^{j+1}, \qquad \mathbf{P}^i\mathbf{g}^j\tau\Delta h_1e_0^2 \supseteq \tau\mathbf{g}^j \cdot \mathbf{P}^i\Delta h_1e_0 \cdot e_0,$$
$$\mathbf{P}^i\mathbf{g}^j d_0l \supseteq \mathbf{P}^id_0 \cdot \mathbf{g}^jl, \qquad \mathbf{P}^i\mathbf{g}^j\tau e_0^3 \supseteq \tau\mathbf{g}^j \cdot \mathbf{P}^ie_0^3,$$
$$\mathbf{P}^i\mathbf{g}^j\tau\Delta h_1d_0e_0 \supseteq \tau\mathbf{g}^j \cdot \mathbf{P}^i\Delta h_1e_0 \cdot d_0, \qquad \mathbf{P}^i\mathbf{g}^j d_0er \supseteq \mathbf{P}^id_0 \cdot \mathbf{g}^jr \cdot e_0.$$

The set $\tau\mathbf{g}^j \cdot \mathbf{P}^i\Delta h_1e_0$ consists of all products $x \cdot y$ in which x is an element of $\tau\mathbf{g}^j$ and y is an element of $\mathbf{P}^i\Delta h_1e_0$. The same interpretation is applied for other sets on the right hand side.

The sets on the right hand side are all non-empty because of Lemmas 3.11, 3.17 and 3.19. For example, since $\tau\mathbf{g}^j$ and $\mathbf{P}^i\Delta h_1e_0$ are non-empty,

$$\mathbf{P}^i\mathbf{g}^j\tau\Delta h_1e_0 \supseteq \tau\mathbf{g}^j \cdot \mathbf{P}^i\Delta h_1e_0$$

is non-empty. Therefore, the sets on the left are all non-empty.

Several values of λ remain.

Now consider $\lambda = \tau g^2$. The set $\mathbf{P}^i\mathbf{g}^j(\tau g^2)$, or $\tau\mathbf{P}^i\mathbf{g}^{j+2}$, is non-empty by Lemma 3.20.

Next consider $\lambda = gr$. The case $i = 0$ is established in Lemma 3.17. We consider $i > 0$. Since $\mathbf{g}^jr$ is non-empty by Lemma 3.17, we consider any element x in $\mathbf{g}^jr$. Since $h_1^3\phi(x) = h_1^3n^2 \cdot g^j = 0$ [**9**, Theorem 4.13], then $\mathbf{P}^i\mathbf{g}^j\lambda = \mathbf{P}^i\mathbf{g}x$ is non-empty by Lemma 3.12. The same argument is applied for $\lambda = gm$.

Finally, consider $\lambda = \tau\Delta h_1g$. The case $i = 0$ is established in Lemma 3.18. We consider $i > 0$. When $j > 0$, since $\tau\Delta h_1\mathbf{g}^j$ is non-empty by Lemma 3.18, we consider any element x in $\tau\Delta h_1\mathbf{g}^j$. Since $h_1^3\phi(x) = h_1^3\tau\Delta h_1 \cdot g^j = 0$ [**9**, Theorem 4.13], then $\mathbf{P}^i\mathbf{g}^j\lambda = \mathbf{P}^i\mathbf{g}x$ is non-empty by Lemma 3.12. When $j = 0$, since

$$\phi(\tau P^{i-1}d_0 \cdot d_0\Delta h_1) = \tau P^{i-1}d_0^2\Delta h_1 = \tau P^{i-1}(Pg + h_1^3\Delta h_1)\Delta h_1 = P^i\tau\Delta h_1g,$$

$\mathbf{P}^i\tau\Delta h_1g$ contains $\tau P^{i-1}d_0 \cdot \Delta h_1d_0$, so it is non-empty.

Therefore, the set $\tau^k\mathbf{P}^i\mathbf{g}^j\lambda$ is non-empty. The non-triviality of elements in $\tau^k\mathbf{P}^i\mathbf{g}^j\lambda$ is obtained by comparison to $\mathrm{Ext}_{\mathbf{A}(2)}$. □

The multiplicative structure of the wedge family

We are mostly interested in the $\mathbb{M}_2$-module structure of the wedge. However, it also has the multiplicative structure. To be precise, the wedge is closed under the multiplication.

Proposition 3.22. *For $i \geqslant 0, j \geqslant 0, k \geqslant 0$ and $\lambda \in \mathbf{\Lambda}$, the set*

$$\{x \in \mathrm{Ext} : x \in \tau^k\mathbf{P}^i\mathbf{g}^j\lambda\}$$

is closed under the multiplication.

Proof. It is sufficient to prove that for $i \geqslant 0, j \geqslant 0, k \geqslant 0$ and $\lambda \in \mathbf{\Lambda}$ the subset

$$\{\tau^kP^ig^j\phi(\lambda)\}$$

of $\mathrm{Ext}_{\mathbf{A}(2)}$ is closed under the multiplication. This is done using [**9**, Theorem 4.13]. □

4. The $e_0^t\mathbf{g}^k$ and $\Delta h_1 e_0^t\mathbf{g}^k$ families

The wedge family is not optimal in the sense that there exist elements of weight greater than the weight of elements in $\mathbf{\Lambda}$. For example, the wedge element $\tau e_0^2 g$ in $\mathbf{\Lambda}$ being of weight 31 is not optimal because the element $e_0^2 g$ in Ext is of weight 32. Table 4 lists all such elements in $\mathbf{\Lambda}$. The analyzing of elements in $\mathbf{\Lambda}$ leads us to the study of two families $e_0^t\mathbf{g}^k$ and $\Delta h_1 e_0^t\mathbf{g}^k$. Unfortunately, these two families have complicated behavior. We state two conjectures on them. If these two conjectures are correct, then we obtain other two systematic phenomena in Ext.

element of the wedge	weight	element of higher weight	weight
$\tau\Delta h_1 e_0$	22	$\Delta h_1 e_0$	23
τe_0^3	29	e_0^3	30
$\tau\Delta h_1 d_0 e_0$	30	$\Delta h_1 d_0 \cdot e_0$	31
$\tau e_0^2 g$	31	$e_0 \cdot e_0 g$	32
$\tau\Delta h_1 e_0^2$	32	$\Delta h_1 e_0 \cdot e_0$	33

Table 4

Remark 4.1. By $\mathbf{g}^j$ we mean $\mathbf{g}^j(1)$ which is understood in the sense of Definition 3.1.

Lemma 4.2. *The set $\mathbf{g}^j$ is empty for all $j \geqslant 0$.*

Proof. We prove the statement via contradiction. Suppose that $\mathbf{g}^j$ is non-empty. Consider any element x in $\mathbf{g}^j$. Since x maps to the non-zero element g^j in $\mathrm{Ext}_{\mathbf{A}(2)}$, x is non-zero. Furthermore, because x has Chow degree zero, x corresponds to a classical element at degree $(8j, 4j)$ in $\mathrm{Ext}_{\mathrm{cl}}$ via the Chow degree zero isomorphism. However, $\mathrm{Ext}_{\mathrm{cl}}$ is zero in degrees $(8j, 4j)$ for all non-negative integers j [**1**]. Therefore, x does not exist. □

Lemma 4.3. *The set $d_0\mathbf{g}$ contains e_0^2.*

Proof. We have $\phi(e_0^2) = e_0^2 = gd$. The last identity is because $e_0^2 = gd$ in $\mathrm{Ext}_{\mathbf{A}(2)}$. □

We now study the behavior of the sets $\mathbf{P}^i\mathbf{g}^j\tau e_0^3$, $\mathbf{P}^i\mathbf{g}^j\tau\Delta h_1 e_0$, $\mathbf{P}^i\mathbf{g}^j\tau\Delta h_1 d_0 e_0$, $\mathbf{P}^i\mathbf{g}^j\tau e_0^2 g$ and $\mathbf{P}^i\mathbf{g}^j\tau\Delta h_1 e_0^2$ for $i \geqslant 0$ and $j \geqslant 0$.

Theorem 4.4. *The sets $\mathbf{P}^i\mathbf{g}^j\tau e_0^3$, $\mathbf{P}^i\mathbf{g}^j\tau\Delta h_1 e_0$, $\mathbf{P}^i\mathbf{g}^j\tau\Delta h_1 d_0 e_0$, $\mathbf{P}^i\mathbf{g}^j\tau e_0^2 g$ and $\mathbf{P}^i\mathbf{g}^j\tau\Delta h_1 e_0^2$ contain an element divisible by τ if*

$i \geqslant 0$ and $j = 0$, or

$i \geqslant j \geqslant 1$, or

$1 \leqslant i < j \leqslant 3i$.

Proof. Consider $\mathbf{P}^i\mathbf{g}^j\tau e_0^3$. When $i = 0$ and $j = 0$, the element τe_0^3 is divisible by τ. When $i \geqslant 1$ and $j = 0$, the set $\tau\mathbf{P}^i e_0^3$ contains the element $\tau \cdot P^i e_0^3$ which is divisible by τ. Apply Example 3.21 and Lemma 4.3 to get:

When $i \geqslant j \geqslant 1$, the set $\tau\mathbf{P}^i\mathbf{g}^j e_0^3$ contains the element $\tau \cdot d_0^{2j} \cdot P^{i-j} e_0^3$ which is divisible by τ.

- When $1 \leqslant i < j \leqslant 3i$, the set $\tau \mathbf{P}^i \mathbf{g}^j e_0^3$ contains the element $\tau \cdot d_0^{3i-j} \cdot e_0^{2(j-i)+3}$ which is divisible by τ.

The same argument can be used for $\mathbf{P}^i \mathbf{g}^j \tau \Delta h_1 e_0$, $\mathbf{P}^i \mathbf{g}^j \tau \Delta h_1 d_0 e_0$, $\mathbf{P}^i \mathbf{g}^j \tau e_0^2 g$ and $\mathbf{P}^i \mathbf{g}^j \tau \Delta h_1 e_0^2$. □

Remark 4.5. A wedge element being divisible by τ is not optimal.

There are unknown cases from Theorem 4.4. Consider $\mathbf{P}^i \mathbf{g}^j \tau e_0^3$. When $i = 0$ and $j \geqslant 1$, the set $\tau e_0^3 \mathbf{g}^j$ is not known fully. When $3i < j$, the set $\mathbf{P}^i \mathbf{g}^j \tau e_0^3$ contains the set $\tau e_0^{4i+3} \mathbf{g}^{j-3i}$ which is not known fully. The same story happens to $\mathbf{P}^i \mathbf{g}^j \tau \Delta h_1 e_0$, $\mathbf{P}^i \mathbf{g}^j \tau \Delta h_1 d_0 e_0$, $\mathbf{P}^i \mathbf{g}^j \tau e_0^2 g$ and $\mathbf{P}^i \mathbf{g}^j \tau \Delta h_1 e_0^2$ which leads us to the study of two families $\Delta h_1 e_0^t \mathbf{g}^k$ and $e_0^t \mathbf{g}^k$ for $k \geqslant 0$ and $t \geqslant 1$ which are not known fully.

Remark 4.6. By "not known fully" we mean that we do not know if the sets $\Delta h_1 e_0^t \mathbf{g}^k$ and $e_0^t \mathbf{g}^k$ are non-empty in general. The low dimension calculations show that they are empty with some values of t and k and non-empty with some other values.

Remark 4.7. Since Δh_1 is not an element of Ext, $\Delta h_1 \mathbf{g}^k$ is not defined. Therefore, we do not consider the set $\Delta h_1 e_0^t \mathbf{g}^k$ when $t = 0$. We do not consider the set $e^t \mathbf{g}^k$ when $t = 0$ either because the set $\mathbf{g}^k$ is known to be empty by Lemma 4.2.

4.1. The $e_0^t \mathbf{g}^k$ family

Lemma 4.8. *If $e_0^t \mathbf{g}^k$ is non-empty, then $e_0^t \mathbf{g}^k$ consists of elements which are non-zero in the h_1-localization* $\mathrm{Ext}[h_1^{-1}]$.

Proof. For any element x in $e_0^t \mathbf{g}^k$ and any non-negative integer n, we have $\phi(h_1^n x) = h_1^n e_0^t g^k$ which is non-zero in $\mathrm{Ext}_{\mathbf{A}(2)}$ [**9**, Theorem 4.13]. Consequently, $h_1^n x$ is non-zero in Ext. In other words, x is non-zero in the h_1-localization $\mathrm{Ext}_{\mathbf{A}}[h_1^{-1}]$. □

Proposition 4.9. *Let t and k be non-negative integers. If $\alpha(t+k) > t$, then $e_0^t \mathbf{g}^k$ is empty.*

Proof. (Via contradiction) Suppose that $e_0^t \mathbf{g}^k$ is non-empty. As a result, its elements survive the h_1-localization by Lemma 4.8. Note that elements of $e_0^t \mathbf{g}^k$ have Chow degree t and coweight $(7t + 8k)$. By Theorem 2.5, after considering Chow degrees, any element of $e_0^t \mathbf{g}^k$ maps to a summation of monomials of the form

$$v_1^{4n} v_2^m \prod_{i=1}^{t-4n-m} v_{m_i}$$

in $\mathrm{Ext}_{\mathbf{A}}[h_1^{-1}]$ for some n, m and $m_i \geqslant 3$. By comparing coweights, we have

$$7t + 8k = 4n + 3m + \sum_{i=1}^{t-4n-m} (2^{m_i} - 1).$$

Then

$$8t + 8k = 8n + 4m + \sum_{i=1}^{t-4n-m} 2^{m_i}.$$

Since $m_i \geqslant 3$, m has to be even, i.e., $m = 2m'$ for some non-negative integer m'. We obtain

$$t + k = n + m' + \sum_{i=1}^{t-4n-m} 2^{m_i - 3}.$$

By Lemma 2.9,

$$\alpha(t+k) \leqslant \alpha(n) + \alpha(m') + t - 4n - m = t + (\alpha(n) - 4n) + (\alpha(m') - 2m') \leqslant t. \quad \square$$

Corollary 4.10. *If $e_0\mathbf{g}^k$ is non-empty, then $k = 2^n - 1$ for some non-negative integer n.*

Proof. Since $e_0\mathbf{g}^k$ is non-empty, $\alpha(1+k) \leqslant 1$ by Proposition 4.9. Then $1 + k = 2^n$ for some non-negative integer n. □

We state the following conjecture.

Conjecture 4.11. *The set $e_0\mathbf{g}^k$ is non-empty if and only if $k = 2^n - 1$ for some non-negative integer n.*

We mention some evidence supporting the conjecture. The elements $e_0 g$ and $e_0 g^3$ survive in Ext (by explicit computations). Also, the conjecture fits nicely with the properties of the h_1-localization of Ext [**7**].

Theorem 4.12. *Suppose that $e_0\mathbf{g}^{2^n-1}$ is non-empty for every non-negative integer n. Then $e_0^t\mathbf{g}^k$ is non-empty if and only if $k = (\sum_{i=1}^t 2^{n_i}) - t$ for some non-negative integers n_i.*

Proof. If $e_0^t\mathbf{g}^k$ is non-empty, then by Proposition 4.9 we have $\alpha(k+t) \leqslant t$. As a result, $k + t = \sum_{i=1}^t 2^{n_i}$ for some non-negative integers n_i. In other words,

$$k = (\sum_{i=1}^{t} 2^{n_i}) - t.$$

Conversely, if $k = (\sum_{i=1}^t 2^{n_i}) - t$, since $e_0\mathbf{g}^{2^{n_i}-1}$ is non-empty for all n_i then

$$e_0^t\mathbf{g}^k \supseteq e_0\mathbf{g}^{2^{n_1}-1} \cdot e_0\mathbf{g}^{2^{n_2}-1} \cdots e_0\mathbf{g}^{2^{n_t}-1}$$

is non-empty. □

Remark 4.13. The condition $k = (\sum_{i=1}^t 2^{n_i}) - t$ is equivalent to $\alpha(k+t) \leqslant t$. In practice, we use the latter condition rather than the former one.

4.2. The $\Delta h_1 e_0^t \mathbf{g}^k$ family

Proposition 4.14. *If $\alpha(1+k+t) > t$ for $t \geqslant 1$ and $k \geqslant 0$, then the set $\Delta h_1 e_0^t\mathbf{g}^k$ is empty.*

Proof. (Via contradiction) Recall the following commutative diagram [**7**]

$$\begin{array}{ccc} \mathrm{Ext} & \xrightarrow{\phi} & \mathrm{Ext}_{\mathbf{A}(2)} \\ \downarrow L & & \downarrow L \\ \mathrm{Ext}[h_1^{-1}] & \xrightarrow{\phi} & \mathrm{Ext}_{\mathbf{A}(2)}[h_1^{-1}] \end{array}$$

Suppose that $\Delta h_1 e_0^t\mathbf{g}^k$ is non-empty. Then it contains an element x. The element x maps to the element $\Delta h_1 e_0^t g^k$ in $\mathrm{Ext}_{\mathbf{A}(2)}$, surviving h_1-localization. The element

$\Delta h_1 e_0^t g^k$ maps to

$$h_1^{-2k-5} v_1^4 a_1^{2+2k+t} v_2^t + h_1^{-2k+1} v_2^{4+t} a_1^{2k+t}$$

in $\mathrm{Ext}_{\mathbf{A}(2)}[h_1^{-1}]$ via L.

Since $\alpha(1+k+t) > t$, the term $h_1^{-2k-5} v_1^4 a_1^{2+2k+t} v_2^t$ is not in the image of $\phi\colon \mathrm{Ext}[h_1^{-1}] \longrightarrow \mathrm{Ext}_{\mathbf{A}(2)}[h_1^{-1}]$ by Lemma 2.11. □

Lemma 4.15. *For any integer $k \geqslant 0$, there is no element x in* Ext *such that $\phi(x) = \Delta h_1 g^k$ in $\mathrm{Ext}_{\mathbf{A}(2)}$.*

Proof. We apply the same argument as in Proposition 4.14. □

By Proposition 4.14, a necessary condition for the set $\Delta h_1 e_0 \mathbf{g}^j$ to be non-empty is $\alpha(2+j) \leqslant 1$, or $j = 2^n - 2$ for some non-negative integer n. Unfortunately, we do not know if it is sufficient. We state the following conjecture.

Conjecture 4.16. *The set $\Delta h_1 e_0 \mathbf{g}^j$ is non-empty if and only if $j = 2^n - 2$ for some non-negative integer n.*

Theorem 4.17. *Suppose that $e_0 \mathbf{g}^{2^n-1}$ and $\Delta h_1 e_0 \mathbf{g}^{2^n-2}$ are non-empty for every non-negative integer n. Then $\Delta h_1 e_0^t \mathbf{g}^k$ is non-empty if and only if $k = (\sum_{i=1}^t 2^{n_i}) - t - 1$ for some non-negative integers n_i.*

Proof. If $\Delta h_1 e_0^t \mathbf{g}^k$ is non-empty, then $\alpha(1+k+t) \leqslant t$ or $k = (\sum_{i=1}^t 2^{n_i}) - t - 1$ for some non-negative integers n_i.

Conversely, if $k = (\sum_{i=1}^t 2^{n_i}) - t - 1$ for some non-negative integers n_i, then $\Delta h_1 e_0^t \mathbf{g}^k$ contains the set

$$\Delta h_1 e_0 \mathbf{g}^{2^{n_{i_1}}-2} \cdot e_0 \mathbf{g}^{2^{n_{i_2}}-1} \cdots e_0 \mathbf{g}^{2^{n_{i_t}}-1}$$

which is non-empty. □

4.3. The wedge at filtrations $f = 4k$ and $f = 4k+1$ for $k \geqslant 2$

At filtrations $f = 4k+2$ and $f = 4k+3$ for $k \geqslant 2$, the wedge is optimal in the sense that all elements are of the greatest weight. At filtrations $f = 4k$ and $f = 4k+1$ for $k \geqslant 2$, the wedge is not optimal in the sense that there exist elements in Ext which are of weight greater than the weight of the wedge elements.

Theorem 4.18. *Suppose that $e_0 \mathbf{g}^{2^n-1}$ is non-empty for every non-negative integer n. Then at filtration $f = 4k$ for $k \geqslant 2$ the set $\tau e_0^s \mathbf{g}^{k-s}$ contains an element divisible by τ if $s \geqslant \alpha(k)$ and does not contain any element divisible by τ if $s < \alpha(k)$.*

Proof. If $s \geqslant \alpha(k)$, by Theorem 4.12 and Remark 4.13 the set $e_0^s \mathbf{g}^{k-s}$ contains an element x. Then $\tau e_0^s \mathbf{g}^{k-s}$ contains the element $\tau \cdot x$ divisible by τ.

If $s < \alpha(k)$, we suppose that $\tau e_0^s \mathbf{g}^{k-s}$ contains an element $\tau \cdot y$ divisible by τ. The element $\tau \cdot y$ maps to $\tau e_0^s g^{k-s}$ in $\mathrm{Ext}_{\mathbf{A}(2)}$. Then y maps to $e_0^s g^{k-s}$ in $\mathrm{Ext}_{\mathbf{A}(2)}$. In other words, y is an element of the set $e_0^s \mathbf{g}^{k-s}$. However, since $s < \alpha(k)$, the set $e_0^s \mathbf{g}^{k-s}$ is empty by Proposition 4.9. □

Theorem 4.19. *Suppose that $e_0 \mathbf{g}^{2^n-1}$ and $\Delta h_1 e_0 \mathbf{g}^{2^n-2}$ are non-empty for every non-negative integer n. Then at filtration $f = 4k+1$ for $k \geqslant 2$ the set $\tau \Delta h_1 e_0^s \mathbf{g}^{k-s-1}$ contains an element divisible by τ if $s \geqslant \alpha(k)$ and does not contain any element divisible by τ if $s < \alpha(k)$.*

Proof. If $s \geqslant \alpha(k)$, then by Theorem 4.17 the set $\Delta h_1 e_0^s \mathbf{g}^{k-s-1}$ contains an element x. Then $\tau\Delta h_1 e_0^s \mathbf{g}^{k-s-1}$ contains the element $\tau \cdot x$ divisible by τ.

If $s < \alpha(k)$, we suppose that $\tau\Delta h_1 e_0^s \mathbf{g}^{k-s-1}$ contains an element $\tau \cdot y$ divisible by τ. The element $\tau \cdot y$ maps to $\tau\Delta h_1 e_0^s g^{k-s-1}$ in $\mathrm{Ext}_{\mathbf{A}(2)}$. Then the element y maps to $\Delta h_1 e_0^s g^{k-s-1}$ in $\mathrm{Ext}_{\mathbf{A}(2)}$. In other words, y is an element of the set $\Delta h_1 e_0^s \mathbf{g}^{k-s-1}$. However, since $s < \alpha(k)$, the set $\Delta h_1 e_0^s \mathbf{g}^{k-s-1}$ is empty by Proposition 4.14. □

Corollary 4.20. *At filtration $f = 4k+1$ and $f = 4k$ for $k \geqslant 2$ and $s < \alpha(k)$, all elements in the sets $\tau e_0^s \mathbf{g}^{k-s}$ and $\tau\Delta h_1 e_0^s \mathbf{g}^{k-s-1}$ are optimal.*

4.4. The wedge chart

Figure 1 shows the wedge from its vertex to stem 70.

- Dots and open circles indicate copies of $\mathbb{M}_2$.
- Dots indicate elements which behave irregularly, as in Propositions 4.9 and 4.14.
- Open circles indicate elements which behave regularly.

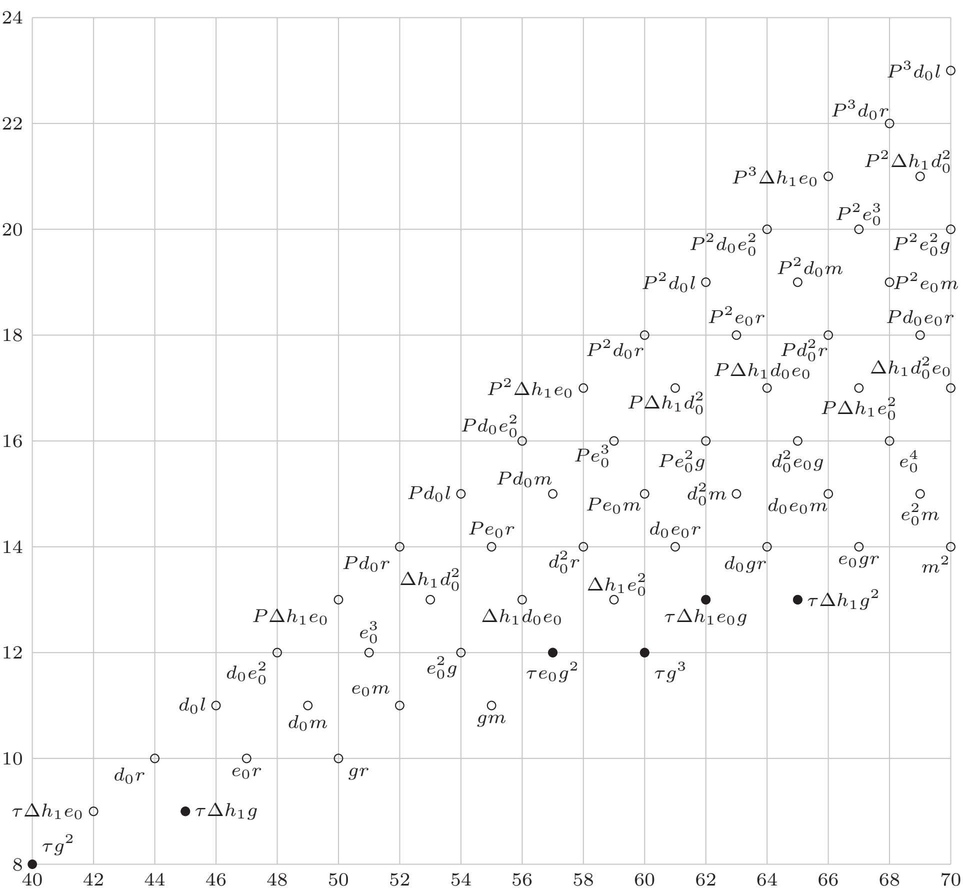

Figure 1: The $\mathbb{C}$-motivic wedge through the 70-stem.

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