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A SIMPLICIAL CONSTRUCTION FOR NONCOMMUTATIVE SETTINGS

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Abstract

In this paper we present a general construction that can be used to define the higher order Hochschild homology for a noncommutative algebra. We also discuss other examples where this construction can be used.

1. Introduction

Higher order Hochschild homology, $H_n^{\mathbf{X}_\bullet}(A,M)$, was introduced by Pirashvili in [**13**]. It is associated to a commutative k-algebra A, a symmetric bimodule M, and a simplicial set $\mathbf{X}_\bullet$. When the simplicial set $\mathbf{X}_\bullet$ models S^1 with the usual simplicial structure, one recovers the usual Hochschild homology. The cohomology version of this construction was introduced by Ginot in [**5**].

Secondary (co)homology of a triple (A,B,ε) was introduced in [**14**]. In order for the construction to work we must have that the morphism $\varepsilon\colon B\to A$ gives a B-algebra structure on A, and, in particular, B must be commutative.

As noted above, higher order Hochschild (co)homology is defined only for commutative k-algebras, while Hochschild (co)homology is defined for any k-algebra. The problem comes from the fact that for a general simplicial set $(\mathbf{X}_\bullet, d_i, s_i)$ we do not have a natural order on the fibers of the maps d_i. This means that there is a choice to be made when we define the pre-simplicial k-module corresponding to higher order Hochschild (co)homology. One possible approach for this problem is to restrict ourselves to those simplicial sets that do have a natural order on the fibers of d_i. However, this approach does not provide a lot of new examples since any such simplicial set must be of dimension one (see [**1**]).

A similar problem appears when we want to define the secondary (co)homology of a triple (A,B,ε), and the k-algebra B is not commutative. There is a choice to be made when one wants to write the formulas for the simplicial maps, and none of those choices give a simplicial module (unless B is commutative).

In this paper we present a construction that allows us to extend several homological constructions to noncommutative settings. For this we use the simplicial nature of the higher order Hochschild (co)homology. First, we show that to a so called $\mathbf{\Lambda}$-system we can associate a unique maximal pre-simplicial module. Then we construct several natural examples of $\mathbf{\Lambda}$-systems. In particular, we associate one such $\mathbf{\Lambda}$-system

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to a simplicial set $\mathbf{X}_\bullet$, a k-algebra A and an A-bimodule M. Then we consider the associated pre-simplicial module and take its homology. When A is commutative and M is A-symmetric we recover the usual higher Hochschild homology $H_n^{\mathbf{X}_\bullet}(A,M)$. Our construction can also be used to define a secondary homology in the noncommutative setting.

We discuss in detail the case when $\mathbf{X}_\bullet$ is modeled by S^1. We show that if A is a commutative k-algebra and M is a symmetric A-bimodule, then $H_n^{S^1}(A,M) \simeq H_n^{S^1}(M_l(A), M_l(M))$, and therefore we have Morita invariance for this case. In the last section we give an account of other related problems and some open questions.

2. Preliminaries

In this paper k is a field, $\otimes$ is $\otimes_k$, all maps are k-linear, and all algebras have multiplicative unit. Furthermore, we set the notation $\mathbb{N} = \{0,1,2,\dots\}$. We recall a few facts and definitions that will be useful in the upcoming sections.

We say that $(\mathbf{X}_\bullet, d_i)$ is a pre-simplicial object (others say semi-simplicial) in a category $\mathcal{C}$ if for every $n \in \mathbb{N}$, we have an object $X_n \in \mathcal{C}$, and for all $0 \leqslant i \leqslant n$ we have morphisms $\delta_i \colon X_{n+1} \to X_n$ that satisfy the following relation:

$$\delta_i\delta_j = \delta_{j-1}\delta_i \quad \text{if } i < j.$$

When $\mathcal{C}$ is the category of vector spaces over k, we say that $(\mathbf{X}_\bullet, d_i)$ is a pre-simplicial k-module.

Let A be a k-algebra (not necessarily commutative), and M be an A-bimodule. We consider the pre-simplicial module $(C_n(A,M), d_i)$ that is used to define Hochschild homology. That is $C_n(A,M) = M \otimes A^{\otimes n}$ and

$$d_i(x_0 \otimes \cdots \otimes x_n) = \begin{cases} x_0x_1 \otimes x_2 \otimes \cdots \otimes x_n & \text{if } i = 0, \\ x_0 \otimes \cdots \otimes x_{i-1} \otimes x_ix_{i+1} \otimes x_{i+2} \otimes \cdots \otimes x_n & \text{if } 1 \leqslant i \leqslant n-1, \\ x_nx_0 \otimes x_1 \otimes \cdots \otimes x_{n-1} & \text{if } i = n. \end{cases}$$

For more results concerning Hochschild (co)homology, we refer to [**3**], [**4**], [**9**], and [**12**].

We recall from [**13**] the construction of the higher order Hochschild homology. Assume that A is a commutative k-algebra, and M a symmetric A-bimodule.

Let V be a finite pointed set such that $|V| = v+1$. We define $\mathcal{L}(A,M)(V) = M \otimes A^{\otimes v}$. For $\phi \colon V \to W$ we define

$$\mathcal{L}(A,M)(\phi) \colon \mathcal{L}(A,M)(V) \to \mathcal{L}(A,M)(W)$$

determined as follows:

$$\mathcal{L}(A,M)(\phi)(a_0 \otimes a_1 \otimes \cdots \otimes a_v) = b_0m \otimes b_1 \otimes \cdots \otimes b_w,$$

where

$$b_i = \prod_{\{j \in V \mid \phi(j) = i\}} a_j.$$

Take $\mathbf{X} = (\mathbf{X}_\bullet, d_i, s_i)$ to be a finite pointed simplicial set, and define

$$C_n^{\mathbf{X}_\bullet}(A,M) = \mathcal{L}(A,M)(X_n).$$

For each $d_i \colon X_n \to X_{n-1}$ we take $(d_i)_* = \mathcal{L}(A,M)(d_i) \colon C_n^{\mathbf{X}_\bullet}(A,M) \to C_{n-1}^{\mathbf{X}_\bullet}(A,M)$

and take $\partial_n\colon C_n^{\mathbf{X}_\bullet}(A,M) \to C_{n-1}^{\mathbf{X}_\bullet}(A,M)$ defined as $\partial_n = \sum_{i=0}^{n}(-1)^i(d_i)_*$.

The homology of this complex is denoted by $H_\bullet^{\mathbf{X}}(A,M)$ and is called the higher order Hochschild homology. When $\mathbf{X}_\bullet$ is modeled by S^1 with the usual simplicial structure, one recovers the complex that defines Hochschild homology. For more results concerning higher order Hochschild (co)homology we refer to [**5**], [**6**], [**7**], and [**13**].

Secondary cohomology was introduced in [**14**] in order to study B-algebra structures on $A[[t]]$. The homology version and the associated cyclic (co)homology were introduced and studied in [**11**]. The relation between the secondary and higher order Hochschild cohomology was established in [**2**].

3. A simplicial construction for noncommutative settings

In this section we give a general construction that is designed to construct pre-simplicial modules in noncommutative settings. Its practical relevance will become apparent in the next section, when we use it to define several (co)homology theories for noncommutative algebras. Fixing the notation $\mathbb{N}^* = \{1,2,3,\ldots\}$, we first consider several definitions.

Definition 3.1. Suppose that for each $n \in \mathbb{N}^*$ and for each $0 \leqslant i \leqslant n$, we have a finite nonempty set Λ_n^i. We call such a collection a **Δ-indexing set**, and we denote it by $\boldsymbol{\Lambda} = \{\Lambda_n^i \mid n \in \mathbb{N}^*,\ i = 0,\ldots,n\}$.

Definition 3.2. Let $\boldsymbol{\Lambda} = \{\Lambda_n^i \mid n \in \mathbb{N}^*,\ i = 0,\ldots,n\}$ be a Δ-indexing set. We call $\mathcal{M} = (M_n, d_i^\alpha)$ a **$\boldsymbol{\Lambda}$-system** if it consists of a collection of k-vector spaces $\{M_n\}_{n=0}^{\infty}$, and a collection of k-linear morphisms $d_i^\alpha\colon M_n \to M_{n-1}$ for all $\alpha \in \Lambda_n^i$.

Note that if we have a pre-simplicial k-module (M_n, d_i) then we can get $\boldsymbol{\Lambda}$-system (M_n, d_i^α) by taking $|\Lambda_n^i| = 1$ for all $n \in \mathbb{N}^*$, and all $0 \leqslant i \leqslant n$ and defining $d_i^\alpha = d_i$. However, in general, a $\boldsymbol{\Lambda}$-system does not automatically define a pre-simplicial k-module or a chain complex. The plan is to prove that every $\boldsymbol{\Lambda}$-system contains a unique maximal pre-simplicial k-module.

Definition 3.3. Let $\mathcal{M} = (M_n, d_i^\alpha)$ be a $\boldsymbol{\Lambda}$-system. We call $A_\bullet = (A_n)_{n\geqslant 0}$ a **λ-subcomplex** of the $\boldsymbol{\Lambda}$-system $\mathcal{M}$ if A_n is a sub-vector-space of M_n for every n, and for $0 \leqslant i \leqslant n$ we have

(i) $d_i^\alpha|_{A_n} = d_i^\beta|_{A_n}$ for all $\alpha,\beta \in \Lambda_n^i$, with this common restriction denoted d_i^A,

(ii) $d_i^A(A_n) \subseteq A_{n-1}$, and

(iii) $d_i^A d_j^A = d_{j-1}^A d_i^A$ for $i < j$.

Remark 3.4. Notice that (ii) and (iii) imply that (A_n, d_i^A) is a pre-simplicial module and, in particular, we get a chain complex (hence the name λ-subcomplex).

Remark 3.5. Let $\mathcal{M}$ be a $\boldsymbol{\Lambda}$-system, and $\mathcal{S}$ denote the collection of all λ-subcomplexes. Since $\{0\} \subseteq \mathcal{S}$, it is clear that $\mathcal{S} \neq \varnothing$. We impart a partial order on $\mathcal{S}$ by saying $A_\bullet \leqslant B_\bullet$ if there exists inclusions $A_n \subseteq B_n$ in every dimension n. Notice that both $d_i^A = d_i^\alpha|_{A_n}$ and $d_i^B = d_i^\alpha|_{B_n}$, so since $A_n \subseteq B_n$, we have $d_i^B|_{A_n} = d_i^A$.

Theorem 3.6. *Let $\mathcal{M}$ be a $\mathbf{\Lambda}$-system. Then $\mathcal{M}$ has a unique maximal λ-subcomplex. In particular, we have a unique maximal pre-simplicial k-module $\Theta(\mathcal{M})$.*

Proof. First, we show the existence of a maximal λ-subcomplex. To do this, we shall use Zorn's Lemma. Consider a countable, totally ordered subset of $\mathcal{S}$, i.e. $\{A^m_\bullet\}_{m=0}^\infty$ with $A^m_\bullet \leqslant A^{m+1}_\bullet$.

Claim: $\Omega_\bullet := \bigcup_{m=1}^\infty A^m_\bullet$ (where $\Omega_n = \bigcup\limits_{m\geqslant 1} A^m_n$) is a λ-subcomplex.

(i): Indeed, if $a \in \Omega_n$, then $a \in A^m_n$ for some m. Then for all $\alpha, \beta \in \Lambda^i_n$, $d^\alpha_i(a) = d^\beta_i(a)$ since $A^m_\bullet$ is a λ-subcomplex. Thus (i) is satisfied.

(ii): Similarly, as $a \in \Omega_n$ is also in some A^m_n, $d^\Omega_i(a) = d^{A^m}_i(a) \in A^m_{n-1} \subseteq \Omega_{n-1}$. Hence (ii) is satisfied.

(iii): As above, take $a \in \Omega_n$. Then $a \in A^m_n$ for some m, so

$$d^\Omega_i(d^\Omega_j(a)) = d^{A_m}_i(d^{A_m}_j(a)) = d^{A_m}_{j-1}(d^{A_m}_i(a)) = d^\Omega_{j-1}(d^\Omega_i(a)),$$

therefore (iii) holds.

Thus, $\Omega_\bullet$ is a λ-subcomplex (i.e. $\Omega_\bullet \in \mathcal{S}$). Now, being the union of all $A^m_\bullet$, each $A^m_\bullet \leqslant \Omega_\bullet$, so this is indeed an upper bound of the totally ordered subset $\{A^m_\bullet\}_{m=0}^\infty$. Thus, by Zorn's Lemma, there exists a maximal element of $\mathcal{S}$.

Now we show there is a unique maximal λ-subcomplex. Suppose there are two maximal λ-subcomplexes, $C_\bullet$ and $D_\bullet$. Consider $Y_\bullet := C_\bullet + D_\bullet$, where Y_n as a k-vector space is $C_n + D_n = \{y \in M_n \mid y = c + d, \text{ for some } c \in C_n,\ d \in D_n\}$. We show that $Y_\bullet$ is a λ-subcomplex.

(i): Take $y \in Y_n$. Then $y = c + d$ for some $c \in C_n$ and $d \in D_n$. So for all $\alpha, \beta \in \Lambda^i_n$, we have

$$d^\alpha_i(y) = d^\alpha_i(c + d) = d^\alpha_i(c) + d^\alpha_i(d) = d^\beta_i(c) + d^\beta_i(d) = d^\beta_i(c + d) = d^\beta_i(y).$$

This shows (i).

(ii): If $y \in Y_n$, then $y = c + d$ for some $c \in C_n$ and $d \in D_n$, so

$$d^Y_i(y) = d^Y_i(c + d) = d^Y_i(c) + d^Y_i(d) = d^C_i(c) + d^D_i(d) \in C_{n-1} + D_{n-1} = Y_{n-1}.$$

Hence (ii) holds.

(iii): Let $i < j$, and take $y \in Y_n$ with $y = c + d$ for some $c \in C_n$ and $d \in D_n$. Since $C_\bullet$ and $D_\bullet$ satisfy (iii) and using the observation in the proof for (ii), we have:

$$\begin{aligned} d^Y_i(d^Y_j(y)) &= d^Y_i(d^Y_j(c + d)) = d^Y_i(d^C_j(c) + d^D_j(d)) = d^Y_i(d^C_j(c)) + d^Y_i(d^D_j(d)) \\ &= d^C_i(d^C_j(c)) + d^D_i(d^D_j(d)) = d^C_{j-1}(d^C_i(c)) + d^D_{j-1}(d^D_i(d)) \end{aligned}$$

and on the other hand,

$$\begin{aligned} d^Y_{j-1}(d^Y_i(y)) &= d^Y_{j-1}(d^Y_i(c + d)) = d^Y_{j-1}(d^C_i(c) + d^D_i(d)) \\ &= d^Y_{j-1}(d^C_i(c)) + d^Y_{j-1}(d^D_i(d)) = d^C_{j-1}(d^C_i(c)) + d^D_{j-1}(d^D_i(d)). \end{aligned}$$

Thus, $d^Y_i d^Y_j = d^Y_{j-1} d^Y_i$, so (iii) is satisfied.

Therefore, $Y_\bullet$ is a λ-subcomplex. Clearly there are injections $C_n \hookrightarrow Y_n$ and $D_n \hookrightarrow Y_n$, but they were chosen to be maximal, so it must be that $C_\bullet = Y_\bullet = D_\bullet$. Hence, a maximal λ-subcomplex is unique, and we denote it by $\Theta(\mathcal{M})$. □

Definition 3.7. Let $\mathcal{M}$ be a $\mathbf{\Lambda}$-system with unique maximal λ-subcomplex $\Theta(\mathcal{M})$. We call the homology of $\Theta(\mathcal{M})$ the **$\mathbf{\Lambda}$-homology group of** $\mathcal{M}$, and we denote it by $H_n(\mathcal{M}) := H_n(\Theta(\mathcal{M})_\bullet)$.

Next, we need to talk about morphisms between $\mathbf{\Lambda}$-systems.

Definition 3.8. Take $\mathbf{\Gamma}$ and $\mathbf{\Lambda}$ to be two Δ-indexing sets, $\mathcal{M} = (M_n, d_i^\beta)$ a $\mathbf{\Gamma}$-system, and $\mathcal{N} = (N_n, \delta_i^\alpha)$ a $\mathbf{\Lambda}$-system. A **λ-morphism** from $\mathcal{M}$ to $\mathcal{N}$ is a collection of k-linear maps $f_n\colon M_n \to N_n$ for all $n \in \mathbb{N}$, such that if $n \geqslant 1$, then for all $0 \leqslant i \leqslant n$, and all $\alpha \in \Lambda_n^i$ there exists a $\beta \in \Gamma_n^i$ such that $\delta_i^\alpha f_n = f_{n-1} d_i^\beta$.

We have the following result.

Lemma 3.9. *Take $\mathbf{\Gamma}$ and $\mathbf{\Lambda}$ to be two Δ-indexing sets, $\mathcal{M} = (M_n, d_i^\beta)$ a $\mathbf{\Gamma}$-system, and $\mathcal{N} = (N_n, \delta_i^\alpha)$ a $\mathbf{\Lambda}$-system. If $f\colon \mathcal{M} \to \mathcal{N}$ is a λ-morphism then f induces a morphism of pre-simplicial modules $f\colon \Theta(\mathcal{M}) \to \Theta(\mathcal{N})$.*

Proof. First we show that $f_n(\Theta(\mathcal{M})_n) \subseteq \Theta(\mathcal{N})_n$. If $n = 0$, then this is obvious since $\Theta(\mathcal{N})_0 = N_0$. Define $A_s \subseteq N_s$ determined by

$$A_s = f_s(\Theta(\mathcal{M})_s) \quad \text{for all } s \geqslant 0.$$

We want to show that $A_\bullet = (A_s)_{s\geqslant 0}$ defines a λ-subcomplex in $\mathcal{N}$.

Because $(\Theta(\mathcal{M})_s)_{s\geqslant 0}$ is a λ-subcomplex of $\mathcal{M}$, then for all β_1, β_2 in Γ_s^i we have that $d_i^{\beta_1} = d_i^{\beta_2}$ on $\Theta(\mathcal{M})_s$. We will denote this map by d_i (suppressing the s index).

Take $n \geqslant 1$ and $0 \leqslant i \leqslant n$. Since f is a λ-morphism then for α_1, $\alpha_2 \in \Lambda_n^i$ we can find $\beta_1, \beta_2 \in \Gamma_n^i$ such that $\delta_i^{\alpha_1} f_n = f_{n-1} d_i^{\beta_1}$ and $\delta_i^{\alpha_2} f_n = f_{n-1} d_i^{\beta_2}$. Take $x \in A_n$ with $x = f_n(c)$ for some $c \in \Theta(\mathcal{M})_n$. We have

$$\delta_i^{\alpha_1}(x) = \delta_i^{\alpha_1}(f_n(c)) = f_{n-1}(d_i^{\beta_1}(c)) = f_{n-1}(d_i(c)) = f_{n-1}(d_i^{\beta_2}(c)) = \delta_i^{\alpha_2}(f_n(c)) = \delta_i^{\alpha_2}(x),$$

which means that $\delta_i^{\alpha_1} = \delta_i^{\alpha_2}$ on A_n for all α_1, $\alpha_2 \in \Lambda_n^i$. And so we have condition (i) from Definition 3.3. We denote the common restriction by δ_i^A.

Take $x \in A_n$ with $x = f_n(c)$ for some $c \in \Theta(\mathcal{M})_n$. Then we have

$$\delta_i^A(x) = \delta_i^A f_n(c) = \delta_i^\alpha f_n(c) = f_{n-1} d_i^\beta(c) = f_{n-1} d_i(c) \in A_{n-1}$$

for some $\alpha \in \Lambda_n^i$ (and the corresponding $\beta \in \Gamma_n^i$). This means that

$$\delta_i^A(A_n) \subseteq A_{n-1},$$

and so we have condition (ii) from Definition 3.3.

Finally, for all $i < j$, and $x = f_n(c)$ for some $c \in \Theta(\mathcal{M})_n$ we have

$$\delta_i^A \delta_j^A(x) = \delta_i^A \delta_j^A(f_n(c)) = f_{n-2}(d_i d_j(c)) = f_{n-2}(d_{j-1} d_i(c)) = \delta_{j-1}^A \delta_i^A(f_n(c)) = \delta_{j-1}^A \delta_i^A(x).$$

Thus $(A_s)_{s\geqslant 0}$ defines an λ-subcomplex, and so we get that $A_n = f_n(\Theta(\mathcal{M})_n) \subseteq \Theta(\mathcal{N})_n$ for all $n \in \mathbb{N}$.

We already noticed that $\delta_i^A f_n = f_{n-1} d_i$, which means that f is a morphism of pre-simplicial modules from $\Theta(\mathcal{M})$ to $\Theta(\mathcal{N})$. □

4. A few examples

4.1. Higher order Hochschild homology

Let A be a k-algebra (not necessarily commutative), and M be an A-bimodule. As a warm-up, we define higher order Hochschild homology when the simplicial set models the sphere S^2. We will use the description from [**10**] as a point of reference.

Example 4.1. Set $\Lambda_n^0 = \{\rho \mid \rho \in \widehat{S_n}\}$, where we let $\widehat{S_n}$ be the set of permutations of $\{0, 2, \dots, n\}$. For $0 < i < n$, set

$$\Lambda_n^i = \{\sigma = (\sigma_1, \dots, \sigma_{n-1}) \mid \sigma_j \in \{1, \tau\}\},$$

and $\Lambda_n^n = \{\rho \mid \rho \in S_n\}$, where S_n is the set of permutations of $\{0, \dots, n-1\}$. Notice that $\sigma_j = 1$ represents usual multiplication and $\sigma_j = \tau$ represents transpose multiplication, as seen below.

Let ρ act on a tensor product of length n by permuting the elements according to ρ and then taking the product (i.e. $\rho(x_0 \otimes x_2 \otimes \cdots \otimes x_n) = x_{\rho(0)} x_{\rho(2)} \cdots x_{\rho(n)}$). Take

$$\sigma_j(x \otimes y) = \begin{cases} xy & \text{if } \sigma_j = 1, \\ yx & \text{if } \sigma_j = \tau. \end{cases}$$

Observe that σ is an $(n-1)$-tuple consisting of usual or transpose multiplication rules (the σ_j's). We define a $\mathbf{\Lambda}$-system $\mathcal{F}$ by taking $F_n = M \otimes A^{\otimes \frac{n(n-1)}{2}}$, and $d_i^\sigma \colon M \otimes A^{\otimes \frac{n(n-1)}{2}} \to M \otimes A^{\otimes \frac{(n-1)(n-2)}{2}}$ defined as follows:

$$d_0^\rho(m_0 \otimes \begin{pmatrix} 1 & a_{1,2} & \cdots & a_{1,n} \\ & \ddots & \ddots & \vdots \\ & & 1 & a_{n-1,n} \\ & & & 1 \end{pmatrix}) = \rho(m_0 \otimes a_{1,2} \otimes \cdots \otimes a_{1,n}) \otimes \begin{pmatrix} 1 & a_{2,3} & \cdots & a_{2,n} \\ & \ddots & \ddots & \vdots \\ & & 1 & a_{n-1,n} \\ & & & 1 \end{pmatrix}.$$

For $1 \leqslant i \leqslant n-1$,

$$d_i^\sigma(m_0 \otimes \begin{pmatrix} 1 & a_{1,2} & \cdots & a_{1,n} \\ & \ddots & \ddots & \vdots \\ & & 1 & a_{n-1,n} \\ & & & 1 \end{pmatrix}) = \sigma_i(m_0 \otimes a_{i,i+1})$$

$$\otimes \begin{pmatrix} 1 & a_{1,2} & \cdots & \sigma_1(a_{1,i} \otimes a_{1,i+1}) & a_{1,i+2} & \cdots & a_{1,n} \\ & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ & & 1 & \sigma_{i-1}(a_{i-1,i} \otimes a_{i-1,i+1}) & a_{i-1,i+2} & \cdots & a_{i-1,n} \\ & & & 1 & \sigma_{i+1}(a_{i,i+2} \otimes a_{i+1,i+2}) & \cdots & \sigma_{n-1}(a_{i,n} \otimes a_{i+1,n}) \\ & & & & \ddots & \ddots & \vdots \\ & & & & & 1 & a_{n-1,n} \\ & & & & & & 1 \end{pmatrix}.$$

Finally,

$$d_n^\rho(m_0 \otimes \begin{pmatrix} 1 & a_{1,2} & \cdots & a_{1,n} \\ & \ddots & \ddots & \vdots \\ & & 1 & a_{n-1,n} \\ & & & 1 \end{pmatrix}) = \rho(m_0 \otimes a_{1,n} \otimes \cdots \; a_{n-1,n}) \otimes \begin{pmatrix} 1 & a_{1,2} & \cdots & a_{1,n-1} \\ & \ddots & \ddots & \vdots \\ & & 1 & a_{n-2,n-1} \\ & & & 1 \end{pmatrix}.$$

Notice that when A is commutative and M is A-symmetric we get the usual higher order Hochschild homology $H_n^{S^2}(A, M)$.

Next, we want to define higher order Hochschild homology for a general simplicial set.

Example 4.2. Let $\mathbf{X} = (\mathbf{X}_\bullet, d_i, s_i)$ be a pointed simplicial set. Consider the Δ-indexing set $\mathbf{\Lambda}^{\mathbf{X}_\bullet}$ defined by

$$\Lambda_n^i = \prod_{j \in X_{n-1}} S_{Z_n^i(j)},$$

where S_Z is the symmetric group on the set Z, and for $j \in X_{n-1}$ we set $Z_n^i(j) = d_i^{-1}(j)$ where $d_i \colon X_n \to X_{n-1}$.

Let A be a k-algebra and M an A-bimodule. We define the $\mathbf{\Lambda}^{\mathbf{X}_\bullet}$-system $\mathcal{C}^{\mathbf{X}_\bullet}(A, M)$ as follows. For each n define $C_n^{\mathbf{X}_\bullet}(A, M) = M \otimes A^{\otimes x_n}$ where $x_n = |X_n| - 1$. For $\sigma = (\sigma_0, \sigma_1, \dots, \sigma_{x_{n-1}}) \in \Lambda_n^i$ we define $d_i^\sigma : C_n^{\mathbf{X}_\bullet}(A, M) \to C_{n-1}^{\mathbf{X}_\bullet}(A, M)$ determined by

$$d_i^\sigma(a_0 \otimes a_1 \otimes \cdots \otimes a_{x_n}) = b_0^\sigma \otimes b_1^\sigma \otimes \cdots \otimes b_{x_{n-1}}^\sigma,$$

where for $j \in X_{n-1}$ we define

$$b_j^\sigma = \prod_{\{s \in X_n \ | \ d_i(s) = j\}} a_{\sigma(s)}.$$

In the last formula the product is ordered over s. Notice that the order that we pick on $Z_n^i(j)$ is not important, we just want to make sure that we cover all the possible ordered products.

As one expects, if A is commutative and M is a symmetric A-bimodule we get the usual higher order Hochschild homology $H_\bullet^{\mathbf{X}}(A, M)$.

Example 4.3. Take A a commutative k-algebra, and M a symmetric A-bimodule. Take $e \in M_l(A)$, and $m \in M_l(B)$ such that $e^2 = e$, and $em = me = m$. Consider the element

$$W_n(e, m) = m \otimes e^{\otimes x_n} \in C_n^{\mathbf{X}_\bullet}(A, M).$$

Notice that $d_i^\alpha(W_n(e, m)) = W_{n-1}(e, m)$, which means that if we define $C_n = kW_n(e, m)$ we get a λ-subcomplex, and so $W_n(e, m) \in \Theta(\mathcal{C}^{\mathbf{X}_\bullet}(A, M))_n$.

Remark 4.4. Notice that the $\mathbf{\Lambda}$-system from Example 4.2 is completely determined by A, M and the simplicial set $\mathbf{X}$. We denote the homology groups $H_n(\Theta(\mathcal{C}^{\mathbf{X}}(A, M)))$ by $H_n^{\mathbf{X}}(A, M)$. When A is commutative and M is a symmetric A-bimodule, we recover the higher order Hochschild homology, so this notation is consistent with [**13**]. When the simplicial set models the sphere S^2 with the usual simplicial structure (see [**10**]), we recover Example 4.1.

4.2. Secondary Hochschild homology

The next example is associated with the secondary Hochschild homology, denoted $HH_\bullet(A, B, \varepsilon)$. Recall that in [**11**] we need A to be a B-algebra, and, in particular, B must be commutative. Using the construction from the previous section, we are able to drop that condition.

Example 4.5. Let A and B be k-algebras, and $\varepsilon\colon B \to A$ be a k-algebra morphism. Here we do not assume B is commutative. Take $\mathbf{\Lambda} = \{\Lambda_n^i\}$ as follows: for $0 \leqslant i \leqslant n-1$

$$\Lambda_n^i = \{\sigma = (\sigma_0, \sigma_1, \dots, \sigma_i, \dots, \sigma_{n-1}) \mid \sigma_j \in \{1, \tau\} \text{ for } j \neq i,\ \sigma_i \in \{l, c, r\}\},$$

and

$$\Lambda_n^n = \{\sigma = (\sigma_0, \sigma_1, \dots, \sigma_i, \dots, \sigma_{n-1}) \mid \sigma_j \in \{1, \tau\} \text{ for } j \neq 0,\ \sigma_0 \in \{l, c, r\}\}.$$

We define a $\mathbf{\Lambda}$-system $\mathcal{E}(A, B, \varepsilon)$ where we set

$$E_n(A, B, \varepsilon) = A^{\otimes n+1} \otimes B^{\otimes \frac{n(n+1)}{2}}.$$

For $0 \leqslant i \leqslant n-1$ and $\sigma \in \Lambda_n^i$ define $d_i^\sigma\colon A^{\otimes n+1} \otimes B^{\otimes \frac{n(n+1)}{2}} \to A^{\otimes n} \otimes B^{\otimes \frac{n(n-1)}{2}}$ given by

$$d_i^\sigma(\otimes \begin{pmatrix} a_0 & b_{0,1} & \cdots & b_{0,n-1} & b_{0,n} \\ & a_1 & \cdots & b_{1,n-1} & b_{1,n} \\ & & \ddots & \vdots & \vdots \\ & & & a_{n-1} & b_{n-1,n} \\ & & & & a_n \end{pmatrix}) =$$

$$\otimes \begin{pmatrix} a_0 & \cdots & b_{0,i-1} & \sigma_0(b_{0,i} \otimes b_{0,i+1}) & b_{0,i+2} & \cdots & b_{0,n} \\ & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ & & a_{i-1} & \sigma_{i-1}(b_{i-1,i} \otimes b_{i-1,i+1}) & b_{i-1,i+2} & \cdots & b_{i-1,n} \\ & & & \sigma_i(a_i \otimes a_{i+1} \otimes b_{i,i+1}) & \sigma_{i+1}(b_{i,i+2} \otimes b_{i+1,i+2}) & \cdots & \sigma_{n-1}(b_{i,n} \otimes b_{i+1,n}) \\ & & & & a_{i+2} & \cdots & b_{i+2,n} \\ & & & & & \ddots & \vdots \\ & & & & & & a_n \end{pmatrix}.$$

Where, for $j \neq i$ we have

$$\sigma_j(b_1 \otimes b_2) = \begin{cases} b_1 b_2 & \text{if } \sigma_j = 1 \in \Lambda_n^i, \\ b_2 b_1 & \text{if } \sigma_j = \tau \in \Lambda_n^i \end{cases}$$

for all $b_1, b_2 \in B$, and

$$\sigma_i(a_1 \otimes a_2 \otimes b) = \begin{cases} \varepsilon(b) a_1 a_2 & \text{if } \sigma_i = l \in \Lambda_n^i, \\ a_1 \varepsilon(b) a_2 & \text{if } \sigma_i = c \in \Lambda_n^i, \\ a_1 a_2 \varepsilon(b) & \text{if } \sigma_i = r \in \Lambda_n^i \end{cases}$$

for all $a_1, a_2 \in A$ and $b \in B$. Finally, for $i = n$ we have

$$d_n^\sigma(\otimes \begin{pmatrix} a_0 & b_{0,1} & \cdots & b_{0,n-1} & b_{0,n} \\ & a_1 & \cdots & b_{1,n-1} & b_{1,n} \\ & & \ddots & \vdots & \vdots \\ & & & a_{n-1} & b_{n-1,n} \\ & & & & a_n \end{pmatrix}) =$$

$$\otimes \begin{pmatrix} \sigma_0(a_n \otimes a_0 \otimes b_{0,n}) & \sigma_1(b_{0,1} \otimes b_{1,n}) & \cdots & \sigma_{n-2}(b_{0,n-2} \otimes b_{n-2,n}) & \sigma_{n-1}(b_{0,n-1} \otimes b_{n-1,n}) \\ & a_1 & \cdots & b_{1,n-2} & b_{1,n-1} \\ & & \ddots & \vdots & \vdots \\ & & & a_{n-2} & b_{n-2,n-1} \\ & & & & a_{n-1} \end{pmatrix}.$$

Remark 4.6. We denote the homology of $\Theta(\mathcal{E}(A,B,\varepsilon))$ by $HH_\bullet(A,B,\varepsilon)$ and call it the secondary homology of the triple (A,B,ε). Notice that if B is commutative and $\varepsilon(B)\subseteq\mathcal{Z}(A)$, we recover the usual secondary Hochschild homology $HH_*(A,B,\varepsilon)$ as defined in [**11**].

Example 4.7. Take B to be a commutative k-algebra, A to be a k-algebra, $\varepsilon\colon B\to A$ to be a morphism of k-algebras such that $\varepsilon(B)\subseteq\mathcal{Z}(A)$, and $\iota\colon M_l(B)\to M_l(A)$ to be the induced k-algebra morphism. Take $e\in M_l(A)$, and $f\in M_l(B)$ such that $e^2=e$, $f^2=f$, and $ef=fe=e$. Consider the element

$$T_n(e,f)=\otimes\begin{pmatrix} e & f & \cdots & f & f\\ & e & \cdots & f & f\\ & & \ddots & \vdots & \vdots\\ & & & e & f\\ & & & & e\end{pmatrix}\in E_n(M_l(A),M_l(B),\iota).$$

Notice that $d_i^\alpha(T_n(e,f))=T_{n-1}(e,f)$. This means that if we define $C_n=kT_n(e,f)$, we get a λ-subcomplex. In particular, $T_n(e,f)\in\Theta(\mathcal{E}(M_l(A),M_l(B),\iota))_n$.

5. Back to Hochschild homology

In this section we take A to be a commutative k-algebra, and M is a symmetric A-bimodule. For the matrix algebra $M_l(A)$ we have two possible different ways of defining Hochschild homology. We have the classical $H_n(M_l(A),M_l(M))$ (as in the preliminary section), and $H_n^{S^1}(M_l(A),M_l(M))$ (as in the previous section). We will show that the two constructions agree.

Recall the simplicial structure on S^1. Take $X_0=\{*_0\}$ and $X_n=\{*_n\}\cup\{I_b^a\mid a+b+1=n\}$ with

$$d_i(*_n)=*_{n-1},$$
$$d_i(I_b^a)=\begin{cases} *_{a+b} & \text{if } a=0 \text{ and } i=0,\\ I_b^{a-1} & \text{if } a\neq 0 \text{ and } i\leqslant a,\\ I_{b-1}^a & \text{if } b\neq 0 \text{ and } i>a,\\ *_{a+b} & \text{if } b=0 \text{ and } i=n=a+1,\end{cases}$$
$$s_i(*_n)=*_{n+1},$$
$$s_i(I_b^a)=\begin{cases} I_b^{a+1} & \text{if } i\leqslant a,\\ I_{b+1}^a & \text{if } i>a.\end{cases}$$

Next we give the details for the Δ-indexing set $\mathbf{\Lambda}^{S^1}$, as well as the $\mathbf{\Lambda}^{S^1}$-system $\mathcal{C}^{S^1}(M_l(A),M_l(M))$, as described in Example 4.2.

One can see that $|\mathbf{\Lambda}_n^i|=2$, so we can identify $\mathbf{\Lambda}_n^i$ with the set $\{1,\tau\}$. For all $n\in\mathbb{N}$ we have

$$C_n^{S^1}(M_l(A),M_l(M))=M_l(M)\otimes M_l(A)^{\otimes n}.$$

For $0\leqslant i\leqslant n$ and $\alpha\in\mathbf{\Lambda}_n^i$, we have $\delta_i^\alpha\colon C_n^{S^1}(M_l(A),M_l(M))\to C_{n-1}^{S^1}(M_l(A),M_l(M))$

determined by

$$\delta_i^1(x_0 \otimes \cdots \otimes x_n) = \begin{cases} x_0x_1 \otimes x_2 \otimes \cdots \otimes x_n & \text{if } i = 0, \\ x_0 \otimes \cdots \otimes x_{i-1} \otimes x_ix_{i+1} \otimes x_{i+2} \otimes \cdots \otimes x_n & \text{if } 1 \leqslant i \leqslant n-1, \\ x_nx_0 \otimes x_1 \otimes \cdots \otimes x_{n-1} & \text{if } i = n, \end{cases}$$

and

$$\delta_i^\tau(x_0 \otimes \cdots \otimes x_n) = \begin{cases} x_1x_0 \otimes x_2 \otimes \cdots \otimes x_n & \text{if } i = 0, \\ x_0 \otimes \cdots \otimes x_{i-1} \otimes x_{i+1}x_i \otimes x_{i+2} \otimes \cdots \otimes x_n & \text{if } 1 \leqslant i \leqslant n-1, \\ x_0x_n \otimes x_1 \otimes \cdots \otimes x_{n-1} & \text{if } i = n. \end{cases}$$

We are now ready to state the following result.

Proposition 5.1. *Let A be a commutative k-algebra and M a symmetric A-bimodule. Then we have*

$$H_n^{S^1}(A,M) \simeq H_n(A,M) \simeq H_n(M_l(A), M_l(M)) \simeq H_n^{S^1}(M_l(A), M_l(M)).$$

Proof. Since A is commutative and M is symmetric, the first isomorphism is known from [**13**]. Also, it is well known from [**12**] that the maps $i_A\colon A \to M_l(A)$ and $i_M\colon M \to M_l(M)$ determined by

$$x \to \begin{pmatrix} x & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

can be extended to a morphism of pre-simplicial modules

$$\iota_2\colon (C_n(A,M), d_i) \to (C_n(M_l(A), M_l(M)), d_i),$$

which induced an isomorphism at the level of homology. Thus Hochschild homology is Morita invariant, that is $H_n(A,M) \simeq H_n(M_l(A), M_l(M))$.

Take $\boldsymbol{\Upsilon}$ the trivial Δ-indexing set (i.e. $|\Upsilon_n^i| = 1$ for all n and all $0 \leqslant i \leqslant n$). Then every pre-simplicial module is a $\boldsymbol{\Upsilon}$-system. In particular, one observes that $\mathcal{C}(A,M)$ and $\mathcal{C}(M_l(A), M_l(M))$ are the $\boldsymbol{\Upsilon}$-systems associated to $(C_n(A,M), d_i)$ and $(C_n(M_l(A), M_l(M)), d_i)$, respectively.

Since A is commutative and M is symmetric, the maps $i_A\colon A \to M_l(A)$ and $i_M\colon M \to M_l(M)$ induce a λ-morphism $\mathcal{C}(A,M) \to \mathcal{C}^{S^1}(M_l(A), M_l(M))$ (as in Definition 3.8). By Lemma 3.9 we obtain a morphism of pre-simplicial modules

$$\iota_0\colon (C_n(A,M), d_i) \to (\Theta(\mathcal{C}^{S^1}(M_l(A), M_l(M)))_n, \delta_i).$$

It easy to check that the identity map $\mathcal{C}^{S^1}(M_l(A), M_l(M)) \to \mathcal{C}((M_l(A), M_l(M)))$ is a λ-morphism (as in Definition 3.8). Again by Lemma 3.9 this gives a morphism of pre-simplicial k-modules

$$\iota_1\colon (\Theta(\mathcal{C}^{S^1}(M_l(A), M_l(M)))_n, \delta_i) \to (C_n((M_l(A), M_l(M))), d_i).$$

Finally, we have that $\iota_2 = \iota_1\iota_0$, and since ι_2 induces an isomorphism in homology we get that ι_1 also induces an isomorphism $H_n^{S^1}(M_l(A), M_l(M)) \simeq H_n(M_l(A), M_l(M))$, which finishes the proof. □

6. Final remarks

The setting in Theorem 3.6 is quite general, and if applied to poorly chosen $\mathbf{\Lambda}$-systems, the theorem is not likely to give interesting results. One has to balance between Δ-indexing sets that are too big or too small.

The results from the previous section show that when $\mathbf{X}_\bullet$ is modeled by S^1, our construction of higher order Hochschild homology for noncommutative algebras behaves as one would hope. However, the proof depends heavily on the already known existence and properties of Hochschild homology for noncommutative algebras.

If A is a commutative k-algebra, M a symmetric A-bimodule, and $\mathbf{X}_\bullet$ a simplicial set one can show that we have a morphism $H_n^{\mathbf{X}_\bullet}(A, M) \to H_n^{\mathbf{X}_\bullet}(M_l(A), M_l(M))$. It would be interesting to prove that this morphism is actually an isomorphism (i.e. we have Morita invariance).

One can easily check the functoriality of $H^{\mathbf{X}_\bullet}(A, M)$. It would be interesting to see if the construction of $H_\bullet(\mathcal{M}^{\mathbf{X}}(A, M))$ is invariant under the homotopy equivalence of the simplicial set $\mathbf{X}$. Notice that we did not use the degeneracy maps of the simplicial set $\mathbf{X}$, but that information could be easily incorporated in some variation of Theorem 3.6 (that would deal with maximal simplicial modules instead of maximal pre-simplicial modules).

Similar constructions can be done if one wants to define higher order Hochschild cohomology, or for the generalized higher Hochschild (co)homology (see [**2**] or [**8**]).

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