Homology, Homotopy and Applications, vol. 22(2), 2020, pp.387–399

CONSTRUCTIONS OF SELF-MAPS OF SU(4) VIA POSTNIKOV TOWERS

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(*communicated by Donald M. Davis*)

Abstract

Cohomology operations restrict the degree of a self-map of SU(4) to be either odd or a multiple of 8; we find self-maps realizing these possible degrees. The notion of the degree of a self-map can be refined to a notion of *multidegree* which records the effect of the self-map on each of the generators of $H^\star(\mathrm{SU}(4))$. We find restrictions on the possible multidegrees of self-maps of SU(4) and, via Postnikov towers, build self-maps stage-by-stage realizing the possible multidegrees.

1. Introduction

For closed oriented n-dimensional manifolds M and N, let $D(M, N)$ be the set of degrees of maps $M \to N$, i.e.,

$$D(M, N) := \{\deg f \mid f\colon M \to N\}.$$

Often we are interested in $D(M) = D(M, N)$, the set of degrees of self-maps of M.

Much work has gone into studying $D(M, N)$ for high-dimensional manifolds, e.g., for certain six-manifolds [**2**] and for highly-connected manifolds [**4**]. The case of SU(3) is considered in recent work [**6, 15**] where it is shown that $d \in D(\mathrm{SU}(3))$ exactly when d is odd or a multiple of 4. This paper studies $D(\mathrm{SU}(4))$. Self-maps of SU(4) are not as well-studied as those of H-spaces with fewer cells, but there has been some prior work on self-maps SU(4), e.g., see [**11**] on the nilpotency of the group of self-maps $[\mathrm{SU}(4), \mathrm{SU}(4)]$ and see [**13**, Corollary 5.7] which exhibits self-maps of degree $8^m(2\ell + 1)$.

The structure of the cohomology ring of $\mathrm{SU}(n)$ makes it a particularly interesting choice when studying the degree of self-maps. Specifically, $H^\star(\mathrm{SU}(n); \mathbb{Z})$ is exterior on generators $x_3, x_5, \ldots, x_{2n-1}$. Therefore, a self-map $f\colon \mathrm{SU}(n) \to \mathrm{SU}(n)$ has a *multi-degree*.

Definition 1.1. For map $f\colon \mathrm{SU}(n) \to \mathrm{SU}(n)$, the multidegree of f is the tuple

$$(t_3, t_5, \ldots, t_{2n-1}),$$

where $f^\star(x_i) = t_i\, x_i$.

Received August 15, 2019; published on May 20, 2020.
2010 Mathematics Subject Classification: 55M25, 57T10, 57N65.
Key words and phrases: mapping degree sets, Postnikov towers.
Article available at `http://dx.doi.org/10.4310/HHA.2020.v22.n2.a23`

An investigation of $D(\mathrm{SU}(4))$ can therefore be refined to an investigation of the set of possible *multidegrees* of self-maps of $\mathrm{SU}(n)$.

Theorem A. *There is a map* $\phi\colon \mathrm{SU}(4) \to \mathrm{SU}(4)$ *of multidegree* (t_3, t_5, t_7) *if and only if* $t_3 \equiv t_5 \pmod 2$ *and* $t_3 \equiv t_7 \pmod 6$.

Observing that $\deg f = \prod_i t_i$ and performing some elementary number theory yields a corollary.

Corollary B. *There is a map* $\phi\colon \mathrm{SU}(4) \to \mathrm{SU}(4)$ *of degree d exactly when d is odd or a multiple of 8.*

In Section 2, we sketch the necessity of the congruences in Theorem A. Then Section 3 introduces the machinery that faciliates building maps stage-by-stage, e.g., Lemma 3.5 and Lemma 3.7. The proof of Theorem A is presented in Section 4, and Corollary B is discussed in Section 5. The paper ends by highlighting some future directions.

Conventions

Throughout the paper, "space" means a 1-connected CW-complex, and homology and cohomology with unspecified coefficients means the coefficient ring is $\mathbb{Z}$.

Acknowledgments

The authors warmly thank Jean-François Lafont for many helpful conversations.

2. Necessity

We sketch why the congruence condition in Theorem A is necessary. The cohomology ring $H^\star(\mathrm{SU}(4))$ is generated by classes x_3, x_5, x_7 having $\deg x_i = i$. Nonzero cohomology groups occur in dimensions 0, 3, 5, 7, 8, 10, 12, and 15.

Suppose $\phi\colon \mathrm{SU}(4) \to \mathrm{SU}(4)$ has multidegree (t_3, t_5, t_7), meaning $\phi^\star(x_i) = t_i\, x_i$. Here, and throughout the paper, we write ρ_m for the coefficient homomorphism on cohomology induced by reduction $\mathbb{Z} \to \mathbb{Z}_{/m}$. Note that

$$\mathrm{Sq}^2\colon H^3(\mathrm{SU}(4); \mathbb{Z}_{/2}) \to H^5(\mathrm{SU}(4); \mathbb{Z}_{/2})$$

is an isomorphism, so by naturality,

$$\mathrm{Sq}^2 \rho_2 \phi^\star x_3 = \phi^\star \rho_2 x_5,$$

and thus $t_3 \equiv t_5 \bmod 2$.

The reduced power operation

$$P^1\colon H^3(\mathrm{SU}(4); \mathbb{Z}_{/3}) \to H^7(\mathrm{SU}(4); \mathbb{Z}_{/3})$$

satisfies $P^1 \rho_3 x_3 = \rho_3 x_7$. We check this by with a formula involving the Chern classes c_2, c_4 in $H^\star(\mathrm{BSU}(4))$. Specifically,

$$P^1 \rho_3 c_2 = (\rho_3 c_2)^2 + \rho_3 c_4,$$

and the square vanishes after desuspending. Calculations of P^1 in this context can be found in [**3**]. Consequently, $t_3 \equiv t_7 \bmod 3$.

Finally, we check $t_3 \equiv t_7 \bmod 2$. One may have hoped to use

$$\mathrm{Sq}^2\colon H^5(\mathrm{SU}(4);\mathbb{Z}_{/2}) \to H^7(\mathrm{SU}(4);\mathbb{Z}_{/2})$$

but that maps is trivial. Instead, we verify that $t_3 \equiv t_7 \bmod 2$ by invoking a secondary operation; this is described in Subsection 4.2.

3. Building maps stage-by-stage

We show the condition in Theorem A is sufficient by exhibiting self-maps of SU(4). The tool we use to construct such maps are Postnikov towers [**12**]. Indeed, we wish to construct the maps stage-by-stage, meaning we desire to deal with stages of Postnikov towers and the maps between those stages in the absence of a total space. Postnikov *systems* are convenient for such reasoning. But Postnikov systems are less well-known than Postnikov towers, so we recall the key facts we need in the sequel. As we climb the Postnikov tower, cohomology operations (Subsection 3.4) and H-space structures (Subsection 3.5) will help us continue to climb to the next rung.

3.1. Postnikov systems

Definition 3.1. A *Postnikov system* P_n is a sequence $((\pi_i, k_i))_{i=2}^n$ of groups π_i and elements k_i defined recursively so that

- the empty sequence ($n = 1$) is a Postnikov system,
- the data $((\pi_i, k_i))_{i=2}^{n-1}$ constitutes a Postnikov system,
- the abelian group π_n is finitely generated, and
- the "k-invariant" k_n lives in the appropriate home, i.e., $k_n \in H^{n+1}(X_{n-1}; \pi_n)$.

We permit Postnikov systems with infinitely many stages, which we denote by $((\pi_i, k_i))_i$.

Given $((\pi_i, k_i))_i$ the total space can be recovered, and the Postnikov tower of that total space has the expected stages. To build the stages, start with $X_1 = \{*\}$. Then for $n \geqslant 2$ let X_n be the homotopy fiber of the classifying map $k_n\colon X_{n-1} \to K(\pi_n, n+1)$ which fits into a fiber sequence

$$K(\pi_n(X), n) \xrightarrow{i_n} X_n \xrightarrow{q_n} X_{n-1} \xrightarrow{k_n} K(\pi_n(X), n+1).$$

These stages assemble to the desired total space.

3.2. Morphisms of Postnikov systems

Having introduced objects, we introduce morphisms.

Definition 3.2 (cf. [**14**])**.** For Postnikov systems $P = ((\pi_i, k_i))_i$ and $P' = ((\pi'_i, k'_i))_i$, a *morphism of Postnikov systems* is a collection of group homomorphisms $f_i\colon \pi_i \to \pi'_i$ with the recursive property: the collection of maps $(f_i)_{i=2}^{n-1}$ defines a continuous map $\phi_{n-1}\colon X_{n-1} \to X'_{n-1}$ in such a way that the maps $\phi^\star_{n-1}\colon H^{n+1}(X'_{n-1}; \pi'_n) \to H^{n+1}(X_{n-1}; \pi'_n)$ and $f_{n*}\colon H^{n+1}(X_{n-1}; \pi_n) \to H^{n+1}(X_{n-1}; \pi'_n)$ satisfy the coherence condition

$$\phi^\star_{n-1}(k'_n) = f_{n*}(k_n). \qquad (\star)$$

This coherence condition is precisely what makes it possible, given a map on the $n-1$ stage, to produce a map on the next stage.

Lemma 3.3. *Consider Postnikov systems $((\pi_i, k_i))_i$ and $((\pi'_i, k'_i))_i$ with $n-1$ stage X_{n-1} and X'_{n-1}, respectively. If the morphism $\phi_{n-1}\colon X_{n-1} \to X'_{n-1}$ satisfies the coherence condition ($\star$), then ϕ_{n-1} extends to a morphism $\phi_n\colon X_n \to X'_n$.*

Proof. The coherence condition ($\star$) is equivalent to the commutativity of the right-hand square in

$$\begin{array}{ccccc} X_n & \longrightarrow & X_{n-1} & \xrightarrow{k_n} & K(\pi_n, n+1) \\ \downarrow{\scriptstyle \phi_n} & & \downarrow{\scriptstyle \phi_{n-1}} & & \downarrow{\scriptstyle f_n} \\ X'_n & \longrightarrow & X'_{n-1} & \xrightarrow{k'_n} & K(\pi'_n, n+1) \end{array}$$

because $H^{n+1}(K(\pi_n, n+1); \pi'_n)$ can be naturally identified with $\mathrm{Hom}(\pi_n, \pi'_n)$. Since the right square commutes and X_n and X'_n are the homotopy fibers of k_n and k'_n, respectively, there is an induced map $\phi_n\colon X_n \to X'_n$. ☐

By Lemma 3.3, the data of a morphism of Postnikov systems permits us to build maps $\phi_n\colon X_n \to X'_n$. Often $X = X'$; we vary the notation to make clear whether we are speaking of the source or target.

3.3. Whitehead sequence

We will occasionally make use of Whitehead's "certain exact sequence" [**16**].

Definition 3.4. For a space X with n-skeleton $X^{(n)}$, the *Whitehead sequence*

$$\cdots \longrightarrow H_{n+1}(X) \xrightarrow{b_{n+1}} \Gamma_n(X) \xrightarrow{j_n} \pi_n(X) \xrightarrow{h_n} H_n(X) \longrightarrow \cdots,$$

is exact and natural. Here, $\Gamma_n(X) = \mathrm{im}\left(i_\star\colon \pi_n(X^{(n-1)}) \to \pi_n(X^{(n)})\right)$.

That this is exact was shown by Whitehead.

The significance of the Whitehead sequence to us lies largely with its relationship to Postnikov towers. For instance, if X_{n-1} is a Postnikov stage for X, there is a natural identification

$$\Gamma_n(X) \equiv H_{n+1}(X_{n-1}).$$

This is proved in [**1**, Section 2.7].

Moreover, in certain situations the map j_n can be related to a k-invariant. Baues shows that k_n can be naturally identified with j_n if $\mathrm{Ext}(H_n(X_{n-1}, \pi_n(X))) = 0$, and k_n can be identified with the extension $\pi_n(X) \to H_n(X) \to H_n(X_{n-2})$ if

$$H_{n+1}(X_{n-1}) = \pi_{n-1}(X) = 0.$$

With this in mind, here is a simple but useful lemma for disposing of some nonzero k-invariants.

Lemma 3.5. *Let $\phi_{n-1}\colon X_{n-1} \to X'_{n-1}$ be a map of Postnikov stages, and assume that*

(i) the homology groups $H_n(X_{n-1})$ and $H_n(X'_{n-1})$ are free,

(ii) *the maps* $b_{n+1}\colon H_{n+1}(X) \to H_{n+1}(X_{n-1})$ *and* b'_{n+1} *are both zero, and*

(iii) *the Hurewicz maps* $h_n\colon \pi_n(X) \to H_n(X)$ *and* $h'_n\colon \pi_n(X') \to H_n(X')$ *are both zero.*

Then ϕ_{n-1} *extends to* $\phi_n\colon X_n \to X'_n$.

Proof. The k-invariants to extend ϕ_{n-1} to ϕ_n are found in $H^{n+1}(X_{n-1};\pi_n(X))$ and $H^{n+1}(X'_{n-1};\pi_n(X'))$. Via universal coefficients we see

$$H^{n+1}(X_{n-1};\pi_n(X)) \cong \mathrm{Hom}(H_{n+1}(X_{n-1}),\pi_n(X)) \oplus \mathrm{Ext}(H_n(X_{n-1}),\pi_n(X)),$$

and the Ext term is zero by (i). Hence the k-invariants are exactly the maps $j\colon H_{n+1}(X_{n-1}) \to \pi_n(X)$ and $j'\colon H_{n+1}(X'_{n-1}) \to \pi_n(X')$. Since $b_{n+1} = 0$ by (ii) and $h_n = 0$ by (iii), the map j is an isomorphism, and similarly for j'. By Lemma 3.3, ϕ_n exists if a suitable map $f_n\colon \pi_n(X) \to \pi_n(X')$ exists, and in this case we can take $f_n = j'\phi_{n-1}j^{-1}$. □

3.4. Cohomology operations

The congruence conditions in Theorem A, e.g., $t_3 \equiv t_5 \pmod 2$, are the numerological shadow of cohomology operations. In general, cohomology operations are not additive, but we will only consider operations which are.

Definition 3.6. A primary, secondary, or higher cohomology operation Θ is said to be a *linear operation* on a space X if

1. $\Theta(x+y) = \Theta(x) + \Theta(y)$ for all $x, y \in \mathrm{Def}(\Theta, X)$, and
2. $\mathrm{Ind}(\Theta, X) = 0$.

All stable primary operations are linear. With this definition in mind, we are ready for Lemma 3.7. It plays a key role in the sequel. Here $p_{n-1}\colon X \to X_{n-1}$ is the Postnikov section. We denote $X = \mathrm{SU}(4)$. The map $p_{n-1}\colon X \to X_{n-1}$ is the Postnikov section. And finally, $d_1 = 3$ and $d_2 = 5$ and $d_3 = 7$ so $H^\star(X)$ is generated by $\{x_{d_i}\}$.

Lemma 3.7. *Suppose the cohomology of* X *is generated by* $\{x_{d_i}\}$ *with* $1 \leqslant i \leqslant r$, *and fix* $\ell = d_i < d_j = n$ *for some* $i, j \leqslant r$. *Assume that*

1. *there is a map* $\phi_{n-1}\colon X_{n-1} \to X_{n-1}$,
2. *the Whitehead sequence for* X *contains an exact fragment*
$$\cdots \xrightarrow{0} \pi_n(X) \xrightarrow{h_n} H_n(X) \xrightarrow{b_n} H_n(X_{n-2}) \xrightarrow{0} \cdots,$$
where $H_n(X) \cong \mathbb{Z}$,
3. *the group* $H^{n+1}(X_{n-1};\pi_n(X))$ *is naturally isomorphic to* $\mathrm{Ext}(H_n(X_{n-2}),\pi_n(X))$,
4. *there is a cohomology class* $\chi_\ell \in H^\ell(X_{n-1};\pi_\ell(X))$ *such that* $p^\star_{n-1}(\chi_\ell) = x_\ell$,
5. *there is an isomorphism* $\pi_\ell(X) \cong \mathbb{Z}$, *and*
6. *there is a linear cohomology operation* Θ *such that* $k_n = \Theta(\chi_\ell)$.

Then ϕ_{n-1} *may be extended to* ϕ_n *if and only if the degree* t_n *satisfies* $t_n \equiv t_\ell \pmod m$ *where* m *is the smallest positive integer such that* $m\Theta(\chi_\ell) = 0$.

Proof. We will need to distinguish between π_n of the source and the target, so to avoid confusion we will call the target X'; every fact deduced about X obviously also holds for X'. Since $p_{n-1}\colon X \to X_{n-1}$ is an isomorphism on cohomology in dimension ℓ, $\phi^\star_{n-1}(\chi'_\ell) = t_\ell \chi_\ell$. On the other hand, by (2), $H_n(X) \cong \mathbb{Z}$ and h_n is injective, so $\pi_n(X)$ is either $\mathbb{Z}$ or 0. If $\pi_n(X) = 0$ then there is no obstruction to extending ϕ_{n-1}, so we may assume $\pi_n(X) \cong \mathbb{Z}$ and h_n is multiplication by c for some $c \in \mathbb{Z}$, and hence that

$$H_n(X_{n-2}) \cong H^{n+1}(X_{n-1}; \pi_n(X)) \cong \mathbb{Z}_{/c}$$

by (3) and (4). Therefore, in the diagram

$$\begin{array}{ccccc} \pi_n(X) & \xrightarrow{h_n} & H_n(X) & \xrightarrow{b_n} & H_n(X_{n-2}) \\ \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle t_n} & & \downarrow{\scriptstyle (\phi_{n-1})_\star} \\ \pi_n(X') & \xrightarrow{h'_n} & H_n(X') & \xrightarrow{b'_n} & H_n(X'_{n-2}) \end{array}$$

we must have that f_n is multiplication by t_n, and $(\phi_{n-1})_\star$ is multiplication by t_n mod c. We then apply those facts to the following diagram obtained from (5) and (6) illustrating both the operation Θ and the matching condition $f_{n*}(k_n) = \phi^\star_{n-1}(k'_n)$.

$$\begin{array}{ccccc} H^\ell(X_{n-1}; \pi_\ell(X)) & \xrightarrow{t_\ell} & H^\ell(X_{n-1}; \pi_\ell(X')) & \xleftarrow{t_\ell} & H^\ell(X'_{n-1}; \pi_\ell(X')) \\ \downarrow{\scriptstyle \Theta} & & \downarrow{\scriptstyle \Theta} & & \downarrow{\scriptstyle \Theta} \\ H^{n+1}(X_{n-1}; \pi_n(X)) & \underset{f_{n*}}{\xrightarrow{t_n}} & H^{n+1}(X_{n-1}; \pi_n(X')) & \underset{\phi^\star_{n-1}}{\xleftarrow{t_n}} & H^{n+1}(X'_{n-1}; \pi_n(X')) \end{array}$$

On the bottom row, multiplication by t_n is taken to be modulo c. The map ϕ_{n-1} extends to ϕ_n if the above diagram can be made to commute by an appropriate choice of f_n, which is precisely a choice of t_n. But the condition for commutativity of each square is $t_\ell \Theta(\chi_\ell) = t_n \Theta(\chi'_\ell)$, which implies that the diagram commutes and the map extends if and only if $t_\ell \equiv t_n \pmod{m}$, where m is the order of $\Theta(\chi_\ell)$, which divides but may not necessarily equal c. □

3.5. H-spaces

We will occasionally use the fact that the spaces we consider have H-space structures. For us an "*H*-space structure" means a multiplication $\mu\colon X \times X \to X$ and a strict identity element $e \in X$ such that the $\mu(x, e) = \mu(e, x) = x$.

Definition 3.8 ([**5**])**.** Let (X, μ) be an H-space and $x \in H^n(X; R)$. Consider the natural splitting

$$H^n(X \times X; R) \cong H^n(X \vee X; R) \oplus H^n(X \wedge X; R),$$

where $X \wedge X$ is the smash product $(X \times X)/(X \vee X)$. Let $\bar{p}$ be projection onto the second term above. Then x is *R-primitive* if $\bar{p}\mu^\star(x) = 0$.

For H-spaces, the morphisms are H-maps $f\colon (X, \mu) \to (Y, \nu)$ meaning maps for which $\nu(f \times f) \simeq f\mu$. Zabrodsky provides a useful criterion for "potential" H-maps. If Y is an H-space, $[X, Y]$ is an algebraic loop in which we define $D(f, g) \in [X, Y]$ for maps $f, g\colon X \to Y$ to be the (unique) element satisfying $D(f, g) + g = f$. The *H-deviation* of an arbitrary map $f\colon X \to Y$ is the element $D_f \in [X \wedge X, Y]$ defined so that

$$D_f p = D(f\mu, \nu(f \times f))$$

and is trivial if and only if f is an H-map.

Lemma 3.9. *Let (X, μ) and (Y, ν) be H-spaces, and let $f\colon X \to Y$ be a map. Then Y admits another multiplication ν' such that $f\colon (X, \mu) \to (Y, \nu')$ is an H-map if and only if the H-deviation D_f lies in the image of the map*

$$(f \wedge f)^\star\colon [Y \wedge Y, Y] \to [X \wedge X, Y].$$

Proof. See [**17**, Proposition 1.5.1(a)]. □

Viewing x as a map $X \to K(R, n)$, the H-deviation D_x of x is in $[X \wedge X, K(R, n)] = H^n(X \wedge X; R)$, and, in fact, is the difference between $\mu^\star(x)$ and $i_1^\star(x) + i_2^\star(x)$, where $i_j\colon X \to X \vee X$ is inclusion to the jth factor. Hence x is primitive if and only if its classifying map is an H-map. Applying this fact to the context of Postnikov towers, Kahn [**5**] proves that primitive k-invariants are, in fact, exactly what is necessary to give X an H-space structure.

Theorem 3.10. *Let X be a 1-connected space with homotopy groups π_n for $n \geqslant 2$. Then X is an H-space if and only if it has a Postnikov tower in which each Postnikov invariant k_n is π_n-primitive. Furthermore, if X is an H-space, all its Postnikov stages X_n are H-spaces.*

Proof. The first assertion as applied to spaces with only finitely many homotopy groups is [**5**, Theorem 3.2] and can be extended to arbitrary X by an argument of Barratt [**5**, p. 450]. The second assertion is [**5**, Corollary 3.1]. □

To extend an H-map of Postnikov stages $\phi_{n-1}\colon X_{n-1} \to X'_{n-1}$ to an H-map of the next stages $\phi_n\colon X_n \to X'_n$, we use the following lemma.

Lemma 3.11. *Let X and X' be H-spaces, let $\phi_{n-1}\colon X_{n-1} \to X'_{n-1}$ be an H-map, and set $\pi_n = \pi_n(X)$ and $\pi'_n = \pi_n(X')$ so that $f_n\colon \pi_n \to \pi'_n$ induces a map $\phi_n\colon X_n \to X'_n$.*

If the induced map

$$(\phi_n \wedge \phi_n)^\star\colon H^n(X'_n \wedge X'_n; \pi'_n) \to H^n(X_n \wedge X_n; \pi'_n)$$

is surjective, then there is an H-structure on X'_n such that ϕ_n is an H-map.

Proof. By Lemma 3.9, there will be an H-structure on X'_n that makes ϕ_n an H-map if the H-deviation D_{ϕ_n} is in the image of the map

$$(\phi_n \wedge \phi_n)^\#\colon [X'_n \wedge X'_n, X'_n] \to [X_n \wedge X_n, X'_n].$$

Here we have used $^\#$ rather than * to distinguish from the map on cohomology in the hypothesis. Let $K_n = K(\pi_n, n+1)$ and $K'_n = K(\pi'_n, n+1)$ to arrive at the following diagram.

$$\begin{array}{ccccccc}
\Omega K_n & \xrightarrow{i_n} & X_n & \xrightarrow{q_n} & X_{n-1} & \xrightarrow{k_n} & K_n \\
\downarrow{\scriptstyle \Omega f_n} & & \downarrow{\scriptstyle \phi_n} & & \downarrow{\scriptstyle \phi_{n-1}} & & \downarrow{\scriptstyle f_n} \\
\Omega K'_n & \xrightarrow{i'_n} & X'_n & \xrightarrow{q'_n} & X'_{n-1} & \xrightarrow{k'_n} & K'_n
\end{array}$$

Since X and X' are H-spaces, k_n and k'_n are both H-maps, which implies that q_n and q'_n are also H-maps, so $\phi_{n-1}q_n$ is also an H-map. From the fiber sequence $\Omega K' \to$

$X'_n \to X'_{n-1}$ we get a commuting diagram of Puppe sequences.

$$\begin{array}{ccccc} [X'_n \wedge X'_n, \Omega K'] & \xrightarrow{i'_n} & [X'_n \wedge X'_n, X'_n] & \xrightarrow{q'_n} & [X'_n \wedge X'_n, X'_{n-1}] \\ \downarrow{\scriptstyle (\phi_n \wedge \phi_n)^\star} & & \downarrow{\scriptstyle (\phi_n \wedge \phi_n)^\#} & & \downarrow \\ [X_n \wedge X_n, \Omega K'] & \xrightarrow{i'_n} & [X_n \wedge X_n, X'_n] & \xrightarrow{q'_n} & [X_n \wedge X_n, X'_{n-1}] \end{array}$$

Since $\phi_{n-1} q_n \colon X_n \to X'_{n-1}$ is an H-map, the image of D_{ϕ_n} under q'_n is trivial, so D_{ϕ_n} is in the image of i'_n. As the homotopy sets furthest left are exactly the cohomology groups of the hypothesis, and the left square above commutes, surjectivity of $(\phi_n \wedge \phi_n)^\star$ is enough to imply that $D_{\phi_n} \in \operatorname{im}(\phi_n \wedge \phi_n)^\#$ as desired. □

4. Proof of Theorem A

We now prove Theorem A. We rely on Lemma 3.7 to translate the congruence conditions into initial progress climbing the Postnikov tower; after we have gotten started, we rely on Lemma 3.5 to climb the rest of the way.

To be more precise, the proof will be accomplished in the following steps. Using the condition $t_3 \equiv t_5 \pmod 2$, we construct ϕ_5 in Subsection 4.1. With $t_3 \equiv t_7 \pmod 6$, we extend this to ϕ_7 in Subsection 4.2. We arrange for t_3, t_5, and t_7 to be units modulo 30 in Subsection 4.3, which facilitates the argument (Subsection 4.4) that ϕ_8 can be constructed to be an H-map. Then Subsection 4.5 extends ϕ_8 to ϕ_{10}, and the latter can be taken to be an H-map (Subsection 4.6) and can be extended to ϕ_{14} (Subsection 4.7) and finally, extended to SU(4) in Subsection 4.8.

4.1. Construct ϕ_5

Since $X_3 = K(\mathbb{Z}, 3)$, we construct $\phi_3 \colon X_3 \to X'_3$ via $t_3 \iota_3 \in H^3(X_3) \cong \mathbb{Z}$. Here, and in the remainder of the paper, write ι_n for the generator of $H^n(K(G, n); G)$ when G is clear from context.

Because $\pi_4(\mathrm{SU}(4))$ vanishes, we have $\phi_4 = \phi_3$. To construct ϕ_5, we apply Lemma 3.7 with the stable and primary (hence linear) operation $\tilde{\beta}\overline{\mathrm{Sq}}^2$. Here (and in the remainder of the paper) denote the Bockstein of the coefficient sequence $\mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \to \mathbb{Z}_{/n}$ by $\tilde{\beta}$ to distinguish it from the Bockstein β of the mod-p Steenrod algebra.

4.2. Extend ϕ_5 to ϕ_7

Now we extend ϕ_5 to ϕ_7, the k-invariant for which lies in $H^8(X_6; \pi_7)$. Applying the relative Hurewicz theorem to the pair $(\mathrm{SU}(4), \mathrm{SU}(3))$ and inspecting the long exact sequence for the fibration $\mathrm{SU}(3) \to \mathrm{SU}(4) \to S^7$, we see that the Hurewicz map $h_7 \colon \pi_7 \to H_7(X)$ in the Whitehead sequence is exactly the map $\pi_7(\mathrm{SU}(4)) \to \pi_7(S^7)$ in the long exact sequence; hence the map $\pi_7 \to H_7(X)$ is multiplication by 6 and, since $\pi_6 = 0$, we have

$$H_7(X_5) = H_7(X_6) = \mathbb{Z}_{/6}.$$

This also implies that the map $H_8(X_6) \to \pi_7$ is zero, which combined with the facts that $H_8(X) = \mathbb{Z}$ and π_8 is finite implies that $H_8(X_6) = \mathbb{Z}$. In conclusion, we have

$$H^8(X_6; \pi_7) = \mathrm{Hom}(H_8(X_6), \mathbb{Z}) \oplus \mathrm{Ext}(H_7(X_5), \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}_{/6},$$

where the splitting *is* natural since the $\mathrm{Hom}(H_8(X_6), \mathbb{Z})$ is free. Furthermore, since $H^7(X_6; \mathbb{Z}) = 0$, the long exact sequence for the coefficients $\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_{/6}$ implies that the group $H^7(X_6; \mathbb{Z}_{/6})$ must be $\mathbb{Z}_{/6}$, and therefore that the Bockstein

$$\tilde{\beta} \colon H^7(X_6; \mathbb{Z}_{/6}) \to H^8(X_6; \mathbb{Z})$$

identifies it with the $\mathbb{Z}_{/6}$ of the latter group.

Now we need to identify the generators. We first identify the $\mathbb{Z}_{/2}$ piece; for brevity, we write $\overline{\mathrm{Sq}}^2$ for $\mathrm{Sq}^2 \rho_2$. Consider the Leray-Serre spectral sequence for the fibration $K(\mathbb{Z}, 5) \to X_5 \to K(\mathbb{Z}, 3)$, with $\mathbb{Z}$ coefficients. Since the fibration is nontrivial, the transgression takes ι_5 to the generator of $H^6(K(\mathbb{Z}, 3))$, which is $\tilde{\beta}\overline{\mathrm{Sq}}^2 \iota_3$. So we compute that

$$\begin{aligned} \tau(\iota_5) &= \beta\overline{\mathrm{Sq}}^2 \iota_3, \\ \tau(\overline{\mathrm{Sq}}^2 \iota_5) &= \overline{\mathrm{Sq}}^2 \tilde{\beta}\overline{\mathrm{Sq}}^2 \iota_3 = 0, \\ \tau(\tilde{\beta}\overline{\mathrm{Sq}}^2 \iota_5) &= 0, \text{ and} \\ d_6(\iota_3\iota_5) &= \iota_3\tilde{\beta}\overline{\mathrm{Sq}}^2 \iota_3. \end{aligned}$$

Then $H^7(X_6; \mathbb{Z}_{/2})$ is generated by an element λ_7 for which $i_5^\star \lambda_7 = \overline{\mathrm{Sq}}^2 \iota_5$, which defines a secondary operation Φ based on the relation

$$\overline{\mathrm{Sq}}^2 \circ \tilde{\beta}\overline{\mathrm{Sq}}^2 = 0$$

in dimension 3. In general, for a CW-complex Y, we have

$$\begin{aligned} \mathrm{Def}(\Phi, Y) &= \{x \in H^3(Y; \mathbb{Z}) : \tilde{\beta}\overline{\mathrm{Sq}}^2 x = 0\}, \\ \mathrm{Ind}(\Phi, Y) &= \overline{\mathrm{Sq}}^2 H^5(Y; \mathbb{Z}), \end{aligned}$$

and, in particular, Φ is a linear cohomology operation for $Y = \mathrm{SU}(4)$ (since it is the looping of an operation on $\mathrm{BSU}(4)$), taking values in $H^7(\mathrm{SU}(4); \mathbb{Z}_{/2})$. It follows that $\tilde{\beta}(\lambda_7)$ generates the $\mathbb{Z}_{/2}$ portion of $H^8(X_6; \mathbb{Z})$.

The $\mathbb{Z}_{/3}$ piece is much more straightforward. The k-invariant attaching $K(\mathbb{Z}, 5)$ to $K(\mathbb{Z}, 3)$ is trivial modulo 3, so we get $H^7(X_5; \mathbb{Z}_{/3}) \cong \mathbb{Z}_{/3}$, generated by $q_5^\star P^1 \rho_3 \iota_3$. The k-invariant for constructing X_7 is nontrivial modulo 3, and the generator for the $\mathbb{Z}$ piece of $H^8(X_5; \mathbb{Z})$ is $2(q_5^\star \iota_3)\iota_5$, which is not primitive. But k_7, as a k-invariant for an H-space, must be primitive, so it follows that

$$k_7 = \tilde{\beta}(\lambda_7 + q_5^\star P^1 \rho_3 \iota_3).$$

Since this has order 6, applications of Lemma 3.7 modulo 2 and 3 show that ϕ_5 can be extended to ϕ_7 if and only if $t_3 \equiv t_7 \pmod 6$.

4.3. Modify t_3, t_5, t_7

The algebraic loop $[\mathrm{SU}(4), \mathrm{SU}(4)]$ has a homomorphism to $\mathbb{Z}^3$ given by the multi-degree; since the identity $(1, 1, 1)$ is clearly a valid map of $\mathrm{SU}(4)$ and by assumption $t_3 \equiv t_5 \equiv t_7 \pmod 2$, we may change (t_3, t_5, t_7) to $(t_3 + 15, t_5 + 15, t_7 + 15)$ if necessary to ensure the t_i are all units modulo 2 without changing their class modulo 3 or 5. Similarly, since $t_3 \equiv t_7 \pmod 3$, we may add a multiple of $(10, 10, 10)$ to make the t_i all units modulo 3, and a multiple of $(6, 6, 6)$ to do it modulo 5.

4.4. Ensure ϕ_8 is an H-map

We first check that ϕ_3, ϕ_5, and ϕ_7 are H-maps. The map ϕ_3 is a group homomorphism of $K(\mathbb{Z},3)$, so is certainly an H-map. For the latter two, the sets in which their H-deviations lie are surjected onto by the groups

$$H^5(X_5 \wedge X_5;\mathbb{Z}) = 0,$$
$$H^7(X_7 \wedge X_7;\mathbb{Z}) = 0,$$

so ϕ_5 and ϕ_7 are both H-maps (independent of the modification from Subsection 4.3). Next, ϕ_7 can be extended to ϕ_8 by Lemma 3.5.

By Lemma 3.9, ϕ_8 can be modified to be an H-map if

$$(\phi_8 \wedge \phi_8)^\star \colon H^8(X'_8 \wedge X'_8;\mathbb{Z}_{/24}) \to H^8(X_8 \wedge X_8;\mathbb{Z}_{/24})$$

is a surjection. A simple calculation shows that the groups on each side are

$$H^8(X_8 \wedge X_8;\mathbb{Z}_{/24}) \cong H^3(X_8;\mathbb{Z}_{/24}) \otimes H^5(X_8;\mathbb{Z}_{/24}) \oplus H^5(X_8;\mathbb{Z}_{/24}) \otimes H^3(X_8;\mathbb{Z}_{/24})$$

and similarly with primes. Since t_3 and t_5 are units modulo 2 and 3, $(\phi_8 \wedge \phi_8)^\star$ is multiplication by a unit of $\mathbb{Z}_{/24} \oplus \mathbb{Z}_{/24}$, hence a surjection. So ϕ_8 can be taken to be an H-map, possibly after modifying the H-structure of X'_8.

4.5. Extend ϕ_8 to ϕ_{10}

To extend ϕ_8 to ϕ_{10}, we first determine $H^{10}(X_8;\pi_9)$. In the Whitehead sequence, $H_{10}(X_8)$ sits in the middle of the short exact sequence

$$\mathbb{Z} = H_{10}(X) \to H_{10}(X_8) \to \pi_9 = \mathbb{Z}_{/2}$$

so $H_{10}(X_8)$ is either $\mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}_{/2}$. The k-invariant group $H^{10}(X_8;\pi_9)$ is therefore either $\mathbb{Z}_{/2}$ or $\mathbb{Z}_{/2} \oplus \mathbb{Z}_{/2}$. But $H^{10}(X_8;\pi_9)$ must contain a nonzero multiple of a combination of pullbacks of $\iota_3\iota_7$, which is not primitive, whereas the k-invariant is primitive (and nonzero). Therefore $H^{10}(X_8;\pi_9) \cong \mathbb{Z}_{/2} \oplus \mathbb{Z}_{/2}$ generated by k_9 and a version of $\iota_3\iota_7$. It is then clear that the coherence condition $\phi_8^\star(k'_9) = f_{9*}(k_9)$ is satisfied since ϕ_8 is an H-map, and hence must take a primitive element to a primitive element.

Once ϕ_9 has been constructed, ϕ_{10} follows immediately by Lemma 3.5.

4.6. Ensure ϕ_{10} is an H-map

We have built ϕ_{10} but we do not yet know it is an H-map. Arranging for it to be such will be similar to the procedure for ϕ_8. Note that ϕ_9 is an H-map since

$$H^9(X_9 \wedge X_9;\mathbb{Z}_{/2}) = 0$$

so there is no H-deviation. For ϕ_{10}, we examine the map $(\phi_{10} \wedge \phi_{10})^\star$, which involves the groups

$$H^{10}(X_{10} \wedge X_{10};I) \cong H^3(X_{10};I) \otimes H^7(X_{10};I) \oplus H^7(X_{10};I) \otimes H^3(X_{10};I)$$

and similarly with X'_{10}, where the coefficient group is $I = \pi_{10}(X) = \mathbb{Z}_{/120} \oplus \mathbb{Z}_{/2}$. In Subsection 4.3 we arranged t_3 and t_7 to be units modulo 2, 3, and 5, so $(\phi_{10} \wedge \phi_{10})^\star$ is multiplication by a unit and hence a surjection, meaning ϕ_{10} can be an H-map.

4.7. Extend ϕ_{10} to ϕ_{14}

Extending ϕ_{10} to ϕ_{11} is the same argument as in Subsection 4.5 but with $\pi_{11} = \mathbb{Z}_{/4}$ instead of $\pi_9 = \mathbb{Z}_{/2}$ and the other parts carrying through *mutatis mutandis*.

Extending ϕ_{11} to ϕ_{12} is an application of Lemma 3.5, and ϕ_{12} can be taken to be an H-map by logic analogous to Subsection 4.6 using the fact that t_3, t_5, t_7 are all units modulo 60.

Extending to ϕ_{13} follows by Lemma 3.5, and it can be made an H-map since

$$H^{13}(X_{13} \wedge X_{13}; \pi_{13}) = 0$$

and so construction of ϕ_{14} is again analogous to Subsection 4.5.

4.8. Construct the self-map of SU(4)

Note that SU(4) is a 15-dimensional space. So constructing ϕ_{14} is enough by applying the following result with $X = X' = \mathrm{SU}(4)$.

Proposition 4.1. *Suppose X and X' are n-dimensional spaces. If $\phi_{n-1}\colon X_{n-1} \to X'_{n-1}$ is a map of their respective Postnikov stages, then there is a map $\phi\colon X \to X'$ for which*

$$\begin{array}{ccc} H^i(X'_{n-1}) & \xrightarrow{\phi^\star_{n-1}} & H^i(X_{n-1}) \\ \downarrow & & \downarrow \\ H^i(X') & \xrightarrow{\phi^\star} & H^i(X) \end{array}$$

commutes for $0 \leqslant i \leqslant n-1$

Proof. We may assume that X_{n-1} has the same n-skeleton as X, and similarly for X'_{n-1} and X'. We may also assume ϕ_{n-1} is cellular. The restriction of ϕ_{n-1} to $X^{(n)}_{n-1}$ is therefore the desired map ϕ, and induces the same maps on cohomology for $i \leqslant n-1$ since $H^i(X)$ can be naturally identified with $H^i(X_{n-1})$ for $i \leqslant n-1$ (and similarly for X'). □

We have proved Theorem A.

5. Explicit self-maps

The significance of Theorem A is its ability to produce self-maps with specified multidegree. If the reader is interested merely in degree and not multidegree, there are easier routes to prove Corollary B directly.

The challenging part is constructing maps of all the required degrees. There are explicit sources of self-maps $\mathrm{SU}(4) \to \mathrm{SU}(4)$ of odd-degree [**13**]. For odd k, let $\psi_k\colon \mathrm{SU}(4) \to \mathrm{SU}(4)$ be such a map with $\deg \psi_k = k$.

By the congruence condition Theorem A, the multidegree of $\psi_3\colon \mathrm{SU}(4) \to \mathrm{SU}(4)$ must be either $(1, 3, 1)$ or $(-1, 3, -1)$ but exactly which choice occurs depends on the chosen construction. Adapting the construction from [**6**, Lemma 4.5] it possible to be opinionated about the multidegree of ψ_k in certain cases. Let $f_k\colon S^7 \to S^7$ be a map of degree k. Then pull-back the bundle $\mathrm{SU}(4) \to S^7$ to obtain $f_k^\star(\mathrm{SU}(4))$, an SU(3)-bundle over S^7, and a degree k map $\psi_k\colon f_k^\star(\mathrm{SU}(4)) \to \mathrm{SU}(4)$. It remains to check that $f_k^\star(\mathrm{SU}(4)) \cong \mathrm{SU}(4)$. Principal bundles over S^7 with fiber SU(3) are classified by $S^7 \to \mathrm{BSU}(3)$ which is $\pi_6(\mathrm{SU}(3)) \cong \mathbb{Z}_{/6}$. The bundle SU(4) is classified by the generator of $\mathbb{Z}_{/6}$ [**9**, Section 3], so by this method we produce maps with degree

$k \equiv 1 \bmod 6$ and multidegree $(1,1,k)$. Of special importance is the map ψ_7 which we have now arranged to have multidegree $(1,1,7)$.

With maps of odd degree ψ_k, it remains to produce self-maps of SU(4) with degree each multiple of 8. The power map $p_k\colon \mathrm{SU}(4) \to \mathrm{SU}(4)$ defined by $p_k(A) = A^k$ has $\deg p_k = k^3$, so $\deg p_2 = 8$. (Not coincidentally, the cubes in $\mathbb{Z}_{/8}$ are $\{0,1,3,5,7\}$, which are exactly the residue classes $D(\mathrm{SU}(4))$ mod 8.) For self-maps $f, g\colon \mathrm{SU}(4) \to \mathrm{SU}(4)$, define a new self-map $f \blacksquare g$ by the rule $(f \blacksquare g)(A) = f(A) \cdot g(A)$. With respect to degree, this behaves quite differently from composition. Specifically, since the cohomology of SU(4) is primitively generated, the multidegree of $f \blacksquare g$ is the termwise sum of the multidegrees of f and g. We arranged for multidegree of ψ_7 to be $(1,1,7)$, and consequently, $\psi_7 \blacksquare p_{-3}$ has multidegree $(-2,-2,4)$ which has degree 16. Table 1 shows maps of other multiples of 8; that degree is multiplicative under the usual composition $\circ$ completes this alternate proof of Corollary B.

map	degree
p_2	8
$\psi_7 \blacksquare p_{-3}$	16
$\psi_3 \circ p_2$	24
$\psi_7 \blacksquare \mathrm{id}$	32
$\psi_5 \circ p_2$	40
$\psi_3 \circ (\psi_7 \blacksquare p_{-3})$	48
$\psi_7 \circ p_2$	56
$p_2 \circ p_2$	64

Table 1: There is a self-map of SU(4) having degree $d = 8, 16, \ldots, 64$.

6. Next steps

A natural improvement would be to hope that $\phi\colon \mathrm{SU}(4) \to \mathrm{SU}(4)$ is an H-map. We can arrange for ϕ_{14} to be an H-map by requiring t_3, t_5, t_7 to be units modulo 7. But this requirement is not sufficient to ensure the final map $\phi\colon \mathrm{SU}(4) \to \mathrm{SU}(4)$ is an H-map because we have ignored the H-structure when collapsing the higher cells of X_{14} to get X. Indeed, one would need to do an analysis similar to Lemma 3.11 with $X = \mathrm{SU}(4) \wedge \mathrm{SU}(4)$ and $X' = \mathrm{SU}(4)$ in order to construct an H-map on SU(4) itself. See [**7, 8, 10**] for more discussion of existence or non-existence of nontrivial self-H-maps.

The techniques of this paper can be applied to other Lie groups. The case of $\mathrm{SU}(2) \cong S^3$ is easy, recent work [**6, 15**] covers the case of SU(3), and having just worked through SU(4), a natural next step is $\mathrm{SU}(n)$ for $n \geqslant 5$. Based on our preliminary investigations, it appears plausiable that self-maps of $\mathrm{SU}(n)$ are controlled merely by additional congruences.

Conjecture 6.1. *For each n there are indices i_k and j_k and moduli $m_k \in \mathbb{Z}$ so that there is a self-map $\mathrm{SU}(n) \to \mathrm{SU}(n)$ of multidegree $(t_3, t_5, \ldots, t_{2n-1})$ exactly when the congruence conditions $t_{i_k} \equiv t_{j_k} \pmod{m}_k$ are satisfied.*

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