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AN ELEMENTARY COMPUTATION OF THE COHOMOLOGY OF THE FOMIN–KIRILLOV ALGEBRA WITH 3 GENERATORS

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Abstract

We give an elementary computation of the algebra structure of the Yoneda algebra of the Fomin–Kirillov algebra FK(3) over a field of characteristic different from 2 and 3. The computation is based on a new bootstrap technique we introduce which is built upon the (nonacyclic) Koszul complex of FK(3).

1. Introduction

In the mini-workshop "Cohomology of Hopf Algebras and Tensor Categories" organized in Oberwolfach in March 2019, Sarah Witherspoon asked about new techniques to compute the algebraic structure of the Yoneda algebra of Hopf algebras, for example different from the well-known ones based on multiplicative spectral sequences. Moreover, in the same workshop Nicolás Andruskiewitsch emphasized the problem of the computation of the cohomology of the Fomin–Kirillov algebras FK(n), for $n = 3, 4, 5$. The latter are Nichols algebras (see [**14, 7**]), which appear in the classification of finite dimensional pointed Hopf algebras (see [**1**]), and they were also intensively studied due to their connection to the Schubert calculus of flag manifolds (see [**5, 10, 11**])

The cohomology of the Fomin–Kirillov algebra FK(3) was recently computed by Ştefan and Vay[1] in [**18**]. However, their computation is highly involved, based on clever calculations using the heavy machinery of spectral sequences. Our computation of the cohomology of FK(3) is different from that of [**18**] and is based on a new bootstrap technique we introduce in this article. The main ingredient is to compute the homology of the Koszul complex of FK(3), which is the only "heavy" calculation in this article (see the Appendix). The reason to do so is based on the fact that the minimal projective resolution of $\Bbbk$ in the category of bounded-below graded modules over any quadratic algebra contains the (possibly nonacyclic) Koszul complex as a subcomplex (see Proposition 2.2), and sometimes (as in the case of FK(3)) the latter complex is enough to easily construct the whole minimal projective resolution (see Proposition 3.3), as well as the algebraic structure of the Yoneda algebra (see Theorem 3.5). Indeed, the minimal projective resolution of $\Bbbk$ is obtained by "repairing" the

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[1]It was also independently computed by A. Conner and P. Goetz, but they did not publish their results.

degrees where the Koszul complex is not acyclic by means of the Horseshoe Lemma, and for FK(3) this repair process is performed using only the same Koszul complex one started with, giving a simple bootstrap mechanism. This repairing mechanism can be applied in principle to any nice family of subcomplexes of minimal projective resolutions of modules over any nonnegatively graded connected algebra to obtain the corresponding complete projective resolutions but for such general algebras the computations quickly become very complicated, since much more repairing is needed. It is for this reason that we refrain from explaining the procedure in the general situation here. In any case, we hope that this new point of view will be useful in the computation of the cohomology of other quadratic algebras, *e.g.* FK(n) for $n = 4, 5$.

In Section 2, we recall the basic terminology and results on quadratic algebras (see Subsection 2.2), the Fomin–Kirillov algebras (see Subsection 2.3), and some basic results on the Yoneda algebra of a bialgebra in a braided monoidal category (see Subsection 2.4). The only possibly new result in this section is Proposition 2.2, which is the starting point for our bootstrap technique for computing the minimal projective resolution of $\Bbbk$ in the category of bounded-below graded modules over the Fomin–Kirillov algebra FK(3).

The main result of this work, namely, a simple computation of the cohomology of FK(3) over a field of characteristic different from 2 and 3, is included in Section 3 (see Proposition 3.3 and Theorem 3.5). To highlight the simplicity of the proof of the previous results, we leave the technical details concerning the computation of the homology of the Koszul complex of FK(3) to the Appendix. The homology of the Koszul complex of FK(3) was mentioned (without proof) in [**17**], but we did not find an explicit computation anywhere in the literature.

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2. Preliminaries

2.1. Basic notation

In this article, we work over a fixed field $\Bbbk$. We write $\mathbb{N}$ for the set of (strictly) positive integers and $\mathbb{N}_0$ for the set of nonnegative integers. Given $i, j \in \mathbb{Z}$ such that $i \leqslant j$, we will denote by $[\![i, j]\!]$ the set $\{m \in \mathbb{Z} \colon i \leqslant m \leqslant j\}$.

All morphisms between vector spaces will be $\Bbbk$-linear (satisfying further requirements if they are so decorated). All unadorned tensor products $\otimes$ would be over $\Bbbk$. We will consider (bi)graded vector spaces $M = \oplus_{i,j\in\mathbb{Z}} M^{i,j}$, where i denotes the cohomological degree, following the usual Koszul sign rule, and j is the internal (or Adams) degree, that does not give rise to any signs. We will also write $M = \oplus_{i\in\mathbb{Z}} M^i$, where $M^i = \oplus_{j\in\mathbb{Z}} M^{i,j}$. If M is bigraded, a shift in the cohomological degree will be denoted by $M[1]$, *i.e.* $M[1]^{i,j} = M^{i+1,j}$, for all $i, j \in \mathbb{Z}$, whereas a shift in the internal degree will be denoted by $M(1)$, *i.e.* $M(1)^{i,j} = M^{i,j+1}$, for all $i, j \in \mathbb{Z}$. As usual, we will write $M_{-i,j} = M^{i,j}$, if changing from homological to cohomological notation, where the Adams degree remains unchanged. The morphisms in the category of Adams graded A-modules over an Adams graded algebra are homogeneous of degree zero, unless otherwise stated. We will also consider in some cases homogeneous morphisms of a fixed degree, and even the internal space of sums of homogeneous morphisms, but we will explicitly say so. Given a complex $(C_\bullet, \partial_\bullet)_{\bullet\in\mathbb{N}_0}$, where $\partial_n : C_{n+1} \to C_n$ for all $n \in \mathbb{N}_0$, an **augmentation** will be a morphism $\epsilon: P_0 \to M$ such that $\epsilon \circ \partial_0 = 0$, or, equivalently, such that ϵ is a morphism of complexes from $P_\bullet$ to the complex given by M concentrated in homological degree zero.

2.2. Basics on quadratic algebras

All of the following results are classical and can be found in [**15**], with the possible exception of the last proposition. Recall that a unitary (associative) $\Bbbk$-algebra A is said to be **nonnegatively graded** if $A = \oplus_{n\in\mathbb{N}_0} A_n$ is a direct sum decomposition (as vector spaces) such that $A_n \cdot A_m \subseteq A_{n+m}$, for all $n, m \in \mathbb{N}_0$, and $1_A \in A_0$. The grading will be called **internal** (or **Adams**) to emphasize that it does not intervene in the Koszul sign rule. A is said to be **connected** if we also have $A_0 = \Bbbk$. Let $A_{>0} = \oplus_{n\in\mathbb{N}} A_n$ and let V be a graded vector subspace of $A_{>0}$ such that the restriction of the canonical projection $A_{>0} \to A_{>0}/(A_{>0} \cdot A_{>0})$ to V is a bijection. We will assume for the rest of this subsection that V is finite dimensional. We say in this case that A is a **finitely generated** algebra. Then, the canonical map $TV \to A$ induced by the inclusion $V \subseteq A$ is surjective. We will usually write the product of TV by juxtaposition. A nonnegatively graded connected algebra A is said to be **quadratic** if there is a vector subspace $R \subseteq V^{\otimes 2}$ such that the kernel of $TV \to A$ is the (two-sided) ideal generated by R. By abuse of terminology, we will identify the quadratic algebra A with its presentation (V, R), where $A = TV/\langle R\rangle$.

Let V^* be the dual vector space of V and for every integer $n \geqslant 2$ define the pairing $\gamma_n : (V^*)^{\otimes n} \otimes V^{\otimes n} \to \Bbbk$ by $\gamma_n(f_1 \otimes \cdots \otimes f_n, v_1 \otimes \cdots \otimes v_n) = f_1(v_1) \cdots f_n(v_n)$, for all $v_1, \ldots, v_n \in V$ and $f_1, \ldots, f_n \in V^*$. Set $R^\perp \subseteq V^* \otimes V^*$ to be the vector subspace orthogonal to R for γ_2, *i.e.*

$$R^\perp = \Big\{\alpha \in (V^*)^{\otimes 2} : \gamma_2(\alpha, r) = 0, \text{for all } r \in R\Big\}.$$

The **quadratic dual** $A^!$ of a quadratic algebra $A = TV/\langle R\rangle$ is the algebra given by $T(V^*)/\langle R^\perp\rangle$. The induced internal grading is denoted by $A^! = \oplus_{n\in-\mathbb{N}_0} A^!_n$. Note that $A^!_0 = \Bbbk$ and $A^!_{-1} = V^*$. Moreover, for any integer $n \geqslant 2$, the composition of the isomorphism $(V^*)^{\otimes n} \xrightarrow{\sim} (V^{\otimes n})^*$ induced by the pairing γ_n and the dual of the canonical inclusion $\cap_{i=0}^{n-2} V^{\otimes i} \otimes R \otimes V^{\otimes(n-i-2)} \to V^{\otimes n}$ induces a canonical isomorphism of vector spaces

$$A^!_{-n} \xrightarrow{\sim} \left(\bigcap_{i=0}^{n-2} V^{\otimes i} \otimes R \otimes V^{\otimes(n-i-2)}\right)^*. \tag{1}$$

Recall that the graded dual $(A^!)^\# = \oplus_{n\in\mathbb{N}_0} (A^!_{-n})^*$ is a graded bimodule over $A^!$

via $(a \cdot f \cdot b)(c) = f(bca)$, for all $a, b, c \in A^!$ and $f \in (A^!)^{\#}$. Note, in particular, that $v \cdot f \in (A^!_{-n})^*$, for all $f \in (A^!_{-n-1})^*$, $v \in V^*$ and $n \in \mathbb{N}_0$. Since $V^* \otimes V \simeq \mathrm{End}(V)$, there is a unique element $\iota \in V^* \otimes V$ whose image under the previous isomorphism is the identity of V. It is easy to prove that, if $\{v_i\}_{i\in I}$ is a basis of V and $\{f_i\}_{i\in I}$ is the dual basis of V^*, then $\iota = \sum_{i\in I} f_i \otimes v_i$. For $n \in \mathbb{N}_0$ set $K_n(A) = (A^!_{-n})^* \otimes A$, provided with the regular (right) A-module structure, and $d_n \colon K_{n+1}(A) \to K_n(A)$ as the multiplication by ι on the left. Furthermore, let $\epsilon \colon K_0(A) \to \Bbbk$ be the canonical projection from A onto $A_0 = \Bbbk$. It is easy to see that $d_{n-1} \circ d_n = 0$, for all $n \in \mathbb{N}$, and $\epsilon \circ d_0 = 0$. The complex $(K_\bullet(A), d_\bullet)_{\bullet\in\mathbb{N}_0}$ is called the **(right) Koszul complex of** A. As usual, we can consider the Koszul complex as a complex indexed by $\mathbb{Z}$, with $K_n(A) = 0$ and $d_n = 0$, for all $n \in \mathbb{Z}_{<0}$. Equivalently, if we use the composition of the canonical isomorphism $V^{\otimes n} \xrightarrow{\sim} (V^{\otimes n})^{**}$ and the dual of (1) for $A^!_{-n}$, then $d_n \colon K_{n+1}(A) \to K_n(A)$ is the restriction of the map $\tilde{d}_n \colon V^{\otimes(n+1)} \otimes A \to V^{\otimes n} \otimes A$ determined by

$$(v_1 \otimes \cdots \otimes v_{n+1}) \otimes a \mapsto (v_1 \otimes \cdots \otimes v_n) \otimes v_{n+1}a, \tag{2}$$

for all $v_1, \ldots, v_{n+1} \in V$, $a \in A$ and $n \in \mathbb{N}_0$.

The quadratic algebra A is said to be **Koszul** if the complex $(K_\bullet(A), d_\bullet)_{\bullet\in\mathbb{N}_0}$ is acyclic in positive homological degrees, *i.e.* $\mathrm{Ker}(d_n) = \mathrm{Im}(d_{n+1})$, for all $n \in \mathbb{N}_0$. It is easy to prove that $H_0(K_\bullet(A), d_\bullet) = A/A_{>0} \simeq \Bbbk$ and $H_1(K_\bullet(A), d_\bullet) = 0$, for any quadratic algebra. Moreover, the complex

$$K_2(A) \xrightarrow{d_1} K_1(A) \xrightarrow{d_0} K_0(A) \xrightarrow{\epsilon} \Bbbk \to 0 \tag{3}$$

is the beginning of a minimal projective resolution of $\Bbbk$ in the category of bounded-below graded (right) A-modules. Recall that a complex $(C_\bullet, \partial_\bullet)_{\bullet\in\mathbb{N}_0}$ of bounded-below free (or projective) graded A-modules is **minimal** if $\partial_n \otimes_A \mathrm{id}_\Bbbk$ vanishes for all $n \in \mathbb{N}_0$, and that minimal projective resolutions exist in the category of bounded-below graded (right) A-modules and are unique up to nonunique isomorphism (see [**15**, Ch. 1, Section 4]).

If A is a quadratic algebra with finite dimensional $\mathrm{Tor}^A_n(\Bbbk, \Bbbk)$ for all $n \in \mathbb{N}_0$, the Yoneda algebra $\mathrm{Ext}^\bullet_A(\Bbbk, \Bbbk)$ given by deriving the functor $\mathrm{Hom}_A(-, -)$ of all homomorphisms of A-modules coincides with the one given by deriving the functor $\mathcal{H}om_A(-, -)$ of sums of homogeneous homomorphisms of A-modules. This follows from the fact that both functors coincide $\mathrm{Hom}_A(M, N) = \mathcal{H}om_A(M, N)$ for all graded A-modules M and N such that M is finitely generated. In consequence, the (internal) grading of A induces an extra-grading on $\mathrm{Ext}^\bullet_A(\Bbbk, \Bbbk)$, and we will denote the space of elements of cohomological degree i and internal degree j by $\mathrm{Ext}^{i,j}_A(\Bbbk, \Bbbk)$. The following result is well known (see [**15**, Ch. 1, Prop. 3.1]).

Proposition 2.1. *Let A be a quadratic algebra. Then, there is a natural map of graded algebras (for the internal grading)*

$$i \colon A^! \to \mathrm{Ext}^\bullet_A(\Bbbk, \Bbbk)$$

whose image is precisely the subalgebra $\oplus_{i\in\mathbb{N}_0} \mathrm{Ext}^{i,-i}_A(\Bbbk, \Bbbk)$ of $\mathrm{Ext}^\bullet_A(\Bbbk, \Bbbk)$. Moreover, if $p \colon \mathrm{Ext}^\bullet_A(\Bbbk, \Bbbk) \to \oplus_{i\in\mathbb{N}_0} \mathrm{Ext}^{i,-i}_A(\Bbbk, \Bbbk)$ denotes the quotient by the ideal generated by the terms $\mathrm{Ext}^{i,-j}_A(\Bbbk, \Bbbk)$, where $i < j$, then $p \circ i$ is an isomorphism.

The morphism p in the previous statement can also be derived from the next result. We believe that it must be well known to the experts, but we could not find it in the literature. It essentially states that, if we want to construct the minimal projective resolution of a quadratic algebra, the Koszul complex is a starting point.

Proposition 2.2. *Let A be a quadratic algebra and let $(P_\bullet, \delta_\bullet)$ be a projective resolution of $\Bbbk$ in the category of bounded-below graded A-modules. Then, there is an injective morphism of (augmented) complexes $\iota_\bullet\colon K_\bullet(A) \to P_\bullet$, such that the cokernel is a complex of free graded modules, so $P_\bullet$ is the mapping cone of a morphism of complexes from the cokernel of $\iota_\bullet$ to $K_\bullet(A)[1]$.*

Proof. It suffices to prove the result in case $(P_\bullet, \delta_\bullet)$ is a minimal projective resolution of $\Bbbk$ (in the category of bounded-below graded A-modules), since any other projective resolution of $\Bbbk$ in that category can be written as a direct sum of the minimal projective resolution and an acyclic complex (see [15, Ch. 1, Prop. 4.2]). Moreover, it suffices to prove that there is a decomposition $P_i \simeq K_i(A) \oplus C_i$ of free graded right A-modules, for all $i \in \mathbb{N}_0$, such that the differential of $P_\bullet$ sends $K_{i+1}(A)$ to $K_i(A)$, and this restriction coincides with the differential of $K_\bullet(A)$.

Set $C = \oplus_{n\in\mathbb{N}_0} C_n$ as $C_0 = \Bbbk$, $C_1 = V$ and C_n is given by the dual of the right member of (1) if $n \geqslant 2$. It is a coaugmented graded coalgebra (where the internal degree coincides with the homological degree) for the deconcatenation coproduct, and the morphism p in the previous proposition can be identified with the dual of a morphism of coaugmented graded coalgebras $\boldsymbol{i}\colon C \to \operatorname{Tor}^A_\bullet(\Bbbk,\Bbbk)$ whose image is the diagonal coalgebra $\oplus_{i\in\mathbb{N}_0} \operatorname{Tor}^A_{i,i}(\Bbbk,\Bbbk)$, where we are writing $\operatorname{Tor}^A_\bullet(\Bbbk,\Bbbk) = \oplus_{i,j\in\mathbb{N}_0} \operatorname{Tor}^A_{i,j}(\Bbbk,\Bbbk)$, the degree i is homological and j is the induced internal degree. By a theorem by B. Keller, $D = \operatorname{Tor}^A_\bullet(\Bbbk,\Bbbk)$ has a unique (up to noncanonical weak equivalence) minimal coaugmented A_∞-coalgebra structure (extending the previous coalgebra structure on $\operatorname{Tor}^A_\bullet(\Bbbk,\Bbbk)$) such that there is quasi-isomorphism f_τ of dg algebras from the cobar construction of D to A, and the minimal projective resolution of $\Bbbk$ in the category of bounded-below graded A-modules is obtained as the twisted tensor product $D \otimes_\tau A$, where $\tau\colon D \to A$ is the twisting cochain induced by f_τ (see [8, Thm. 4.7]). Moreover, the higher comultiplications $\{\Delta_n\}_{n\geqslant 2}$ of D can be chosen to preserve the internal degree. We will write $D_i = \operatorname{Tor}^A_i(\Bbbk,\Bbbk)$ and $D_{i,j} = \operatorname{Tor}^A_{i,j}(\Bbbk,\Bbbk)$ for all $i, j \in \mathbb{N}_0$. Recall that $D_{i,j} = 0$ if $i > j$ (see [15, Prop. 3.1]).

Since A is concentrated in homological degree zero and τ has cohomological degree 1, τ vanishes on $\operatorname{Tor}^A_i(\Bbbk,\Bbbk)$ for $i \neq 1$. Using the linear isomorphism $\operatorname{Tor}^A_1(\Bbbk,\Bbbk) \simeq V$ given by (3), we can consider its only possibly nonzero component $\bar\tau = \tau|_V\colon V \to A$. The fact that D is minimal and f_τ is a quasi-isomorphism implies that $\bar\tau$ is injective. Moreover, by internal degree reasons the equation defining the twisting cochain τ given by

$$\sum_{i\in\mathbb{N}} (-1)^{i(i+1)/2} \mu_A^{(i)} \circ \tau^{\otimes i} \circ \Delta_i = 0$$

(see [9, p. 133]) reduces to $\mu_A \circ \bar\tau^{\otimes 2} \circ \Delta_2|_R = 0$. Indeed, note that $\tau^{\otimes i} \circ \Delta_i|_{D_{i',i}} = 0$ if $i' \neq 2(i-1)$. Moreover, since $2(i-1) > i$ for $i > 3$, in which case $D_{2(i-1),i}$ vanishes, the only remaining case is $i = 2$, giving the mentioned equation by using the identification $\operatorname{Tor}^A_2(\Bbbk,\Bbbk) \simeq R$. Since $\Delta_2|_R\colon R \to V^{\otimes 2}$ can be chosen to be the inclusion (see [12, Thm. A]), $\mu_A \circ \bar\tau^{\otimes 2} \circ \Delta_2|_R = 0$ is then tantamount to $\bar\tau^{\otimes 2}(R) \subseteq R$, which means that the unique isomorphism of algebras $TV \to TV$ induced by $\bar{\tau}$ gives an isomorphism of algebras $g_\tau\colon A \to A$. As a consequence, by composing with g_τ^{-1}, we may assume without loss of generality that $\bar{\tau}$ is the canonical inclusion of V inside of A. In this case, the differential

$$\sum_{i\in\mathbb{N}}(-1)^{i(i+1)/2}(\mathrm{id}_D \otimes \mu_A^{(i+1)}) \circ (\mathrm{id}_D \otimes \tau^{\otimes i} \otimes \mathrm{id}_A) \circ (\Delta_{i+1} \otimes \mathrm{id}_A) \tag{4}$$

of the minimal projective resolution $D \otimes_\tau A$ (see [**9**, p. 135]) preserves the graded vector subspace $C \otimes A$. Indeed, by grading considerations we see that

$$\big((\mathrm{id}_D \otimes \tau^{\otimes i}) \circ \Delta_{i+1}\big)(D_{j,j}) \subseteq D_{j-1,j-i} \otimes V^{\otimes i}$$

for all $i \in \mathbb{N}$ and $j \in \mathbb{N}_0$, which vanishes if $i \neq 1$, since $D_{j-1,j-i} = 0$ for all $i > 1$ and $j \in \mathbb{N}_0$. Moreover, the restriction of the mapping (4) to $C \otimes A$ gives precisely the map $-(\mathrm{id}_D \otimes \mu_A) \circ (\mathrm{id}_D \otimes \tau \otimes \mathrm{id}_A) \circ (\Delta_2 \otimes \mathrm{id}_A)$, which coincides with minus the differential of the Koszul complex $K_\bullet(A)$. We have thus proved that there is a canonical injection $\iota_\bullet$ from the Koszul complex $K_\bullet(A)$ to the minimal projective resolution $P_\bullet$ of $\Bbbk$ in the category of bounded-below graded A-modules. Furthermore, the remaining part of the statement follows easily, since each projective module $P_i = D_i \otimes A$ decomposes as a direct sum $(D_{i,i} \otimes A) \oplus (D_{i,>i} \otimes A)$, where $D_{i,>i} = \oplus_{j>i} D_{i,j}$, and the differential of $P_i = D_i \otimes A$ sends $D_{i,i} \otimes A$ to $D_{i-1,i-1} \otimes A$. □

2.3. Basics on Fomin–Kirillov algebras

All the following results are classical and can be found in [**5**] (see also [**10, 6**]). For this section we fix $n \in \mathbb{N}_{\geqslant 2}$. Define the vector space

$$V(n) = \mathrm{span}_\Bbbk \Big\langle \big\{[i,j] \colon i,j \in [\![1,n]\!], i \neq j\big\} \Big\rangle \Big/ \Big\langle \big\{[i,j] + [j,i] \colon i,j \in [\![1,n]\!], i \neq j\big\} \Big\rangle,$$

where we recall that, given $i,j \in \mathbb{Z}$ such that $i \leqslant j$, $[\![i,j]\!] = \{m \in \mathbb{Z} \colon i \leqslant m \leqslant j\}$. We will denote the class of $[i,j]$ also by $[i,j]$. Let $\mathbb{S}_n$ be the group of permutations of $\{1,\dots,n\}$, and for $i,j \in [\![1,n]\!]$ different, let $(i,j) \in \mathbb{S}_n$ be the unique transposition interchanging i and j (and fixing all other elements of $[\![1,n]\!]$). It is clear that $V(n)$ has a (left) action of $\Bbbk\mathbb{S}_n$ given by $\sigma \cdot [i,j] = [\sigma(i),\sigma(j)]$, for all $\sigma \in \mathbb{S}_n$ and $[i,j] \in V(n)$. Moreover, $V(n)$ is also a (left) comodule over $\Bbbk\mathbb{S}_n$ via the left coaction $\delta\colon V(n) \to \Bbbk\mathbb{S}_n \otimes V(n)$ defined as $\delta([i,j]) = (i,j) \otimes [i,j]$. Recall that, for a group G, a left $\Bbbk G$-comodule (V,δ) is equivalent to a G-decomposition $V = \oplus_{g\in G} V^g$, by means of $\delta(v) = g \otimes v$, for all $v \in V^g$ and $g \in G$. Let ${}^{\Bbbk\mathbb{S}_n}_{\Bbbk\mathbb{S}_n}\mathscr{YD}$ be the category of (left) Yetter–Drinfeld modules over the Hopf algebra $\Bbbk\mathbb{S}_n$ (in the symmetric monoidal category of vector spaces). We recall that a Yetter–Drinfeld module over a group Hopf algebra $\Bbbk G$ is just a G-module V together with a G-decomposition $V = \oplus_{g\in G} V^g$ such that $g \cdot v \in V^{ghg^{-1}}$, for $g,h \in G$ and $v \in V^h$. We remark that ${}^{\Bbbk G}_{\Bbbk G}\mathscr{YD}$ is a braided monoidal category, with the tensor product $\otimes$ (and the usual G-action and G-coaction for tensor products), the unit $\Bbbk$ (with the trivial G-action and G-coaction) and the braiding $c_{V,W}\colon V \otimes W \to W \otimes V$ of the form $v \otimes w \mapsto g \cdot w \otimes v$, for $v \in V^g$, $g \in G$ and $w \in W$ (see [**16**, 11.6]). It is easy to verify that $V(n)$ is, in fact, a Yetter–Drinfeld module for the previous structures. In particular, the braiding $c_n\colon V(n)^{\otimes 2} \to V(n)^{\otimes 2}$ on $V(n)$ is of the form

$$c_n\big([i,j] \otimes [k,\ell]\big) = (i,j) \cdot [k,\ell] \otimes [i,j],$$

for all $[i,j],[k,\ell] \in V(n)$.

Let $TV(n)$ be the tensor algebra with the product given by concatenation, but which will be denoted simply by juxtaposition. Define the subsets of $V(n)^{\otimes 2}$ given by

$$\mathscr{R}_2(n) = \Big\{[i,j]^2 \colon \text{for all } i,j \in [\![1,n]\!] \text{ with } \#\{i,j\} = 2\Big\},$$

as well as

$$\mathscr{R}_3(n)=\Big\{[i,j][j,k] + [j,k][k,i] + [k,i][i,j] \colon \text{for all } i,j,k \in [\![1,n]\!] \text{ with } \#\{i,j,k\} = 3\Big\}$$

and

$$\mathscr{R}_4(n) = \Big\{[i,j][k,\ell] - [k,\ell][i,j] \colon \text{for all } i,j,k,\ell \in [\![1,n]\!] \text{ with } \#\{i,j,k,\ell\} = 4\Big\}.$$

The **Fomin–Kirillov algebra** $\mathrm{FK}(n)$ is the (unitary) algebra defined as the quotient of $TV(n)$ by the (two-sided) ideal generated by the vector subspace $R(n) \subseteq V(n)^{\otimes 2}$ spanned by $\mathscr{R}_2(n) \cup \mathscr{R}_3(n) \cup \mathscr{R}_4(n)$. By definition, it is a quadratic algebra.

Since ${}^{\Bbbk\mathbb{S}_n}_{\Bbbk\mathbb{S}_n}\mathscr{YD}$ is a braided monoidal category and $\mathrm{FK}(n)$ is a unitary algebra in ${}^{\Bbbk\mathbb{S}_n}_{\Bbbk\mathbb{S}_n}\mathscr{YD}$, for the subspace $R(n)$ of $V(n)^{\otimes 2}$ is a Yetter–Drinfeld submodule (see [**16**, Def. 11.6.4], for the definition), the braiding of the latter category implies that $\mathrm{FK}(n)^{\otimes 2}$ is also a unitary algebra in ${}^{\Bbbk\mathbb{S}_n}_{\Bbbk\mathbb{S}_n}\mathscr{YD}$ (see [**16**, Def. 11.5.2]). Define the map $\Delta\colon \mathrm{FK}(n) \to \mathrm{FK}(n)^{\otimes 2}$ as the unique morphism of unitary algebras satisfying that $\Delta([i,j]) = 1 \otimes [i,j] + [i,j] \otimes 1$, for all $[i,j] \in V(n)$, and $\epsilon\colon \mathrm{FK}(n) \to \Bbbk$ as the unique morphism of unitary algebras satisfying that $\epsilon([i,j]) = 0$, for all $[i,j] \in V(n)$. The following result is well known and easy to prove (see also [**14**, Example 6.2]).

Proposition 2.3. *The Fomin–Kirillov algebra* $\mathrm{FK}(n)$ *provided with the coproduct* Δ *and the counit* ϵ *is a bialgebra in the braided monoidal category* ${}^{\Bbbk\mathbb{S}_n}_{\Bbbk\mathbb{S}_n}\mathscr{YD}$. *It is, moreover, a Hopf algebra in* ${}^{\Bbbk\mathbb{S}_n}_{\Bbbk\mathbb{S}_n}\mathscr{YD}$, *with the unique antipode* $S\colon \mathrm{FK}(n) \to \mathrm{FK}(n)$ *satisfying that* $S([i,j]) = -[i,j]$, *for all* $[i,j] \in V(n)$.

Remark 2.4. The notion of bialgebra makes sense in any braided monoidal category, as well as the ability to naturally endow the tensor product of two algebras with an algebra structure, by making use of the braiding (see [**16**, 11.5]). However, when the braiding is not symmetric several anomalies arise, which do not occur in the symmetric case. It is for this reason that a bialgebra (resp., Hopf algebra) object in a braided monoidal category is usually called a **braided bialgebra** (resp., **braided Hopf algebra**), to emphasize that the braiding of the underlying monoidal category is not symmetric.

The inclusion $\iota_n\colon V(n) \to V(n+1)$ sending $[i,j] \in V(n)$ to the same element considered in $V(n+1)$ induces an algebra morphism $I_n\colon \mathrm{FK}(n) \to \mathrm{FK}(n+1)$. Moreover, there is a retraction $\pi_{n+1}\colon V(n+1) \to V(n)$ of ι_n sending $[i,n+1] \in V(n+1)$ to zero for all $i \in [\![1,n]\!]$. It induces an algebra morphism $\Pi_{n+1}\colon \mathrm{FK}(n+1) \to \mathrm{FK}(n)$ such that $\Pi_{n+1} \circ I_n$ is the identity of $\mathrm{FK}(n)$.

Since the Fomin–Kirillov algebra $\mathrm{FK}(n)$ is quadratic, it is canonically a graded algebra (over $\mathbb{N}_0$), by setting the degree $|[i,j]|$ to be 1, for all $[i,j] \in V(n)$. Recall that the **Hilbert series** of a graded vector space $W = \oplus_{i\in\mathbb{Z}} W_i$ is defined as the formal series $W(t) = \sum_{i\in\mathbb{Z}} \dim(W_i) t^i \in \Bbbk[\![t^{-1},t]\!]$. The Hilbert series of $\mathrm{FK}(n)$ is well known for $n = 2, 3, 4, 5$, and in those cases $\mathrm{FK}(n)$ is also finite dimensional. However, it is not known if $\mathrm{FK}(6)$ is even finite dimensional. For $m \in \mathbb{N}$, denote $[m] = \sum_{i=0}^{m-1} t^i$. Then the Hilbert series of $\mathrm{FK}(2)$ is $[2]$, that of $\mathrm{FK}(3)$ is $[2]^2[3] = 1 + 3t + 4t^2 + 3t^3 + t^4$, that of $\mathrm{FK}(4)$ is $[2]^2[3]^2[4]^2$ and that of $\mathrm{FK}(5)$ is $[4]^4[5]^2[6]^4$ (see [**5**, (2.8)]). Note that the dimension of a graded vector space with Hilbert series $\prod_{i=1}^{j}[d_i]^{r_i}$ is $\prod_{i=1}^{j} d_i^{r_i}$. In consequence, the dimension of $\mathrm{FK}(2)$ is 2, that of $\mathrm{FK}(3)$ is 12, that of $\mathrm{FK}(4)$ is 576 and that of $\mathrm{FK}(5)$ is 8294400.

Remark 2.5. It is easy to show that $\mathrm{FK}(n)$ is a **symmetric algebra** in the sense introduced by Ardizzoni in [**2**, Def. 3.5], *i.e.* it is the universal enveloping algebra of a braided vector space with zero (braided) bracket. In this case, $(V(n), c_n)$ is the braided vector space. A braided vector space with a (braided) bracket is one of the possible definitions of **braided Lie algebra**. However, there is no canonical (co)homology theory associated to such a braided Lie algebra similar to the one obtained from the Chevalley–Eilenberg complex for traditional Lie algebras (in symmetric monoidal categories).

Note the linear isomorphism

$$V(n)^* \simeq \mathrm{span}_{\Bbbk} \Big\langle \big\{ \langle i,j \rangle : i,j \in [\![1,n]\!], i \neq j \big\} \Big\rangle \Big/ \Big\langle \big\{ \langle i,j \rangle + \langle j,i \rangle : i,j \in [\![1,n]\!], i \neq j \big\} \Big\rangle.$$

We will denote the class of $\langle i,j \rangle$ also by $\langle i,j \rangle$. The previous isomorphism comes from considering $\{\langle i,j \rangle\}_{1 \leqslant i < j \leqslant n}$ as the dual basis to $\{[i,j]\}_{1 \leqslant i < j \leqslant n}$ in $V(n)$. The (left) action of $\Bbbk\mathbb{S}_n$ on $V(n)$ induces a (left) action of $\Bbbk\mathbb{S}_n$ on $V(n)^*$ by the usual formula $(\sigma \cdot f)(v) = f(\sigma^{-1} \cdot v)$, for all $v \in V(n)$, $f \in V(n)^*$ and $\sigma \in \Bbbk\mathbb{S}_n$. It is just $\sigma \cdot \langle i,j \rangle = \langle \sigma(i), \sigma(j) \rangle$, for all $\sigma \in \mathbb{S}_n$, and $\langle i,j \rangle \in V(n)^*$. Analogously, $V(n)^*$ is also a (left) comodule over $\Bbbk\mathbb{S}_n$ via the (left) coaction $\delta' \colon V(n)^* \to \Bbbk\mathbb{S}_n^* \otimes V(n)$ given by $\delta'(\langle i,j \rangle) = (i,j) \otimes \langle i,j \rangle$, dual to that of $V(n)$. Recall that the dual V^* of a finite dimensional Yetter–Drinfeld $V \in {}^{\Bbbk G}_{\Bbbk G}\mathscr{YD}$ for the usual G-action $(g \cdot f)(v) = f(g^{-1} \cdot v)$, for $v \in V$, $f \in V^*$ and $g \in G$, and the usual G-coaction given by the decomposition $V^* = \oplus_{g \in G} (V^*)^g$ with $(V^*)^g = (V^{g^{-1}})^*$, is also a Yetter–Drinfeld module over $\Bbbk G$. Hence, $V(n)^*$ is a Yetter–Drinfeld module over $\Bbbk\mathbb{S}_n$ for these structures and the induced braiding $c_n^! \colon (V(n)^*)^{\otimes 2} \to (V(n)^*)^{\otimes 2}$ is

$$c_n^!\big(\langle i,j \rangle \otimes \langle k,\ell \rangle\big) = (i,j) \cdot \langle k,\ell \rangle \otimes \langle i,j \rangle,$$

for all $\langle i,j \rangle, \langle k,\ell \rangle \in V(n)^*$.

The quadratic dual $\mathrm{FK}(n)^!$ of $\mathrm{FK}(n)$ is given as the quotient of $T(V(n)^*)$ by the (two-sided) ideal $\langle R(n)^{\perp} \rangle$ generated by the image of $\mathrm{id}_{(V(n)^*)^{\otimes 2}} + c_n^!$. More concretely, $\langle R(n)^{\perp} \rangle$ is generated by the sets

$$\mathscr{R}_3^!(n) = \big\{ \langle i,j \rangle \langle j,k \rangle + \langle j,k \rangle \langle i,k \rangle : \text{for all } i,j,k \in [\![1,n]\!] \text{ with } \#\{i,j,k\} = 3 \big\}$$

and

$$\mathscr{R}_4^!(n) = \big\{ \langle i,j \rangle \langle k,\ell \rangle + \langle k,\ell \rangle \langle i,j \rangle : \text{for all } i,j,k,\ell \in [\![1,n]\!] \text{ with } \#\{i,j,k,\ell\} = 4 \big\}.$$

Since the vector subspace of $(V(n)^*)^{\otimes 2}$ spanned by $\mathscr{R}_3^!(n)$ and $\mathscr{R}_4^!(n)$ is a Yetter–Drinfeld submodule, $\mathrm{FK}(n)^!$ is a unitary algebra in ${}^{\Bbbk\mathbb{S}_n}_{\Bbbk\mathbb{S}_n}\mathscr{YD}$. These algebras were intensively studied in [**20**].

Remark 2.6. Let A be the Fomin–Kirillov algebra $\mathrm{FK}(n)$. Recall that if V is a Yetter–Drinfeld module over $\Bbbk G$, then the **inverse** Yetter–Drinfeld module V^{inv} is given by the same G-module V but the coaction is induced by $(V^{\mathrm{inv}})^g = V^{g^{-1}}$, for $g \in G$. It is clearly a Yetter–Drinfeld module over $\Bbbk G$. Note that $K_i(A) = ((A^!_{-i})^*)^{\mathrm{inv}} \otimes A$ is naturally a Yetter–Drinfeld module over $\Bbbk\mathbb{S}_n$, and the maps $d_i \colon K_{i+1}(A) \to K_i(A)$ are morphisms of Yetter–Drinfeld modules, for all $i \in \mathbb{N}_0$. The last statement can be proved as follows. First, one notes that the isomorphisms (1) are of Yetter–Drinfeld modules over $\Bbbk\mathbb{S}_n$ if the left member has inverse Yetter–Drinfeld module structure, and then one uses the equivalent description of the Koszul differential given in (2), which is clearly a morphism of $\Bbbk\mathbb{S}_n$-modules and $\Bbbk\mathbb{S}_n$-comodules.[2] We will omit the superscript inv from now on to simplify the notation. As a consequence, the homology $H_\bullet(K_\bullet(A), d_\bullet)$ is also a Yetter–Drinfeld module over $\Bbbk\mathbb{S}_n$.

2.4. More on Yoneda algebras

Recall that a (cohomologically) graded (braided) algebra $A = \oplus_{n\in\mathbb{Z}} A^n$ with product μ_A in a braided monoidal category $\mathscr{C}$ with braiding c is called **braided graded commutative** if $(\mu_A \circ c_{A,A})(a \otimes b) = (-1)^{nm}\mu_A(a \otimes b)$, for all $a \in A^n$ and $b \in A^m$. For example, if the internal grading of $\mathrm{FK}(n)^!$ is set to coincide with minus the cohomological one (*i.e.* $\langle i, j\rangle$ has cohomological degree 1, for all $i \neq j$ in $[\![1, n]\!]$), then $\mathrm{FK}(n)^!$ is a braided graded commutative algebra in the braided monoidal category ${}^{\Bbbk\mathbb{S}_n}_{\Bbbk\mathbb{S}_n}\mathscr{YD}$. This also follows from the next result together with Proposition 2.1, since the map i there is clearly a morphism of Yetter–Drinfeld modules in case $A = FK(n)$. The following result is proved in [**13**, Thm. 3.12].

Proposition 2.7. *Let B be a (braided) bialgebra in a braided closed monoidal category $\mathscr{C}$ with tensor product $\otimes$ and unit e, that is abelian (as a category) with sufficient projectives and such that the tensor product $\otimes$ is exact. Then, the Yoneda algebra $\mathrm{Ext}^\bullet_B(e, e)$, computed as the cohomology of the internal homomorphisms of B-modules from a B-projective resolution of e to e, is a braided graded commutative algebra in $\mathscr{C}$.*

Remark 2.8. The previous result also follows from the Hilton–Eckmann argument explained in [**19**, Thm. 1.7], if one works with a suspended monoidal category (see [**19**, Def. 1.4]) that is enriched in a braided monoidal category $(\mathscr{V}, c)$, a particular case of the notion introduced in [**3**, Def. 3.1]. Indeed, under the extra-hypothesis, the new version of [**19**, Thm. 1.7], implies that the shifted endomorphism algebra of the unit e is a braided graded commutative algebra. The proof is precisely the same, with the minor exception that one should replace $f \otimes g$ in the upper left square of the last diagram in [**19**, p. 2243], by $c(f \otimes g)$. Concerning the proof of our proposition, consider now a braided closed monoidal category $\mathscr{C}$. It is clearly enriched over itself and the associated bounded derived category for the underlying abelian structure of $\mathscr{C}$ is clearly a suspended monoidal category enriched in the braided monoidal

[2] The need of the inverse construction comes from the fact that the isomorphism $(V^{\otimes n})^* \simeq (V^*)^{\otimes n}$ induced by γ_n in Subsection 2.2 is **not** of $\Bbbk\mathbb{S}_n$-comodules. One can also solve this problem by considering instead $\gamma'_n \colon (V^*)^{\otimes n} \otimes V^{\otimes n} \to \Bbbk$ by $\gamma'_n(f_n \otimes \cdots \otimes f_1, v_1 \otimes \cdots \otimes v_n) = f_1(v_1) \cdots f_n(v_n)$, for $v_1, \dots, v_n \in V$ and $f_1, \dots, f_n \in V^*$. This has the disadvantage that we should replace $A^!$ by the opposite algebra in most of the statements. In any case, we decided to use γ_n since it is very common in the literature (see [**15**]).

category $\mathscr{C}$. The modified version of [**19**, Thm. 1.7], gives us now the statement in the proposition. For an even more general statement, see [**3**, Thm. 5.6].

3. The main result: the Yoneda algebra of $\mathrm{FK}(3)$

It is well known that $\mathrm{FK}(n)$ is **not** Koszul for any $n \in \mathbb{N}_{\geqslant 3}$ (see [**17**]). However, we have the following result, which was mentioned without proof in [**17**].

Proposition 3.1. *Let A be the Fomin–Kirillov algebra* $\mathrm{FK}(3)$*. Then, we have the isomorphisms of (bi)graded Yetter–Drinfeld modules over $\Bbbk\mathbb{S}_3$ given by*

$$H_n\big(K_\bullet(A), d_\bullet\big) \simeq \begin{cases} \Bbbk(-2n), & \text{if } n = 0, 3, \\ 0, & \text{else,} \end{cases} \tag{5}$$

where $\Bbbk(2n)$ has the trivial action and coaction over $\Bbbk\mathbb{S}_3$.

Proof. We will prove this result in the Appendix. □

For convenience, we now recall a basic result on homological algebra, called the Horseshoe lemma (see [**4**, Ch. V, Prop. 2.2]).

Lemma 3.2. *Let A be any algebra and let*

$$0 \to M' \xrightarrow{i} M \xrightarrow{p} M'' \to 0 \tag{6}$$

be a short exact sequence of A-modules. If $(P''_\bullet, \partial''_\bullet)_{\bullet \in \mathbb{N}_0}$ is a complex of projective A-modules together with an augmentation $\epsilon'' \colon P'' \to M''$, and $(P'_\bullet, \partial'_\bullet)_{\bullet\in\mathbb{N}_0}$ is an complex of A-modules acyclic in positive homological degrees together with a quasi-isomorphic augmentation $\epsilon' \colon P' \to M'$, then there is a collection of maps $\{f_n \colon P''_n \to P'_{n-1}\}_{n\in\mathbb{N}}$ of A-modules such that $f_n \circ \partial''_n = -\partial'_{n-1} \circ f_{n+1}$ for all $n \in \mathbb{N}$, a map $f_0 \colon P''_0 \to M$ such that $i \circ \epsilon' \circ f_1 = -f_0 \circ \partial''_0$ and $p \circ f_0 = \epsilon''$, a complex $(P_\bullet, \partial_\bullet)_{\bullet\in\mathbb{N}_0}$ with $P_n = P''_n \oplus P'_n$ and $\partial_n(p'' + p') = \partial''_n(p'') + f_{n+1}(p'') + \partial'_n(p')$ for all $p' \in P'_{n+1}$, $p'' \in P''_{n+1}$ and $n \in \mathbb{N}_0$, and an augmentation $\epsilon \colon P_0 \to M$ given by $\epsilon(p'' + p') = f_0(p'') + \epsilon'(p')$, for all $p' \in P'_0$, $p'' \in P''_0$. The canonical inclusion $i_\bullet \colon P'_\bullet \to P_\bullet$ and projection $p_\bullet \colon P_\bullet \to P''_\bullet$ induce a short exact sequence

$$\begin{array}{ccccccccccccc}
 & & 0 & & 0 & & \cdots & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \xrightarrow{\partial'_n} & P'_n & \xrightarrow{\partial'_{n-1}} & P'_{n-1} & \xrightarrow{\partial'_{n-2}} & \cdots & \xrightarrow{\partial'_1} & P'_1 & \xrightarrow{\partial'_0} & P'_0 & \xrightarrow{\epsilon'} & M' \longrightarrow 0 \\
 & & \downarrow{\scriptstyle i_n} & & \downarrow{\scriptstyle i_{n-1}} & & & & \downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle i_0} & & \downarrow{\scriptstyle i} \\
\cdots & \xrightarrow{\partial_n} & P_n & \xrightarrow{\partial_{n-1}} & P_{n-1} & \xrightarrow{\partial_{n-2}} & \cdots & \xrightarrow{\partial_1} & P_1 & \xrightarrow{\partial_0} & P_0 & \xrightarrow{\epsilon} & M \\
 & & \downarrow{\scriptstyle p_n} & & \downarrow{\scriptstyle p_{n-1}} & & & & \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_0} & & \downarrow{\scriptstyle p} \\
\cdots & \xrightarrow{\partial''_n} & P''_n & \xrightarrow{\partial''_{n-1}} & P''_{n-1} & \xrightarrow{\partial''_{n-2}} & \cdots & \xrightarrow{\partial''_1} & P''_1 & \xrightarrow{\partial''_0} & P''_0 & \xrightarrow{\epsilon''} & M'' \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & \cdots & & 0 & & 0 & & 0
\end{array}$$

of augmented complexes extending (6).

The next result is a bootstrap technique for computing the minimal projective resolution of $\Bbbk$ in the category of bounded-below graded FK(3)-modules. It does not appear in [**18**].

Proposition 3.3. *Let A be the Fomin–Kirillov algebra* FK(3). *Then, the minimal projective resolution $(P_\bullet, \delta_\bullet)_{\bullet\in\mathbb{N}_0}$ of $\Bbbk$ in the category of bounded-below graded A-modules is given as follows. For $n \in \mathbb{N}_0$, set*

$$P_n = \bigoplus_{i\in[\![0,\lfloor n/4\rfloor]\!]} \omega_i.K_{n-4i}(A), \tag{7}$$

where $\lfloor z\rfloor = \sup\{n \in \mathbb{Z} \colon n \leqslant z\}$ denotes the (floor) integer part of $z \in \mathbb{R}$, ω_i is a symbol of internal degree $6i$ for all $i \in \mathbb{N}_0$, and the differential $\delta_{n-1}\colon P_n \to P_{n-1}$ is given by

$$\delta_{n-1}\Bigg(\sum_{i\in[\![0,\lfloor n/4\rfloor]\!]} \omega_i.\rho_{n-4i}\Bigg) = \sum_{i\in[\![0,\lfloor n/4\rfloor]\!]} \big(\omega_i.d_{n-4i-1}(\rho_{n-4i}) + \omega_{i-1}.f_{n-4i}(\rho_{n-4i})\big), \tag{8}$$

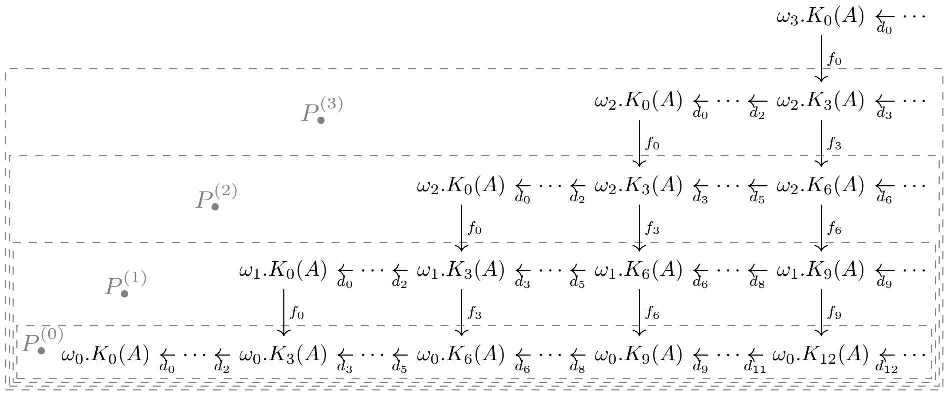
for some homogeneous morphisms $f_j \colon K_j(A) \to K_{j+3}(A)$ of A-modules of (internal) degree 6 for $j \in \mathbb{N}_0$, where $\rho_j \in K_j(A)$, for all $j \in \mathbb{N}_0$ and $\omega_{-1} = 0$. It gives a (minimal) resolution of $\Bbbk$ by means of the augmentation $\epsilon\colon P_0 = K_0(A) \to \Bbbk$ of the Koszul complex. Furthermore, if $\operatorname{char}(\Bbbk) \neq 2, 3$, *then the maps $\{f_\bullet\}_{\bullet\in\mathbb{N}_0}$ can further be chosen so that $(P_\bullet, \delta_\bullet)_{\bullet\in\mathbb{N}_0}$ is a projective resolution of $\Bbbk$ in the category of bounded-below graded A-modules provided with a Yetter–Drinfeld module structure over $\Bbbk\mathbb{S}_3$.*

Proof. Note first that we identify the Koszul complex $(K_\bullet(A), d_\bullet)_{\bullet\in\mathbb{N}_0}$ with the subcomplex $(\omega_0.K_\bullet(A), \delta_\bullet)_{\bullet\in\mathbb{N}_0}$ of $(P_\bullet, \delta_\bullet)_{\bullet\in\mathbb{N}_0}$. Recall that two complexes $(C_\bullet, \partial_\bullet)_{\bullet\in\mathbb{Z}}$ and $(C'_\bullet, \partial'_\bullet)_{\bullet\in\mathbb{Z}}$ coincide up to (homological) degree n if $C_i = C'_i$ for all $i \leqslant n$ and $\partial_i = \partial'_i$ for all $i < n$. We note first that, since $H_n(K_\bullet(A), d_\bullet) = 0$ for $n = 1, 2$ (by Proposition 3.1) and the Koszul complex is minimal, $(K_\bullet(A), d_\bullet)_{\bullet\in\mathbb{N}_0}$ coincides with the minimal projective resolution $(P_\bullet, \delta_\bullet)_{\bullet\in\mathbb{N}_0}$ up to degree 3. Proposition 3.1 gives us the short exact sequence

$$0 \to \operatorname{Im}(d_3) \xrightarrow{\text{inc}} \operatorname{Ker}(d_2) \xrightarrow{\pi} \Bbbk(-6) \to 0$$

of bounded-below graded A-modules, and it tells us that the complex $(C'_\bullet, \partial'_\bullet)_{\bullet\in\mathbb{N}_0}$ given by $C'_n = K_{n+4}(A)$ and $\partial'_n = d_{n+4}$ for all $n \in \mathbb{N}_0$ is a (projective) resolution of $\operatorname{Im}(d_3)$, for the augmentation $d_3\colon K_4(A) \to \operatorname{Im}(d_3)$. On the other hand, the Adams shifted Koszul complex $(K_\bullet(A)(-6), d_\bullet)_{\bullet\in\mathbb{N}_0}$ together with the augmentation map $\epsilon\colon K_0(A)(-6) \to \Bbbk(-6)$ is a (nonacyclic) complex of projective A-modules. By Lemma 3.2, there are morphisms $f_j\colon K_j(A) \to K_{j+3}(A)$ of graded right A-modules of (internal) degree 6 for $j \in \mathbb{N}_0$, such that $d_{j+3} \circ f_{j+1} = -f_j \circ d_j$ for all $j \in \mathbb{N}_0$, $d_2 \circ f_0 = 0$ and $\pi \circ f_0 = \epsilon$. We now consider the projective A-modules $P_\bullet$ given in (7) provided with the maps $\delta_\bullet$ defined in (8) for the maps $f_j\colon K_j(A) \to K_{j+3}(A)$ previously defined for all $j \in \mathbb{N}_0$. Note that this is indeed a complex since $f_{j+3} \circ f_j = 0$, for all $j \in \mathbb{N}_0$, for the top degree of A is 4.

We can depict the complex $(P_\bullet, \delta_\bullet)_{\bullet\in\mathbb{N}_0}$ as follows. It is given as the total complex of the following first quadrant double complex



(9)

where $\omega_i.K_j(A)$ is situated at position $(3i+j, i)$, for $i, j \in \mathbb{N}_0$, and we have omitted the symbols ω_i when writing the components $d_\bullet$ and $f_\bullet$ of the differential $\delta_\bullet$ of $P_\bullet$.

Given $m \in \mathbb{N}_0$ we denote by $(P_\bullet^{(m)}, \delta_\bullet^{(m)})_{\bullet \in \mathbb{N}_0}$ the subcomplex of $(P_\bullet, \delta_\bullet)_{\bullet \in \mathbb{N}_0}$ given by

$$P_n^{(m)} = \bigoplus_{\substack{i \in [\![0, m]\!] \\ i \leqslant \lfloor n/4 \rfloor}} \omega_i.K_{n-4i}(A),$$

for $n \in \mathbb{N}_0$. It is clear that it is a subcomplex of $(P_\bullet, \delta_\bullet)_{\bullet \in \mathbb{N}_0}$. Indeed, $(P_\bullet^{(m)}, \delta_\bullet^{(m)})_{\bullet \in \mathbb{N}_0}$ is precisely the total complex of the double complex given by the lowest $m+1$ rows in (9), *i.e.* the rows indexed from 0 to m (see (9)).

A direct recursive argument based on Proposition 3.1 and Lemma 3.2 shows that

$$H_n\big(P_\bullet^{(m)}, \delta_\bullet\big) \simeq \begin{cases} \Bbbk\left(-6\frac{n(m+1)}{3+4m}\right), & \text{if } n = 0, 3+4m, \\ 0, & \text{else.} \end{cases} \tag{10}$$

Indeed, the previous isomorphism for $m = 0$ is exactly stated in Proposition 3.1, since $(P_\bullet^{(0)}, \delta_\bullet^{(0)})_{\bullet \in \mathbb{N}_0}$ is the Koszul complex $(K_\bullet(A), d_\bullet)_{\bullet \in \mathbb{N}_0}$. Assume that (10) also holds for $0, \dots, m$, with $m \in \mathbb{N}_0$. Consider the short exact sequence of complexes

$$0 \to P_\bullet^{(m)} \to P_\bullet^{(m+1)} \to K_\bullet(A)[4(m+1)]\big(-6(m+1)\big) \to 0,$$

where the first nonzero map is the canonical inclusion. This sequence directly follows from the inclusion of the double complex in (9) given by the lowest $m+1$ rows inside of the double complex given by the lowest $m+2$ rows, and the fact that the cokernel is precisely a shift of the Koszul complex $(K_\bullet(A), d_\bullet)_{\bullet \in \mathbb{N}_0}$. The long exact sequence in homology together with (10) for 0 and m give the purported result for $m+1$, since the connecting homomorphism

$$H_{4(m+1)}\Big(K_\bullet(A)[4(m+1)]\big(-6(m+1)\big)\Big) \to H_{4m+3}(P_\bullet^{(m)})$$

is the identity of $\Bbbk(-6(m+1))$. This last statement can be checked by the usual diagram chasing argument in the Snake lemma. Since filtered colimits are exact in the category of graded modules, $(P_\bullet, \delta_\bullet)_{\bullet \in \mathbb{N}_0}$ is acyclic in positive homological degrees, as was to be shown. Moreover, the complex $(P_\bullet, \delta_\bullet)_{\bullet\in\mathbb{N}_0}$ is minimal, *i.e.* $\delta_\bullet \otimes_A \mathrm{id}_\Bbbk = 0$. This follows from the explicit form of $\delta_\bullet$ given by (8), as $d_\bullet \otimes_A \mathrm{id}_\Bbbk$ vanishes (due to (2)) and $f_\bullet \otimes_A \mathrm{id}_\Bbbk = 0$, for $f_\bullet$ has internal degree 6.

The last part of the statement follows directly from the fact that $(\omega_i)_{i\in\mathbb{N}_0}$ can be taken to be trivial for the action and coaction of $\mathbb{S}_3$ due to Proposition 3.1, together with general nonsense about the construction of the minimal projective resolution of $\Bbbk$ in the category of bounded-below graded A-modules provided with a Yetter–Drinfeld module structure, since the category of Yetter–Drinfeld modules over $\mathbb{S}_3$ is semisimple if $\mathrm{char}(\Bbbk) \neq 2, 3$. □

The following result is a consequence of the first part of the previous proposition and the well-known isomorphism of graded vector spaces $\mathrm{Ext}^n_A(\Bbbk, \Bbbk) \simeq \mathrm{Hom}_A(P_n, \Bbbk)$ for any quadratic algebra A and any minimal projective resolution P_n of $\Bbbk$ in the category of bounded-below graded A-modules. Indeed, consider for $n \in \mathbb{N}_0$ the graded vector space

$$J_n = \bigoplus_{i\in[\![0,\lfloor n/4\rfloor]\!]} \omega_i.C_{n-4i},$$

where $C_0 = \Bbbk$, $C_1 = V$, C_n is the dual of (1) for $n \in \mathbb{N}_{\geqslant 2}$, and zero else. Then, $\mathrm{Hom}_A(P_n, k) \simeq J_n^* \simeq \oplus_{i\in[\![0,\lfloor n/4\rfloor]\!]}(\omega_i.C_{n-4i})^*$, for $n \in \mathbb{N}_0$, giving the next result.

Corollary 3.4. *Let A be the Fomin–Kirillov algebra* FK(3)*. For every $n \in \mathbb{N}_0$, we have the isomorphism of vector spaces*

$$\mathrm{Ext}^{n,i}_A(\Bbbk, \Bbbk) \simeq \begin{cases} \Bbbk, & \textit{if } i \in \{-n-2j \colon j \in [\![0, \lfloor n/4\rfloor]\!]\}, \\ 0, & \textit{else.} \end{cases}$$

We continue to denote the Fomin–Kirillov algebra FK(3) by A. We now show that the algebraic structure of the Yoneda algebra $\mathrm{Ext}^\bullet_A(\Bbbk, \Bbbk)$ follows without much more effort. Recall first that the cohomology of the dg algebra $\mathcal{E}nd_A(P_\bullet)$ with differential given by $g_\bullet \mapsto \delta_\bullet \circ g_\bullet - (-1)^{|g|} g_\bullet \circ \delta_\bullet$ for any homogeneous morphism of graded A-modules $g_\bullet \colon P_\bullet \to P_\bullet$ of cohomological degree $|g|$ is precisely the graded algebra $\mathrm{Ext}^\bullet_A(\Bbbk, \Bbbk)$. Indeed, the morphism $\mathcal{E}nd_A(P_\bullet) \to \mathcal{H}om_A(P_\bullet, \Bbbk)$ given by $g_\bullet \mapsto \epsilon \circ g_\bullet$ is the desired quasi-isomorphism.

We now prove the main result in [**18**], namely their Thm. 4.17.

Theorem 3.5. *Let A be the Fomin–Kirillov algebra* FK(3) *over a field $\Bbbk$ of characteristic different from* 2 *and* 3*. Then, there is an isomorphism of (bi)graded algebras*

$$\mathrm{Ext}^\bullet_A(\Bbbk, \Bbbk) \simeq A^![\omega]$$

in ${}^{\Bbbk\mathbb{S}_3}_{\Bbbk\mathbb{S}_3}\mathcal{YD}$, where ω has cohomological degree 4 *and internal degree* -6 *and it is invariant and coinvariant under $\Bbbk\mathbb{S}_3$, and $A^![\omega]$ denotes the polynomial algebra with coefficients in $A^!$.*

Proof. Recall the canonical injection $i \colon A^! \to \mathrm{Ext}^\bullet_A(\Bbbk, \Bbbk)$ given in Proposition 2.1, which is a morphism of (bi)graded algebras in ${}^{\Bbbk\mathbb{S}_3}_{\Bbbk\mathbb{S}_3}\mathcal{YD}$. To prove the theorem it suffices to find a central regular element $\omega \in \mathrm{Ext}^\bullet_A(\Bbbk, \Bbbk)$ of cohomological degree 4 and internal degree -6 that is invariant and coinvariant under $\Bbbk\mathbb{S}_3$.

Let $\Omega\colon P_\bullet \to P_\bullet[4](-6)$ be the homogeneous morphism of complexes of cohomological degree -4 and internal degree 6 given by $\Omega(\omega_0.\rho_j) = 0$ for all $j \in \mathbb{N}_0$ and $\Omega(\omega_i.\rho_j) = s_{4,-6}(\omega_{i-1}.\rho_j)$, for all $i \in \mathbb{N}$ and $j \in \mathbb{N}_0$, where $s_{4,-6}\colon P_\bullet \to P_\bullet[4](-6)$ is the suspension morphism (in the homological and internal degrees), *i.e.* the homogeneous morphism of cohomological degree -4 and internal degree 6 whose underlying set theoretic map is the identity, and $\rho_j \in K_j(A)$. The kernel of Ω is then $P_\bullet^{(0)}$, which is canonically identified with the Koszul complex $K_\bullet(A)$. Hence, we obtain the short exact sequence

$$0 \to K_\bullet(A) \to P_\bullet \xrightarrow{\Omega} P_\bullet[4](-6) \to 0$$

of complexes of graded A-modules.

Let us denote the composition of Ω with $s_{4,-6}^{-1}$ by $\bar{\Omega}$. Since Ω is a morphism of complexes, $\bar{\Omega}$ is a cycle of cohomological degree 4 and internal degree -6 in the dg algebra $\mathcal{E}nd_A(P_\bullet)$. Note that the map $\mathcal{E}nd_A(P_\bullet) \to \mathcal{E}nd_A(P_\bullet)$ given by $g_\bullet \mapsto g_\bullet \circ \bar{\Omega}$ is injective on cohomology. Indeed, if $\epsilon \circ g_\bullet \in (\omega_i.C_n)^*$ is a nonzero element, for some $i, n \in \mathbb{N}_0$, then $\epsilon \circ g_\bullet \circ \Omega$ is a nonzero element of $(\omega_{i+1}.C_n)^*$. Moreover, the map it induces on cohomology is exactly the right multiplication by the cohomology class of $\bar{\Omega}$, *i.e.* $[f_\bullet] \mapsto [f_\bullet][\bar{\Omega}]$, where the brackets denote here the cohomology class of the corresponding cocycles. Set $\omega = [\bar{\Omega}] \in \mathrm{Ext}_A^{4,-6}(\Bbbk, \Bbbk)$. Since $\bar{\Omega}$ is $\mathbb{S}_3$-equivariant, then ω is invariant under the action of $\Bbbk\mathbb{S}_3$. Moreover, as $\mathrm{Ext}_A^\bullet(\Bbbk, \Bbbk)$ is a braided graded commutative algebra by Propositions 2.3 and 2.7, and $\bar{\Omega} \in \mathrm{Ext}_A^{4,-6}(\Bbbk, \Bbbk)$ is invariant under the action of $\Bbbk\mathbb{S}_3$, it is in the center of $\mathrm{Ext}_A^\bullet(\Bbbk, \Bbbk)$. It is also $\mathbb{S}_3$-coinvariant by Proposition 3.3. The theorem is thus proved. □

The computation of the algebraic structure of the Yoneda algebra of the bosonization $\mathrm{FK}(3)\#\Bbbk\mathbb{S}_3$ from the algebra structure of the Yoneda algebra of FK(3) is then standard, if $\mathrm{char}(\Bbbk) \neq 2, 3$ (see [**18**, Lemma 4.18 and Thm. 4.19]). Moreover, given V any finite dimensional right module over $A = \mathrm{FK}(3)$ with a compatible Yetter–Drinfeld module structure over $\Bbbk\mathbb{S}_3$, one also gets immediately that $\mathrm{Ext}_A^\bullet(\Bbbk, V)$ is a finitely generated module over $\mathrm{Ext}_A^\bullet(\Bbbk, \Bbbk)$, since V is obtained from $\Bbbk$ via a finite sequence of extensions. Since taking $\mathbb{S}_3$-invariants is exact if $\mathrm{char}(\Bbbk) \neq 2, 3$, the module $\mathrm{Ext}_{A\#\Bbbk\mathbb{S}_3}^\bullet(\Bbbk, V)$ over $\mathrm{Ext}_{A\#\Bbbk\mathbb{S}_3}^\bullet(\Bbbk, \Bbbk)$ is also finitely generated.

Appendix A.

We present in this section the detailed proof of Proposition 3.1. This is the only computationally involved part of the article, and can be regarded as "brute force". However, despite its length, it is more or less straightforward provided one has bases of the algebras $A = \mathrm{FK}(3)$ and $A^!$, as well as the structure constants (for the products) on those bases. Moreover, by the periodicity of the behavior of the expression of the differential of the Koszul complex after degree 4, where all morphisms starting at even degrees behave in the same way and all differentials starting at odd degrees too (see (11)), the computations can be checked using a mathematical software.

Recall that the Koszul complex $(K_\bullet(A), d_\bullet)_{\bullet \in \mathbb{N}_0}$ is always acyclic in homological degree 1 and $H_0(K_\bullet(A), d_\bullet) \simeq \Bbbk$ by (7). Moreover, the latter isomorphism is clearly of $\mathbb{S}_3$-modules and $\mathbb{S}_3$-comodules, where $\Bbbk$ has the trivial action and the trivial coaction over $\mathbb{S}_3$. Hence, to prove Proposition 3.1 it suffices to compute $H_n(K_\bullet(A), d_\bullet)$ for $n \geqslant 2$.

Let us write $a = [1,2]$, $b = [2,3]$ and $c = [3,1]$, and let $\mathscr{B}_1^! = \{A, B, C\}$ be the dual basis to the basis $\mathscr{B}_1 = \{a, b, c\}$ of $V(3)$. Then,

$$A \simeq \Bbbk\langle a, b, c\rangle/\langle a^2, b^2, c^2, ab + bc + ca, ba + ac + cb\rangle,$$
$$A^! \simeq \Bbbk\langle A, B, C\rangle/\langle BA - AC, CA - AB, AB - BC, CB - BA\rangle.$$

It is easy to see that

$$\mathscr{B} = \{\underbrace{1}_{\mathscr{B}_0}, \underbrace{a, b, c}_{\mathscr{B}_1}, \underbrace{ab, bc, ba, ac}_{\mathscr{B}_2}, \underbrace{aba, abc, bac}_{\mathscr{B}_3}, \underbrace{abac}_{\mathscr{B}_4}\}$$

is a basis (of homogeneous elements) of A.

Note that the relations appearing in the presentation of $A^!$ do not satisfy the conditions in Bergman's diamond lemma. Moreover, given $\{X, Y, Z\} = \{A, B, C\}$, then $XY = YZ$ and $XY = ZX$ in $A^!$. The next identities follow easily from the previous property.

Fact A.1. *Let $\{X, Y, Z\} = \{A, B, C\}$. Then $X^2 \in \mathscr{Z}(A^!)$. Moreover, for $i \in \mathbb{N}_0$, the following identities hold in $A^!$:*

(i) $X^{2i}Y = Z^{2i}Y$ *and* $X^{2i+1}Y = Z^{2i+1}X$*;*

(ii) $B^{2i+2}A = AB^{2i+2} = A^{2i+1}B^2$, $AB^{2i+1} = A^{2i+1}B$ *and* $B^{2i+1}A = A^{2i+1}C$*;*

(iii) $C^{2i+2}A = AC^{2i+2} = A^{2i+1}B^2$, $AC^{2i+1} = A^{2i+1}C$ *and* $C^{2i+1}A = A^{2i+1}B$*.*

Using the previous result and well chosen algebra morphisms $A^! \to \Bbbk$, one obtains the following (see [**18**, Lemma 4.4], but also [**20**, Thm. 4.10]).

Lemma A.2. *For $n \geqslant 2$, set $\mathscr{B}_n^! = \{A^n, B^n, C^n, A^{n-1}B, A^{n-1}C, A^{n-2}B^2\} \subseteq A^!_{-n}$. Note that $\#(\mathscr{B}_2^!) = 5$ but $\#(\mathscr{B}_n^!) = 6$ for $n \geqslant 3$. Then, $\mathscr{B}_n^!$ is a basis of $A^!_{-n}$ for all $n \in \mathbb{N}$.*

We will now write down the Koszul complex $((A^!_{-\bullet})^* \otimes A, d_\bullet)_{\bullet \in \mathbb{N}_0}$. To do so, it is convenient to write $\epsilon^! \in (A_0^!)^*$ for the augmentation of $A^!$, $\{\alpha_1, \beta_1, \gamma_1\} \subseteq (A^!_{-1})^*$ for the dual basis to $\mathscr{B}_1^!$, $\{\alpha_2, \beta_2, \gamma_2, \alpha_1\beta, \alpha_1\gamma\} \subseteq (A^!_{-2})^*$ for the dual basis to $\mathscr{B}_2^!$, and if $n \geqslant 3$, $\{\alpha_n, \beta_n, \gamma_n, \alpha_{n-1}\beta, \alpha_{n-1}\gamma, \alpha_{n-2}\beta_2\} \subseteq (A^!_{-n})^*$ for the dual basis to $\mathscr{B}_n^!$. We will usually omit the subindex 1, and write thus α instead of α_1, etc. To ease the notation we assume that if the subindex of the previous elements is zero or negative the corresponding element is zero (*e.g.* $\alpha_0\beta = 0$). Moreover, let $\chi_{2\mathbb{Z}} \colon \mathbb{Z} \to \{0, 1\}$ be the characteristic function of $2\mathbb{Z}$ and write χ_n instead of $\chi_{2\mathbb{Z}}(n)$.

The next result is a straightforward computation.

Fact A.3. *The action of $A^!$ on $(A^!)^\#$ is given as follows. First, $A.\alpha = B.\beta = C.\gamma = \epsilon^!$ and the other actions of $\mathscr{B}_1^!$ on the other basis elements of $(A^!_{-1})^*$ vanish. If $n \geqslant 2$, then*

(i) $A.\alpha_n = \alpha_{n-1}$, $A.\beta_n = A.\gamma_n = 0$, $A.\alpha_{n-1}\beta = \chi_n\gamma_{n-1} + \alpha_{n-2}\gamma$,
$A.\alpha_{n-1}\gamma = \chi_n\beta_{n-1} + \alpha_{n-2}\beta$, *and* $A.\alpha_{n-2}\beta_2 = \chi_{n+1}(\beta_{n-1} + \gamma_{n-1}) + \alpha_{n-3}\beta_2$*;*

(ii) $B.\beta_n = \beta_{n-1}$, $B.\alpha_n = B.\gamma_n = 0$, $B.\alpha_{n-1}\beta = \alpha_{n-1} + \chi_{n+1}\gamma_{n-1} + \alpha_{n-3}\beta_2$,
$B.\alpha_{n-1}\gamma = \chi_n\gamma_{n-1} + \alpha_{n-2}\gamma$, *and* $B.\alpha_{n-2}\beta_2 = \alpha_{n-2}\beta$*;*

(iii) $C.\gamma_n = \gamma_{n-1}$, $C.\alpha_n = C.\beta_n = 0$, $C.\alpha_{n-1}\beta = \chi_n\beta_{n-1} + \alpha_{n-2}\beta$, $C.\alpha_{n-1}\gamma = \alpha_{n-1} + \chi_{n+1}\beta_{n-1} + \alpha_{n-3}\beta_2$, *and* $C.\alpha_{n-2}\beta_2 = \alpha_{n-2}\gamma$.

The differential $d_\bullet$ of the Koszul complex $K_\bullet(A)$ is explicitly given as follows. The differential $d_0\colon (A^!_{-1})^* \otimes A \to (A^!_0)^* \otimes A$ sends $\alpha \otimes x$ to $\epsilon^! \otimes a.x$, $\beta \otimes x$ to $\epsilon^! \otimes b.x$, and $\gamma \otimes x$ to $\epsilon^! \otimes c.x$, and if $n \geqslant 2$, then $d_{n-1}\colon (A^!_{-n})^* \otimes A \to (A^!_{-n+1})^* \otimes A$ gives

$$\begin{aligned} \alpha_n|x &\mapsto \alpha_{n-1}|a.x, \quad \beta_n|x \mapsto \beta_{n-1}|b.x, \quad \gamma_n|x \mapsto \gamma_{n-1}|c.x, \\ \alpha_{n-1}\beta|x &\mapsto \big(\chi_n\gamma_{n-1} + \alpha_{n-2}\gamma\big)|a.x + \big(\chi_n\beta_{n-1} + \alpha_{n-2}\beta\big)|c.x \\ &\quad + \big(\alpha_{n-1} + \chi_{n+1}\gamma_{n-1} + \alpha_{n-3}\beta_2\big)|b.x, \\ \alpha_{n-1}\gamma|x &\mapsto \big(\chi_n\beta_{n-1} + \alpha_{n-2}\beta\big)|a.x + \big(\chi_n\gamma_{n-1} + \alpha_{n-2}\gamma\big)|b.x \\ &\quad + \big(\alpha_{n-1} + \chi_{n+1}\beta_{n-1} + \alpha_{n-3}\beta_2\big)|c.x, \\ \alpha_{n-2}\beta_2|x &\mapsto \big(\chi_{n+1}(\beta_{n-1} + \gamma_{n-1}) + \alpha_{n-3}\beta_2\big)|a.x + \alpha_{n-2}\beta|b.x + \alpha_{n-2}\gamma|c.x, \end{aligned} \tag{11}$$

for all $x \in A$, where we have used vertical bars between the elements instead of tensor products to reduce space. We will use this notation for the rest of this section.

Since $d_{n-1}((A^!_{-n})^* \otimes A_m) \subseteq (A^!_{-n+1})^* \otimes A_{m+1}$, it suffices to study the mapping $d_{n-1,m} = d_{n-1}|_{(A^!_{-n})^* \otimes A_m}$, for $m \in \mathbb{N}_0$ and $n \in \mathbb{N}$. Moreover, since $A = \oplus_{m \in [\![0,4]\!]} A_m$, we can restrict ourselves to $m \in [\![0,4]\!]$. Note first that $d_{n-1,4} = 0$, for all $n \in \mathbb{N}$.

Fact A.4. *We have* $\mathrm{Im}(d_{n-1,3}) = (A^!_{-n+1})^* \otimes A_4$ *for* $n \in \mathbb{N}$, *which in turn implies that* $H_n(K_\bullet(A), d_\bullet) \cap ((A^!)^\# \otimes A_4)$ *vanishes for all* $n \in \mathbb{N}_0$ *and*

$$\dim\big(\mathrm{Ker}(d_{n-1,3})\big) = \begin{cases} 3, & \text{if } n = 0, \\ 8, & \text{if } n = 1, \\ 13, & \text{if } n = 3, \\ 12, & \text{if } n = 2 \text{ or } n \geqslant 4. \end{cases} \tag{12}$$

Proof. The first identity follows from (11). Indeed, if $n = 1$, $\epsilon^!|abac = d_0(\alpha|bac)$, and if $n \geqslant 2$, then

$$\begin{aligned} \alpha_{n-1}|abac &= d_{n-1}(\alpha_n|bac), \beta_{n-1}|abac = d_{n-1}(\beta_n|abc), \\ \gamma_{n-1}|abac &= d_{n-1}(-\gamma_n|aba), \\ \alpha_{n-2}\beta|abac &= d_{n-1}(\alpha_{n-1}\gamma|bac - \chi_n\beta_n|abc), \\ \alpha_{n-2}\gamma|abac &= d_{n-1}(\alpha_{n-1}\gamma|abc + \chi_n\gamma_n|aba), \\ \alpha_{n-3}\beta_2|abac &= d_{n-1}\big(\alpha_{n-2}\beta_2|bac - \chi_{n+1}(\beta_n|abc - \gamma_n|aba)\big). \end{aligned}$$

For the last identity we have simply used that $\dim(\mathrm{Ker}(d_{n-1,3}))$ is

$$\dim\big((A^!_{-n})^* \otimes A_3\big) - \dim\big(\mathrm{Im}(d_{n-1,3})\big) = 3\dim(A^!_{-n}) - \dim(A^!_{-n+1}). \qquad \square$$

Lemma A.5. *We have that*

$$\dim\big(\mathrm{Im}(d_{n-1,2})\big) = \begin{cases} 3, & \text{if } n = 1, \\ 8, & \text{if } n = 2, \\ 12, & \text{if } n \geqslant 3, \end{cases} \text{ and } \dim\big(\mathrm{Ker}(d_{n-1,2})\big) = \begin{cases} 4, & \text{if } n = 0, \\ 9, & \text{if } n = 1, \\ 12, & \text{if } n \geqslant 2. \end{cases} \tag{13}$$

From the first identity and (12) *we get that* $H_n(K_\bullet(A), d_\bullet) \cap ((A^!)^\# \otimes A_3) = 0$, *for all* $n \in \mathbb{N}_0$ *such that* $n \neq 3$, *and* $H_3(K_\bullet(A), d_\bullet) \cap ((A^!)^\# \otimes A_3) \simeq \Bbbk(-6)$.

Proof. The first identity follows from computing (11) for a basis of $(A^!_{-n})^* \otimes A_2$ and extracting the linearly dependent elements. The linear independence of the sets below is either trivial or it follows clearly from considering the underlined terms (in the given order). The fact that they are systems of generators for $n \neq 3$ follows from the dimensions computed in (12), and by a simple inspection in case $n = 3$. If $n = 1$, we have the basis of $\mathrm{Im}(d_{0,2})$ given by $\epsilon^!|abc = d_0(\alpha|bc)$, $\epsilon^!|aba = d_0(\alpha|ba)$ and $\epsilon^!|bac = d_0(\beta|ac)$. If $n \geqslant 2$,

$$\begin{Bmatrix} \alpha_{n-1}|abc = d_{n-1}(\alpha_n|bc), \alpha_{n-1}|aba = d_{n-1}(\alpha_n|ba), \beta_{n-1}|aba = d_{n-1}(\beta_n|ab), \\ \beta_{n-1}|bac = d_{n-1}(\beta_n|ac), \gamma_{n-1}|abc = d_{n-1}(\gamma_n|ba), \gamma_{n-1}|bac = d_{n-1}(\gamma_n|ab) \end{Bmatrix} \tag{14}$$

is clearly a linearly independent set of $\mathrm{Im}(d_{n-1,2})$. A basis of $\mathrm{Im}(d_{1,2})$ is given by the union of (14) (for $n = 2$) and

$$\Big\{\beta|abc + \gamma|aba = d_1(\alpha\beta|ba), \alpha|bac - \beta|abc = d_1(\alpha\beta|ac)\Big\}.$$

A basis of $\mathrm{Im}(d_{2,2})$ is given by the union of (14) (for $n = 3$) and

$$\begin{Bmatrix} \underline{\alpha\beta|aba} + \beta_2|abc = d_2(\alpha_2\gamma|ba - \alpha_3|bc), \underline{\alpha\gamma|abc} - \alpha\beta|bac = d_2(\alpha_2\beta|bc), \\ \underline{\alpha\gamma|aba} + \alpha\beta|abc = d_2(\alpha_2\beta|ba), \underline{\alpha_2|bac} - \alpha\beta|abc = d_2(\alpha_2\beta|ac - \gamma_3|ab), \\ \underline{\gamma_2|aba} + \alpha\beta|bac = d_2(\alpha_2\beta|ab - \alpha_3|ba), \underline{\alpha\gamma|bac} - \beta_2|abc = d_2(\alpha_2\gamma|ac + \alpha_3|bc) \end{Bmatrix},$$

whereas a basis of $\mathrm{Im}(d_{n-1,2})$ for $n \geqslant 4$ is given by the union of (14) (for $n \geqslant 4$) and

$$\begin{Bmatrix} \alpha_{n-2}\gamma|bac - \chi_{n+1}\beta_{n-1}|abc - \underline{\alpha_{n-3}\beta_2|abc} = d_{n-1}(\alpha_{n-1}\gamma|ac + \alpha_n|bc - \chi_n\gamma_n|ab), \\ \underline{\alpha_{n-3}\beta_2|bac} + \alpha_{n-1}|bac - \big(\chi_n\beta_{n-1} + \alpha_{n-2}\beta\big)|abc = d_{n-1}(\alpha_{n-1}\beta|ac - \chi_{n+1}\gamma_n|ab), \\ \underline{\alpha_{n-3}\beta_2|aba} + \chi_{n+1}\gamma_{n-1}|aba + \alpha_{n-2}\beta|bac = d_{n-1}(\alpha_{n-1}\beta|ab - \alpha_n|ba - \chi_n\beta_n|ac), \\ \alpha_{n-2}\beta|bac - \underline{\alpha_{n-2}\gamma|abc} = d_{n-1}(\alpha_{n-2}\beta_2|ac), \\ \underline{\alpha_{n-2}\gamma|bac} + \alpha_{n-2}\beta|aba = d_{n-1}(\alpha_{n-2}\beta_2|ab), \\ \underline{\alpha_{n-2}\gamma|aba} + \big(\alpha_{n-2}\beta + \chi_n\beta_{n-1}\big)|abc + \chi_n\gamma_{n-1}|aba = d_{n-1}(\alpha_{n-1}\beta|ba) \end{Bmatrix}.$$

The second identity follows from the first, since $\dim(\mathrm{Ker}(d_{n-1,2}))$ is

$$\dim\big((A^!_{-n})^* \otimes A_2\big) - \dim\big(\mathrm{Im}(d_{n-1,2})\big) = 4\dim(A^!_{-n}) - \dim\big(\mathrm{Im}(d_{n-1,2})\big).$$

The last statement is immediate. □

Lemma A.6. *We have that*

$$\dim\big(\mathrm{Im}(d_{n-1,1})\big) = \begin{cases} 4, & \textit{if } n = 1, \\ 9, & \textit{if } n = 2, \\ 12, & \textit{if } n \geqslant 3, \end{cases} \textit{ and } \dim\big(\mathrm{Ker}(d_{n-1,1})\big) = \begin{cases} 3, & \textit{if } n = 0, \\ 5, & \textit{if } n = 1, \\ 6, & \textit{if } n \geqslant 2. \end{cases} \tag{15}$$

Comparing the first of the previous identities and the second identity of (13) *we conclude that* $H_n(K_\bullet(A), d_\bullet) \cap ((A^!)^\# \otimes A_2) = 0$, *for all* $n \in \mathbb{N}_0$.

Proof. Again, the first identity follows from computing (11) for a basis of the vector space $(A^!_{-n})^* \otimes A_1$ and extracting the linearly dependent elements. As in the previous lemma, the linear independence of the sets below is either trivial or it directly follows from considering the underlined terms (in the given order), and the fact that they are systems of generators follows from the second identity in (13). If $n = 1$, we have the basis of $\mathrm{Im}(d_{0,1})$ given by the elements $\epsilon^!|ab = d_0(\alpha|b)$, $\epsilon^!|ac = d_0(\alpha|c)$, $\epsilon^!|bc = d_0(\beta|c)$ and $\epsilon^!|ba = d_0(\beta|a)$. If $n \geqslant 2$, it is direct that

$$\Big\{\alpha_{n-1}|ab = d_{n-1}(\alpha_n|b), \alpha_{n-1}|ac = d_{n-1}(\alpha_n|c), \beta_{n-1}|ba = d_{n-1}(\beta_n|a), \beta_{n-1}|bc = d_{n-1}(\beta_n|c), \gamma_{n-1}|(ab+bc) = d_{n-1}(-\gamma_n|a), \gamma_{n-1}|(ba+ac) = d_{n-1}(-\gamma_n|b)\Big\} \tag{16}$$

is a linearly independent set of $\mathrm{Im}(d_{n-1,1})$.

A basis of $\mathrm{Im}(d_{1,1})$ is given by the union of (16) (for $n = 2$) and

$$\Big\{\underline{\alpha|ba} - \beta|ab = d_1(\alpha\beta|a + \beta_2|c), \underline{\alpha|bc} + \gamma|ac = d_1(\alpha\beta|c), \underline{\beta|ac} + \gamma|bc = d_1(\alpha\gamma|c)\Big\}.$$

For $n \geqslant 3$, a basis of $\mathrm{Im}(d_{n-1,2})$ is given by the union of (16) (for $n \geqslant 3$) and

$$\begin{aligned}\Big\{ & -\underline{\alpha_{n-1}|bc} - \big(\chi_{n+1}\beta_{n-1} + \alpha_{n-3}\beta_2\big)|ab - \alpha_{n-3}\beta_2|bc + \big(\chi_n\gamma_{n-1} + \alpha_{n-2}\gamma\big)|ba \\ & = d_{n-1}(\alpha_{n-1}\gamma|a + \alpha_n|b + \chi_{n+1}\beta_n|c), \\ & \underline{\alpha_{n-1}|ba} + \big(\chi_{n+1}\gamma_{n-1} + \alpha_{n-3}\beta_2\big)|ba - \big(\alpha_{n-2}\beta + \chi_n\beta_{n-1}\big)|ab - \alpha_{n-2}\beta|bc \\ & = d_{n-1}(\alpha_{n-1}\beta|a + \chi_n\beta_n|c), \\ & \chi_n\beta_{n-1}|ac + \underline{\alpha_{n-2}\beta|ac} + \big(\chi_n\gamma_{n-1} + \alpha_{n-2}\gamma\big)|bc = d_{n-1}(\alpha_{n-1}\gamma|c), \\ & \underline{\alpha_{n-2}\beta|ba} - \alpha_{n-2}\gamma|(ab+bc) = d_{n-1}(\alpha_{n-2}\beta_2|a), \\ & \big(\chi_{n+1}(\beta_{n-1} + \gamma_{n-1}) + \alpha_{n-3}\beta_2\big)|ac + \underline{\alpha_{n-2}\beta|bc} = d_{n-1}(\alpha_{n-2}\beta_2|c), \\ & \big(\chi_{n+1}(\beta_{n-1} + \gamma_{n-1}) + \alpha_{n-3}\beta_2\big)|ab - \underline{\alpha_{n-2}\gamma|(ba+ac)} = d_{n-1}(\alpha_{n-2}\beta_2|b)\Big\}.\end{aligned}$$

The second identity follows from the first, since $\dim(\mathrm{Ker}(d_{n-1,1}))$ is

$$\dim\big((A^!_{-n})^* \otimes A_1\big) - \dim\big(\mathrm{Im}(d_{n-1,1})\big) = 3\dim(A^!_{-n}) - \dim\big(\mathrm{Im}(d_{n-1,1})\big).$$

The last statement is immediate. □

Fact A.7. *We have that*

$$\dim\big(\mathrm{Im}(d_{n-1,0})\big) = \begin{cases} 3, & \text{if } n = 1, \\ 5, & \text{if } n = 2, \\ 6, & \text{if } n \geqslant 3, \end{cases} \tag{17}$$

and $\mathrm{Ker}(d_0) = \Bbbk$. *The latter implies that* $H_0(K_\bullet(A), d_\bullet) \simeq \Bbbk$, *whereas a comparison of* (17) *and the second identity in* (15) *tells us that* $H_n(K_\bullet(A), d_\bullet) \cap ((A^!)^\# \otimes A_1) = 0$, *for all* $n \in \mathbb{N}_0$.

Proof. If $n = 1$, we have the basis of $\mathrm{Im}(d_{0,0})$ given by $\epsilon^!|a = d_0(\alpha|1)$, $\epsilon^!|b = d_0(\beta|1)$, and $\epsilon^!|c = d_0(\gamma|1)$. If $n \geqslant 2$, consider the linearly independent set

$$\Big\{\alpha_{n-1}|a = d_{n-1}(\alpha_n|1), \beta_{n-1}|b = d_{n-1}(\beta_n|1), \gamma_{n-1}|c = d_{n-1}(\gamma_n|1)\Big\} \tag{18}$$

in $(A^!_{-n+1})^* \otimes A_1$. Moreover, the union of the previous set for $n = 2$ with

$$\Big\{\gamma|a + \alpha|b + \beta|c = d_1(\alpha\beta|1), \beta|a + \gamma|b + \alpha|c = d_1(\alpha\gamma|1)\Big\}$$

is a basis of $\mathrm{Im}(d_{1,0})$, and if $n \geqslant 3$, the union of (18) (for $n \geqslant 3$) with

$$\Big\{\underline{\alpha_{n-1}|b} + \big(\chi_n\gamma_{n-1} + \alpha_{n-2}\gamma\big)|a + \big(\chi_n\beta_{n-1} + \alpha_{n-2}\beta\big)|c + \big(\chi_{n+1}\gamma_{n-1} + \alpha_{n-3}\beta_2\big)|b$$
$$= d_{n-1}(\alpha_{n-1}\beta|1),$$
$$\big(\chi_n\beta_{n-1} + \alpha_{n-2}\beta\big)|a + \big(\chi_n\gamma_{n-1} + \alpha_{n-2}\gamma\big)|b + \underline{\alpha_{n-1}|c} + \big(\chi_{n+1}\beta_{n-1} + \alpha_{n-3}\beta_2\big)|c$$
$$= d_{n-1}(\alpha_{n-1}\gamma|1),$$
$$\big(\chi_{n+1}(\beta_{n-1} + \gamma_{n-1}) + \alpha_{n-3}\beta_2\big)|a + \underline{\alpha_{n-2}\beta|b} + \alpha_{n-2}\gamma|c = d_{n-1}(\alpha_{n-2}\beta_2|1)\Big\}$$

is a basis of $\mathrm{Im}(d_{n-1,0})$, since they are clearly linearly independent, as it follows from considering the underlined terms (in the given order). The fact that the previous sets are systems of generators follows directly from the dimension of the domain of $d_{n-1,0}$. This proves the first statement, and the second follows easily. The last statement is now immediate. □

The isomorphism (5) in Proposition 3.1 is now a consequence of Lemmas A.5 and A.6, as well as Facts A.4 and A.7. Finally, it remains to prove that the isomorphism $H_3(K_\bullet(A), d_\bullet) \simeq \Bbbk(-6)$ is compatible with the structures of $\mathbb{S}_3$-modules and $\mathbb{S}_3$-comodules, where $\Bbbk(-6)$ has the trivial action and coaction. This follows from picking an $\mathbb{S}_3$-invariant and $\mathbb{S}_3$-coinvariant element $\bar{\omega}$ in $\mathrm{Ker}(d_{2,3}) \setminus \mathrm{Im}(d_{3,2})$, for instance,

$$\bar{\omega} = 2\Big(\alpha_3|bac + \beta_3|abc - \gamma_3|aba\Big) - \alpha_2\beta|abc + \alpha_2\gamma|aba - \alpha\beta_2|bac.$$

The fact $\bar{\omega} \notin \mathrm{Im}(d_{3,2})$ follows from showing that $\bar{\omega}$ cannot be written as a linear combination of the elements in the basis in the proof of Lemma A.5 for $n = 4$, which is direct to verify.

References

[1] Andruskiewitsch, Nicolás, Schneider, H.-J., *On the classification of finite-dimensional pointed Hopf algebras*, Ann. of Math. (2) **171** (2010), no. 1, 375–417.

[2] Ardizzoni, A., *On primitively generated braided bialgebras*, Algebr. Represent. Theory **15** (2012), no. 4, 639–673.

[3] Batanin, M., Markl, M., *Centers and homotopy centers in enriched monoidal categories*, Adv. Math. **230** (2012), no. 4–6, 1811–1858.

[4] Cartan, H., Eilenberg, S., *Homological algebra*, Princeton Landmarks in Mathematics, Princeton University Press, Princeton, NJ, 1999; With an appendix by David A. Buchsbaum; Reprint of the 1956 original.

[5] Fomin, S., Kirillov, A.N., *Quadratic algebras, Dunkl elements, and Schubert calculus*, *Advances in geometry*, Progr. Math., vol. 172, Birkhäuser Boston, Boston, MA, 1999, pp. 147–182.

[6] Fomin, S., Procesi, C., *Fibered quadratic Hopf algebras related to Schubert calculus*, J. Algebra **230** (2000), no. 1, 174–183.

[7] Graña, M., *Nichols algebras of non-abelian group type: zoo examples*, (2016), available at http://mate.dm.uba.ar/~lvendram/zoo/.

[8] Herscovich, E., *On the multi-Koszul property for connected algebras*, Doc. Math. **18** (2013), 1301–1347.

[9] Herscovich, E., *Using torsion theory to compute the algebraic structure of Hochschild (co)homology*, Homology Homotopy Appl. **20** (2018), no. 1, 117–139.

[10] Kirillov, A.N., *On some quadratic algebras*, L.D. Faddeev's Seminar on Mathematical Physics, Amer. Math. Soc. Transl. Ser. 2, vol. 201, Amer. Math. Soc., Providence, RI, 2000, pp. 91–113.

[11] Kirillov, A.N., *On some quadratic algebras I $\frac{1}{2}$: combinatorics of Dunkl and Gaudin elements, Schubert, Grothendieck, Fuss–Catalan, universal Tutte and reduced polynomials*, SIGMA Symmetry Integrability Geom. Methods Appl. **12** (2016), Paper No. 002,172.

[12] Lu, D.-M., Palmieri, J.H., Wu, Q.-S., Zhang, J.J., *A-infinity structure on Ext-algebras*, J. Pure Appl. Algebra **213** (2009), no. 11, 2017–2037.

[13] Mastnak, M., Pevtsova, J., Schauenburg, P., Witherspoon, S., *Cohomology of finite-dimensional pointed Hopf algebras*, Proc. Lond. Math. Soc. (3) **100** (2010), no. 2, 377–404.

[14] Milinski, A., Schneider, H.-J., *Pointed indecomposable Hopf algebras over Coxeter groups*, New trends in Hopf algebra theory (La Falda, 1999), Contemp. Math., vol. 267, Amer. Math. Soc., Providence, RI, 2000, pp. 215–236.

[15] Polishchuk, A., Positselski, L., *Quadratic algebras*, Univ. Lecture Ser. vol. 37, American Mathematical Society, Providence, RI, 2005.

[16] Radford, D.E., *Hopf algebras*, Ser. Knots and Everything vol. 49, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2012.

[17] Roos, J.-E., *Some non-Koszul algebras*, Advances in geometry, Progr. Math., vol. 172, Birkhäuser Boston, Boston, MA, 1999, pp. 385–389.

[18] Ştefan, D., Vay, C., *The cohomology ring of the 12-dimensional Fomin–Kirillov algebra*, Adv. Math. **291** (2016), 584–620.

[19] Suárez-Álvarez, M., *The Hilton–Eckmann argument for the anti-commutativity of cup products*, Proc. Amer. Math. Soc. **132** (2004), no. 8, 2241–2246.

[20] Walton, C., Zhang, J.J., *On the quadratic dual of the Fomin–Kirillov algebras*, Trans. Amer. Math. Soc. **372** (2019), no. 6, 3921–3945.

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