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CROSSED MODULES IN THE CATEGORY OF LODAY QD-RINEHART ALGEBRAS

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Abstract

In this paper we introduce the notion of Loday QD-Rinehart algebra as an abstraction of Loday QD-algebroids. Additionally, we study cohomology groups, derivations, abelian extensions and crossed modules of these algebraic structures and analyze the relationships between them.

1. Introduction

In recent research, especially in geometry and physics, Loday brackets appear in the form of algebroid brackets, i.e., brackets on sections of vector bundles. As a consequence, Loday algebroids were formalized in many articles [**1, 9, 13, 16, 18, 19, 21, 25, 29, 36, 37**]. Nowadays researchers prefer the terminology Loday algebroid to distinguish them from other general algebroid brackets with two anchors [**17**], called sometimes Leibniz algebroids or Leibniz brackets. These structures have applications in physics, for instance, in the context of nonholonomic constraints [**10, 11, 12, 15, 30**]. Loday QD-algebroids were introduced in [**13**] and called as Loday algebroids in [**12, 17**]. A Loday QD-algebroid on a vector bundle E over a base manifold M is an $\mathbb{R}$-bilinear Loday bracket on the $C^\infty(M)$-module $Sec(E)$ of smooth sections of E for which the adjoint operators ad^l_X and ad^r_X are derivative endomorphisms. As a consequence of the usage of this structure in many branches, it is natural to introduce the algebraic abstraction of Loday QD-algebroids. Since Loday QD-algebroids are a generalization of Lie algebroids, whose algebraic counterpart are Lie-Rinehart algebras, then its abstraction can be thought as a generalization of Lie-Rinehart algebras [**20**], also called (K, A)-Lie algebras in [**35**] or d-Lie rings in [**33**]. Additionally, many algebraic and categorical properties of Lie-Rinehart algebras can be extended or adapted to Loday QD-Rinehart algebras by the techniques used in the transition from Lie algebras to Leibniz algebras.

In this paper we introduce the notion of Loday QD-Rinehart algebra and we explicitly present related algebraic constructions such as derivations, abelian extensions, crossed modules and their cohomology groups, which are particular cases of the corresponding well-known categorical constructions within the framework of

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protomodular, Barr-exact, with finite coproducts categories [**2, 3, 4, 7, 22**]. Additionally, we obtain the relationships between low dimensional cohomology groups and derivations, abelian extensions and crossed modules.

The paper is organized as follows. In section 2, we introduce the structure of Loday QD-Rinehart algebras and study their basic properties, and some constructions such as actions, derivations, abelian extensions and crossed modules. It is worth mentioning that the category of Loday QD-Rinehart algebras is protomodular, Barr-exact and has finite coproducts, but it is not pointed, and thus not semi-abelian. Nevertheless, the categorical framework given in [**4, 7**] can be particularized to Loday QD-Rinehat algebras, which provides the above notions. In subsection 2.1 we establish the equivalence between the categories of crossed modules, cat^1-Loday QD-Rinehart algebras and internal categories. We show the explicit equivalence, but it can be derived from the general categorical framework in [**7**]. In section 3, we define the Leibniz-Rinehart cohomology $HL^*_{QD}(\mathfrak{L}, M)$ as the homology of the cochain complex $CL^*_{QD}(\mathfrak{L}, M) := \mathrm{Hom}_A(\mathfrak{L}^{\otimes_A *}, M)$ of a given Loday QD-Rinehart algebra $(\mathfrak{L}, \alpha_l, \alpha_r)$, with coefficients in a representation M of $\mathfrak{L}$. Moreover, in subsection 3.1 we construct free Loday QD-Rinehart algebras and prove that the Leibniz-Rinehart cohomology vanishes over these objects. Finally, subsections 3.2 and 3.3 are devoted to prove the interpretation of second and third Loday-Rinehart cohomologies by means of abelian and crossed extensions, respectively.

2. Loday QD-Rinehart algebras

Throughout this paper we fix $\mathbb{K}$ as a ground field and A as a commutative $\mathbb{K}$-algebra. We let $\mathrm{Der}(A)$ be the set of all $\mathbb{K}$-derivations of A, i.e., the set of $\mathbb{K}$-linear maps $D\colon A \to A$ such that $D(ab) = aD(b) + D(a)b$. $\mathrm{Der}(A)$ has a structure of Lie $\mathbb{K}$-algebra with respect to the bracket $[D, D'] = D \circ D' - D' \circ D$ and an A-module structure under the product $(aD)(b) = aD(b)$. Moreover, the following identity holds for any $D, D' \in \mathrm{Der}(A)$, $a \in A$:

$$[D, aD'] = a[D, D'] + D(a)D'.$$

Definition 2.1. [**27**] A Leibniz algebra $\mathfrak{L}$ over $\mathbb{K}$ is a $\mathbb{K}$-vector space equipped with a $\mathbb{K}$-bilinear map $[-,-]\colon \mathfrak{L} \times \mathfrak{L} \to \mathfrak{L}$ satisfying the Leibniz identity $[X, [Y, Z]] = [[X, Y], Z] - [[X, Z], Y]$, for all $X, Y, Z \in \mathfrak{L}$.

Definition 2.2. [**27**] Let $\mathfrak{L}$ be a Leibniz algebra and $\varphi := (d, D)$ be a pair of $\mathbb{K}$-linear maps $d, D\colon \mathfrak{L} \to \mathfrak{L}$ such that

$$\begin{aligned} D[X_1, X_2] &= [D(X_1), X_2] - [D(X_2), X_1], \\ d[X_1, X_2] &= [d(X_1), X_2] + [X_1, d(X_2)], \\ [X_1, d(X_2)] &= [X_1, D(X_2)], \end{aligned}$$

for all $X_1, X_2 \in \mathfrak{L}$. The pair $\varphi = (d, D)$ is called a biderivation of $\mathfrak{L}$.

Example 2.3. [**27**] Let $\mathfrak{L}$ be a Leibniz algebra. We fix $X \in \mathfrak{L}$ and we define ad_X, Ad_X as maps from $\mathfrak{L}$ to itself given by $ad_X(Y) = -[Y, X]$, $Ad_X = [X, Y]$, for all $Y \in \mathfrak{L}$. Then $(ad_X,\, Ad_X)$ is a biderivation of $\mathfrak{L}$.

The set of all biderivations of $\mathfrak{L}$ is denoted by $\mathrm{Bider}(\mathfrak{L})$. Following [**27**], $\mathrm{Bider}(\mathfrak{L})$ is endowed with a Leibniz algebra structure with respect to the bracket

$$[\varphi, \varphi'] = (d \circ d' - d' \circ d, D \circ d' - d' \circ D), \tag{1}$$

for all $\varphi = (d, D)$, $\varphi' = (d', D') \in \mathrm{Bider}(\mathfrak{L})$.

Definition 2.4. [**28**] Let $\mathfrak{L}, \mathfrak{L}'$ be Leibniz algebras. An action of $\mathfrak{L}$ on $\mathfrak{L}'$ is a pair of $\mathbb{K}$-bilinear maps, $\mathfrak{L} \times \mathfrak{L}' \to \mathfrak{L}'$, $(X, X') \mapsto [X, X']$, $\mathfrak{L}' \times \mathfrak{L} \to \mathfrak{L}'$, $(X', X) \mapsto [X', X]$, such that

$$\begin{aligned}
[X,[Y,Y']] &= [[X,Y],Y'] - [[X,Y'],Y],\\
[X,[Y',Y]] &= [[X,Y'],Y] - [[X,Y],Y'],\\
[Y',[X,Y]] &= [[Y',X],Y] - [[Y',Y],X],\\
[X,[X',Y']] &= [[X,X'],Y'] - [[X,Y'],X'],\\
[X',[X,Y']] &= [[X',X],Y'] - [[X',Y'],X],\\
[X',[Y',X]] &= [[X',Y'],X] - [[X',X],Y'],
\end{aligned}$$

for all $X, Y \in \mathfrak{L}$, $X', Y' \in \mathfrak{L}'$.

This definition agrees with the general notion of derived action in categories of interest in the sense of Orzech [**31, 32**] (see also [**5, 34**]).

Our aim is to generalize the structure of Lie-Rinehart algebras, which are the algebraic counterpart of Lie algebroids. Inspired by the different generalizations of Lie algebroids introduced in [**14**], we analyze in the sequel the algebraic analog of the so-called Loday QD-algebroids. Hence we have the following:

Definition 2.5. A Loday QD-Rinehart algebra over $(\mathbb{K}, A)$ is a Leibniz $\mathbb{K}$-algebra $\mathfrak{L}$ together with a structure of A-module on $\mathfrak{L}$ and the maps, called left and right anchor maps, $\alpha_l, \alpha_r \colon \mathfrak{L} \to \mathrm{Der}(A)$ which are simultaneously Leibniz algebra and A-module homomorphisms such that

(a) $[aX, Y] = a[X, Y] - \alpha_r(Y)(a)X$,

(b) $[X, aY] = a[X, Y] + \alpha_l(X)(a)Y$,

for all $a \in A$, $X, Y \in \mathfrak{L}$.

Remark 2.6. For a Loday QD-Rinehart algebra $(\mathfrak{L}, \alpha_l, \alpha_r)$ and having in mind (a) and (b) in Definition 2.5, we can write $[aX, bY]$ in two ways. Both of them provide the following equality:

$$[aX, bY] = ab[X, Y] + a\alpha_l(X)(b)Y - b\alpha_r(Y)(a)X, \tag{2}$$

for all $a, b \in A$, X, $Y \in \mathfrak{L}$.

Conversely, if A is unital then equation (2) provides both (a) and (b) in Definition 2.5. Indeed, if A is unital then we have $D(1) = 0$, for all $D \in \mathrm{Der}(A)$, which means

$$\begin{aligned}
[aX, Y] &= [aX, 1Y]\\
&= a1[X, Y] + a\alpha_l(X)(1)Y - 1\alpha_r(Y)(a)X\\
&= a[X, Y] - \alpha_r(Y)(a)X
\end{aligned}$$

and by a similar calculation $[X, aY] = a[X, Y] + \alpha_l(X)(a)Y$, for all $a \in A$, $X, Y \in \mathfrak{L}$.

Definition 2.7. Let $(\mathfrak{L},\alpha_l,\alpha_r),(\mathfrak{L}',\alpha_l',\alpha_r')$ be Loday QD-Rinehart algebras. A homomorphism of Loday QD-Rinehart algebras $f\colon(\mathfrak{L},\alpha_l,\alpha_r)\to(\mathfrak{L}',\alpha_l',\alpha_r')$ is a map $f\colon\mathfrak{L}\to\mathfrak{L}'$ which is simultaneously a homomorphism of Leibniz $\mathbb{K}$-algebras and a homomorphism of A-modules such that $\alpha_l'\circ f=\alpha_l$, $\alpha_r'\circ f=\alpha_r$.

Consequently, we have the category of Loday QD-Rinehart algebras over $(\mathbb{K},A)$ which will be denoted here by $\mathfrak{LQDR}(A)$. This category is not semi-abelian, since it is not pointed, but it is protomodular, Barr-exact, and has finite coproducts (these facts can be verified with an approach similar to that given in [**7**, Section 3.1] for Lie-Rinehart algebras), so it fits in the general framework of [**2, 7**]. Therefore the explicit classical definitions given hereafter are adaptations of the corresponding general notions.

Example 2.8.

(a) If $\alpha_l=0=\alpha_r$ then a Loday QD-Rinehart algebra $(\mathfrak{L},\alpha_l,\alpha_r)$ is a Leibniz A-algebra.

(b) If $A=\mathbb{K}$, then $\mathrm{Der}(A)=0$, so a Loday QD-Rinehart algebra $(\mathfrak{L},\alpha_l,\alpha_r)$ is a Leibniz algebra.

(c) Every Lie-Rinehart algebra is a Loday QD-Rinehart algebra, in fact, there is an inclusion functor $inc\colon\mathfrak{LR}(A)\hookrightarrow\mathfrak{LQDR}(A)$ that assigns to a Lie-Rinehart algebra $(\mathcal{L},\alpha)$ the Loday QD-Rinehart algebra $(\mathfrak{L},\alpha_l=\alpha,\alpha_r=\alpha)$, which is left adjoint to the Liezation functor that assigns to a Loday QD-Rinehart algebra $(\mathfrak{L},\alpha_l,\alpha_r)$ the Lie-Rinehart algebra $\mathfrak{L}_{\mathrm{Lie}}=\mathfrak{L}/\mathfrak{L}^{\mathrm{ann}}$, where $\mathfrak{L}^{\mathrm{ann}}=\langle\{[X,X]\colon X\in\mathfrak{L}\}\rangle$, and anchor map $\tilde{\alpha}\colon\mathfrak{L}_{\mathrm{Lie}}\to\mathrm{Der}(A)$ induced by α_l or α_r, that is, $\tilde{\alpha}(\bar{X})=\alpha_l(X)$ or $\tilde{\alpha}(\bar{X})=-\alpha_r(X)$.

(d) Left-right NP-algebras over a ring $\mathbb{K}$ [**6**] are associative and Leibniz algebras P satisfying the identities

$$[b,a.c]=a.[b,c]+[b,a].c,\quad [a.b,c]=a.[b,c]+[a,c].b,$$

for all $a,b,c\in P$.
Let P be a left-right NP-algebra over a field $\mathbb{K}$ satisfying the condition $[P,[P,P]]=-[[P,P],P]$. Define $\alpha_l\colon P\to\mathrm{Der}(P)$, $\alpha_r\colon P\to\mathrm{Der}(P)$, by $p\mapsto[p,-]$ and $p\mapsto-[-,p]$ for all $p\in P$, respectively. Then (P,α_l,α_r) is a Loday QD-Rinehart algebra.

(e) A Loday QD-algebroid on a vector bundle E over a base manifold M is an $\mathbb{R}$-bilinear Loday bracket on the $C^\infty(M)$-module $Sec(E)$ of smooth sections of E for which the adjoint operators ad_X^l and ad_X^r are derivative endomorphisms. Loday QD-algebroids, introduced in [**13**] and called Loday algebroids in [**12, 17**], are the main examples of Loday QD-Rinehart algebras.

(f) Let $(\mathfrak{L},\alpha_l,\alpha_r)$ be a Loday QD-Rinehart algebra with $\alpha_l\circ\alpha_r=\alpha_l\circ\alpha_l=\alpha_r\circ\alpha_l$, for instance $\alpha_l=\alpha_r$. Then $A\rtimes\mathfrak{L}$ is a Loday QD-Rinehart algebra with respect to the operations:

$$\begin{aligned}a'(a,X)&=(a'a,a'X),\\ [(a,X),(a',X')]&=(\alpha_l(X)(a')+\alpha_r(X')(a),[X,X']),\end{aligned}$$

and anchor maps $\tilde{\alpha}_l(a,X)=\alpha_l(X)$, $\tilde{\alpha}_r(a,X)=\alpha_r(X)$, for all (a,X), $(a',X')\in A\rtimes\mathfrak{L}$.

(g) Let $T_l\colon \mathrm{Der}(A) \to \mathrm{Der}(A)$ and $T_r\colon \mathrm{Der}(A) \to \mathrm{Der}(A)$ be $\mathbb{K}$-linear A-module homomorphisms satisfying

$$\begin{aligned} T_l(D_1) \circ T_l(D_2) &= T_l(T_l(D_1) \circ D_2), \\ T_r(D_1) \circ T_r(D_2) &= T_r(T_r(D_1) \circ D_2), \\ T_l(D_2) \circ D_1 &= T_r(D_2) \circ D_1, \end{aligned}$$

for all D_1, $D_2 \in \mathrm{Der}(A)$. For instance, fix $D \in \mathrm{Der}(A)$ in the center of the associative algebra $\mathrm{Der}(A)$ and consider $T_l(D') = D \circ D'$, $T_r(D') = D' \circ D$, for all $D' \in \mathrm{Der}(A)$.
Then $\mathrm{Der}(A)$ is endowed with a Leibniz $\mathbb{K}$-algebra structure (which is different from the usual Lie algebra structure) with respect to the bracket

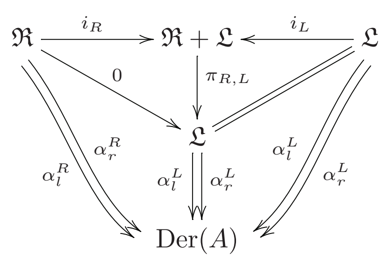
$$\begin{aligned} [\![\,,]\!]\colon \mathrm{Der}(A) \times \mathrm{Der}(A) &\longrightarrow \mathrm{Der}(A), \\ (D_1, \quad D_2) &\longmapsto T_l(D_1) \circ D_2 - T_r(D_2) \circ D_1. \end{aligned}$$

Moreover, this bracket is preserved by T_l and T_r.
On the other hand, $\mathrm{Der}(A)$ has an A-module structure with respect to the usual operation $(aD)(X) = aD(X)$, $a, X \in A$, and the following equalities hold:

$$\begin{aligned} [\![D_1, aD_2]\!] &= a[\![D_1, D_2]\!] + T_l(D_1)(a) \circ D_2, \\ [\![aD_1, D_2]\!] &= a[\![D_1, D_2]\!] - T_r(D_2)(a) \circ D_1. \end{aligned}$$

for all $a \in A, D_1, D_2 \in \mathrm{Der}(A)$.
Hence $(\mathrm{Der}(A), T_l, T_r)$ is a Loday QD-Rinehart algebra with anchor maps T_l and T_r.

Remark 2.9. Before dealing with actions and semi-direct products, let us note that since the category of Loday QD-Rinehart algebras has no zero object, to define an action of a Loday QD-Rinehart $\mathfrak{L}$ algebra on another Loday QD-Rinehart algebra $\mathfrak{R}$ it is necessary to define the homomorphism $\pi_{R,L}\colon \mathfrak{R} + \mathfrak{L} \to \mathfrak{L}$, the coproduct formed by the identity map on $\mathfrak{L}$ and the zero map from $\mathfrak{R}$ to $\mathfrak{L}$, but the commutativity of the following diagram



implies that $\alpha_l^R = \alpha_r^R = 0$, therefore $\mathfrak{R}$ should be a Leibniz A-algebra.

To get more details about this observation, the note after Definition 2.1.4 in [**7**] mentions that if $\mathcal{C}$ is a category with pullbacks, initial object and finite coproducts, properties satisfied by the category of Loday QD-Rinehart algebras, the internal action definition of an object of $\mathcal{C}$ can only be applied to an object $\mathcal{C}/0$, which in the case of Loday QD-Rinehart algebras means that the second object has zero anchor maps, consequently, it is a Leibniz A-algebra.

Definition 2.10. Let $(\mathfrak{L}, \alpha_l, \alpha_r)$ be a Loday QD-Rinehart algebra and R be a Leibniz A-algebra. An action of $(\mathfrak{L}, \alpha_l, \alpha_r)$ on R is a pair of $\mathbb{K}$-bilinear maps

$$\begin{array}{ll} \mathfrak{L} \times R \longrightarrow R, & R \times \mathfrak{L} \longrightarrow R, \\ (X, r) \longmapsto [X, r], & (r, X) \longmapsto [r, X], \end{array}$$

which define a Leibniz action of $\mathfrak{L}$ on R such that

(a) $[r, aX] = a[r, X]$,

(b) $[aX, r] = a[X, r]$,

(c) $[X, ar] = a[X, r] + \alpha_l X(a)r$,

(d) $[ar, X] = a[r, X] - \alpha_r X(a)r$,

for all $a \in A$, $X \in \mathfrak{L}, r \in R$.

When $(\mathfrak{L}, \alpha_l, \alpha_r)$ acts over an abelian Leibniz A-algebra R (i.e., a Leibniz algebra with trivial bracket), then it is said that R is endowed with a representation structure over $(\mathfrak{L}, \alpha_l, \alpha_r)$.

We denote the category of Loday QD-Rinehart representations over $(\mathfrak{L}, \alpha_l, \alpha_r)$ by $\mathcal{REP}_{(\mathfrak{L},A)}$.

According to [**4**], an extension in a protomodular category $\mathcal{C}$ is a diagram in $\mathcal{C}$ of the form

$$K \xrightarrow{\chi} A \xrightarrow{\alpha} B,$$

where α is an effective descent morphism, and χ is a kernel of α. The category of such extensions when B and K are fixed is denoted by $\mathrm{Ext}_{\mathcal{C}}(B, K)$.

The abelian objects of $\mathrm{Ext}_{\mathcal{C}}(B, K)$ when we consider the category $\mathcal{C}$ as the category of Loday QD-Rinehart algebras give rise to the following definition.

Definition 2.11. Let $(\mathfrak{L}, \alpha_l, \alpha_r)$ be a Loday QD-Rinehart algebra and R be a representation over $(\mathfrak{L}, \alpha_l, \alpha_r)$. An abelian extension of $(\mathfrak{L}, \alpha_l, \alpha_r)$ by R is a sequence

$$R \xrightarrow{i} \mathfrak{L}' \xrightarrow{\pi} \mathfrak{L},$$

where $(\mathfrak{L}', \alpha'_l, \alpha'_r)$ is a Loday QD-Rinehart algebra, π is a surjective homomorphism of Loday QD-Rinehart algebras and i is the kernel of π. Moreover, i is an A-linear map and the following identities hold:

$$\begin{aligned} [i(r), i(r')] &= 0, \\ [X', i(r)] &= [\pi(X'), r], \\ [i(r), X'] &= [r, \pi(X')], \end{aligned}$$

for all $r, r' \in R, X' \in \mathfrak{L}'$.

An abelian extension is called A-split if π has an A-linear section.

Definition 2.12. Let $(\mathfrak{L}, \alpha_l, \alpha_r)$ be a Loday QD-Rinehart algebra and R be a Leibniz A-algebra with an action of $(\mathfrak{L}, \alpha_l, \alpha_r)$ on R. Its semi-direct product $(R \rtimes \mathfrak{L}, \tilde{\alpha}_l, \tilde{\alpha}_r)$ is the Loday QD-Rinehart algebra with underlying vector space $R \oplus \mathfrak{L}$ endowed with the operations

$$a(r,X) = (ar, aX),$$
$$[(r,X),(r',X')] = ([r,r'] + [X,r'] + [r,X'], [X,X']),$$

for all $r, r' \in R$, $X, X' \in \mathfrak{L}$ and anchor maps

$$\tilde{\alpha}_l \colon R \oplus \mathfrak{L} \longrightarrow \mathrm{Der}(A),\ \tilde{\alpha}_l(r,X) = \alpha_l(X),$$
$$\tilde{\alpha}_r \colon R \oplus \mathfrak{L} \longrightarrow \mathrm{Der}(A),\ \tilde{\alpha}_r(r,X) = \alpha_r(X).$$

If R is abelian, then the canonical embeddings $i_R \colon R \to R \rtimes \mathfrak{L}$, $i_{\mathfrak{L}} \colon \mathfrak{L} \to R \rtimes \mathfrak{L}$, as well as the canonical projection $p_{\mathfrak{L}} \colon R \rtimes \mathfrak{L} \to \mathfrak{L}$ are homomorphisms of Loday QD-Rinehart algebras. Consequently, we have the abelian extension

$$R \xrightarrow{i_R} R \rtimes \mathfrak{L} \xrightarrow{p_{\mathfrak{L}}} \mathfrak{L}, \tag{3}$$

which is A-split by $i_{\mathfrak{L}} \colon \mathfrak{L} \to R \rtimes \mathfrak{L}$. The induced representation structure on R provided from this sequence in the standard way coincides with the previous one.

Definition 2.13. Let $(\mathfrak{L}, \alpha_l, \alpha_r)$ be a Loday QD-Rinehart algebra and R be a representation of $(\mathfrak{L}, \alpha_l, \alpha_r)$. A derivation from $\mathfrak{L}$ to R consists of a $\mathbb{K}$-linear map $\delta \colon \mathfrak{L} \to R$ such that

$$\delta(aX) = a\delta(X),$$
$$\delta([X,Y]) = [\delta(X), Y] + [X, \delta(Y)],$$

for all $a \in A$, $X, Y \in \mathfrak{L}$.

The set of all derivations from a Loday QD-Rinehart algebra $(\mathfrak{L}, \alpha_l, \alpha_r)$ to a representation R is denoted by $\mathrm{Der}_A(\mathfrak{L}, R)$.

Example 2.14.

(a) The map $p_R \colon R \rtimes \mathfrak{L} \to R, p_R(r,X) = r$, in sequence (3) is a derivation, where the representation structure from $(R \rtimes \mathfrak{L}, \tilde{\alpha}_l, \tilde{\alpha}_r)$ over R comes through $p_{\mathfrak{L}}$.

(b) Let $(\mathfrak{L}, \alpha_l, \alpha_r)$ be a Loday QD-Rinehart algebra which acts on a Leibniz A-algebra R. The map $ad_r \colon \mathfrak{L} \to R, ad_r(X) = [X, r], X \in \mathfrak{L}, r \in R$, is a derivation.

Theorem 2.15. *There is a 1-1 correspondence between elements of* $\mathrm{Der}_A(\mathfrak{L}, R)$ *and homomorphisms of Loday QD-Rinehart algebras* $\sigma \colon \mathfrak{L} \to \mathfrak{L} \rtimes R$*, for which* $p_{\mathfrak{L}} \circ \sigma = id_{\mathfrak{L}}$*.*

Proof. A homomorphism $\sigma \colon \mathfrak{L} \to \mathfrak{L} \rtimes R$ satisfying $p_{\mathfrak{L}} \circ \sigma = id_{\mathfrak{L}}$ gives rise to a derivation $\delta_\sigma \colon p_R \circ \sigma \colon \mathfrak{L} \to R$. On the other hand, given a derivation $\delta \colon \mathfrak{L} \to R$, we have the homomorphism of Loday QD-Rinehart algebras $\sigma_\delta \colon \mathfrak{L} \to R \rtimes \mathfrak{L}$, $\sigma_\delta(X) = (\delta(X), X)$, for all $X \in \mathfrak{L}$. The maps $\sigma \mapsto \delta_\sigma$, $\delta \mapsto \sigma_\delta$ are inverse to each other, as required. □

Let $(\mathfrak{L}, \alpha_l, \alpha_r)$ be a Loday QD-Rinehart algebra, R be a Leibniz A-algebra and

$$\overline{\mathrm{Bider}(R)} := \{(d, D) \in \mathrm{Bider}(R) \colon d \circ d' = d \circ D' = D \circ D' = D \circ d', \text{ for all } (d', D') \in \mathrm{Bider}(R)\}.$$

For example, any $r \in Z(R)$ gives rise to $(ad_r, Ad_r) \in \overline{\mathrm{Bider}(R)}$.

Let $DO(A, \mathfrak{L}, R)$ be the vector space of pairs (φ, X), where $\varphi = (d, D) \in \overline{\mathrm{Bider}(R)}$ and $X \in \mathfrak{L}$, such that

$$d(ar) = ad(r) + \alpha_r(X)(a)r, \qquad D(ar) = aD(r) + \alpha_l(X)(a)r,$$

for all $a \in A$, $r \in R$. Then the componentwise operations (see (1)) endow $DO(A, \mathfrak{L}, R)$ with a structure of A-module and with a structure of Leibniz $\mathbb{K}$-algebra. In addition, $DO(A, \mathfrak{L}, R)$ is a Loday QD-Rinehart algebra where the anchor maps are defined by the compositions

$$\tilde{\alpha}_l \colon DO(A, \mathfrak{L}, R) \xrightarrow{pr} \mathfrak{L} \xrightarrow{\alpha_l} \mathrm{Der}(A), \qquad \tilde{\alpha}_r \colon DO(A, \mathfrak{L}, R) \xrightarrow{pr} \mathfrak{L} \xrightarrow{\alpha_r} \mathrm{Der}(A).$$

Let R be a Leibniz A-algebra satisfying $\mathrm{Ann}(R) = 0$ or $[R, R] = 0$, and $(\mathfrak{L}, \alpha_l, \alpha_r)$ be a Loday QD-Rinehart algebra with an action on R, then the map $f \colon \mathfrak{L} \to DO(A, \mathfrak{L}, R)$ given by $f(X) = (\varphi^X, X) = ((d^X, D^X), X)$, $D^X(r) = [X, r]$ and $d^X(r) = -[r, X]$, is a homomorphism of Loday QD-Rinehart algebras which makes commutative the following diagram:

$$\begin{array}{ccc} DO(A, \mathfrak{L}, R) & \xrightarrow{p} & \mathfrak{L}. \\ {\scriptstyle f}\uparrow & \diagup\!\!\diagup & \\ \mathfrak{L} & & \end{array}$$

Example 2.16. Recall from [**8**, Definition 5.2] that a Leibniz algebra L is said to be semisimple if $rad(L) = Leib(L)$. From [**8**, Corollary 5.5], any semisimple Leibniz A-algebra satisfies the condition $[L, L] = L$.

An example of Leibniz A-algebra satisfying the above conditions is the 5-dimensional Leibniz algebra $L = \mathrm{span}\{h, e, f, x_0, x_1\}$ with nontrivial multiplications given by:

$$\begin{array}{lllll} [h, e] = 2e, & [h, f] = -2f, & [e, f] = h, & [e, h] = -2e, & [f, h] = 2f, \\ [f, e] = -h, & [h, x_0] = x_0, & [f, x_0] = x_1, & [h, x_1] = -x_1, & [e, x_1] = -x_0. \end{array}$$

From [**8**, Example 5.7], L is a semisimple Leibniz A-algebra (here $A = \mathbb{K}$). Moreover, $\mathrm{Ann}(L) = 0$.

Conversely, a homomorphism of Loday QD-Rinehart algebras $f \colon \mathfrak{L} \to DO(A, \mathfrak{L}, R)$, $X \mapsto (\varphi^X, X) = ((d^X, D^X), X)$, making commutative the above diagram gives rise to an action from $(\mathfrak{L}, \alpha_l, \alpha_r)$ over R, where $[X, r] := D^X(r)$ and $[r, X] := -d^X(r)$.

Indeed,

$$\begin{aligned} [X, ar] &= D^X(ar) = a\ D^X(r) + \alpha_l(X)(a)r = a[X, r] + \alpha_l(X)(a)r, \\ [ar, X] &= -d^X(ar) = -(a\ d^X(r) + \alpha_r(X)(a)r) = -a\ d^X(r) - \alpha_r(X)(a)r \\ &= a[r, X] - \alpha_r(X)(a)r. \end{aligned}$$

On the other hand, since

$$(d^{aX}, D^{aX}) = f(aX) = af(X) = a(d^X, D^X) = (a\ d^X, a\ D^X),$$

we have

$$[r, aX] = -d^{aX}(r) = -a\ d^X(r) = a[r, X]$$

and similarly

$$[aX, r] = a[X, r],$$

for all $a \in A$, $r \in R, X \in \mathfrak{L}$, as required.

With a similar approach to [**7**] adapted to the case of Loday QD-Rinehart algebras, we have the following notion of crossed module.

Definition 2.17. A crossed module $\partial\colon R \to \mathfrak{L}$ of Loday QD-Rinehart algebras consists of a Loday QD-Rinehart algebra $(\mathfrak{L}, \alpha_l, \alpha_r)$ and a Leibniz A-algebra R together with an action of $(\mathfrak{L}, \alpha_l, \alpha_r)$ on R and the Leibniz $\mathbb{K}$-algebra homomorphism ∂ such that the following identities hold:

(a) $\partial[X, r] = [X, \partial(r)]$,
(b) $\partial[r, X] = [\partial(r), X]$,
(c) $[\partial(r'), r] = [r', r] = [r', \partial(r)]$,
(d) $\partial(ar) = a\partial(r)$,
(e) $\alpha_l(\partial(r))(a) = 0$,
(f) $\alpha_r(\partial(r))(a) = 0$,

for all $r, r' \in R$, $X \in \mathfrak{L}$, $a \in A$.

Example 2.18.

(a) Let $(\mathfrak{L}, \alpha_l, \alpha_r)$ be a Loday QD-Rinehart algebra. An ideal $\mathcal{I}$ of $\mathfrak{L}$ is a normal monomorphism $\mathcal{I} \hookrightarrow \mathfrak{L}$ in the protomodular category $\mathfrak{LQD}\mathfrak{R}(A)$ [**3**], which means that $\mathcal{I}$ is an ideal of $\mathfrak{L}$ as Leibniz $\mathbb{K}$-algebra, it has an A-module structure and a Loday QD-Rinehart algebra structure with anchor maps given by the restrictions of α_l, α_r, such that the following compositions

$$\mathcal{I} \hookrightarrow \mathfrak{L} \xrightarrow{\alpha_l} \mathrm{Der}(A), \qquad \mathcal{I} \hookrightarrow \mathfrak{L} \xrightarrow{\alpha_r} \mathrm{Der}(A)$$

are trivial. Then $(\mathcal{I}, \mathfrak{L}, \mathfrak{inc}.)$ is a crossed module with the actions of $\mathfrak{L}$ on $\mathcal{I}$ given by the Leibniz bracket.

(b) Let R be a representation over $(\mathfrak{L}, \alpha_l, \alpha_r)$. Then the zero morphism $0\colon R \to \mathfrak{L}$ is a crossed module of Loday QD-Rinehart algebras.

(c) Let $(\mathfrak{L}, \alpha_l, \alpha_r)$ be a Loday QD-Rinehart algebra, $\theta\colon R \to R'$ be a homomorphism of representations over $(\mathfrak{L}, \alpha_l, \alpha_r)$. There is an action of $R' \rtimes \mathfrak{L}$ on R defined by $[(r', X), r] = [X, r]$, $[r, (r', X)] = [r, X]$, for all $X \in \mathfrak{L}$, $r \in R$ and $r' \in R'$. Define

$$\begin{aligned} \partial\colon R &\longrightarrow R' \rtimes \mathfrak{L}, \\ r &\longmapsto (\theta(r), 0). \end{aligned}$$

Then $(R, R' \rtimes \mathfrak{L}, \partial)$ is a crossed module of Loday QD-Rinehart algebras.

(d) Let $(\mathfrak{L}, \alpha_l, \alpha_r)$ be a Loday QD-Rinehart algebra. A central surjective homomorphism $\partial\colon R \to \mathfrak{L}$ (i.e., $\mathrm{Ker}(\partial) \subseteq Z(R)$) from a Leibniz A-algebra R to a Loday QD-Rinehart algebra $\mathfrak{L}$ is a Loday QD-Rinehart crossed module where the action from $(\mathfrak{L}, \alpha_l, \alpha_r)$ over R is given by $[X, r] = [r', r]$, $[r, X] = [r, r']$, such that $\partial(r') = X$.

(e) Let $f\colon (\mathfrak{L}, \alpha_l, \alpha_r) \to (\mathfrak{L}', \alpha'_l, \alpha'_r)$ be a homomorphism of Loday QD-Rinehart algebras, then $inc\colon \mathrm{Ker}(f) \to \mathfrak{L}$ is a crossed module of Loday QD-Rinehart algebras.

Definition 2.19. A homomorphism between two crossed modules of Loday QD-Rinehart algebras $\partial\colon R \to \mathfrak{L}$ and $\partial'\colon R' \to \mathfrak{L}'$ consists of a tuple $(f, \phi)\colon (R, \mathfrak{L}, \partial) \to (R', \mathfrak{L}', \partial')$, where $f\colon R \to R'$ is an A-algebra homomorphism and $\phi\colon \mathfrak{L} \to \mathfrak{L}'$ is a homomorphism of Loday QD-Rinehart algebras such that

(a) $\phi \circ \partial = \partial' \circ f$,

(b) $f[X,r] = [\phi(X), f(r)]; \qquad f[r,X] = [f(r), \phi(X)], r \in R, X \in \mathfrak{L}$.

Having crossed modules of Loday QD-Rinehart algebras as objects and homomorphisms of crossed modules of Loday QD-Rinehart algebras as morphisms, we obtain the category of crossed modules of Loday QD-Rinehart algebras, which we will denote by $\mathfrak{X}\mathfrak{L}\mathrm{QD}\mathfrak{R}$.

Proposition 2.20. *Let $\partial\colon R \to \mathfrak{L}$ be a crossed module of Loday QD-Rinehart algebras and $\mathcal{I} = \partial(R)$. Then the following statements hold:*

(a) $\mathrm{Im}(\partial)$ *is an ideal of* $\mathfrak{L}$ *and A-submodule.*

(b) $\mathrm{Ker}(\partial) \trianglelefteq R$.

(c) $\mathrm{Ker}(\partial)$ *is an $\mathfrak{L}/\mathcal{I}$-module.*

Proof. Direct checking. □

2.1. Equivalence with cat^1-Loday QD-Rinehart algebras and internal categories

The following definition mimics the original notion of cat^1-groups in [**26**].

Definition 2.21. A cat^1-Loday QD-Rinehart algebra is a triple $(\mathfrak{L}, w_0, w_1)$ consisting of a Loday QD-Rinehart algebra $\mathfrak{L}$ with two additional unary operations $w_0, w_1\colon \mathfrak{L} \to \mathfrak{L}$ such that

$$w_0 \circ w_1 = w_1, \qquad w_1 \circ w_0 = w_0$$
$$[\mathrm{Ker}(w_0), \mathrm{Ker}(w_1)] = 0 = [\mathrm{Ker}(w_1), \mathrm{Ker}(w_0)]$$

where w_0, w_1 are homomorphisms of Loday QD-Rinehart algebras.

A homomorphism between two cat^1-Loday QD-Rinehart algebras is a homomorphism of Loday QD-Rinehart algebras compatible with the unary operations. The resulting category will be denoted by $\mathfrak{Cat}^1$-$\mathfrak{L}\mathrm{QD}\mathfrak{R}(A)$.

Let $\partial\colon R \to \mathfrak{L}$ be a crossed module. We have the functor $C\colon \mathfrak{X}\mathfrak{L}\mathrm{QD}\mathfrak{R} \to \mathfrak{Cat}^1$-$\mathfrak{L}\mathrm{QD}\mathfrak{R}(A)$ defined by $C(\partial\colon R \to \mathfrak{L}) := (R \rtimes \mathfrak{L}, w_0, w_1)$, where $w_0(r,l) = (0,l)$, $w_1(r,l) = (0, \partial(r) + l)$, for all $(r,l) \in R \rtimes \mathfrak{L}$.

Conversely, we have the functor $X\colon \mathfrak{Cat}^1$-$\mathfrak{L}\mathrm{QD}\mathfrak{R}(A) \to \mathfrak{X}\mathfrak{L}\mathrm{QD}\mathfrak{R}$ defined by $X(\mathfrak{L}, w_0, w_1) := (\partial\colon \mathrm{Ker}(w_0) \to \mathrm{Im}(w_0))$, where $\partial = w_{1|\mathrm{Ker}(w_0)}$.

Proposition 2.22. *The adjoint functors*

$$\mathfrak{Cat}^1 - \mathfrak{L}\mathrm{QD}\mathfrak{R}(A) \underset{C}{\overset{X}{\rightleftarrows}} \mathfrak{X}\mathfrak{L}\mathrm{QD}\mathfrak{R}$$

give rise to a natural equivalence of categories.

Definition 2.23. An internal category in $\mathfrak{L}\mathrm{QD}\mathfrak{R}(A)$ is a diagram

$$\mathfrak{L}_1 \underset{s}{\overset{d_1, d_0}{\rightrightarrows}} \mathfrak{L}_0$$

in $\mathfrak{L}\mathrm{QD}\mathfrak{R}(A)$ such that $d_0 \circ s = d_1 \circ s = 1_{\mathfrak{L}_0}$, with an operation $m\colon \mathfrak{L}_1 \times_{\mathfrak{L}_0} \mathfrak{L}_1 \to \mathfrak{L}_1$ satisfying the usual axioms of a category, namely, $d_0 \circ m = d_0 \circ \pi_1$, $d_1 \circ m = d_1 \circ \pi_2$, where $\pi_i\colon \mathfrak{L}_1 \times_{\mathfrak{L}_0} \mathfrak{L}_1 \to \mathfrak{L}_1, i = 1, 2$, is the projection in the corresponding component, $m \circ (1 \times m) = m \circ (m \times 1)\colon \mathfrak{L}_1 \times_{\mathfrak{L}_0} \mathfrak{L}_1 \times_{\mathfrak{L}_0} \mathfrak{L}_1 \to \mathfrak{L}_1$, $m \circ (1 \times s) = \pi_1$ and $m \circ (s \times 1) = \pi_2$. Objects of such a category will be denoted by $(\mathfrak{L}_1, \mathfrak{L}_0, d_0, d_1, s, m)$.

A homomorphism between internal categories is a pair $F := (F_1, F_0)\colon (\mathfrak{L}_1, \mathfrak{L}_0, d_0, d_1, s, m) \to (\mathfrak{L}'_1, \mathfrak{L}'_0, d'_0, d'_1, s', m')$ of homomorphisms $F_1\colon \mathfrak{L}_1 \to \mathfrak{L}'_1$, $F_0\colon \mathfrak{L}_0 \to \mathfrak{L}'_0$ compatible with the structure maps and operations. In other words, F_0 and F_1 make the following diagrams commutative

$$\begin{array}{ccc} \mathfrak{L}_1 & \underset{s}{\overset{d_1,\, d_0}{\rightrightarrows\!\!\leftarrow}} & \mathfrak{L}_0 \\ {\scriptstyle F_1}\downarrow & & \downarrow{\scriptstyle F_0} \\ \mathfrak{L}'_1 & \underset{s'}{\overset{d'_1,\, d'_0}{\rightrightarrows\!\!\leftarrow}} & \mathfrak{L}'_0 \end{array} \qquad \begin{array}{ccc} \mathfrak{L}_1 \times_{\mathfrak{L}_0} \mathfrak{L}_1 & \xrightarrow{m} & \mathfrak{L}_1 \\ {\scriptstyle (F_1,F_1)}\downarrow & & \downarrow{\scriptstyle F_1} \\ \mathfrak{L}'_1 \times_{\mathfrak{L}'_0} \mathfrak{L}'_1 & \xrightarrow[m']{} & \mathfrak{L}'_1 \end{array}$$

The resulting category will be denoted by $\mathfrak{IC}(\mathfrak{L}\mathrm{QD}\mathfrak{R})(A)$.

Given an internal category $(\mathfrak{L}_1, \mathfrak{L}_0, d_0, d_1, s, m)$ in $\mathfrak{L}\mathrm{QD}\mathfrak{R}(A)$, then we have an action of $\mathfrak{L}_0$ on $\mathrm{Ker}(d_0)$ defined by $[l_0, l_1] = [s(l_0), l_1]$ and $[l_1, l_0] = [l_1, s(l_0)]$, from which we get the crossed module $\partial\colon \mathrm{Ker}(d_0) \to \mathfrak{L}_0$ with $\partial = d_{1|_{\mathrm{Ker}(d_0)}}$. This construction gives rise to the functor $\psi\colon \mathfrak{IC}(\mathfrak{L}\mathrm{QD}\mathfrak{R})(A) \to \mathfrak{X}\mathfrak{L}\mathrm{QD}\mathfrak{R}$.

On the other hand, for a given crossed module $\partial\colon \mathfrak{L}_1 \to \mathfrak{L}_0$ we have the internal category $(\mathfrak{L}_1 \rtimes \mathfrak{L}_0, \mathfrak{L}_0, d_0, d_1, s, m)$ where $d_0(l_1, l_0) = l_0$, $d_1(l_1, l_0) = \partial(l_1) + l_0$, $s(l_0) = (0, l_0)$. The multiplication $m\colon (\mathfrak{L}_1 \rtimes \mathfrak{L}_0) \times_{\mathfrak{L}_0} (\mathfrak{L}_1 \rtimes \mathfrak{L}_0) \to \mathfrak{L}_1 \rtimes \mathfrak{L}_0$ is defined by $m((l, p), (l', p + \partial(l))) = (l + l', p)$. Here we must observe that the elements of $(\mathfrak{L}_1 \rtimes \mathfrak{L}_0) \times_{\mathfrak{L}_0} (\mathfrak{L}_1 \rtimes \mathfrak{L}_0)$ are of the form $((l, p), (l', p + \partial(l)))$. Consequently, we get the functor $\Phi\colon \mathfrak{X}\mathfrak{L}\mathrm{QD}\mathfrak{R} \to \mathfrak{IC}(\mathfrak{L}\mathrm{QD}\mathfrak{R})(A)$.

Proposition 2.24. *The adjoint functors*

$$\mathfrak{X}\mathfrak{L}\mathrm{QD}\mathfrak{R} \underset{\psi}{\overset{\Phi}{\rightleftarrows}} \mathfrak{IC}(\mathfrak{L}\mathrm{QD}\mathfrak{R})(A)$$

give rise to a natural equivalence of categories.

Remark 2.25. Janelidze in [**22**] studies the notion of (pre)crossed module and the equivalence between internal categories and crossed modules in semi-abelian categories. This study was extended to the framework of protomodular, Barr-exact, with finite coproducts categories in [**7**], therefore an adaptation of chapter 3 in [**7**] to the category $\mathfrak{L}\mathrm{QD}\mathfrak{R}(A)$ gives rise to the definition of internal crossed module, which is equivalent to Definition 2.17, and provides directly the equivalence between internal categories and crossed modules.

3. Cohomology of Loday QD-Rinehart algebras

Let $(\mathfrak{L}, \alpha_l, \alpha_r)$ be a Loday QD-Rinehart algebra and M be a representation over $(\mathfrak{L}, \alpha_l, \alpha_r)$. Let

$$CL^n_{QD}(\mathfrak{L}, M) := \mathrm{Hom}_A(\mathfrak{L}^{\otimes_A n}, M),\ n \geqslant 0,$$

where $V^{\otimes_A n}$ denotes the tensor algebra over A generated by an A-module V.

Let be the coboundary map

$$\delta^n \colon CL^n_{QD}(\mathfrak{L}, M) \longrightarrow CL^{n+1}_{QD}(\mathfrak{L}, M)$$

given by

$$\begin{aligned}(\delta^n f)(X_1,\dots, X_{n+1}) :=& \\ & [X_1, f(X_2,\dots,X_{n+1})] + \sum_{i=2}^{n+1}(-1)^i[f(X_1,\dots,\widehat{X_i},\dots,X_{n+1}), X_i] \\ & + \sum_{1\leqslant i<j\leqslant n+1}(-1)^{j+1} f(X_1,\dots,X_{i-1},[X_i,X_j],X_{i+1},\dots,\widehat{X_j},\dots,X_{n+1}),\end{aligned}$$

where $X_1,\dots,X_{n+1} \in \mathfrak{L}$, $f \in CL^n_{QD}(\mathfrak{L}, M)$.

The cochain complex $(CL^*_{QD}(\mathfrak{L}, M), \delta^*)$ is well defined, that is $\delta^{n+1} \circ \delta^n = 0$, $n \geqslant 0$. Indeed, if we define for any $Y \in \mathfrak{L}$ and $n \in \mathbb{N}$, two A-linear maps,

$$\theta^n(Y) \colon CL^n_{QD}(\mathfrak{L}, M) \longrightarrow CL^n_{QD}(\mathfrak{L}, M)$$

given by

$$\theta^n(Y)(f)(X_1,\dots,X_n) = -[f(X_1,\dots,X_n), Y] + \sum_{i=1}^{n} f(X_1,\dots,[X_i,Y],\dots,X_n)$$

and

$$i^{n+1}(Y) \colon CL^{n+1}_{QD}(\mathfrak{L}, M) \longrightarrow CL^n_{QD}(\mathfrak{L}, M)$$

given by

$$i^{n+1}(Y)(f)(X_1,\dots,X_n) = f(X_1,\dots,X_n,Y)$$

then the following formulas hold.

Proposition 3.1 (Cartan's formulas)**.** *The following identities hold:*

(a) $\delta^{n-1} \circ i^n(Y) + i^{n+1}(Y) \circ \delta^n = \theta^n(Y)$*, for* $n \geqslant 0$*;*
(b) $\theta^n(X) \circ \theta^n(Y) - \theta^n(Y) \circ \theta^n(X) = -\theta^n([X,Y])$*, for* $n \geqslant 0$*;*
(c) $\theta^{n-1}(X) \circ i^n(Y) - i^n(Y) \circ \theta^n(X) = i([X,Y])$*, for* $n > 0$*;*
(d) $\theta^{n+1}(Y) \circ \delta^n = \delta^n \circ \theta^n(Y)$*, for* $n \geqslant 0$*;*
(e) $\delta^{n+1} \circ \delta^n = 0$*, for* $n \geqslant 0$.

Proof. The proof follows routinely by induction, so we leave it for the reader. □

The homology of the complex $(CL^*_{QD}(\mathfrak{L}, M), \delta^*)$ is the Leibniz-Rinehart cohomology of the Loday QD-Rinehart algebra $(\mathfrak{L}, \alpha_l, \alpha_r)$ with coefficients in the representation M, i.e.

$$HL^*_{QD}(\mathfrak{L}, M) := H^*(CL^*_{QD}(\mathfrak{L}, M), \delta^*).$$

Obviously, if $A = \mathbb{K}$ then this definition provides the Leibniz cohomology of a Leibniz algebra [**28**].

A direct check on the cochain complex shows that

$$HL^0_{QD}(\mathfrak{L}, M) = \{m \in M \mid [X,m] = 0, \text{for all } X \in \mathfrak{L}\}.$$

A 1-cocycle is a derivation from $\mathfrak{L}$ to M (see Definition 2.13). Additionally, a 1-coboundary is also a map of the form $ad_m(X) = [X,m]$, which is a derivation (see

Example 2.14 *(b)*), called inner derivation. We denote by $\mathrm{IDer}_A(\mathfrak{L}, M)$ the set of all inner derivations from $\mathfrak{L}$ to M, which is an A-subbimodule of $\mathrm{Der}_A(\mathfrak{L}, M)$. So

$$HL^1_{QD}(\mathfrak{L}, M) = \mathrm{Der}_A(\mathfrak{L}, M)/\mathrm{IDer}_A(\mathfrak{L}, M).$$

As a consequence we get the exact sequence

$$0 \longrightarrow HL^0_{QD}(\mathfrak{L}, M) \longrightarrow M \longrightarrow \mathrm{Der}_A(\mathfrak{L}, M) \longrightarrow HL^1_{QD}(\mathfrak{L}, M) \longrightarrow 0.$$

3.1. Free Loday QD-Rinehart algebras

From now on we assume that A is unital.

First we recall the construction of free Leibniz algebras from [**28**]. Let V be a $\mathbb{K}$-vector space and let $\bar{T}V = V \oplus V^{\otimes 2} \oplus V^{\otimes 3} \oplus \cdots$ be the reduced tensor $\mathbb{K}$-vector space with the bracket defined inductively by

$$[x, v] = x \otimes v,$$
$$[x, y \otimes v] = [x, y] \otimes v - [x \otimes v, y],$$

for all $x, y \in \bar{T}(V)$, $v \in V$.

Definition 3.2. Let V be a $\mathbb{K}$-vector space and $\alpha_l, \alpha_r \colon V \to \mathrm{Der}(A)$ be $\mathbb{K}$-linear morphisms. The triple (V, α_l, α_r) is called a double anchored $\mathbb{K}$-vector space.

A morphism of double anchored $\mathbb{K}$-vector spaces $\Phi \colon (V, \alpha_l, \alpha_r) \to (V', \alpha'_l, \alpha'_r)$ is just a $\mathbb{K}$-linear map $\Phi \colon V \to V'$ such that $\alpha'_l \circ \Phi = \alpha_l$ and $\alpha'_r \circ \Phi = \alpha_r$.

Consequently, we have the category $\mathfrak{DoubleAnc}(\mathbb{K})$ of double anchored $\mathbb{K}$-vector spaces and linear morphisms.

Example 3.3. Let V be a $\mathbb{K}$-vector space and $\gamma_l, \gamma_r \colon V \to \mathrm{Der}(A)$ be $\mathbb{K}$-linear maps. Then $(A \otimes V, \alpha_l, \alpha_r)$ is a double anchored $\mathbb{K}$-vector space where $\alpha_l(a \otimes v)(a') = a\gamma_l(v)(a')$ and $\alpha_r(a \otimes v)(a') = a\gamma_r(v)(a')$. Additionally $A \otimes V$ is an A-module with respect to the operation $a.(a' \otimes v) = (aa') \otimes v, a, a' \in A, v \in V$.

One has the forgetful functor

$$U \colon \mathfrak{L}\mathrm{QD}\mathfrak{R}(A) \longrightarrow \mathfrak{DoubleAnc}(\mathbb{K}),$$

which assigns (V, α_l, α_r) to a Loday QD-Rinehart algebra $(\mathfrak{L}, \alpha_l, \alpha_r)$, where V is the underlying $\mathbb{K}$-vector space of $\mathfrak{L}$. Now we construct the functor

$$F \colon \mathfrak{DoubleAnc}(\mathbb{K}) \longrightarrow \mathfrak{L}\mathrm{QD}\mathfrak{R}(A)$$

as follows: let (V, α_l, α_r) be a double anchored $\mathbb{K}$-vector space, $(A \otimes V, \alpha_l, \alpha_r)$ be the corresponding one defined in Example 3.3 and $\bar{T}(A \otimes V)$ be the free Leibniz algebra over $A \otimes V$. Then the morphisms $\alpha_l, \alpha_r \colon A \otimes V \to \mathrm{Der}(A)$ can be extended to morphisms $\bar{T}(A \otimes V) \to \mathrm{Der}(A)$, which are still denoted by α_l, α_r. Then $(\bar{T}(A \otimes V), \alpha_l, \alpha_r)$ is a Loday QD-Rinehart algebra, where $\bar{T}(A \otimes V)$ is endowed with the A-module structure defined inductively by the relations

$$x_1 \otimes a(x_2 \otimes \cdots \otimes x_n) = a(x_1 \otimes x_2 \otimes \cdots \otimes x_n) + \alpha_l(x_1)(a)(x_2 \otimes \cdots \otimes x_n),$$
$$a(x_1 \otimes \cdots \otimes x_{n-1}) \otimes x_n = a(x_1 \otimes x_2 \otimes \cdots \otimes x_{n-1} \otimes x_n) - \alpha_r(x_n)(a)(x_1 \otimes \cdots \otimes x_{n-1}),$$

for $x_i = a_i \otimes v_i \in A \otimes V, i = 1, \ldots, n$.

Theorem 3.4. *F is left adjoint to U.*

Proof. Similar to the proof of Theorem 5.3 in [**23**]. □

Definition 3.5. $(\bar{T}(A \otimes V), \alpha_l, \alpha_r)$ is called the free Loday QD-Rinehart algebra over the double anchored $\mathbb{K}$-vector space (V, α_l, α_r).

Theorem 3.6. *Let $(\mathfrak{L}, \alpha_l, \alpha_r)$ be a free Loday QD-Rinehart algebra, then*

$$HL^i_{QD}(\mathfrak{L}, -) = 0, \quad \text{for } i \geqslant 2.$$

Proof. Let $(\mathfrak{L}, \alpha_l, \alpha_r)$ be a free Loday QD-Rinehart algebra over the double anchored $\mathbb{K}$-vector space (V, α_l, α_r) and M be a representation of $\mathfrak{L}$. Let $L(V)$ be the free Leibniz algebra over the underlying $\mathbb{K}$-vector space V of $\mathfrak{L}$. It is straightforward to check that

$$CL^n_{QD}(\mathfrak{L}, M) = \mathrm{Hom}_A(\mathfrak{L}^{\otimes_A n}, M) \overset{(*)}{\cong} \mathrm{Hom}(L(V)^{\otimes n}, M) = CL^n_{\mathrm{Leib}}(L(V), M),$$

where $CL^n_{\mathrm{Leib}}(L(V), M)$ denotes the Leibniz cochain complex of the Leibniz algebra $L(V)$ with coefficients in M [**28**]. The isomorphism of cochain complexes $(*)$ is given as follows: for any $f \in \mathrm{Hom}_A(\mathfrak{L}, M)$, the corresponding $\varphi \in \mathrm{Hom}(L(V), M)$ is given by $\varphi(x) = f(1 \otimes x)$; conversely, for any $\varphi \in \mathrm{Hom}(L(V), M)$ the corresponding $f \in \mathrm{Hom}_A(\mathfrak{L}, M)$ is defined by $f(a \otimes x) = a\varphi(x)$. From this isomorphism immediately derives the isomorphism

$$HL^*_{QD}(\mathfrak{L}, M) = HL^*_{\mathrm{Leib}}(L(V), M).$$

Then $HL^i_{QD}(\mathfrak{L}, M) = 0$, $i \geqslant 2$, by [**28**, Corollary 3.5]. □

3.2. Abelian extensions

Definition 3.7. Let M be a representation of a Loday QD-Rinehart algebra $(\mathfrak{L}, \alpha_l, \alpha_r)$. We say that two abelian extensions of $(\mathfrak{L}, \alpha_l, \alpha_r)$ by M (see Definition 2.11), $M \xrightarrow{i'} \mathfrak{L}' \xrightarrow{\pi'} \mathfrak{L}$ and $M \xrightarrow{i''} \mathfrak{L}'' \xrightarrow{\pi''} \mathfrak{L}$, are equivalent if there exists a homomorphism of Loday QD-Rinehart algebras $\varphi\colon \mathfrak{L}' \to \mathfrak{L}''$ making the following diagram commutative:

$$\begin{array}{ccccc} M & \xrightarrow{i'} & \mathfrak{L}' & \xrightarrow{\pi'} & \mathfrak{L} \\ \| & & \downarrow{\scriptstyle\varphi} & & \| \\ M & \xrightarrow{i''} & \mathfrak{L}'' & \xrightarrow{\pi''} & \mathfrak{L} \end{array}$$

Remark 3.8. Note that φ is necessarily an isomorphism and, consequently, the previous relation is an equivalence relation. We denote by $\mathrm{Ext}_{QD}(\mathfrak{L}, M)$ the set of equivalence classes of abelian extensions of $(\mathfrak{L}, \alpha_l, \alpha_r)$ by M. This set is non empty, since it contains, at least, the abelian extension (3).

Let $(\mathfrak{L}, \alpha_l, \alpha_r)$ be a Loday QD-Rinehart algebra, M be a representation of $(\mathfrak{L}, \alpha_l, \alpha_r)$ and $\omega \in ZL^2_{QD}(\mathfrak{L}, M) = \mathrm{Ker}(\delta^2)$, that is a 2-cocycle, then we can construct the abelian extension of $(\mathfrak{L}, \alpha_l, \alpha_r)$ by M

$$M \xrightarrow{\chi} M \oplus_\omega \mathfrak{L} \xrightarrow{p} \mathfrak{L},$$

where the operations on $M \oplus_\omega \mathfrak{L}$ are given by

$$\begin{aligned} a(m,X) &= (am, aX), \\ [(m,X),(m',X')] &= ([X,m'] + [m,X'] + \omega(X,X'), [X,X']), \end{aligned}$$

for all $m, m' \in M; X, X' \in \mathfrak{L}$, and the anchor maps are

$$\begin{aligned} &\tilde{\alpha}_l \colon M \oplus_\omega \mathfrak{L} \longrightarrow \mathrm{Der}(A), \qquad \tilde{\alpha}_l(m,X) = \alpha_l(X), \\ &\tilde{\alpha}_r \colon M \oplus_\omega \mathfrak{L} \longrightarrow \mathrm{Der}(A), \qquad \tilde{\alpha}_r(m,X) = \alpha_r(X). \end{aligned}$$

$(M \oplus_\omega \mathfrak{L}, \tilde{\alpha}_l, \tilde{\alpha}_r)$ is endowed with a structure of Loday QD-Rinehart algebra if and only if ω satisfies the following equation:

$$\begin{aligned} &\omega(X,[X',X'']) - \omega([X,X'],X'') + \omega([X,X''],X') \\ &+ [X, \omega(X',X'')] - [\omega(X,X'),X''] + [\omega(X,X''),X'] = 0, \end{aligned} \tag{4}$$

for all $X, X', X'' \in \mathfrak{L}$.

Equation (4) means that ω is a 2-cocycle.

Proposition 3.9. *Let $(\mathfrak{L}, \alpha_l, \alpha_r)$ be a Loday QD-Rinehart algebra and M be a representation of $(\mathfrak{L}, \alpha_l, \alpha_r)$. Every class of abelian extensions with an A-linear section in $\mathrm{Ext}_{QD}(\mathfrak{L}, M)$ can be represented by an abelian extension of the form $M \xrightarrow{\chi} M \oplus_\omega \mathfrak{L} \xrightarrow{p} \mathfrak{L}$.*

We will denote by $\mathrm{Ext}^{\mathrm{sp}}_{QD}(\mathfrak{L}, M)$ the set of equivalence classes of abelian extensions of $(\mathfrak{L}, \alpha_l, \alpha_r)$ by M with an A-linear section.

Proof. Let $M \xrightarrow{i} \mathfrak{L}' \xrightarrow{\pi} \mathfrak{L}$ be an abelian extension and $\sigma \colon \mathfrak{L} \to \mathfrak{L}'$ be a linear A-section of the short exact sequence, that is, $\sigma \colon \mathfrak{L} \to \mathfrak{L}'$ is an A-linear map such that $\pi \circ \sigma = 1_{\mathfrak{L}}$.

We define the A-linear map

$$\omega \colon \mathfrak{L} \otimes_A \mathfrak{L} \longrightarrow M$$

by

$$\omega(X \otimes Y) = i^{-1}([\sigma(X), \sigma(Y)] - \sigma[X,Y]).$$

It is an easy task to check that $\omega \in ZL^2_{QD}(\mathfrak{L}, M)$, so we can construct the abelian extension $M \xrightarrow{\chi} M \oplus_\omega \mathfrak{L} \xrightarrow{p} \mathfrak{L}$.

Finally, we want to show that this abelian extension is equivalent to the abelian extension $M \xrightarrow{i} \mathfrak{L}' \xrightarrow{\pi} \mathfrak{L}$, which means that we must find a homomorphism $\phi \colon M \oplus_\omega \mathfrak{L} \to \mathfrak{L}'$ making the following diagram commutative

$$\begin{array}{ccccc} M & \xrightarrow{\chi} & M \oplus_\omega \mathfrak{L} & \xrightarrow{p} & \mathfrak{L} \\ \| & & {\scriptstyle\phi}\downarrow & & \| \\ M & \xrightarrow{i} & \mathfrak{L}' & \xrightarrow{\pi} & \mathfrak{L} \end{array}$$

It is an easy task to check that $\phi \colon M \oplus_\omega \mathfrak{L} \to \mathfrak{L}'$ given by $\phi(m,X) = i(m) + \sigma(X)$ satisfies all the required conditions, consequently, both abelian extensions are equivalent. ☐

Proposition 3.10. *Two abelian extensions* $M \xrightarrow{\chi} M \oplus_\omega \mathfrak{L} \xrightarrow{p} \mathfrak{L}$ *and* $M \xrightarrow{\chi'} M \oplus_{\omega'} \mathfrak{L} \xrightarrow{p'} \mathfrak{L}$ *are equivalent if and only if* ω *and* ω' *are cohomologous.*

Proof. Suppose there exists a homomorphism $\phi\colon M \oplus_\omega \mathfrak{L} \to M \oplus_{\omega'} \mathfrak{L}$ such that the following diagram is commutative

$$\begin{array}{ccccc} M & \xrightarrow{\chi} & M \oplus_\omega \mathfrak{L} & \xrightarrow{p} & \mathfrak{L} \\ \| & & \downarrow{\scriptstyle \phi} & & \| \\ M & \xrightarrow{\chi'} & M \oplus_{\omega'} \mathfrak{L} & \xrightarrow{p'} & \mathfrak{L} \end{array} \tag{5}$$

ϕ is necessarily of the form $\phi(m, X) = (m + \theta(X), X)$, where $\theta\colon \mathfrak{L} \to M$ is an A-linear map. Moreover, since ϕ is a homomorphism of Loday QD-Rinehart algebras, then $\phi[(m, X), (m', X')] = [\phi(m, X), \phi(m', X')]$ implies that

$$\omega(X, X') - \omega'(X, X') = [X, \theta(X')] + [\theta(X), X'] - \theta[X, X'], \tag{6}$$

for all $X, X' \in \mathfrak{L}$.

From (6), it follows that $\omega(X, X') - \omega'(X, X') = \delta^1\theta(X, X')$, that is, $\omega - \omega' \in BL^1_{QD}(\mathfrak{L}, M) = \mathrm{Im}(\delta^1)$; in other words, ω and ω' are cohomologous.

Conversely, if ω and ω' are cohomologous, then there exists an A-linear map $\theta\colon \mathfrak{L} \to M$ such that $\omega - \omega' = \delta^1(\theta)$.

If we define $\phi\colon M \oplus_\omega \mathfrak{L} \to M \oplus_{\omega'} \mathfrak{L}$ by $\phi(m, X) = (m + \theta(X), X)$, then ϕ is a homomorphism of Loday QD-Rinehart algebras making diagram (5) commutative. □

Theorem 3.11. *Let* $(\mathfrak{L}, \alpha_l, \alpha_r)$ *be a Loday QD-Rinehart algebra and* M *be a representation of* $(\mathfrak{L}, \alpha_l, \alpha_r)$*. Then there exists a bijection*

$$\mathrm{Ext}^{\mathrm{sp}}_{QD}(\mathfrak{L}, M) \cong HL^2_{QD}(\mathfrak{L}, M).$$

3.3. Crossed extensions

Let $(\mathfrak{X}, \alpha'_l, \alpha'_r)$ be a Loday QD-Rinehart algebra and M be a representation over $(\mathfrak{X}, \alpha'_l, \alpha'_r)$. Consider the sequences

$$M \xrightarrow{\iota} R \xrightarrow{\mu} (\mathfrak{L}, \alpha_{\mathfrak{l}}, \alpha_{\mathfrak{r}}) \xrightarrow{\vartheta} (\mathfrak{X}, \alpha'_l, \alpha'_r),$$

where $\mu\colon R \to \mathfrak{L}$ is a crossed module of Loday QD-Rinehart algebras, $\iota\colon M \to R$ is an injective homomorphism, $\vartheta\colon \mathfrak{L} \to \mathfrak{X}$ is a surjective homomorphism, the canonical maps $\mathrm{Coker}(\mu) \to \mathfrak{X}$ and $M \to \mathrm{Ker}(\mu)$ are isomorphisms of Loday QD-Rinehart algebras and representations, respectively, $\mathrm{Ker}(\vartheta) = \mathrm{Im}(\mu)$ and the homomorphisms $\vartheta\colon (\mathfrak{L}, \alpha_{\mathfrak{l}}, \alpha_{\mathfrak{r}}) \to (\mathfrak{X}, \alpha'_l, \alpha'_r)$ and $R \to \mathrm{Im}(\mu)$ have A-linear sections. We refer to this kind of exact sequences as crossed extensions of Loday QD-Rinehart algebras.

Morphisms between crossed extensions are commutative diagrams

$$\begin{array}{ccccccc} M & \xrightarrow{\iota} & R & \xrightarrow{\mu} & \mathfrak{L} & \xrightarrow{\vartheta} & \mathfrak{X} \\ \| & & {\scriptstyle \beta_1}\downarrow & & \downarrow{\scriptstyle \beta_0} & & \| \\ M & \xrightarrow{\iota'} & R' & \xrightarrow{\mu'} & \mathfrak{L}' & \xrightarrow{\vartheta'} & \mathfrak{X} \end{array}$$

where β_0 is an A-split homomorphism of Loday QD-Rinehart algebras, β_1 is an A-split homomorphism of Leibniz A-algebras and the pair (β_1, β_0) is a homomorphism of crossed modules.

Consequently, we have defined the category of crossed extensions of Loday QD-Rinehart algebras, which is denoted by $\mathfrak{Cross}(\mathfrak{X}, M)$.

Let $\vartheta\colon \mathfrak{L} \to \mathfrak{X}$ be a surjective homomorphism of Loday QD-Rinehart algebras with an A-linear section. If M is an $\mathfrak{X}$-representation, then we can define the cochain complex $CL^*_{QD}(\mathfrak{X}, \mathfrak{L}, M)$ via the exact sequence

$$0 \longrightarrow CL^*_{QD}(\mathfrak{X}, M) \xrightarrow{\vartheta^*} CL^*_{QD}(\mathfrak{L}, M) \xrightarrow{\kappa^*} CL^*_{QD}(\mathfrak{X}, \mathfrak{L}, M) \longrightarrow 0.$$

The cohomology of the complex $CL^*_{QD}(\mathfrak{X}, \mathfrak{L}, M)$ will be denoted by $HL^{*+1}_{QD}(\mathfrak{X}, \mathfrak{L}, M)$. Define the subcategory $\mathfrak{Cross}(\mathfrak{X}, \mathfrak{L}, M)$ of $\mathfrak{Cross}(\mathfrak{X}, M)$ whose objects are the crossed extensions

$$\varepsilon\colon M \xrightarrow{\iota} R \xrightarrow{\mu} \mathfrak{L} \xrightarrow{\vartheta} \mathfrak{X},$$

with fixed ϑ. Morphism in $\mathfrak{Cross}(\mathfrak{X}, \mathfrak{L}, M)$ are commutative diagrams

$$\begin{array}{ccccccc} M & \xrightarrow{\iota} & R & \xrightarrow{\mu} & \mathfrak{L} & \xrightarrow{\vartheta} & \mathfrak{X} \\ \| & & \downarrow{\scriptstyle\beta_1} & & \| & & \| \\ M & \xrightarrow{\iota'} & R' & \xrightarrow{\mu'} & \mathfrak{L} & \xrightarrow{\vartheta} & \mathfrak{X} \end{array}$$

Observe that this subcategory is a groupoid.

Theorem 3.12. *For any fixed surjection $\vartheta\colon \mathfrak{L} \to \mathfrak{X}$ with an A-linear section, there exists a natural bijection between the connected components of $\mathfrak{Cross}(\mathfrak{X}, \mathfrak{L}, M)$ and $HL^3_{QD}(\mathfrak{X}, \mathfrak{L}, M)$.*

Proof. See subsection 3.4. □

Theorem 3.13. *Let $\mathfrak{X}$ be a Loday QD-Rinehart algebra and M be an $\mathfrak{X}$-representation. If $\mathfrak{X}$ is a projective A-module, then there exists a natural bijection between the class of the connected components of the category $\mathfrak{Cross}(\mathfrak{X}, M)$ and $HL^3_{QD}(\mathfrak{X}, M)$.*

Proof. Follows from Theorem 3.12 by using the fact that $HL^n_{QD}(\mathfrak{X}, M)$ vanishes in injective representations for $n \geqslant 2$. □

3.4. Proof of Theorem 3.12

Let $\mu\colon R \to \mathfrak{L}$ be a crossed module of Loday QD-Rinehart algebras. By the assumption, let $\sigma\colon \mathfrak{X} \to \mathfrak{L}$ and $\rho\colon \mathrm{Im}(\mu) \to R$ be the A-linear sections. Thus $\vartheta \circ \sigma = 1_{\mathfrak{X}}$ and $\mu \circ \rho = 1_{\mathrm{Im}(\mu)}$. Define $g\colon \mathfrak{X} \otimes \mathfrak{X} \to R$ by $g(X \otimes Y) = \rho([\sigma X, \sigma Y] - \sigma[X, Y])$. Since ϑ is a Loday QD-Rinehart algebra homomorphism, we have $\alpha_l(Z)(a) = \alpha'_l \circ \vartheta(Z)(a)$, $\alpha_r(Z)(a) = \alpha'_r \circ \vartheta(Z)(a)$, for any $Z \in \mathfrak{L}$, $a \in A$, from which we get $g(aX, Y) = ag(X, Y) = g(X, aY)$. In other words, g is A-linear.

Now define $h_\varepsilon\colon \mathfrak{L} \otimes \mathfrak{L} \to R$ by

$$h_\varepsilon(Z, Z') = g(\vartheta(Z), \vartheta(Z')) + [\psi(Z), Z'] + [Z, \psi(Z')] - [\psi(Z), \psi(Z')] - \psi[Z, Z'],$$

where $\psi\colon \mathfrak{L} \to R$ is an A-linear map given by

$$\psi(Z) = \rho(Z - \sigma\vartheta(Z)).$$

Lemma 3.14. *h_ε has values in M and $h_\varepsilon(aZ, Z') = ah_\varepsilon(Z, Z') = h_\varepsilon(Z, aZ')$, for all $a \in A$, $Z, Z' \in \mathfrak{L}$.*

Proof. An easy computation shows that $\mu(h_\varepsilon(Z, Z')) = 0$.

$$\begin{aligned} h_\varepsilon(aZ, Z') = {} & ag(\vartheta(Z), \vartheta(Z')) + a[\psi(Z), Z'] - \alpha_r(Z')(a)\psi(Z) + a[Z, \psi(Z')] \\ & -a[\psi(Z), \psi(Z')] - a\psi[Z, Z'] + \alpha_r(Z')(a)\psi(Z) = ah_\varepsilon(Z, Z'). \end{aligned}$$

Similarly $h_\varepsilon(Z, aZ') = ah_\varepsilon(Z, Z')$. □

From the above computation, $h_\varepsilon \in CL^2_{QD}(\mathfrak{L}, M)$ and $\kappa^*(h_\varepsilon)$ in $CL^2_{QD}(\mathfrak{X}, \mathfrak{L}, M)$ is a cocycle whose class in $HL^3_{QD}(\mathfrak{X}, \mathfrak{L}, M)$ does not depend on the sections. The proof of this fact is quite similar to the one given in [**24**], so we omit it. Consequently, we get the map

$$\pi_0(\mathfrak{Cross}(\mathfrak{X}, \mathfrak{L}, M)) \longrightarrow HL^3_{QD}(\mathfrak{X}, \mathfrak{L}, M).$$

Conversely, for any $f \in CL^2(\mathfrak{L}, M)$, $\kappa^*(f)$ is a cocycle in $CL^2_{QD}(\mathfrak{X}, \mathfrak{L}, M)$ which means that $\mu \circ f = \vartheta^*(\kappa)$, for some $\kappa \in CL^3_{QD}(\mathfrak{X}, M)$. Define $R := M \oplus \mathrm{Ker}(\vartheta)$ as an A-module, which is a Leibniz A-algebra with respect to the bracket

$$[(m, X), (m', X')] = (f([X, X']), [X, X']),$$

$m, m' \in M; X, X' \in \mathrm{Ker}(\vartheta)$. Since f is A-linear and the kernel of a Loday QD-Rinehart algebra homomorphism is a Leibniz A-algebra, then the defined bracket is A-bilinear.

Define the maps $[-,-]\colon \mathfrak{L} \times R \to R$, $[-,-]\colon R \times \mathfrak{L} \to R$ by

$$\begin{aligned} [Y, (m, X)] &= ([\vartheta(Y), m] + f(Y, X), [Y, X]), \\ [(m, X), Y] &= ([m, \vartheta(Y)] + f(X, Y), [X, Y]), \end{aligned}$$

$X \in \mathrm{Ker}(\vartheta), m \in M, Y \in \mathfrak{L}$. These maps define a Leibniz action from the Loday QD-Rinehart $\mathfrak{L}$ over R.

Finally, define $\mu\colon R \to \mathfrak{L}$ by $\mu(m, X) = X$, for all $(m, X) \in R$, which is a Leibniz $\mathbb{K}$-algebra homomorphism. Moreover, with the above Leibniz actions, $\mu\colon R \to \mathfrak{L}$ is a crossed module of Loday QD-Rinehart algebras since μ is A-linear and $\mathrm{Im}(\mu) = \mathrm{Ker}(\vartheta)$.

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References

[1] D. Baraglia, Leibniz algebroids, twistings and exceptional generalized geometry, *J. Geom. Phys.* **62** (5) (2012), 903–934.

[2] F. Borceux, G. Janelidze and G. M. Kelly, Internal object actions, *Comment. Math. Univ. Carolin.* **46** (2) (2005), 235–255.

[3] D. Bourn, Normal subobjects and abelian objects in protomodular categories, *J. Algebra* **228** (1) (2000), 143–164.

[**4**] D. Bourn and G. Janelidze, Extensions with abelian kernels in protomodular categories, *Georgian Math. J.* **11** (4) (2004), 645–654.

[**5**] J.M. Casas, T. Datuashvili and M. Ladra, Universal strict general actors and actors in categories of interest, *Appl. Categ. Structures* **18** (1) (2010), 85–114.

[**6**] J.M. Casas, T. Datuashvili and M. Ladra, Left-right noncommutative Poisson algebras, *Cent. Eur. J. Math.* **12** (1) (2014), 57–78.

[**7**] J. Couchman, *Crossed modules and internal categories of Lie-Rinehart algebras*, MRes Thesis, Macquarie University, Sydney, Australia (2017).

[**8**] İ. Demir, K.C. Misra and E. Stitzinger, On some structures of Leibniz algebras in "*Recent advances in representation theory, quantum groups, algebraic geometry, and related topics*", 41–54, *Contemp. Math.*, **623**, Amer. Math. Soc., Providence, RI, 2014.

[**9**] D. García-Beltrán and J.A. Vallejo, An aproach to omni-Lie algebroids using quasi-derivations, *J. Gen. Lie Theory Appl.* **5** (2011), Art. ID G100801, 10 pp.

[**10**] K. Grabowska and J. Grabowski, Variational calculus with constraints on general algebroids, *J. Phys. A* **41** (17) (2008), 175204, 25 pp.

[**11**] K. Grabowska and J. Grabowski, Dirac algebroids in Lagrangian and Hamiltonian Mechanics, *J. Geom. Phys.* **61** (11) (2011), 2233–2253.

[**12**] K. Grabowska, J. Grabowski and P. Urbański, Geometrical Mechanics on algebroids, *Int. J. Geom. Methods Mod. Phys.* **3** (3) (2006), 559–575.

[**13**] J. Grabowski, Quasi-derivations and QD-algebroids, *Rep. Math. Phys.* **52** (3) (2003), 445–451.

[**14**] J. Grabowski, D. Khudaverdyan and N. Poncin, The supergeometry of Loday algebroids, *J. Geom. Mech.* **5** (2) (2013), 185–213.

[**15**] J. Grabowski, M. de León, D. Martín de Diego and J.C. Marrero, Nonholonomic constraints: a new viewpoint, *J. Math. Phys.* **50** (1) (2009), 013520, 17 pp.

[**16**] J. Grabowski and G. Marmo, Non-antisymmetric versions of Nambu-Poisson and Lie algebroid bracket, *J. Phys. A* **34** (18) (2001), 3803–3809.

[**17**] J. Grabowski and P. Urbański, Algebroids-general differential calculi on vector bundles, *J. Geom. Phys.* **31** (2–3) (1999), 111–141.

[**18**] Y. Hagiwara, Nambu-Dirac manifolds, *J. Phys. A* **35** (5) (2002), 1263–1281.

[**19**] Y. Hagiwara and T. Mizutani, Leibniz algebras associated with foliations, *Kodai Math. J.* **25** (2) (2002), 151–165.

[**20**] J. Huebschmann, Poisson cohomology and quantization, *J. Reine Angew. Math.* **408** (1990), 57–113.

[**21**] R. Ibáñez, M. de León, J.C. Marrero and E. Padrón, Leibniz algebroid associated with a Nambu-Poisson structure, *J. Phys. A, Math. Gen.* **32** (46) (1999), 8129–8144.

[**22**] G. Janelidze, Internal crossed modules, *Georgian Math. J.* **10** (1) (2003), 99–114.

[**23**] B. Jubin, N. Poncin and K. Uchino, Free Courant and derived Leibniz pseudoalgebras, *J. Geom. Mech.* **8** (1) (2016), 71–97.

[**24**] C. Kassel and J.-L. Loday, Extensions centrales d'algèbres de Lie, *Ann. Inst. Fourier (Grenoble)* **32** (4) (1982), 119–142.

[25] A. Kotov and T. Strobl, Generalizing geometry-algebroids and sigma models, in "*Handbook of pseudo-Riemannian geometry and supersymmetry*", 209–262, *IRMA Lect. Math. Theor. Phys.*, **16**, Eur. Math. Soc., Zürich, 2010.

[26] J.-L. Loday, Spaces with finitely many non-trivial homotopy groups, *J. Pure Appl. Algebra* **24** (2) (1982), 179–202.

[27] J.-L. Loday, Une version non commutative des algèbres de Lie: les algèbres de Leibniz, *Enseign. Math.* (2) **39** (3–4) (1993), 269–292.

[28] J.-L. Loday and T. Pirashvili, Universal enveloping algebras of Leibniz algebras and (co)homology, *Math. Ann.* **296** (1) (1993), 139–158.

[29] K. Mikami and T. Mizutani, Algebroids associated with a pre-Poisson structures, in "*Noncommutative geometry and physics*" 2005, 71–96, World Sci. Publ., Hackensack, NJ, 2007.

[30] J.P. Ortega and V. Planas-Bielsa, Dynamics on Leibniz manifolds, *J. Geom. Phys.* **52** (1) (2004), 1–27.

[31] G. Orzech, Obstruction theory in algebraic categories, I, *J. Pure Appl. Algebra* **2** (1972), 287–314.

[32] G. Orzech, Obstruction theory in algebraic categories, II, *J. Pure Appl. Algebra* **2** (1972), 315–340.

[33] R.S. Palais, The cohomology of Lie rings, in "*Proc. Sympos. Pure Math.*", Vol. III, American Mathematical Society, Providence, R.I., 1961, pp. 130–137.

[34] T. Porter, Extensions, crossed modules and internal categories in categories of groups with operations, *Proc. Edinburgh Math. Soc.* (2) **30** (3) (1987), 373–381.

[35] G.S. Rinehart, Differential forms on general commutative algebras, *Trans. Amer. Math. Soc.* **108** (1963), 195–222.

[36] M. Stiénon and P. Xu, Modular classes of Loday algebroids, *C. R. Math. Acad. Sci. Paris* **346** (3–4) (2008), 193–198.

[37] A. Wade, On some properties of Leibniz algebroids in "*Infinite dimensional Lie groups in geometry and representation theory*" (Washington, DC, 2000), 65–78, World Sci. Publ., River Edge, NJ, 2002.

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