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A SIMPLE PROOF OF CURTIS' CONNECTIVITY THEOREM FOR LIE POWERS

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Abstract

We give a simple proof of Curtis' theorem: if $A_\bullet$ is a k-connected free simplicial abelian group, then $L^n(A_\bullet)$ is a $k+\lceil \log_2 n\rceil$-connected simplicial abelian group, where L^n is the n-th Lie power functor. In the proof we do not use Curtis' decomposition of Lie powers. Instead we use the Chevalley–Eilenberg complex for the free Lie algebra.

1. Introduction

In [**5**] Curtis constructed a spectral sequence that converges to the homotopy groups $\pi_*(X)$ of a simply connected space X. It was described in the language of simplicial groups. This spectral sequence was an early version of the unstable Adams spectral sequence (see [**6**, §9], [**2**]). Recall that a simplicial group $G_\bullet$ is called n-connected if $\pi_i(G_\bullet)=0$ for $i\leqslant n$. For a group G we denote by $\gamma_n(G)$ the n-th term of its lower central series. In order to prove the convergence of this spectral sequence, Curtis proved a theorem, which we call Curtis' connectivity theorem for lower central series. It can be formulated as follows.

Theorem ([**5**]). *If $G_\bullet$ is a k-connected free simplicial group for $k\geqslant 0$, then the simplicial group $\gamma_n(G_\bullet)$ is $k+\lceil \log_2 n\rceil$-connected.*

Curtis gave a tricky proof of this theorem using some delicate calculations with generators in free groups. Later Rector [**11**] described a mod-p analogue of this spectral sequence where the lower central series is replaced by the mod-p lower central series. Then Quillen [**10**] found a more conceptual way to prove the connectivity theorem for the mod-p lower central series using simplicial profinite groups. This result was enough to prove the convergence of the mod-p version of the spectral sequence. Quillen reduced this connectivity theorem to an earlier result of Curtis, which we call Curtis' connectivity theorem for Lie powers. Denote by $L^n\colon \mathsf{Ab}\to\mathsf{Ab}$ the n-th Lie power functor. Then the theorem can be formulated as follows.

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Theorem ([**4**])**.** *If $A_\bullet$ is a k-connected free simplicial abelian group, then the simplicial abelian group $L^n(A_\bullet)$ is $k+\lceil \log_2 n\rceil$-connected.*

Curtis' proof of this theorem is quite complicated and takes up most of the paper (see [**4**, §4–7]). He used the so-called "decomposition of Lie powers" into smaller functors. The decomposition is a kind of filtration on the functor L^n (see [**4**, §4]). The goal of this paper is to give a simpler proof of this theorem without the decomposition. Instead of this we use the Chevalley–Eilenberg complex for the free Lie algebra. We also generalize the statement to the case of modules over arbitrary commutative rings.

Let R be a commutative ring. We say that a functor $\mathcal{F}\colon \mathsf{Mod}(R)\to\mathsf{Mod}(R)$ is *n-connected* if, for any $k\geqslant 0$ and any k-connected free simplicial module $A_\bullet$, the simplicial module $\mathcal{F}(A_\bullet)$ is $k+n$-connected. In these terms we prove the following.

Theorem. *The Lie power functor $L^n\colon \mathsf{Mod}(R)\to\mathsf{Mod}(R)$ is $\lceil\log_2 n\rceil$-connected.*

We also note that this estimate of the connectivity of L^n is best possible for $n=2^m$. This is an easy corollary of the description of homotopy groups of 2-restricted Lie powers in the language of lambda-algebra given in [**6**] and [**2**] (see also [**9, 8**]).

Proposition. *If $R=\mathbb{Z}$ or $R=\mathbb{Z}/2$ the functor $L^{2^n}\colon \mathsf{Mod}(R)\to\mathsf{Mod}(R)$ is not $n+1$-connected.*

Note that our proof of the main theorem is quite elementary. However, the proposition is a corollary of some non-elementary results about the lambda-algebra.

Assume that $\mathfrak{g}$ is a Lie algebra which is free as a module over the commutative ground ring R. By the Chevalley–Eilenberg complex of $\mathfrak{g}$ we mean the chain complex whose components are the exterior powers $\Lambda^i\mathfrak{g}$ and whose homology is homology of the Lie algebra with trivial coefficients $H_*(\mathfrak{g})$. We consider the free Lie algebra as a functor from the category of free modules to the category of Lie algebras. The free Lie algebra has a natural grading whose components are the Lie powers $L^*(A)=\bigoplus_{n\geqslant 1}L^n(A)$. Here we treat Lie powers as functors from the category of free modules $L^n\colon \mathsf{FMod}(R)\to\mathsf{Mod}(R)$. The grading on the free Lie algebra induces a grading on the Chevalley–Eilenberg complex whose components give exact sequences of functors on the category of free modules:

$$0\to\Lambda^2\to L^2\to 0,$$
$$0\to\Lambda^3\to \mathrm{Id}\otimes L^2\to L^3\to 0,$$
$$0\to\Lambda^4\to\Lambda^2\otimes L^2\to(\mathrm{Id}\otimes L^3)\oplus\Lambda^2L^2\to L^4\to 0,$$
$$\dots$$
$$0\to\Lambda^n\to\cdots\to\bigoplus_{\substack{k_1+\cdots+k_n=i\\ k_1\cdot 1+k_2\cdot 2+\cdots+k_n\cdot n=n}}\Lambda^{k_1}L^1\otimes\Lambda^{k_2}L^2\otimes\cdots\otimes\Lambda^{k_n}L^n\to\cdots\to L^n\to 0$$

(see Corollary 2.2), where $\Lambda^{k_s}L^s$ denotes the composition of the Lie power functor and the exterior power functor. We use these complexes for induction in the proof of the main result.

2. Graded Chevalley–Eilenberg complex

Throughout the paper R denotes a commutative ring. All algebras, modules, simplicial modules, tensor products and exterior powers are assumed to be over R.

Let $\mathfrak{g}$ be a Lie algebra which is free as a module. If we tensor the Chevalley–Eilenberg resolution $V_\bullet(\mathfrak{g})$ (see [**3**, XIII §7–8]) on the trivial module R, we obtain a complex $C_\bullet(\mathfrak{g}) \cong R \otimes_{U\mathfrak{g}} V_\bullet(\mathfrak{g})$ that we call the Chevalley–Eilenberg complex. Its components are exterior powers of the Lie algebra $C_i(\mathfrak{g}) = \Lambda^i \mathfrak{g}$ and the differential is given by the formula

$$d(x_1 \wedge \dots \wedge x_i) = \sum_{s<t} (-1)^{s+t} [x_s, x_t] \wedge x_1 \wedge \dots \wedge \hat{x}_s \wedge \dots \wedge \hat{x}_t \wedge \dots \wedge x_i.$$

The homology of this complex is isomorphic to the homology of the Lie algebra $\mathfrak{g}$ with trivial coefficients

$$H_i(\mathfrak{g}, R) = H_i(C_\bullet(\mathfrak{g})).$$

Let $\mathfrak{g}$ be a graded Lie algebra $\mathfrak{g} = \bigoplus_{n \geqslant 1} \mathfrak{g}_n$. By a graded Lie algebra we mean a traditional Lie algebra (not a Lie superalgebra) $\mathfrak{g}$ together with a decomposition into a direct sum of modules $\mathfrak{g} = \bigoplus_{n \geqslant 1} \mathfrak{g}_n$ such that $[\mathfrak{g}_n, \mathfrak{g}_m] \subseteq \mathfrak{g}_{n+m}$ for all $n, m \geqslant 1$. The degree of a homogeneous element $x \in \mathfrak{g}_n$ is denoted by $|x| = n$.

For $n \geqslant 1$ we consider a submodule $C_i^{(n)}(\mathfrak{g})$ of $C_i(\mathfrak{g})$ spanned by elements $x_1 \wedge \dots \wedge x_i$, where $x_1, \dots, x_i$ are homogeneous and $|x_1| + \dots + |x_i| = n$.

$$C_i^{(n)}(\mathfrak{g}) = \operatorname{span}\{x_1 \wedge \dots \wedge x_i \in \Lambda^i \mathfrak{g} \mid |x_1| + \dots + |x_i| = n\}.$$

It is easy to see that $d(C_i^{(n)}(\mathfrak{g})) \subseteq C_{i-1}^{(n)}(\mathfrak{g})$, and hence we obtain a subcomplex $C_\bullet^{(n)}(\mathfrak{g})$ of $C_\bullet(\mathfrak{g})$.

Proposition 2.1. *Let $\mathfrak{g} = \bigoplus_{n \geqslant 1} \mathfrak{g}_n$ be a graded Lie algebra, where $\mathfrak{g}_n$ is free as a module for each n. Then the Chevalley–Eilenberg complex $C_\bullet(\mathfrak{g})$ has a natural grading*

$$C_\bullet(\mathfrak{g}) = \bigoplus_{n \geqslant 1} C_\bullet^{(n)}(\mathfrak{g}),$$

and there is a natural isomorphism

$$C_i^{(n)}(\mathfrak{g}) \cong \bigoplus_{\substack{k_1 + \dots + k_n = i \\ k_1 \cdot 1 + k_2 \cdot 2 + \dots + k_n \cdot n = n}} \Lambda^{k_1} \mathfrak{g}_1 \otimes \Lambda^{k_2} \mathfrak{g}_2 \otimes \dots \otimes \Lambda^{k_n} \mathfrak{g}_n.$$

Here the sum runs over the set of ordered n-tuples of non-negative integers $(k_1, \dots, k_n)$ such that $k_1 + \dots + k_n = i$ and $k_1 \cdot 1 + k_2 \cdot 2 + \dots + k_n \cdot n = n$.

Proof. For any modules A, B there is an isomorphism $\Lambda^i(A \oplus B) \cong \bigoplus_{k+l=i} \Lambda^k(A) \otimes \Lambda^l(B)$. By induction we obtain the isomorphism

$$\Lambda^i \left(\bigoplus_{s=1}^{N} A_s \right) \cong \bigoplus_{k_1 + \dots + k_N = i} \Lambda^{k_1} A_1 \otimes \dots \otimes \Lambda^{k_N} A_N.$$

Using the fact that the exterior power commutes with direct limits, we obtain the isomorphism, for any infinite sequence of modules $A_1, A_2, \dots,$

$$\Lambda^i \left(\bigoplus_{s=1}^{\infty} A_i\right) \cong \bigoplus_{k_1+k_2+\cdots=i} \Lambda^{k_1} A_1 \otimes \Lambda^{k_2} A_2 \otimes \cdots.$$

Here we consider only those sequences of non-negative integers $k_1, k_2, \dots$ with only finitely many non-zero elements, and hence, each summand in the sum is a finite tensor product.

Take $A_n = \mathfrak{g}_n$. If we have an element $x_1 \wedge \cdots \wedge x_i$ with homogeneous $x_s \in \mathfrak{g}$ from the R-submodule corresponding to a summand $\Lambda^{k_1}\mathfrak{g}_1 \otimes \Lambda^{k_2}\mathfrak{g}_2 \otimes \cdots$, then $|x_1| + \cdots + |x_n| = k_1 \cdot 1 + k_2 \cdot 2 + \cdots$. The assertion follows. □

Let A be a free module. We denote by $L^*(A)$ the free Lie algebra generated by A For any basis (a_s) of A, $L^*(A)$ is isomorphic to the free Lie algebra generated by the family (a_s). The Lie algebra $L^*(A)$ is free as a module (see [**13**], [**12**, Cor. 0.10]). Its enveloping algebra is the tensor algebra $T^*(A)$. The map $L^*(A) \to T^*(A)$ is injective [**12**, Cor. 0.3]. Hence, $L^*(A)$ can be described in terms of the tensor algebra. Consider the tensor algebra $T^*(A)$ as a Lie algebra with respect to the commutator. Then $L^*(A)$ can be described as the Lie subalgebra of $T^*(A)$ generated by A (see also [**6**, §7.4]).

The Lie algebra $L^*(A)$ has a natural grading

$$L^*(A) = \bigoplus_{n=1}^{\infty} L^n(A),$$

where $L^n(A)$ is generated by n-fold commutators. Equivalently $L^n(A)$ can be described using the embedding into the tensor algebra as $L^n(A) = L(A) \cap T^n(A)$. The homology of the free Lie algebra $L^*(A)$ can be described as follows $H_i(L^*(A)) = 0$ for $i > 1$ and $H_1(L^*(A)) = A$. For simplicity we set

$$\mathsf{C}^{(n)}_\bullet(A) := C^{(n)}_\bullet(L^*(A)).$$

All these constructions are natural in A. Denote by L^n the n-th Lie power functor from the category of free modules to the category of modules

$$L^n \colon \mathsf{FMod}(R) \longrightarrow \mathsf{Mod}(R).$$

Moreover, we treat $\mathsf{C}^{(n)}_\bullet$ as a complex in the category of functors $\mathsf{FMod}(R) \to \mathsf{Mod}(R)$. Then Proposition 2.1 implies the following corollary.

Corollary 2.2. *For $n \geqslant 2$ the complex $\mathsf{C}^{(n)}_\bullet$ of functors $\mathsf{FMod}(R) \to \mathsf{Mod}(R)$ is acyclic and has the following components*

$$\mathsf{C}^{(n)}_i = \bigoplus_{\substack{k_1+\cdots+k_n=i \\ k_1\cdot 1+k_2\cdot 2+\cdots+k_n\cdot n=n}} \Lambda^{k_1} L^1 \otimes \Lambda^{k_2} L^2 \otimes \cdots \otimes \Lambda^{k_n} L^n,$$

where $\Lambda^{k_s} L^s$ denotes the composition of the Lie power functor and the exterior power functor. Here the sum runs over the set of ordered n-tuples of non-negative integers $(k_1, \dots, k_n)$ such that $k_1 + \cdots + k_n = i$ and $k_1 \cdot 1 + k_2 \cdot 2 + \cdots + k_n \cdot n = n$.

Remark 2.3. Note that $\mathsf{C}_i^{(n)} = 0$ for $i \notin \{1, \dots, n\}$, and that there are isomorphisms $\mathsf{C}_n^{(n)} = \Lambda^n$ and $\mathsf{C}_1^{(n)} = L^n$. In other words $\mathsf{C}_\bullet^{(n)}$ is an exact sequence that connects Λ^n and L^n

$$\mathsf{C}_\bullet^{(n)}: \qquad 0 \to \Lambda^n \to \cdots \to L^n \to 0.$$

3. Connectivity of functors

For $n \geqslant 0$ we say that a simplicial module $A_\bullet$ is n-connected, if $\pi_i(A_\bullet) = 0$ for $i \leqslant n$.

Lemma 3.1. *Let $A_\bullet$ be an n-connected simplicial module and $B_\bullet$ an m-connected free simplicial module. Then $A_\bullet \otimes B_\bullet$ is $n+m+1$-connected.*

Proof. Consider their component-wise tensor product $A_\bullet \otimes B_\bullet$. The Eilenberg–Zilber theorem implies that $\pi_i(A_\bullet \otimes B_\bullet) \cong H_i(NA_\bullet \otimes NB_\bullet)$, where $NC_\bullet$ denotes the Moore complex of $C_\bullet$. Since $N_iB_\bullet$ is a direct summand of B_i, it is a projective module. This gives the following variant of the Künneth spectral sequence:

$$E^2_{pq} = \bigoplus_{s+t=q} \mathsf{Tor}_p^R(\pi_s(A_\bullet), \pi_t(B_\bullet)) \Rightarrow \pi_{p+q}(A_\bullet \otimes B_\bullet).$$

If $s + t \leqslant n + m + 1$, then either $s < n + 1$ or $t < m + 1$. Hence $E^2_{pq} = 0$ for $p + q \leqslant n + m + 1$. Therefore, $A_\bullet \otimes B_\bullet$ is $n + m + 1$-connected. □

A functor from the category of modules to itself

$$\mathcal{F}\colon \mathsf{Mod}(R) \longrightarrow \mathsf{Mod}(R)$$

is said to be n-connected if, for any $k \geqslant 0$ and any k-connected *free* simplicial module $A_\bullet$, the simplicial module $\mathcal{F}(A_\bullet)$ is $k + n$-connected.

Lemma 3.2. *Let $\mathcal{F}\colon \mathsf{Mod}(R) \to \mathsf{Mod}(R)$ be an n-connected functor and $\mathcal{G}\colon \mathsf{Mod}(R) \to \mathsf{Mod}(R)$ an m-connected functor. Assume that $\mathcal{G}$ sends free modules to free modules. Then the composition $\mathcal{F}\mathcal{G}$ is $n+m$-connected and the tensor product $\mathcal{F} \otimes \mathcal{G}$ is $n+m+1$-connected.*

Proof. The fact about the composition is obvious. The fact about the tensor product follows from Lemma 3.1. □

Lemma 3.3. *Let*

$$0 \to \mathcal{F}_n \to \cdots \to \mathcal{F}_1 \to \mathcal{F}_0 \to \mathcal{G} \to 0$$

be an exact sequence of functors such that $\mathcal{F}_i$ is $n - i$-connected. Then $\mathcal{G}$ is n-connected.

Proof. The proof is by induction. For $n = 0$ this is obvious. Assume that $n \geqslant 1$ and that the statement holds for smaller numbers. Set $\mathcal{H} := \mathsf{Ker}(\mathcal{F}_0 \to \mathcal{G})$. Then by the induction hypothesis $\mathcal{H}$ is $n - 1$-connected. The long exact sequence

$$\cdots \to \pi_i(\mathcal{H}(A_\bullet)) \to \pi_i(\mathcal{F}_0(A_\bullet)) \to \pi_i(\mathcal{G}(A_\bullet)) \to \pi_{i-1}(\mathcal{H}(A_\bullet)) \to \cdots$$

implies that $\mathcal{G}$ is n-connected. □

Proposition 3.4. *The exterior power functor* Λ^n *is* $n-1$*-connected.*

Proof. The décalage formula [**7**, Prop. 4.3.2.1] for exterior and divided powers Λ^n, Γ^n gives a homotopy equivalence, for any free simplicial module $B_\bullet$,

$$\Lambda^n(B_\bullet[1]) \sim \Gamma^n(B_\bullet)[n].$$

Any 0-connected free simplicial module $A_\bullet$ is homotopy equivalent to a simplicial module of the form $B_\bullet[1]$, where $B_\bullet$ is also a free simplicial module (it follows from the same fact for non-negatively graded chain complexes). Moreover, if $A_\bullet$ is k-connected, we can chose $B_\bullet$ so that $B_i = 0$ for $i \leqslant k-1$. Hence $\pi_i(\Lambda^n(A_\bullet)) = \pi_i(\Lambda^n(B_\bullet[1])) = \pi_{i-n}(\Gamma^n(B_\bullet)) = 0$ for $i \leqslant k+n-1$. □

Lemma 3.5. *For any two sequences of positive integer numbers* $u_1, \dots, u_m$ *and* $v_1, \dots, v_m$ *the following inequality holds*

$$\sum_{s=1}^{m}(u_s + \log_2 v_s) \geqslant 1 + \log_2\left(\sum_{s=1}^{m} u_s v_s\right).$$

Proof. It is easy to prove by induction that $\prod_{s=1}^m 2^{u_s} v_s \geqslant 2\sum_{s=1}^m u_s v_s$. If we apply logarithms, we obtain the required statement. □

Theorem 3.6. *The Lie power functor* L^n *is* $\lceil \log_2 n \rceil$*-connected.*

Proof. The proof is by induction. For $n = 1$ we have $L^1 = \mathrm{Id}$ and this is obvious. Assume that $n \geqslant 2$ and that the statement holds for all smaller numbers. Consider the acyclic chain complex $\mathsf{C}_\bullet^{(n)}$ (Corollary 2.2). Using Lemma 3.3 we obtain that it is enough to check that the functor

$$\mathsf{C}_i^{(n)} = \bigoplus_{\substack{k_1+\dots+k_n=i \\ k_1\cdot 1+k_2\cdot 2+\dots+k_n\cdot n=n}} \Lambda^{k_1}L^1 \otimes \Lambda^{k_2}L^2 \otimes \dots \otimes \Lambda^{k_n}L^n$$

is $\lceil \log_2 n \rceil$-connected for $i \geqslant 2$. It is enough to prove this for each summand.

Fix an n-tuple of $(k_1, \dots, k_n)$ such that $k_1 + \dots + k_n = i \geqslant 2$ and $k_1 \cdot 1 + k_2 \cdot 2 + \dots + k_n \cdot n = n$. Note that $i \geqslant 2$ implies $k_n = 0$. Some of the numbers k_j are equal to zero. Denote by $j_1, \dots, j_m$ the indexes corresponding to non-zero numbers $k_{j_s} \neq 0$. By Lemma 3.2 the functor $\Lambda^{k_j}L^j$ is $k_j - 1 + \lceil \log_2 j \rceil$-connected for $j < n$. Then again by Lemma 3.2 the tensor product

$$\Lambda^{k_1}L^1 \otimes \Lambda^{k_2}L^2 \otimes \dots \otimes \Lambda^{k_n}L^n$$

is $\sum_{s=1}^m \left(k_{j_s} - 1 + \lceil \log_2 j_s \rceil\right) + m - 1$-connected. Using Lemma 3.5 we obtain

$$\sum_{s=1}^{m}\left(k_{j_s} - 1 + \log_2 j_s\right) + m - 1 = \sum_{s=1}^{m}\left(k_{j_s} + \log_2 j_s\right) - 1 \geqslant \log_2 n.$$

The assertion follows. □

4. Connectivity of L^{2^n}

For $k \geqslant 0$ we denote by $R[k+1]$ the chain complex concentrated in degree $k+1$, whose $k+1$-st component is equal to R. The corresponding Dold–Kan simplicial module is denoted by $K(R, k+1)$. Note that $K(R, k+1)$ is a k-connected free simplicial module.

Proposition 4.1. *Let $R = \mathbb{Z}$ or $R = \mathbb{Z}/2$ and $n \geqslant 0$. Then L^{2^n} is not $n+1$-connected. Moreover, for any $k \geqslant 0$*

$$\pi_{n+1+k}(L^{2^n}(K(R, k+1)) \neq 0.$$

Proof. (1) Let $R = \mathbb{Z}/2$. Fix k and set $V_\bullet = K(\mathbb{Z}/2, k+1)$. Denote by $L^n_{\mathsf{res}}\colon \mathsf{Vect}(\mathbb{Z}/2) \to \mathsf{Vect}(\mathbb{Z}/2)$ the 2-restricted Lie power functor (see [**1**, §2.7], [**6**, §7.5]). The homotopy groups $\pi_*(L^{2^n}_{\mathsf{res}}(V_\bullet))$ are described in terms of the lambda-algebra in [**6**, Th. 8.8] (see also [**2**] and the discussion after Theorem 7.11 in [**6**]):

$$\pi_{i+k+1}(L^{2^n}_{\mathsf{res}}(V_\bullet)) \cong \Lambda^{i,n}(k+2),$$

where $\Lambda^{i,n}(k+2)$ denotes the vector sub-space of the lambda algebra Λ with the basis given by compositions $\lambda_{i_1} \cdots \lambda_{i_n}$, where $i_{s+1} \leqslant 2i_s$, $i_1 + \cdots + i_n = i$ and $i_1 \leqslant k+2$. In particular, $\lambda_1^n \in \Lambda^{n,n}(k+2) \neq 0$. Hence

$$\pi_{n+1+k}(L^{2^n}_{\mathsf{res}}(V_\bullet)) \neq 0.$$

For an arbitrary simplicial Lie algebra $\mathfrak{g}_\bullet$ and $t \geqslant 1$ we define the map $\tilde{\lambda}_1\colon \mathfrak{g}_t \to \mathfrak{g}_{t+1}$ by the formula $\tilde{\lambda}_1(x) = [s_0x, s_1x]$, where s_0, s_1 are the degeneracy maps. Denote by $\mathfrak{i}_{k+1}$ the unit of $(V_\bullet)_{k+1} = R$. Then $\tilde{\lambda}_1^n(\mathfrak{i}_{k+1}) \in (L(V_\bullet))_{n+k+1}$ is the element representing $\lambda_1^n \in \Lambda^{n,n}(k+2) \cong \pi_{n+1+k}(L^{2^n}_{\mathsf{res}}(V_\bullet))$ (see [**6**, Prop. 8.6]). By the definition of $\tilde{\lambda}_1$, the element $\tilde{\lambda}_1^n(\mathfrak{i}_{k+1})$ lies in the unrestricted part $L^{2^n}(V_\bullet)$ of $L^{2^n}_{\mathsf{res}}(V_\bullet)$. Therefore $\tilde{\lambda}_1^n(\mathfrak{i}_{k+1})$ represents a nontrivial element of $\pi_{n+1+k}(L^{2^n}(V_\bullet))$ and hence

$$\pi_{n+1+k}(L^{2^n}(V_\bullet)) \neq 0.$$

(2) Now assume that $R = \mathbb{Z}$ and set $A_\bullet = K(\mathbb{Z}, k+1)$. We denote by $L^*_{\mathbb{Z}/2}$ the Lie power functor over $\mathbb{Z}/2$, which we already discussed, and by $L^*_{\mathbb{Z}}$ the Lie power functor over $\mathbb{Z}$. Then for any free abelian group A we have $L^*_{\mathbb{Z}}(A) \otimes \mathbb{Z}/2 \cong L^*_{\mathbb{Z}/2}(A \otimes \mathbb{Z}/2)$. The universal coefficient theorem gives the following short exact sequence

$$0 \longrightarrow \pi_i(L^{2^n}_{\mathbb{Z}}(A_\bullet)) \otimes \mathbb{Z}/2 \longrightarrow \pi_i(L^{2^n}_{\mathbb{Z}/2}(V_\bullet)) \longrightarrow \mathsf{Tor}_1^{\mathbb{Z}}(\pi_{i-1}(L^{2^n}_{\mathbb{Z}}(A_\bullet)), \mathbb{Z}/2) \longrightarrow 0.$$

Since the functor $L^{2^n}_{\mathbb{Z}}$ is n-connected, $\pi_{n+k}(L^{2^n}_{\mathbb{Z}}(A_\bullet)) = 0$. Therefore

$$\pi_{n+1+k}(L^{2^n}_{\mathbb{Z}}(A_\bullet)) \otimes \mathbb{Z}/2 \cong \pi_{n+1+k}(L^{2^n}_{\mathbb{Z}/2}(V_\bullet)).$$

We already proved that $\pi_{n+1+k}(L^{2^n}_{\mathbb{Z}/2}(V_\bullet)) \neq 0$. Hence $\pi_{n+1+k}(L^{2^n}_{\mathbb{Z}}(A_\bullet)) \neq 0$. □

Remark 4.2. Proposition 4.1 can be also deduced from results of D. Leibowitz [**8**] or from unpublished results of R. Mikhailov [**9**], where he describes all derived functors in the sense of Dold–Puppe of Lie powers in the case $R = \mathbb{Z}$.

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