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EQUIVARIANT STEINBERG SUMMANDS

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(*communicated by Nicholas J. Kuhn*)

Abstract

We construct Steinberg summands of G-equivariant spectra with $\mathrm{GL}_n(\mathbb{F}_p)$-action. We prove a lemma about their fixed points when G is a p-group, and then use this lemma to compute the fixed points of the Steinberg summand of the equivariant classifying space of $(\mathbb{Z}/p)^n$. These results will be used in a companion paper to study the layers in the mod p symmetric power filtration for $H\underline{\mathbb{F}}_p$.

1. Introduction

This paper establishes two results regarding Steinberg summands of equivariant spectra. Namely, let G be a finite p-group and let $\mathrm{GL}_n = \mathrm{GL}_n(\mathbb{F}_p)$. Then,

1. (Theorem 3.3) For any pointed $(G \times \mathrm{GL}_n)$-space Y, there is a natural homotopy equivalence $e_n(Y^G) \to (e_n Y)^G$ from the Steinberg summand of the fixed points to the fixed points of the Steinberg summand.
2. (Theorem 4.2) Let $B_G(\mathbb{Z}/p)^n$ denote the equivariant classifying space of $(\mathbb{Z}/p)^n$. Let $\mathcal{C}$ denote the set of normal subgroups $H \subseteq G$ such that G/H is an elementary abelian p-group. Then the fixed points of the Steinberg summand $e_n(B_G(\mathbb{Z}/p)^n_+)$ decompose into a wedge sum of spectra
$$(e_n(B_G(\mathbb{Z}/p)^n_+)^G \simeq \bigvee_{H \in \mathcal{C}} E_n(H).$$
More explicitly, if G/H is elementary abelian of rank d, then the summand $E_n(H)$ is
$$E_n(H) \simeq e_{n-d} B(\mathbb{Z}/p)^{n-d}_+ \wedge \Sigma^{1-d} \mathbf{B}^{\diamond}_d \wedge B(\mathbb{Z}/p)^d_+,$$
where $\mathbf{B}^{\diamond}_d$ is the unreduced suspension of the flag complex of $\mathbb{F}_p^d$ (Definition 2.8).

Along the way we gather certain results about (non-equivariant) Steinberg summands that are scattered in the literature. We also prove a result relating Steinberg summands and Stiefel varieties $V_d(\mathbb{F}_p^n)$ (Proposition 4.4), which is an important step in the proof of Proposition 4.7.

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1.1. Background and context

The Steinberg module [14] is an irreducible, unipotent representation which plays a special role in the representation and character theory of a Chevalley group G over the field $\mathbb{F}_p$ (see [7] for a survey). In this paper, we only consider the Steinberg module for the group $\Gamma = \mathrm{GL}_n(\mathbb{F}_p)$, but more generally one has a Steinberg module when G is any Chevalley or p-adic Lie group, and the Steinberg module can be realized topologically using the Bruhat–Tits building of the group Γ (as we have done here).

Stable splittings of spectra have long been used in homotopy theory to construct new classes of maps, particularly in the homotopy groups of the spheres. The Steinberg idempotent associated to $\mathrm{GL}_n(\mathbb{F}_p)$ was used by Mitchell–Priddy to split certain interesting summands $M(n)$ off of the classifying spaces of elementary abelian p-groups [11, 12]. The cohomology of the summands $M(n)$ reflects the length filtration in the mod p Steenrod algebra. These spectra were a crucial ingredient in work of Kuhn, Mitchell, and Priddy on the Whitehead conjecture [9, 8], were used by Mitchell to prove the Conner–Floyd conjecture [10], and have chromatic type n [15]. More recently, Steinberg idempotents for the symplectic group Sp_{2n} appear in a conjectural bu-analogue of the Whitehead conjecture [2].

We adapt the Steinberg idempotent of $\mathrm{GL}_n(\mathbb{F}_p)$ to G-equivariant homotopy theory. This paper is a companion to a larger work [13] in which the layers in the mod p symmetric power filtration are calculated, with a view to understanding $H\underline{\mathbb{F}}_p \wedge H\underline{\mathbb{F}}_p$. In the larger paper, we observe that the genuine G-spectrum $H\underline{\mathbb{F}}_p$ is the infinite mod p symmetric power of the equivariant sphere spectrum $\Sigma^{\infty G} S^0$, and the layers of the filtration

$$\begin{aligned}\Sigma^{\infty G} S^0 = \mathrm{Sp}^1_{\mathbb{Z}/p}(\Sigma^{\infty G} S^0) \subset \mathrm{Sp}^p_{\mathbb{Z}/p}(\Sigma^{\infty G} S^0) \subset \mathrm{Sp}^{p^2}_{\mathbb{Z}/p}(\Sigma^{\infty G} S^0) \subset \cdots \\ \cdots \subset \mathrm{Sp}^{\infty}_{\mathbb{Z}/p}(\Sigma^{\infty G} S^0) = H\underline{\mathbb{F}}_p\end{aligned}$$

are the n-fold suspensions of the Steinberg summands $e_n B_G(\mathbb{Z}/p)^n_+$.

1.2. Conventions and notation

In this paper, all spaces are pointed CW complexes with finitely many cells in each dimension. Thus, 'homotopy equivalence' and 'weak equivalence' are interchangeable.

All groups used are finite. When G is a group, the term 'G-space' is used to mean a G-CW complex. A G-map $f\colon X \to Y$ of G-CW complexes is an equivalence if $f^H\colon X^H \to Y^H$ is an equivalence for every subgroup $H \subseteq G$.

We use the term *spectrum* to mean a sequence of pointed CW-complexes $X_0, X_1, X_2, \dots$ with structure maps $f_i\colon \Sigma X_i \to X_{i+1}$. Such an object is often called a *prespectrum* in the literature, while the term 'spectrum' is reserved for the special case where $X_i \cong \Omega X_{i+1}$. But all spectra considered in this paper are of the form $S^{-n} \wedge \Sigma^\infty X$ for some integer n and pointed space X, so we do not worry about the distinction and use the simpler term. All spectra considered are connective ($\mathrm{colim}_{i\to\infty}(\pi_{n+i} X_i) \cong 0$ for $n < 0$) and of finite type (for fixed i and n variable, the number of $(n+i)$-cells in X_n is bounded). A map of spectra $\{X_i \to Y_i\}_{i\geqslant 0}$ is an *equivalence* if for every $n \geqslant 0$, the map $X_i \to Y_i$ is an isomorphism in homotopy groups up through dimension $n+i$ for $i \gg n$.

All G-spectra are *naïve G-spectra*, meaning they are spectrum objects in the category of G-spaces. That is, we mean a sequence of G-CW complexes $X_0, X_1, X_2, \dots$

with G-maps $f_i \colon \Sigma X_i \to X_{i+1}$. For any subgroup $H \subseteq G$, the H-fixed point spectrum X^H has i-th space X_i^H. A map $f \colon X \to Y$ of naïve G-spectra is an equivalence if $f^H \colon X^H \to Y^H$ is an equivalence of spectra for every subgroup $H \subseteq G$. We refer the reader to [**5**], section 2 for an introduction to equivariant spectra.

2. Steinberg summands

In this section, we construct Steinberg summands and prove some of their basic properties. The results of this section are not original work of the author, but they are scattered throughout the literature so we collect and prove the results important to us.

In subsection 2.1, we define the Steinberg idempotent and Steinberg representation. In subsection 2.2 we define product maps relating these idempotents. The Steinberg summand in topology does not appear until subsection 2.3. There, we give its definition (Definition 2.8) in terms of the flag complex. Several properties of the flag complex are proven, which will later be useful to us.

2.1. The Steinberg idempotent

Classical sources for the material of this subsection are [**4, 14**]. Fix a prime p, and let $\mathbb{F}_p$ denote the field of p elements. Let n be a positive integer, and write $\mathrm{GL}_n = \mathrm{GL}_n(\mathbb{F}_p)$ for brevity. Tensor products will be taken over $\mathbb{Z}_{(p)}$ unless otherwise specified.

Let $\Sigma_n \subset \mathrm{GL}_n$ be the subgroup of permutation matrices, and let $B_n \subset \mathrm{GL}_n$ be the Borel subgroup of upper triangular matrices. Associated to these two subgroups are elements $\overline{\Sigma}_n, \overline{B}_n$ in the group algebra $\mathbb{Z}_{(p)}[\mathrm{GL}_n]$ defined by

$$\overline{\Sigma}_n := \sum_{\sigma \in \Sigma_n} (-1)^\sigma \sigma, \qquad \overline{B}_n := \sum_{b \in B_n} b.$$

Lemma 2 of [**14**] states that $\overline{\Sigma}_n \overline{B}_n \overline{\Sigma}_n \overline{B}_n = c_n \cdot \overline{\Sigma}_n \overline{B}_n$, where c_n is the constant

$$c_n = \prod_{i=1}^{n} (p^i - 1).$$

The number c_n is invertible in $\mathbb{Z}_{(p)}$ and therefore the element

$$e_n = \frac{1}{c_n} \cdot \overline{\Sigma}_n \overline{B}_n$$

is an idempotent in $\mathbb{Z}_{(p)}[\mathrm{GL}_n]$.

Definition 2.1. The element $e_n = \frac{1}{c_n} \cdot \overline{\Sigma}_n \overline{B}_n$ is called the *Steinberg idempotent*. For any left $\mathbb{Z}_{(p)}[\mathrm{GL}_n]$-module M, the $\mathbb{Z}_{(p)}$-submodule

$$e_n M = \{e_n m : m \in M\} \subset M$$

is called the *Steinberg summand* of M. The construction $M \mapsto e_n M$ is a functor from left $\mathbb{Z}_{(p)}[\mathrm{GL}_n]$-modules to $\mathbb{Z}_{(p)}$-modules.

Note that $\mathbb{Z}_{(p)}[\mathrm{GL}_n]$ is both a left module and a right module over itself. Therefore, $\mathbb{Z}_{(p)}[\mathrm{GL}_n] e_n$ is a left $\mathbb{Z}_{(p)}[\mathrm{GL}_n]$-submodule of $\mathbb{Z}_{(p)}[\mathrm{GL}_n]$.

Definition 2.2. The left $\mathbb{Z}_{(p)}[\mathrm{GL}_n]$-module $\mathbb{Z}_{(p)}[\mathrm{GL}_n]e_n$ is denoted by St_n and is called the *Steinberg module*. For any left $\mathbb{Z}_{(p)}[\mathrm{GL}_n]$-module M, there is a natural isomorphism of $\mathbb{Z}_{(p)}$-modules

$$\begin{aligned}(\mathrm{St}_n \otimes M)_{\mathrm{GL}_n} &\xrightarrow{\cong} e_n M,\\ Ae_n \otimes m &\mapsto e_n(A^{-1}m).\end{aligned}$$

The Steinberg module has dimension $p^{\binom{n}{2}}$ over $\mathbb{Z}_{(p)}$ – this fact is a direct corollary of Propositions 2.6 and 2.7, proven in a later section.

Definition 2.3. The element $\hat{e}_n = \frac{1}{c_n} \cdot \overline{B}_n \overline{\Sigma}_n$ is called the *conjugate Steinberg idempotent*. For any left $\mathbb{Z}_{(p)}[\mathrm{GL}_n]$-module M, the $\mathbb{Z}_{(p)}$-submodule $\hat{e}_n M$ is called the *conjugate Steinberg summand* of M.

The following two maps of $\mathbb{Z}_{(p)}$-modules are inverse isomorphisms

$$\begin{aligned}e_n M &\to \hat{e}_n M,\\ \overline{\Sigma}_n \overline{B}_n m &\mapsto \overline{B}_n \overline{\Sigma}_n \overline{B}_n m,\end{aligned} \qquad \begin{aligned}\hat{e}_n M &\to e_n M,\\ \overline{B}_n \overline{\Sigma}_n m &\mapsto \overline{\Sigma}_n \overline{B}_n \overline{\Sigma}_n m.\end{aligned}$$

because composing them in either order induces multiplication by the unit $c_n \in \mathbb{Z}_{(p)}^{\times}$. Therefore $e_n M$ and $\hat{e}_n M$ are isomorphic $\mathbb{Z}_{(p)}$-modules.

2.2. Products on Steinberg summands

Let i and j be positive integers. The block inclusion

$$\begin{aligned}\mathrm{GL}_i \times \mathrm{GL}_j &\to \mathrm{GL}_{i+j},\\ (A, B) &\mapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix},\end{aligned}$$

gives a map $\mathbb{Z}_{(p)}[\mathrm{GL}_i \times \mathrm{GL}_j] \to \mathbb{Z}_{(p)}[\mathrm{GL}_{i+j}]$ of left $\mathbb{Z}_{(p)}[\mathrm{GL}_i \times \mathrm{GL}_j]$-modules. We denote by $e_i \boxtimes e_j$ the image under this map of the idempotent $e_i \otimes e_j$. The idempotent $e_i \boxtimes e_j$ has the following relation to the idempotent e_{i+j}. Let $U_{i,j}$ denote the group of $(i+j) \times (i+j)$ matrices of the form $\begin{pmatrix} I_i & * \\ 0 & I_j \end{pmatrix}$. Let $\Sigma_{\mathrm{shuf}}(i, j)$ denote the set of $\binom{i+j}{i}$ permutations σ with the property that

$$1 \leqslant a < b \leqslant i \implies \sigma(a) < \sigma(b) \quad \text{and} \quad i+1 \leqslant a < b \leqslant i+j \implies \sigma(a) < \sigma(b).$$

Such permutations are known as (i, j)-shuffle permutations. Define

$$\overline{U}_{i,j} = \sum_{u \in U_{i,j}} u, \qquad \overline{\Sigma_{\mathrm{shuf}}(i,j)} = \sum_{\sigma \in \Sigma_{\mathrm{shuf}}(i,j)} (-1)^{\sigma} \sigma.$$

Then the following identities in the group algebra $\mathbb{Z}_{(p)}[\mathrm{GL}_{i+j}]$ can be checked:

$$\begin{aligned}\overline{U}_{i,j} \cdot \overline{B_i \times B_j} &= \overline{B_i \times B_j} \cdot \overline{U}_{i,j} = \overline{B}_{i+j},\\ \overline{\Sigma_{\mathrm{shuf}}(i,j)} \cdot \overline{\Sigma_i \times \Sigma_j} &= \overline{\Sigma_i \times \Sigma_j} \cdot \overline{\Sigma_{\mathrm{shuf}}(i,j)} = \overline{\Sigma}_{i+j},\\ \overline{U}_{i,j} \cdot \overline{\Sigma_i \times \Sigma_j} &= \overline{\Sigma_i \times \Sigma_j} \cdot \overline{U}_{i,j}.\end{aligned}$$

Therefore,

$$\overline{\Sigma_{\mathrm{shuf}}(i,j)} \cdot \overline{U}_{i,j}(e_i \boxtimes e_j) = \frac{c_{i+j}}{c_i c_j} \cdot e_{i+j}. \tag{2.1}$$

Definition 2.4. The homomorphism of left $\mathbb{Z}_{(p)}[\mathrm{GL}_i \times \mathrm{GL}_j]$-modules

$$\begin{aligned}\mathrm{St}_i \otimes_{\mathbb{Z}_{(p)}} \mathrm{St}_j = \mathbb{Z}_{(p)}[\mathrm{GL}_i \times \mathrm{GL}_j](e_i \boxtimes e_j) &\to \mathbb{Z}_{(p)}[\mathrm{GL}_{i+j}]e_{i+j} = \mathrm{St}_{i+j},\\ A(e_i \boxtimes e_j) &\mapsto A\overline{\Sigma_{\mathrm{shuf}}(i,j)} \cdot \overline{U}_{i,j}(e_i \boxtimes e_j),\end{aligned}$$

is called the *Steinberg product.* From the point of view that the Steinberg module represents the functor $(\mathrm{St}_n \otimes (-))_{\mathrm{GL}_n} \cong e_n(-)$, the Steinberg product represents the natural transformation arising from the projection

$$\begin{aligned}(e_i \boxtimes e_j)M &\to e_{i+j}M,\\ x &\mapsto \overline{\Sigma_{\mathrm{shuf}}(i,j)} \cdot \overline{U}_{i,j}x.\end{aligned}$$

The following two properties can be checked, where all maps are $\mathbb{Z}_{(p)}$-module maps.

- (Associativity) The following diagram commutes.

$$\begin{array}{ccc} (e_i \boxtimes e_j \boxtimes e_k)M & \longrightarrow & (e_{i+j} \boxtimes e_k)M \\ \downarrow & & \downarrow \\ (e_i \boxtimes e_{j+k})M & \longrightarrow & e_{i+j+k}M \end{array}$$

- (Commutativity) Let $\sigma \in \Sigma_{i+j}$ be the shuffle permutation that increases every number by j modulo $i+j$. Then $e_i \boxtimes e_j = \sigma^{-1}(e_j \boxtimes e_i)\sigma$, and therefore we have inverse isomorphisms $\sigma\colon (e_i \boxtimes e_j)M \to (e_j \boxtimes e_i)M$ and $\sigma^{-1}\colon (e_j \boxtimes e_i)M \to (e_i \boxtimes e_j)M$. The following diagram commutes.

$$\begin{array}{ccccc} (e_i \boxtimes e_j)M & \xrightarrow{\sigma} & (e_j \boxtimes e_i)M & \xrightarrow{\sigma^{-1}} & (e_i \boxtimes e_j)M \\ & \searrow & \downarrow & \swarrow & \\ & & e_{i+j}M & & \end{array}$$

For any fixed n, one should think of the various idempotents $\{ e_{i_1} \boxtimes \cdots \boxtimes e_{i_k} \}_{i_1+\cdots+i_k=n}$ as functors from the category of left R_n-modules to the category of $\mathbb{Z}_{(p)}$-modules. The Steinberg product defines natural transformations among these functors, starting from the initial functor $e_1 \boxtimes \cdots \boxtimes e_1$ and going to the final functor e_n.

Proposition 2.5. *Let $f\colon e_{i+j}M \to (e_i \boxtimes e_j)M$ be the $\mathbb{Z}_{(p)}$-linear map*

$$f(e_{i+j}m) = \frac{c_i c_j}{c_{i+j}} \cdot (e_i \boxtimes e_j)e_{i+j}m.$$

Then the composition

$$e_{i+j}M \xrightarrow{f} (e_i \boxtimes e_j)M \longrightarrow e_{i+j}M$$

of f with the Steinberg product is the identity map.

Proof. This proposition is a direct result of Equation 2.1 and the fact that $e_{i+j}^2 = e_{i+j}$. □

2.3. The flag complex

Suppose that X is a pointed topological space with GL_n-action. In this section, we construct a spectrum e_nX in a way that mirrors the algebra of the previous section. We will see that there is a splitting in the homotopy category of p-local spectra

$$\Sigma^\infty X \simeq e_nX \vee (1-e_n)X,$$

where $\vee$ denotes the wedge sum. The reason we must pass to spectra is because our construction involves desuspending spaces.

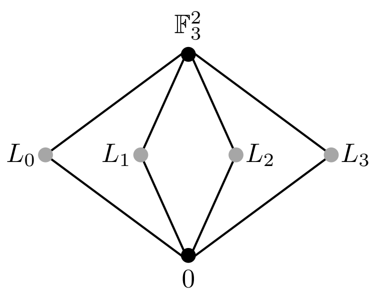
Fix a positive integer n, and let $\mathbb{F}_p^n$ denote a fixed n-dimensional vector space over the field $\mathbb{F}_p$. Let $\mathbf{B}_n$ denote the nerve of the poset of subspaces of $\mathbb{F}_p^n$ that do not equal 0 or $\mathbb{F}_p^n$. This poset, and therefore the associated nerve $\mathbf{B}_n$, carries a left action of GL_n. The following properties are well known, but are proved for the sake of completeness. All supporting proofs are deferred to the end of this section.

Proposition 2.6. *The space $\mathbf{B}_n$ has the homotopy type of a wedge of $p^{\binom{n}{2}}$ spheres of dimension $n-2$.*

Proposition 2.7. *There is an isomorphism of $\mathbb{Z}_{(p)}[\mathrm{GL}_n]$-modules $\mathrm{St}_n \cong H_{n-2}(\mathbf{B}_n; \mathbb{Z}_{(p)})$, given as follows. Let $A \in \mathrm{GL}_n$ be an $n\times n$ matrix with columns $v_1, \ldots, v_n$. Then the element $Ae_n \in \mathrm{St}_n$ is identified with the simplicial chain (which is a cycle)*

$$s_A = \sum_{\sigma\in\Sigma_n} (-1)^\sigma (\langle v_{\sigma(1)}\rangle \subset \langle v_{\sigma(1)}, v_{\sigma(2)}\rangle \subset \cdots).$$

As constructed, $\mathbf{B}_n$ is not a pointed space. Let $\mathbf{B}_n^\diamond$ denote the unreduced suspension of $\mathbf{B}_n$. The space $\mathbf{B}_n^\diamond$ is the geometric realization of a simplicial set where the k-simplices are flags $[W_0 \subseteq \cdots \subseteq W_k]$ of subspaces of $\mathbb{F}_p^n$ with the property that either $W_0 = 0$ or $W_k = \mathbb{F}_p^n$ but not both.



For example, let $p=3$. There are four one-dimensional subspaces of $\mathbb{F}_3^2$, which we denote by L_0, L_1, L_2, and L_3. Pictured to the left is the topological space $\mathbf{B}_2^\diamond$. As a pointed space, it is homotopy equivalent to $\bigvee_3 S^1$. The gray points alone are $\mathbf{B}_2$, which is homotopy equivalent to $\bigvee_3 S^0$.

Then $\mathbf{B}_n^\diamond$ is a pointed space with the 0-simplex $[0]$ as the basepoint. Its $\mathbb{Z}_{(p)}$-homology is as follows

$$\tilde{H}_*(\mathbf{B}_n^\diamond; \mathbb{Z}_{(p)}) \simeq \begin{cases} \mathrm{St}_n & * = n-1, \\ 0 & * \neq n-1. \end{cases}$$

If we smash the space $\mathbf{B}_n^\diamond$ by the negative sphere $S^{-(n-1)}$, we obtain the spectrum $\Sigma^{1-n}\mathbf{B}_n^\diamond$ whose homology is concentrated in degree 0. The spectrum $\Sigma^{1-n}\mathbf{B}_n^\diamond$ should be thought of as a topological analogue to the Steinberg module.

Definition 2.8. Let X be a spectrum with GL_n-action. Then the **Steinberg summand** of X, denoted $e_n X$, is defined as

$$e_n X = (\Sigma^{1-n}\mathbf{B}_n^{\diamond} \wedge X) \wedge_{\mathrm{GL}_n} (E\mathrm{GL}_n)_+.$$

When Y is any pointed space or spectrum with GL_n-action, we henceforth use $Y_{h\mathrm{GL}_n}$ to denote the homotopy orbit space

$$Y_{h\mathrm{GL}_n} := Y \wedge_{\mathrm{GL}_n} (E\mathrm{GL}_n)_+.$$

As an example, let us compute the $\mathbb{F}_p$-homology of $e_n X$, and show that it is equal to the Steinberg summand of the $\mathbb{Z}_{(p)}[\mathrm{GL}_n]$-module $H_*(X;\mathbb{F}_p)$. The Hochschild–Serre spectral sequence associated to the fiber sequence

$$(\Sigma^{1-n}\mathbf{B}_n^{\diamond} \wedge X) \to (\Sigma^{1-n}\mathbf{B}_n^{\diamond} \wedge X)_{h\mathrm{GL}_n} \to B\mathrm{GL}_n$$

has E^2-page

$$E^2_{i,j} = H_i(B\mathrm{GL}_n; H_j(\Sigma^{1-n}\mathbf{B}_n^{\diamond} \wedge X;\mathbb{F}_p)) \implies H_{i+j}((\Sigma^{1-n}\mathbf{B}_n^{\diamond} \wedge X)_{h\mathrm{GL}_n};\mathbb{F}_p).$$

The homology group $H_0(\Sigma^{1-n}\mathbf{B}_n^{\diamond};\mathbb{F}_p) \cong \mathrm{St}_n$ is a projective $\mathbb{F}_p[\mathrm{GL}_n]$-module, and therefore flat. It follows by the Künneth formula that

$$H_j(\Sigma^{1-n}\mathbf{B}_n^{\diamond} \wedge X;\mathbb{F}_p) \cong \mathrm{St}_n \otimes_{\mathbb{Z}_{(p)}} H_j(X;\mathbb{F}_p).$$

Provided that $H_j(X;\mathbb{F}_p)$ is finite-dimensional over $\mathbb{F}_p$, the $\mathbb{F}_p[\mathrm{GL}_n]$-module $\mathrm{St}_n \otimes_{\mathbb{Z}_{(p)}} H_j(X;\mathbb{F}_p)$ is projective, and so it has no higher GL_n-homology. Thus, our E^2-page is

$$H_0(B\mathrm{GL}_n; \mathrm{St}_n \otimes_{\mathbb{Z}_{(p)}} H_*(X;\mathbb{F}_p)) = \mathrm{St}_n \otimes_{\mathbb{Z}_{(p)}[\mathrm{GL}_n]} H_*(X;\mathbb{F}_p),$$

which is by definition the Steinberg summand $e_n H_*(X;\mathbb{F}_p)$. The E^2-page is concentrated on a single vertical line and therefore the spectral sequence collapses.

This argument is functorial in the pointed space X, and therefore implies the following diagram of functors commutes up to natural isomorphism. Here, $\mathrm{GL}_n\mathrm{Sp}$ denotes the category of spectra of finite type with naïve GL_n-action, and $\mathrm{GrMod}_{\mathbb{F}_p[\mathrm{GL}_n]}$ denotes the category of graded left $\mathbb{F}_p[\mathrm{GL}_n]$-modules.

$$\begin{array}{ccc} \mathrm{GL}_n\mathrm{Sp} & \xrightarrow{e_n(-)} & \mathrm{Sp} \\ {\scriptstyle H_*(-;\mathbb{F}_p)}\downarrow & & \downarrow{\scriptstyle H_*(-;\mathbb{F}_p)} \\ \mathrm{GrMod}_{\mathbb{F}_p[\mathrm{GL}_n]} & \xrightarrow[e_n(-)]{} & \mathrm{GrMod}_{\mathbb{F}_p} \end{array}$$

As defined, we have no reason to believe the promise that $e_n X$ is a summand of $\Sigma^\infty X$. Proposition 2.9 below implies that there are natural transformations between elements of $\mathrm{Fun}(\mathrm{GL}_n\mathrm{Sp}, \mathrm{Sp})$ (where Id denotes the functor which forgets the GL_n-action on a spectrum),

$$e_n(-) \to \mathrm{Id}, \qquad \mathrm{Id} \to e_n(-),$$

such that the composition $e_n X \to \mathrm{Id}(X) \to e_n X$ is an isomorphism on $H\mathbb{F}_p$-homology. Given that we have assumed X is connective and of finite type, the composition above is an equivalence on $\mathbb{Z}_{(p)}$-homology groups, and therefore a p-local equivalence.

Therefore, if we define the spectrum $(1-e_n)X$ as the homotopy fiber

$$(1-e_n)X := \mathrm{hofib}(\Sigma^\infty X \to e_n X)$$

then the cofiber sequence $(1-e_n)X \to \Sigma^\infty X \to e_n X$ splits via the map described above.

Proposition 2.9. *Let X be a spectrum with GL_n-action. There are maps of spectra which are functorial in X*

$$e_n X \to X, \qquad\qquad X \to e_n X,$$

such that in $\mathbb{F}_p$-homology, the composition $e_n X \to X \to e_n X$ is multiplication by a unit in $\mathbb{F}_p$.

There are product maps as well. For any finite dimensional $\mathbb{F}_p$-vector space V, let $\mathbf{B}_V$ denote the nerve of the poset of subspaces of V which do not equal 0 or V. If $\dim(V) = n$, then $\mathbf{B}_V \simeq \mathbf{B}_n$. Let $\tilde{\mathbf{B}}_V$ denote the nerve of the poset of subspaces of V, including 0 and V itself. The space $\tilde{\mathbf{B}}_V$ is contractible, because the poset of subspaces of V has an initial element 0. There is an obvious inclusion of simplicial sets $\mathbf{B}^\diamond_V \subset \tilde{\mathbf{B}}_V$. If $V \simeq V' \oplus V''$, then there is a product map

$$\begin{aligned}\tilde{\mathbf{B}}_{V'} \times \tilde{\mathbf{B}}_{V''} &\to \tilde{\mathbf{B}}_V,\\ (W', W'') &\mapsto W' \oplus W''.\end{aligned}$$

When the above map is restricted to either $\tilde{\mathbf{B}}_{V'} \times \mathbf{B}^\diamond_{V''}$ or $\mathbf{B}^\diamond_{V'} \times \tilde{\mathbf{B}}_{V''}$, it lands in the subspace $\mathbf{B}^\diamond_V$. Therefore the product above restricts

$$(\tilde{\mathbf{B}}_{V'} \times \mathbf{B}^\diamond_{V''}) \cup_{(\mathbf{B}^\diamond_{V'} \times \mathbf{B}^\diamond_{V''})} (\mathbf{B}^\diamond_{V'} \times \tilde{\mathbf{B}}_{V''}) \to \mathbf{B}^\diamond_V.$$

But since $\tilde{\mathbf{B}}_{V'}$ and $\tilde{\mathbf{B}}_{V''}$ are both contractible, the union above is homotopy equivalent to the unreduced join $\mathbf{B}^\diamond_{V'} \star \mathbf{B}^\diamond_{V''}$. We have constructed a product on flag complexes, namely

$$\Sigma\mathbf{B}^\diamond_{V'} \wedge \Sigma\mathbf{B}^\diamond_{V''} \to \Sigma\mathbf{B}^\diamond_V.$$

If we choose isomorphisms $V' \cong \mathbb{F}_p^i$ and $V'' \cong \mathbb{F}_p^j$, and the isomorphism $V' \oplus V'' \cong V$ is given by block inclusion, then it is easily checked when we take the top homology of the above product on flag complexes, we recover the Steinberg product $\mathrm{St}_i \otimes \mathrm{St}_j \to \mathrm{St}_{i+j}$ (Definition 2.4) under the $\mathbb{Z}_{(p)}$-module isomorphism $\mathrm{St}_n \cong H_n(\Sigma\mathbf{B}^\diamond_n; \mathbb{Z}_{(p)})$ of Proposition 2.7.

Proposition 2.10. *Let V be a finite dimensional $\mathbb{F}_p$-vector space, and let W be a subspace. Let $P_W \subset \mathrm{GL}(V)$ denote the parabolic subgroup of matrices preserving W. Let $\mathcal{S}_W$ denote the P_W-set*

$$\mathcal{S}_W = \{W' \subset V : W + W' = V \text{ and } W \cap W' = 0\}.$$

Then the P_W-equivariant product map

$$\Sigma\mathbf{B}^\diamond_W \wedge \bigvee_{W' \in \mathcal{S}_W} \Sigma\mathbf{B}^\diamond_{W'} \to \Sigma\mathbf{B}^\diamond_V$$

is a homotopy equivalence of pointed spaces.

As promised, here are the proofs of the propositions in this section.

Proof of Proposition 2.6.[1] We use induction on n. The case $n = 1$ is obvious. Suppose that $n \geqslant 2$. Let $H \subset \mathbb{F}_p^n$ be a subspace of dimension $n-1$, and let $\mathcal{P} \subset \mathbf{B}_n$ be the nerve of the poset of subspaces which intersect H nontrivially. The space $\mathcal{P}$ is contractible, because it has a self map $W \mapsto H \cap W$ which is homotopic to both the constant map at H and to the identity map. Therefore, $\mathbf{B}_n \simeq \mathbf{B}_n/\mathcal{P}$. Note that any subspace of $\mathbb{F}_p^n$ of dimension 2 or greater automatically intersects H nontrivially, and so the only simplices which remain in $\mathbf{B}_n/\mathcal{P}$ are those flags whose bottom space is a line transverse to H. Thus, $\mathbf{B}_n/\mathcal{P}$ decomposes as a wedge sum

$$\mathbf{B}_n/\mathcal{P} \simeq \bigvee_{L \perp H} (\mathbf{B}_n)_{\geqslant L}/(\mathbf{B}_n)_{>L},$$

where $(\mathbf{B}_n)_{\geqslant L}$ (resp. $(\mathbf{B}_n)_{>L}$) denotes the nerve of the poset of subspaces containing L (resp. strictly containing L). The space $(\mathbf{B}_n)_{\geqslant L}$ is contractible, and $(\mathbf{B}_n)_{>L} \simeq \mathbf{B}_{n-1}$. Thus, $\mathbf{B}_n/\mathcal{P} \simeq \bigvee_{L \perp H} \Sigma\mathbf{B}_{n-1}$. The induction is now complete by the observation that there are p^{n-1} lines transverse to H. □

Proof of Proposition 2.7. The group of simplicial chains $C_{n-2}(\mathbf{B}_n)$ is the free $\mathbb{Z}$-module over the set of maximal flags, which is $\mathbb{Z}[\mathrm{GL}_n/B_n] \cong \mathbb{Z}[\mathrm{GL}_n]\overline{B}_n$. For any matrix $A \in \mathrm{GL}_n$, the chain s_A as defined is equal to $A\overline{\Sigma}_n\overline{B}_n$. Thus, the homomorphism $\mathrm{St}_n \to C_{n-2}(\mathbf{B};\mathbb{Z}_{(p)})$ defined by $Ae_n \mapsto s_A$ is a monomorphism.

We claim that s_A is a cycle, namely, $\partial(s_A) = 0$. Let $w_1, \ldots, w_n$ be any permutation of $v_1, \ldots, v_n$. Then any $(n-3)$-simplex of the form $(\cdots \subset \langle w_1, \ldots, w_{i-1}\rangle \subset \langle w_1, \ldots, w_{i+1}\rangle \subset \cdots)$ is on the boundary of exactly the following two different $(k-2)$-simplices.

$$(\cdots \subset \langle w_1, \ldots, w_{i-1}\rangle \subset \langle w_1, \ldots, w_{i-1}, w_i\rangle \subset \langle w_1, \ldots, w_{i+1}\rangle \subset \cdots) \quad \text{and}$$
$$(\cdots \subset \langle w_1, \ldots, w_{i-1}\rangle \subset \langle w_1, \ldots, w_{i-1}, w_{i+1}\rangle \subset \langle w_1, \ldots, w_{i+1}\rangle \subset \cdots),$$

and this implies that $\partial(s_A) = 0$.

We next claim that the set $\{s_A\}_{A\in\mathrm{GL}_n}$ spans $H_{n-2}(\mathbf{B}_n)$. This will complete the proof. Fix a complete flag $\mathcal{F} = (\mathcal{F}_1 \subset \cdots \subset \mathcal{F}_{n-1})$. Suppose that $(W_1 \subset \cdots \subset W_{n-1})$ is a complete flag which is *transverse* to $\mathcal{F}$, *i.e.*, $W_i \cap \mathcal{F}_{n-i} = 0$ for $i = 1, \ldots, n-1$. For each $i = 1, \ldots, n$, $W_i \cap \mathcal{F}_{n-i-1}$ is 1-dimensional, and so we may pick a sequence of nonzero vectors $w_1, \ldots, w_n$ so that $\langle w_i\rangle = W_i \cap \mathcal{F}_{n-i-1}$. The w_i's have two important properties.

- Observe that $w_i \in \mathcal{F}_{n-i-1}$ and therefore $w_i \notin W_{i-1}$. Thus, by induction on i, $\langle w_1, \ldots, w_i\rangle = W_i$ for $i = 1, 2, \ldots, n-1$, and $\langle w_1, \ldots, w_k\rangle = \mathbb{F}_p^n$.
- Suppose that $\sigma \in \Sigma_n$ is a permutation such that $\sigma(i) = j$, where $j > i$. Then $w_j \in \mathcal{F}_{n-(j-1)} \implies w_j \in \mathcal{F}_{n-i}$, and because $w_{\sigma(i)} = w_j$, we have $\langle w_{\sigma(1)}, \ldots, w_{\sigma(i)}\rangle \cap \mathcal{F}_{n-i} \neq 0$. Thus, for any nontrivial permutation $\sigma \in \Sigma_n$, the flag $(\langle w_{\sigma(1)}\rangle \subset \langle w_{\sigma(1)}, w_{\sigma(2)}\rangle \subset \cdots)$ is not transverse to $\mathcal{F}$.

[1] An alternate proof is given as [**3**, Theorem 6.8.5].

Let A be the matrix whose columns are $w_1, \ldots, w_n$. The two properties above imply that

$$s_A = (W_1 \subset \cdots \subset W_{n-1}) + \sum_{\sigma \neq \mathrm{id}} (-1)^\sigma (\text{non-}\mathcal{F}\text{-transverse flags}).$$

It follows that $\dim(\mathrm{span}\{s_A : A \in \mathrm{GL}_n\})$ is at least as large as the number of complete flags transverse to $\mathcal{F}$, which is $p^{\binom{n}{2}}$. This is the dimension of the entire space $H_{n-2}(\mathbf{B}_n)$, so the two are equal, as desired. ☐

Proof of Proposition 2.9. For ease of notation, let us write $\mathbf{B} = \mathbf{B}_n^\diamond$. Then $\mathbf{B}$ is a pointed GL_n-CW-complex whose i-cells are in bijection with the GL_n-set of flags $(V_1 \subsetneq \cdots \subsetneq V_i)$, and it has a ($\mathrm{GL}_n$-equivariant) skeletal filtration

$$\mathbf{B}^{(0)} \subseteq \mathbf{B}^{(1)} \subseteq \cdots \subseteq \mathbf{B}^{(n-1)} = \mathbf{B},$$

where $\mathbf{B}^{(i)}$ contains the cells of dimension i and lower. Then the quotient $\mathbf{B}^{(n-1)}/\mathbf{B}^{(n-2)}$ is a wedge of copies of S^{n-1} indexed over maximal flags, i.e. $\mathbf{B}^{(n-1)}/\mathbf{B}^{(n-2)} \cong (\mathrm{GL}_n/B_n)_+ \wedge S^{n-1}$. The inclusion $e_n X \to X$ of the Steinberg summand is constructed by desuspending $n-1$ times the composition

$$\Sigma^{n-1} e_n X = (\mathbf{B} \wedge X)_{h\mathrm{GL}_n} \to (\mathbf{B}^{(n-1)}/\mathbf{B}^{(n-2)} \wedge X)_{h\mathrm{GL}_n} \simeq \Sigma^{n-1} X_{hB_n} \to \Sigma^{n-1} X,$$

where the last map is the Becker–Gottlieb transfer.

To construct the projection $X \to e_n X$, consider the following diagram:

$$\begin{tikzcd}
 & & (\mathbf{B} \wedge X)_{h\mathrm{GL}_n} \arrow[d] \\
\Sigma^{n-1} X \arrow[r, "\overline{\Sigma}_n"] \arrow[urr, dashed, bend left] \arrow[drr, bend right, "f"'] & \Sigma^{n-1} X \arrow[r] & \Sigma^{n-1} X_{hB_n} \simeq (\mathbf{B}^{(n-1)}/\mathbf{B}^{(n-2)} \wedge X)_{h\mathrm{GL}_n} \arrow[d, "\partial"] \\
 & & (\Sigma \mathbf{B}^{(n-2)} \wedge X)_{h\mathrm{GL}_n},
\end{tikzcd}$$

We claim that the composition labeled f is nullhomotopic, and thus the dotted map exists. To prove this claim, we observe that f arises by taking the composition $\tilde{f}$ of GL_n-maps below, and then applying the functor $((-) \wedge X)_{h\mathrm{GL}_n}$:

$$\begin{tikzcd}
\Sigma^{n-1}(\mathrm{GL}_n)_+ \arrow[r, "(-)\cdot\overline{\Sigma}_n"] \arrow[drr, bend right, "\tilde{f}"'] & \Sigma^{n-1}(\mathrm{GL}_n)_+ \arrow[r] & \Sigma^{n-1}(\mathrm{GL}_n/B_n)_+ \simeq \mathbf{B}^{(n-1)}/\mathbf{B}^{(n-2)} \arrow[d, "\partial"] \\
 & & \Sigma \mathbf{B}^{(n-2)}.
\end{tikzcd}$$

It suffices to prove that $\tilde{f}$ is null on underlying points. Since both source and target are a wedge sum of copies of S^{n-1}, it suffices to check that $H_{n-1}(\tilde{f}) = 0$. It was shown in the proof of Proposition 2.7 that if $A \in \mathrm{GL}_n$ is any matrix, then the horizontal composition above sends $A \mapsto s_A \in \tilde{C}_{n-1}(\mathbf{B}^{(n-1)}/\mathbf{B}^{(n-2)})$, and $\partial(s_A) = 0$.

The composition of the two maps we have constructed has the following effect in $\mathbb{F}_p$-homology

$$\begin{array}{ccccc}
\mathrm{St}_n \otimes_{\mathrm{GL}_n} \tilde{H}_*(X) & \longrightarrow & \mathbb{F}_p[\mathrm{GL}_n] \otimes_{\mathrm{GL}_n} \tilde{H}_*(X) & \longrightarrow & \mathrm{St}_n \otimes_{\mathrm{GL}_n} \tilde{H}_*(X), \\
(A\overline{\Sigma}_n\overline{B}_n \otimes x) & \longmapsto & (A\overline{\Sigma}_n\overline{B}_n \otimes x) & \longmapsto & (A\overline{\Sigma}_n\overline{B}_n\overline{\Sigma}_n\overline{B}_n \otimes x).
\end{array}$$

Since $\overline{\Sigma}_n\overline{B}_n\overline{\Sigma}_n\overline{B}_n = c_n\overline{\Sigma}_n\overline{B}_n$ and $c_n \in \mathbb{Z}_{(p)}^{\times}$, the proposition has been proved. □

Proof of Proposition 2.10. Let V have dimension n. Both $\Sigma\mathbf{B}_W^{\diamond} \wedge \bigvee_{W'\perp W} \Sigma\mathbf{B}_{W'}^{\diamond}$ and $\Sigma\mathbf{B}_V^{\diamond}$ have underlying space equivalent to a wedge of copies of S^n, so it suffices to prove that the map is an equivalence on n-th homology groups. Without loss of generality, assume $V \simeq \mathbb{F}_p^n$, and $W \simeq \mathbb{F}_p^i$ is spanned by the first i basis vectors. The P_W-set of subspaces W' which are transverse to W is equivalent to $P_W/(\mathrm{GL}_i \times \mathrm{GL}_{n-i})$. This set has size $p^{i(n-i)}$, and so by Proposition 2.6,

$$\dim(H_n(\Sigma\mathbf{B}_W^{\diamond} \wedge \bigvee_{W'\perp W} \Sigma\mathbf{B}_{W'}^{\diamond})) = p^{\binom{i}{2}+i(n-i)+\binom{n-i}{2}} = p^{\binom{n}{2}} = \dim(H_n(\Sigma\mathbf{B}_V^{\diamond})).$$

So, for dimension reasons, it suffices to show that the given map is a surjection on homology. Recall that, for any j, the top $\mathbb{Z}_{(p)}$-homology group of $\mathbf{B}_j^{\diamond}$ is $\mathbb{Z}_{(p)}[\mathrm{GL}_j]\overline{\Sigma}_j\overline{B}_j$. Therefore, by the Kunneth formula,

$$\begin{aligned} H_n(\Sigma\mathbf{B}_W^{\diamond} \wedge \bigvee_{W'\perp W} \Sigma\mathbf{B}_{W'}^{\diamond}) &\simeq \mathrm{Ind}_{\mathrm{GL}_i\times\mathrm{GL}_{n-i}}^{P_W}\mathbb{F}_{(p)}[\mathrm{GL}_i \times \mathrm{GL}_{n-i}](\overline{\Sigma_i \times \Sigma_{n-i}})(\overline{B_i \times B_{n-i}}) \\ &\simeq \mathbb{Z}_{(p)}[P_W](\overline{\Sigma_i \times \Sigma_{n-i}})(\overline{B_i \times B_{n-i}}). \end{aligned}$$

The map is given by the inclusion $P_W \to \mathrm{GL}_n$. Therefore, in order to show that the map

$$\mathbb{Z}_{(p)}[P_W](\overline{\Sigma_i \times \Sigma_{n-i}})(\overline{B_i \times B_{n-i}}) \longrightarrow \mathbb{Z}_{(p)}[\mathrm{GL}_n]\overline{\Sigma}_n\overline{B}_n$$

is surjective, it is sufficient to show that any invertible $n \times n$ matrix can be written in the form $a\sigma b$, where $a, b \in B_n$ and $\sigma \in \Sigma_n$. This can be shown easily by row reduction. □

3. Fixed points of a Steinberg summand

The Steinberg summand construction (Definition 2.3) may be carried into the equivariant setting. Recall (or learn) the G-equivariant analogue of homotopy orbit construction.

Definition 3.1. If Λ is any finite group, then $E_G\Lambda$ denotes the $(G \times \Lambda)$-space whose fixed points under any subgroup $\Gamma \subset G \times \Lambda$ are

$$(E_G\Lambda)^{\Gamma} \simeq \begin{cases} \star \text{ if } \Gamma \cap \Lambda = 1, \\ \emptyset \text{ if } \Gamma \cap \Lambda \neq 1, \end{cases}$$

and $B_G\Lambda$ is the quotient G-space $(E_G\Lambda)/\Lambda$.

Note that the G-equivariant classifying spaces $B_G\Lambda$ fit into a theory of equivariant principal Λ-bundles [6].

Definition 3.2. Let G be a finite group, and let X be a spectrum with $(G \times \mathrm{GL}_n)$-action. The Steinberg summand e_nX is the naïve G-spectrum

$$e_nX = (\Sigma^{1-n}\mathbf{B}_n^{\diamond} \wedge X) \wedge_{\mathrm{GL}_n} (E_G\mathrm{GL}_n)_+.$$

When X is a pointed $(G \times \mathrm{GL}_n)$-space, we write $e_nX := e_n(\Sigma^\infty X)$.

The naïve G-spectrum X contains $e_n X$ as a summand – we prove this as Corollary 3.6, but the argument is a minor modification of the proof of Proposition 2.9. Taking G-fixed points of the $(G \times \Lambda)$-space $E_G\Lambda$, yields the Λ-space $E\Lambda$. Thus we have an inclusion of Λ-spaces

$$E\Lambda \simeq (E_G\Lambda)^G \hookrightarrow E_G\Lambda.$$

This inclusion produces, for every subgroup $H \subseteq G$, a natural transformation from the composite functor $e_n((-)^H)$ to the composite functor $(e_n(-))^H$.

$$\begin{array}{ccc} \mathrm{Top}_*^{G\times \mathrm{GL}_n} & \xrightarrow{(-)^H} & \mathrm{Top}_*^{\mathrm{GL}_n} \\ {\scriptstyle e_n(-)}\downarrow & \swarrow\!\!\!\!\swarrow & \downarrow{\scriptstyle e_n(-)} \\ \text{Naïve G-spectra} & \xrightarrow[(-)^H]{} & \text{Spectra} \end{array}$$

In this section, we prove that the natural transformation above is a homotopy equivalence when G is a p-group. That is, we prove the following theorem.

Theorem 3.3. *Let G be a group, and let $H \subseteq G$ be any subgroup. Let X be any pointed $(G \times \mathrm{GL}_n)$-space. The inclusion of fixed points $E\mathrm{GL}_n \simeq (E_G\mathrm{GL}_n)^G \hookrightarrow E_G\mathrm{GL}_n$ induces a map*

$$(\mathbf{B}_n^\diamond \wedge X^H) \wedge_{\mathrm{GL}_n} (E\mathrm{GL}_n)_+ \to ((\mathbf{B}_n^\diamond \wedge X) \wedge_{\mathrm{GL}_n} (E_G\mathrm{GL}_n)_+)^H.$$

If G is a p-group, then the map above is an equivalence. It immediately follows that the map $e_n(X^H) \to (e_n X)^H$ is an equivalence of spectra.

To prove this theorem, we must first establish a well-known formula (Equation 3.1) for the fixed points of the equivariant homotopy orbits of a space.

Definition 3.4. Let G and Λ be any finite groups, and let $H \subseteq G$ be a subgroup. For any homomorphism $f\colon H \to \Lambda$, its *graph* is the subgroup of $H \times \Lambda$

$$\Gamma_f := \{(h, f(h)) \ : \ h \in H\}.$$

The group Λ acts on the set $\mathrm{Hom}(H, \Lambda)$ by conjugation, i.e. $f \mapsto \lambda f \lambda^{-1}$. For a homomorphism $f\colon H \to \Lambda$, let $C_\Lambda(\mathrm{im} f) \subseteq \Lambda$ denote the centralizer of the image of f. Note that if $f, f' \in \mathrm{Hom}(H, \Lambda)$ are conjugate homomorphisms, then the centralizers $C_\Lambda(\mathrm{im} f)$ and $C_\Lambda(\mathrm{im} f')$ are conjugate subgroups.

Notice that if $f, f' \in \mathrm{Hom}(H, \Lambda)$ are two different homomorphisms, then the subgroup of $H \times \Lambda$ generated by $\langle \Gamma_f, \Gamma_{f'} \rangle$ is no longer a graph homomorphism. It follows that $(E_G\Lambda)^{\Gamma_f} \cap (E_G\Lambda)^{\Gamma_{f'}} = \emptyset$. Therefore, for any $(G \times \Lambda)$-space Y we have

$$\begin{aligned} (Y \times_\Lambda E_G\Lambda)^H &= \left(\coprod_{f \in \mathrm{Hom}(H,\Lambda)} Y^{\Gamma_f} \times (E_G\Lambda)^{\Gamma_f} \right) /\Lambda \\ &= \left(\coprod_{f \in \mathrm{Hom}(H,\Lambda)} Y^{\Gamma_f} \right) \times_\Lambda E\Lambda \\ &\simeq \coprod_{[f] \in \mathrm{Hom}(H,\Lambda)/\Lambda} (Y^{\Gamma_f})_{hC_\Lambda(\mathrm{im} f)}. \end{aligned} \tag{3.1}$$

(Note that the last is a weak equivalence of spaces because $E\Lambda$ is defined only up to homotopy.)

The map $E\Lambda = (E_G\Lambda)^G \hookrightarrow E_G\Lambda$ of Λ-spaces yields an inclusion map

$$Y^H \times_\Lambda E\Lambda = (Y \times_\Lambda E\Lambda)^H \hookrightarrow (Y \times_\Lambda E_G\Lambda)^H.$$

Under the decomposition of Equation 3.1, the space $Y^H \times_\Lambda E\Lambda$ is the summand corresponding to the zero homomorphism $H \to \Lambda$.

Proof of Theorem 3.3. Apply the pointed analogue of Equation 3.1 with $Y = \mathbf{B}_n^\diamond \wedge X$ and $\Lambda = \mathrm{GL}_n$ to obtain

$$((\mathbf{B}_n^\diamond \wedge X) \wedge_{\mathrm{GL}_n} (E_G\mathrm{GL}_n)_+)^H = \bigvee_{[f]\in \mathrm{Hom}(H,\mathrm{GL}_n)/\mathrm{GL}_n} ((\mathbf{B}_n^\diamond)^{\mathrm{im} f} \wedge X^{\Gamma_f})_{hC_{\mathrm{GL}_n}(\mathrm{im} f)}.$$

We must prove that for every nontrivial homomorphism f, up to conjugacy, the summand $((\mathbf{B}_n^\diamond)^{\mathrm{im} f} \wedge X^{\Gamma_f})_{C_{\mathrm{GL}_n}(\mathrm{im} f)}$ is contractible. It is sufficient to prove that the pointed $C_{\mathrm{GL}_n}(\mathrm{im} f)$-space $(\mathbf{B}_n^\diamond)^{\mathrm{im} f}$ is equivariantly contractible. This will follow from a proof that the unpointed $C_{\mathrm{GL}_n}(\mathrm{im} f)$-space $(\mathbf{B}_n)^{\mathrm{im} f}$ is equivariantly contractible, which follows from Lemma 3.5 below.

The map of spectra $e_n(X^H) \to (e_n X)^H$ is the $(n-1)$-th desuspension of the inclusion $((\mathbf{B}_n^\diamond \wedge X) \wedge_{\mathrm{GL}_n} (E\mathrm{GL}_n)_+)^H \hookrightarrow ((\mathbf{B}_n^\diamond \wedge X) \wedge_{\mathrm{GL}_n} (E_G\mathrm{GL}_n)_+)^H$, and is therefore an equivalence. ∎

Lemma 3.5. *Let V be a finite dimensional vector space over a finite field $\mathbb{F}$ of positive characteristic p. Let $U \subset \mathrm{GL}(V)$ be a nontrivial unipotent subgroup (i.e. order a power of p). The fixed point space $(\mathbf{B}_V)^U$ carries a residual action of the normalizer of U, which we denote by $N_{\mathrm{GL}(V)}(U)$. Then $(\mathbf{B}_V)^U$ is $N_{\mathrm{GL}(V)}(U)$-equivariantly contractible.*

Proof. The action of the group U on the $\mathbb{F}$-vector space V extends linearly to an action of the group ring $\mathbb{F}[U]$. Let $\mathcal{I}$ denote the *augmentation ideal* of $\mathbb{F}[U]$, defined by generators

$$\mathcal{I} := \langle u - 1 \rangle_{u\in U}.$$

Let $V' \subseteq V$ be the subspace annihilated by $\mathcal{I}$. Because U contains at least one non-identity matrix, it must be that $V' \neq V$. The subspace V' is preserved by the action of $N_{\mathrm{GL}(V)}(U)$. We claim that $V' \neq 0$. To prove this, it suffices to show that there is some $k > 0$ such that $\mathcal{I}^k$ annihilates V. In the case where U is a maximal unipotent subgroup of $\mathrm{GL}(V)$, the ideal $\mathcal{I}^{\dim(V)}$ annihilates V, and therefore for any unipotent subgroup U, $\mathcal{I}^k$ annihilates V for some $k \leqslant \dim(V)$.

Let $W \subsetneq V$ be a nonzero subspace that is preserved by U. Because U is unipotent, W has a vector w such that $uw = w$ for every $u \in U$. This is equivalent to saying that $\mathcal{I}w = 0$, and so it follows that the intersection $W \cap V'$ is nonzero. Thus, there is a well-defined $N_{\mathrm{GL}(V)}(U)$-equivariant poset map

$$\begin{aligned} f\colon (\mathbf{B}_V)^U &\to (\mathbf{B}_V)^U, \\ W &\mapsto V' \cap W. \end{aligned}$$

For every subgroup $\Gamma \subset N_{\mathrm{GL}(V)}(U)$, the map f restricts to a map of fixed point spaces $f^\Gamma\colon ((\mathbf{B}_V)^U)^\Gamma \to ((\mathbf{B}_V)^U)^\Gamma$. Because $V' \cap W \subset W$, the map f^Γ is homotopic to the identity map. Because $V' \cap W \subset V'$, the map f^Γ is homotopic to the constant map at V'. Therefore, the fixed point space $((\mathbf{B}_V)^U)^\Gamma$ is contractible for every $\Gamma \subset N_{\mathrm{GL}(V)}(U)$, which completes the proof. □

We finish this section with a short outline of the proof that the equivariant Steinberg summand of X is a summand of X, in a suitable sense.

Corollary 3.6. *Let G be a p-group, and let X be a spectrum with $(G \times \mathrm{GL}_n)$-action. There are maps of spectra which are functorial in X*

$$e_n X \to X, \qquad X \to e_n X,$$

such that for every subgroup $H \subseteq G$, the composition $(e_n X)^H \to X^H \to (e_n X)^H$ is multiplication by a unit in $\mathbb{F}_p$-homology.

Proof. The proof is nearly identical to that of Proposition 2.9 – we sketch the necessary modifications here. To construct the inclusion $e_n X \to X$, observe that there is an equivalence of naïve G-spectra $((\mathrm{GL}_n)_+ \wedge X) \wedge_{\mathrm{GL}_n} (E_G \mathrm{GL}_n)_+ \simeq X$, and use the composition

$$(\mathbf{B} \wedge X)_{h_G \mathrm{GL}_n} \to (\mathbf{B}^{(n-1)}/\mathbf{B}^{(n-2)} \wedge X)_{h_G \mathrm{GL}_n} \to (\Sigma^{n-1}(\mathrm{GL}_n)_+ \wedge X)_{h_G \mathrm{GL}_n} \simeq \Sigma^{n-1} X,$$

where the second arrow is the Becker–Gottlieb transfer The projection $X \to e_n X$ is constructed in the same manner as in Proposition 2.9. The assertion that the composition $(e_n X)^H \to X^H \to (e_n X)^H$ is multiplication by a unit in $\mathbb{F}_p$-homology is an immediate consequence of Proposition 2.9 combined with the formula $(e_n X)^H \simeq e_n(X^H)$ (Theorem 3.3). □

4. Fixed points in equivariant classifying spaces

Let G be a p-group. The goal of this section is to compute the G-fixed points of the Steinberg summand of $B_G(\mathbb{Z}/p)^n_+$. To state the result of this computation, we must make a few definitions.

Definition 4.1. Let $\mathcal{C}$ denote the set of normal subgroups $H \trianglelefteq G$ such that G/H is an elementary abelian p-group. It is easily seen that the set $\mathcal{C}$ is closed under intersections, and thus $\mathcal{C}$ has a minimal element F. As a poset, $\mathcal{C}$ is isomorphic to the poset of sub-$\mathbb{F}_p$-vector spaces of G/F.

For each subgroup $H \in \mathcal{C}$, let $d(H)$ denote the rank of G/H as an $\mathbb{F}_p$-vector space. Two subgroups H and K are called *transverse* if $d(H \cap K) = d(H) + d(K)$.

Theorem 4.2.

1. *There is a decomposition of spectra*

$$(e_n B_G(\mathbb{Z}/p)^n_+)^G \simeq \bigvee_{H \in \mathcal{C}} E_n(H),$$

for spectra $E_n(H)$ with a product structure $E_m(H) \wedge E_n(K) \to E_{m+n}(H \cap K)$ arising from applying G-fixed points to the Steinberg product $e_m B_G(\mathbb{Z}/p)^m_+ \wedge e_n B_G(\mathbb{Z}/p)^n_+ \to e_{m+n} B_G(\mathbb{Z}/p)^{m+n}_+$.

2. *The spectra $E_n(H)$ satisfy the formula*

$$E_n(H) \simeq e_{n-d(H)} B(\mathbb{Z}/p)_+^{n-d(H)} \wedge \Sigma^{1-d(H)} \mathbf{B}_{d(H)}^{\diamond} \wedge B(G/H)_+. \tag{4.1}$$

If H and K are transverse, then the equivalence above respects the product structure on both sides. More explicitly, let m and n be any positive integers. The product on the right side of Equation 4.1 is assembled from the Steinberg product

$$e_{m-d(H)} B(\mathbb{Z}/p)_+^{m-d(H)} \wedge e_{n-d(K)} B(\mathbb{Z}/p)_+^{n-d(K)} \\ \to e_{m+n-d(H\cap K)} B(\mathbb{Z}/p)_+^{m+n-d(H\cap K)},$$

the product on flag complexes

$$\Sigma^{1-d(H)} \mathbf{B}_{d(H)}^{\diamond} \wedge \Sigma^{1-d(K)} \mathbf{B}_{d(K)}^{\diamond} \to \Sigma^{1-d(H\cap K)} \mathbf{B}_{d(H\cap K)}^{\diamond},$$

and the equivalence

$$B(\mathbb{Z}/p)_+^{m-d(H)} \wedge B(\mathbb{Z}/p)_+^{n-d(K)} \simeq B(\mathbb{Z}/p)_+^{m+n-d(H\cap K)}.$$

The spectra $E_n(H)$ are defined in Definition 4.6. The formula

$$E_n(H) \simeq e_{n-d(H)} B(\mathbb{Z}/p)_+^{n-d(H)} \wedge \Sigma^{1-d(H)} \mathbf{B}_{d(H)}^{\diamond} \wedge B(G/H)_+$$

is proven as Proposition 4.7.

4.1. The mod p Stiefel variety

Let n and d be nonnegative integers. Let $V_d(\mathbb{F}_p^n)$ denote the set of $n \times d$ matrices with entries in the field $\mathbb{F}_p$, and with nullspace zero. Then $V_d(\mathbb{F}_p^n)$ is a finite set with an action of the group $\mathrm{GL}_n(\mathbb{F}_p)$. It is a mod p analogue of the Stiefel manifold $V_d(\mathbb{R}^n)$ of orthonormal d-frames in Euclidean n-space. Note that there is an inclusion of $(\mathrm{GL}_m \times \mathrm{GL}_n)$-sets,

$$V_c(\mathbb{F}_p^m) \times V_d(\mathbb{F}_p^n) \hookrightarrow V_{c+d}(\mathbb{F}_p^{m+n}),$$

given by block inclusion of matrices.

Let $\mathcal{F}$ be a functor from the category of finite dimensional mod p vector spaces with isomorphisms, to the homotopy category HoTop_*, such that

- For any finite dimensional mod p vector spaces V and W, there is an equivalence $\mathcal{F}(V \oplus W) \simeq \mathcal{F}(V) \wedge \mathcal{F}(W)$ of $(\mathrm{GL}(V) \times \mathrm{GL}(W))$-spaces.
- There is an equivalence $\mathcal{F}(0) \simeq S^0$.

Example 4.3. One such functor is $\mathcal{F}(\mathbb{F}_p^n) = B(\mathbb{Z}/p)_+^n$. This is the specific application of Lemma 4.4 which we will use in Proposition 4.7.

For every integer $n \geqslant 0$, the pointed space $\mathcal{F}(\mathbb{F}_p^n)$ carries an action of the group $\mathrm{GL}_n(\mathbb{F}_p)$. One may then consider its Steinberg summand $e_n\mathcal{F}(\mathbb{F}_p^n) = (\Sigma^{1-n}\mathbf{B}_n^{\diamond} \wedge \mathcal{F}(\mathbb{F}_p^n))_{h\mathrm{GL}_n}$, which is a spectrum. These spectra are related by product maps

$$e_k\mathcal{F}(\mathbb{F}_p^k) \wedge e_\ell\mathcal{F}(\mathbb{F}_p^\ell) \to e_{k+\ell}\mathcal{F}(\mathbb{F}_p^{k+\ell}),$$

which are built using the product $\Sigma^{1-k}\mathbf{B}_k^{\diamond} \wedge \Sigma^{1-\ell}\mathbf{B}_\ell^{\diamond} \to \Sigma^{1-(k+\ell)}\mathbf{B}_{k+\ell}^{\diamond}$ and the block inclusion $\mathrm{GL}_k \times \mathrm{GL}_\ell \subset \mathrm{GL}_{k+\ell}$. In this section, we will prove the following lemma which relates the mod p Stiefel variety $V_d(\mathbb{F}_p^n)$ to Steinberg summands.

Lemma 4.4. *Let $\mathcal{F}$ be a functor as above. Let n, d be nonnegative integers such that $n \geqslant d$. Then there is an equivalence of spectra*

$$(\Sigma^{1-d}\mathbf{B}_d^{\diamond} \wedge \mathcal{F}(\mathbb{F}_p^d)) \wedge e_{n-d}\mathcal{F}(\mathbb{F}_p^{n-d}) \to e_n(V_d(\mathbb{F}_p^n)_+ \wedge \mathcal{F}(\mathbb{F}_p^n)).$$

Denote the spectrum on the left by $A(n,d)$ and the spectrum on the right by $B(n,d)$. There are obvious product maps $A(n,d) \wedge A(m,c) \to A(m+n, c+d)$ and $B(n,d) \wedge B(m,c) \to B(m+n, c+d)$. Then the equivalence above respects these product maps.

Proof. Let $W_d \subset \mathbb{F}_p^n$ denote the subgroup spanned by the first d coordinates, and let $W_{n-d} \subset \mathbb{F}_p^n$ denote the subgroup spanned by the last $n-d$ coordinates. Let $\mathbf{B}_d^{\diamond} := \mathbf{B}_{W_d}^{\diamond}$ and $\mathbf{B}_{n-d}^{\diamond} := \mathbf{B}_{W_{n-d}}^{\diamond}$. Let $\mathrm{GL}(W_d), \mathrm{GL}(W_{n-d}), \mathrm{GL}(\mathbb{F}_p^n, W_d) \subset \mathrm{GL}_n$ denote the subgroups of matrices

$$\mathrm{GL}(W_d) = \begin{pmatrix} \mathrm{GL}_d & 0 \\ 0 & I_{n-d} \end{pmatrix}, \ \mathrm{GL}(W_{n-d}) = \begin{pmatrix} I_d & 0 \\ 0 & \mathrm{GL}_{n-d} \end{pmatrix}, \ \mathrm{GL}(\mathbb{F}_p^n, W_d) = \begin{pmatrix} I_d & * \\ 0 & \mathrm{GL}_{n-d} \end{pmatrix}.$$

Then $\mathrm{GL}(\mathbb{F}_p^n, W_d)$ is the subgroup of matrices which act by the identity on W_d. Let $\mathcal{S}$ denote the set of subspaces $W \subset \mathbb{F}_p^n$ of dimension $(n-d)$ such that $W \perp W_d$. We have the following two observations:

1. As a $\mathrm{GL}(\mathbb{F}_p^n, W_d)$-torsor, $\mathcal{S} = \mathrm{GL}(\mathbb{F}_p^n, W_d)/\mathrm{GL}(W_{n-d})$.
2. As a GL_n-torsor, $V_d(\mathbb{F}_p^n) = \mathrm{GL}_n/\mathrm{GL}(\mathbb{F}_p^n, W_d)$.

Therefore,

$$\begin{aligned}
&(\Sigma^{1-d}\mathbf{B}_d^{\diamond} \wedge \mathcal{F}(\mathbb{F}_p^d)) \wedge e_{n-d}\mathcal{F}(\mathbb{F}_p^{n-d}) \\
&:= (\Sigma^{1-d}\mathbf{B}_d^{\diamond} \wedge \mathcal{F}(W_d) \wedge \Sigma^{1-n+d}\mathbf{B}_{n-d}^{\diamond} \wedge \mathcal{F}(W_{n-d}))_{h\mathrm{GL}(W_{n-d})} && \text{(by definition))} \\
&\simeq (\Sigma^{1-d}\mathbf{B}_d^{\diamond} \wedge \mathcal{F}(W_d) \wedge \bigvee_{W \in \mathcal{S}} \Sigma^{1-n+d}\mathbf{B}_W^{\diamond} \wedge \mathcal{F}(W))_{h\mathrm{GL}(\mathbb{F}_p^n, W_d)} && \text{(by (1) above)} \\
&\xrightarrow{\simeq} (\Sigma^{1-n}\mathbf{B}_n^{\diamond} \wedge \mathcal{F}(\mathbb{F}_p^n))_{h\mathrm{GL}(\mathbb{F}_p^n, W_d)} && \text{(by Proposition 2.10)} \\
&\simeq (\Sigma^{1-n}\mathbf{B}_n^{\diamond} \wedge V_d(\mathbb{F}_p^n)_+ \wedge \mathcal{F}(\mathbb{F}_p^n))_{h\mathrm{GL}_n} && \text{(by (2) above).}
\end{aligned}$$

The fact that this equivalence respects the product maps is a routine check. ☐

4.2. The fixed points of the Steinberg summand of an equivariant classifying space

Let G be a finite p-group, and let n be a positive integer. Any homomorphism from G to $(\mathbb{Z}/p)^n$ has kernel contained in $\mathcal{C}$.

Definition 4.5. For each subgroup $H \in \mathcal{C}$, let

$$\mathrm{Hom}(G, (\mathbb{Z}/p)^n))[H] \subset \mathrm{Hom}(G, (\mathbb{Z}/p)^n)$$

denote the set of homomorphisms with kernel H.

Then $\mathrm{Hom}(G, (\mathbb{Z}/p)^n)) = \bigsqcup_{H \in \mathcal{C}} \mathrm{Hom}(G, (\mathbb{Z}/p)^n))[H]$. A homomorphism from G to $(\mathbb{Z}/p)^n$ with kernel H is the same as a monomorphism from G/H to $(\mathbb{Z}/p)^n$. Thus, the GL_n-torsor $\mathrm{Hom}(G, (\mathbb{Z}/p)^n))[H]$ is identified with the mod p Stiefel variety $V_{d(H)}(\mathbb{F}_p^n)$ (see 4.4), and so

$$\mathrm{Hom}(G, (\mathbb{Z}/p)^n) = \bigsqcup_{H \in \mathcal{C}} V_{d(H)}(\mathbb{F}_p^n). \tag{4.2}$$

Now let us study the Steinberg summand of the G-fixed points of the equivariant classifying space $B_G(\mathbb{Z}/p)^n_+$. Equation 3.1 tells us that

$$e_n((B_G(\mathbb{Z}/p)^n_+)^G) \simeq e_n\left(\bigvee_{\mathrm{Hom}(G,(\mathbb{Z}/p)^n)} B(\mathbb{Z}/p)^n_+\right)$$
$$\simeq \bigvee_{H\in\mathcal{C}}\left(e_n \bigvee_{\mathrm{Hom}(G,(\mathbb{Z}/p)^n))[H]} B(\mathbb{Z}/p)^n_+\right).$$

Definition 4.6. Let n be a positive integer and $H \in \mathcal{C}$ be a subgroup of G. The spectrum

$$e_n\left(\bigvee_{\mathrm{Hom}(G,(\mathbb{Z}/p)^n))[H]} B(\mathbb{Z}/p)^n_+\right)$$

is called the H*-summand of* $e_n((B_G(\mathbb{Z}/p)^n_+)^G)$. We denote it by $E_n(H)$. For any two positive integers m, n and subgroups $H, K \in \mathcal{C}$, there is a product map

$$E_m(H) \wedge E_n(K) \to E_{m+n}(H \cap K),$$

which is determined by the product maps

$$e_m((B_G(\mathbb{Z}/p)^m_+)^G) \wedge e_n((B_G(\mathbb{Z}/p)^n_+)^G) \to e_{m+n}((B_G(\mathbb{Z}/p)^{m+n}_+)^G).$$

Let m, n be any two positive integers. There is an obvious isomorphism of $(\mathrm{GL}_m \times \mathrm{GL}_n)$-sets

$$\mathrm{Hom}(G, (\mathbb{Z}/p)^m) \times \mathrm{Hom}(G, (\mathbb{Z}/p)^n) \cong \mathrm{Hom}(G, (\mathbb{Z}/p)^{m+n}).$$

Under the identification of Equation 4.2, the isomorphism above yields product maps on the components

$$V_{d(H)}(\mathbb{F}_p^m) \times V_{d(K)}(\mathbb{F}_p^n) \to V_{d(H\cap K)}(\mathbb{F}_p^{m+n}).$$

If H and K are transverse, this product is given by block inclusion of matrices, for an appropriate choice of basis.

Proposition 4.7. *There is an equivalence of spectra*

$$E_n(H) \simeq e_{n-d(H)}B(\mathbb{Z}/p)^{n-d(H)}_+ \wedge \Sigma^{1-d(H)}\mathbf{B}^{\diamond}_{d(H)} \wedge B(G/H)_+.$$

If n, m are two positive integers and $H, K \in \mathcal{C}$ are transverse subgroups, then under the equivalence above, the product map $E_n(H) \wedge E_m(K) \to E_{n+m}(H \cap K)$ is identified with the product on the right hand term that is built using the following three maps

$$e_{n-d(H)}B(\mathbb{Z}/p)^{n-d(H)}_+ \wedge e_{m-d(K)}B(\mathbb{Z}/p)^{m-d(K)}_+ \to e_{m+n-d(H\cap K)}B(\mathbb{Z}/p)^{m+n-d(H\cap K)}_+,$$
$$\Sigma^{1-d(H)}\mathbf{B}^{\diamond}_{d(H)} \wedge \Sigma^{1-d(K)}\mathbf{B}^{\diamond}_{d(K)} \to \Sigma^{1-d(H\cap K)}\mathbf{B}^{\diamond}_{d(H\cap K)},$$
$$B(G/H)_+ \wedge B(G/K)_+ \cong B(G/(H \cap K))_+.$$

Proof. This is immediate from applying Lemma 4.4 with the functor $\mathcal{F}((\mathbb{Z}/p)^n) = B(\mathbb{Z}/p)^n_+$. □

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