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THE NON-NIL-INVARIANCE OF PERIODIC TOPOLOGICAL CYCLIC HOMOLOGY

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Abstract

Periodic topological cyclic homology TP is a topological analogue of periodic cyclic homology HP. It is known that, for R an algebra over a field of characteristic 0 and I a nilpotent ideal of R, the quotient map $R \to R/I$ induces an isomorphism on HP. In this paper, we show that the analogous result for TP does not hold.

1. Introduction

In [**9**], Hesselholt defined a spectrum $\mathrm{TP}(X)$, the periodic topological cyclic homology of a scheme X, using topological Hochschild homology and the Tate construction. That is a topological analogue of Connes–Tsygan periodic cyclic homology HP defined by Hochschild homology and the Tate construction. In [**8**, Theorem II.5.1], Goodwillie proved that for R an algebra over a field of characteristic 0 and I a nilpotent ideal of R, the quotient map $R \to R/I$ induces an isomorphism on HP. In this paper, we show that the analogous result for TP does not hold, that is to say, there is an algebra of positive characteristic and a nilpotent ideal such that the quotient map does not induce an isomorphism on TP, even rationally. More precisely, we prove the following result.

Theorem 1.1. *Let p be a prime number and $k \geqslant 2$ a natural number. Then the canonical map*

$$\mathrm{TP}_*(\mathbb{F}_p[x]/(x^k)) \to \mathrm{TP}_*(\mathbb{F}_p)$$

is not an isomorphism. Moreover, if k is not a p-power, then the map is also not an isomorphism after inverting p.

There is a map $\mathrm{TP} \to \mathrm{HP}$, which is an equivalence for $\mathbb{Q}$-algebras. Therefore, by Goodwillie's theorem, TP is nil-invariant for such algebras. So it is natural to examine whether TP is nil-invariant for $\mathbb{F}_p$-algebras. Note that the ideal (x) is nilpotent in $\mathbb{F}_p[x]/(x^k)$ and the canonical map $\mathbb{F}_p[x]/(x^k) \to \mathbb{F}_p$ is the quotient map of the nilpotent ideal. Thus, to show the non-nil-invariance of TP for $\mathbb{F}_p$-algebras, it is enough to prove that the map is not an isomorphism. Although we do not use it to prove our theorem, [**4**, Proposition 3.1] calculates $\mathrm{HP}(A[x]/(x^k))$ with A any commutative ring.

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In [**9**], Hesselholt gives a cohomological interpretation by TP of the Hasse–Weil zeta function of a scheme smooth and proper over a finite field inspired by [**6**] and [**5**]. Furthermore, in [**1**] and [**2**], it is proved that TP satisfies the Künneth formula for smooth and proper dg-categories over a perfect field of positive characteristic. Therefore, this new cohomology theory TP is an important cohomology theory for p-adic geometry and non-commutative geometry. Our result concerns a fundamental property of this theory. In Theorem 3.3, we calculate the TP-group of $\mathbb{F}_p[x]/(x^k)$ completely.

2. Periodic topological cyclic homology

Periodic topological cyclic homology TP is introduced in [**9**]. In this section, we briefly recall some notions from there. We let $\mathbb{T}$ denote the circle group. The following construction written in the higher categorical language can be found at [**15**, I.4].

Let E be a free $\mathbb{T}$-CW-complex whose underlying space is contractible. Then we consider the following cofibration sequence of pointed $\mathbb{T}$-spaces

$$E_+ \to S^0 \to \tilde{E},$$

where E_+ is the pointed space $E \sqcup \{\infty\}$ and $S^0 = \{0, \infty\}$, and the left hand map sends ∞ to the base point $\infty \in S^0$ and all other points to $0 \in S^0$.

Let X be a $\mathbb{T}$-spectrum. Smashing the internal hom spectrum $[E_+, X]$ with the above diagram and taking fixed points for a subgroup $C \subset \mathbb{T}$, we have the following sequence called Tate cofibration sequence

$$(E_+ \otimes [E_+, X])^C \to ([E_+, X])^C \to (\tilde{E} \otimes [E_+, X])^C.$$

Following [**9**], we write this sequence as

$$\begin{aligned} (E_+ \otimes [E_+, X])^C &= \begin{cases} \mathrm{H}.(C, X), & if\ C \subsetneq \mathbb{T}, \\ \Sigma\mathrm{H}.(C, X), & if\ C = \mathbb{T}, \end{cases} \\ ([E_+, X])^C &= \mathrm{H}^{\cdot}(C, X), \\ (\tilde{E} \otimes [E_+, X])^C &= \hat{\mathrm{H}}^{\cdot}(C, X). \end{aligned}$$

Let X be a scheme. The topological periodic cyclic homology of X is the spectrum given by

$$\mathrm{TP}(X) = \hat{\mathrm{H}}^{\cdot}(\mathbb{T}, \mathrm{THH}(X)),$$

where THH denotes the topological Hochschild homology of X defined in [**7**] and [**3**]. In the present paper, we will only consider affine schemes. For a commutative ring R, there is a conditionally convergent spectral sequence [**14**, §4],

$$E^2_{i,j} = S_{\mathbb{Z}}\{t, t^{-1}\} \otimes \mathrm{THH}_j(R) \Rightarrow \mathrm{TP}_{i+j}(R),$$

where $\deg(t) = (-2, 0)$.

3. Truncated polynomial algebras

Our main result is the following

Theorem 3.1. *Let p be a prime number and $k \geqslant 2$ a natural number. If k is not a p-power, then the canonical map*

$$\mathrm{TP}_*(\mathbb{F}_p[x]/(x^k))[1/p] \to \mathrm{TP}_*(\mathbb{F}_p)[1/p]$$

is not an isomorphism.

Before proving our main result, we recall from [**13**] and [**10**] some calculations concerning $\mathrm{THH}(\mathbb{F}_p[x]/(x^k))$. The following is shown in [**16**, Paper B] in the higher categorical language.

For a natural number $k \geqslant 1$, we give the pointed finite set $\Pi_k = \{0, 1, x, \ldots, x^{k-1}\}$ with the base point 0 the pointed commutative monoid structure, where 1 is the unit, $0 \cdot 1 = 0 \cdot x^i = 0$, $x^i \cdot x^j = x^{i+j}$ and $x^k = 0$. We denote the cyclic bar construction of Π_k by $\mathrm{N}^{\mathrm{cy}}_\bullet(\Pi_k)$. More precisely, the set of l-simplicies is

$$\mathrm{N}^{\mathrm{cy}}_l(\Pi_k) = \Pi_k \wedge \cdots \wedge \Pi_k,$$

where there are $l+1$ smash factors and the structure maps are given by

$$\begin{aligned} d_i(x_0 \wedge \cdots \wedge x_l) &= x_0 \wedge \cdots \wedge x_i x_{i+1} \wedge \cdots \wedge x_l,\ 0 \leqslant i < l, \\ d_l(x_0 \wedge \cdots \wedge x_l) &= x_l x_0 \wedge x_1 \wedge \cdots \wedge x_{l-1}, \\ s_i(x_0 \wedge \cdots \wedge x_l) &= x_0 \wedge \cdots \wedge x_i \wedge 1 \wedge x_{i+1} \wedge \cdots \wedge x_l,\ 0 \leqslant i \leqslant l, \\ t_l(x_0 \wedge \cdots \wedge x_l) &= x_l \wedge x_0 \wedge x_1 \wedge \cdots \wedge x_{l-1}. \end{aligned}$$

We let $\mathrm{N}^{\mathrm{cy}}(\Pi_k)$ denote the geometric realization of the cyclic set $\mathrm{N}^{\mathrm{cy}}_\bullet(\Pi_k)$.

In [**12**, Theorem 7.1], it is proved that there is a natural equivalence of cyclotomic spectra

$$\mathrm{THH}(\mathbb{F}_p[x]/(x^k)) \simeq \mathrm{THH}(\mathbb{F}_p) \otimes \mathrm{N}^{\mathrm{cy}}(\Pi_k). \tag{1}$$

For each positive integer i, we also have the cyclic subset

$$\mathrm{N}^{\mathrm{cy}}_\bullet(\Pi_k, i) \subset \mathrm{N}^{\mathrm{cy}}_\bullet(\Pi_k)$$

generated by the $(i-1)$-simplex $x \wedge \cdots \wedge x$ (i factors), and denote the geometric realization by $\mathrm{N}^{\mathrm{cy}}(\Pi_k, i)$. We let $\mathrm{N}^{\mathrm{cy}}_\bullet(\Pi_k, 0)$ be the cyclic subset generated by the 0-simplex 1 with the geometric realization $\mathrm{N}^{\mathrm{cy}}(\Pi_k, 0)$. The canonical map gives the following wedge decomposition

$$\bigvee_{i \geqslant 0} \mathrm{N}^{\mathrm{cy}}(\Pi_k, i) = \mathrm{N}^{\mathrm{cy}}(\Pi_k),$$

see also [**13**, (2.2.5)].

We consider the complex $\mathbb{T}$-representation, where $d = \lfloor (i-1)/k \rfloor$ is the integer part of $(i-1)/k$ for $i \geqslant 1$,

$$\lambda_d = \mathbb{C}(1) \oplus \mathbb{C}(2) \oplus \cdots \oplus \mathbb{C}(d),$$

where $\mathbb{C}(i) = \mathbb{C}$ with the $\mathbb{T}$ action

$$\mathbb{T} \times \mathbb{C}(i) \to \mathbb{C}(i)$$

defined by $(z, w) \mapsto z^i w$. Then we have the following by [**13**, theorem B], for $i \geqslant 1$ such that $i \notin k\mathbb{N}$, there is a $\mathbb{T}$-equivariant equivalence

$$\mathrm{N}^{\mathrm{cy}}(\Pi_k, i) \simeq S^{\lambda_d} \wedge (\mathbb{T}/C_i)_+, \tag{2}$$

where C_i is the i-th cyclic group and S^{λ_d} is the one point compactification of λ_d.

Let $\mathrm{THH}(\mathbb{F}_p[x]/(x^k),(x))$ denote the fiber of the canonical map

$$\mathrm{THH}(\mathbb{F}_p[x]/(x^k)) \to \mathrm{THH}(\mathbb{F}_p),$$

and we write

$$\mathrm{TP}(\mathbb{F}_p[x]/(x^k),(x)) = \hat{\mathrm{H}}^{\cdot}(\mathbb{T}, \mathrm{THH}(\mathbb{F}_p[x]/(x^k),(x))).$$

Note that there is an isomorphism $\mathrm{TP}_*(\mathbb{F}_p) \cong \mathbb{Z}_p[t,t^{-1}]$, where t has degree -2. In particular, it is p-local but torsion-free. $\mathrm{TP}_*(\mathbb{F}_p[x]/(x^k))$ is an augmented $\mathrm{TP}_*(\mathbb{F}_p)$-algebra, so calculating the relative term is equivalent to calculating $\mathrm{TP}_*(\mathbb{F}_p[x]/(x^k))$. The non-triviality of $\mathrm{TP}(\mathbb{F}_p[x]/(x^k),(x))$ implies that TP is not nil-invariant. In order to obtain the non-triviality, we use the following decomposition.

Lemma 3.2. *There is a canonical equivalence*

$$\mathrm{TP}(\mathbb{F}_p[x]/(x^k),(x)) \simeq \prod_{i\geqslant 1} \hat{\mathrm{H}}^{\cdot}(\mathbb{T}, \mathrm{THH}(\mathbb{F}_p) \otimes \mathrm{N}^{cy}(\Pi_k, i)).$$

Proof. By (1) and the wedge decomposition of $\mathrm{N}^{\mathrm{cy}}(\Pi_k)$ above, we have

$$\Sigma\mathrm{H}_{\cdot}(\mathbb{T}, \mathrm{THH}(\mathbb{F}_p[x]/(x^k),(x))) \simeq \bigoplus_{i\geqslant 1} \Sigma\mathrm{H}_{\cdot}(\mathbb{T}, \mathrm{THH}(\mathbb{F}_p) \otimes \mathrm{N}^{cy}(\Pi_k, i)),$$

since $\Sigma\mathrm{H}_{\cdot}(\mathbb{T},-)$ preserves all homotopy colimits.

Since the connectivity of $\Sigma\mathrm{H}_{\cdot}(\mathbb{T}, \mathrm{THH}(\mathbb{F}_p) \otimes \mathrm{N}^{cy}(\Pi_k, i))$ goes to ∞ as i goes to ∞, the canonical map is an equivalence

$$\bigoplus_{i\geqslant 1} \Sigma\mathrm{H}_{\cdot}(\mathbb{T}, \mathrm{THH}(\mathbb{F}_p) \otimes \mathrm{N}^{cy}(\Pi_k, i)) \simeq \prod_{i\geqslant 1} \Sigma\mathrm{H}_{\cdot}(\mathbb{T}, \mathrm{THH}(\mathbb{F}_p) \otimes \mathrm{N}^{cy}(\Pi_k, i)).$$

Similarly, since $\mathrm{H}^{\cdot}(\mathbb{T},-)$ preserves all homotopy limits, we have

$$\mathrm{H}^{\cdot}(\mathbb{T}, \mathrm{THH}(\mathbb{F}_p[x]/(x^k),(x))) \simeq \prod_{i\geqslant 1} \mathrm{H}^{\cdot}(\mathbb{T}, \mathrm{THH}(\mathbb{F}_p) \otimes \mathrm{N}^{cy}(\Pi_k, i)).$$

Lastly, since $\mathrm{TP}(\mathbb{F}_p[x]/(x^k),(x))$ is the cofiber of the map

$$\Sigma\mathrm{H}_{\cdot}(\mathbb{T}, \mathrm{THH}(\mathbb{F}_p[x]/(x^k),(x))) \to \mathrm{H}^{\cdot}(\mathbb{T}, \mathrm{THH}(\mathbb{F}_p[x]/(x^k),(x))),$$

we get the desired equivalence. □

It is known that, for a $\mathbb{T}$-spectrum X, there is a $\mathbb{T}$-equivalence

$$X \otimes (\mathbb{T}/C_i)_+ \simeq \Sigma[(\mathbb{T}/C_i)_+, X],$$

see for example [**12**, 8.1]. Hence, we have

$$\begin{aligned}
\hat{\mathrm{H}}^{\cdot}(\mathbb{T}, \mathrm{THH}(\mathbb{F}_p) \otimes (\mathbb{T}/C_i)_+) &= (\tilde{E} \otimes [E_+, \mathrm{THH}(\mathbb{F}_p) \otimes (\mathbb{T}/C_i)_+])^{\mathbb{T}} \\
&\simeq \Sigma(\tilde{E} \otimes [E_+, [(\mathbb{T}/C_i)_+, \mathrm{THH}(\mathbb{F}_p)]])^{\mathbb{T}} \\
&\simeq \Sigma(\tilde{E} \otimes [(\mathbb{T}/C_i)_+, [E_+, \mathrm{THH}(\mathbb{F}_p)]])^{\mathbb{T}} \\
&\simeq (\tilde{E} \otimes (\mathbb{T}/C_i)_+ \otimes [E_+, \mathrm{THH}(\mathbb{F}_p)]])^{\mathbb{T}} \\
&\simeq \Sigma([(\mathbb{T}/C_i)_+, \tilde{E} \otimes [E_+, \mathrm{THH}(\mathbb{F}_p)]])^{\mathbb{T}} \\
&\simeq \Sigma(\tilde{E} \otimes [E_+, \mathrm{THH}(\mathbb{F}_p)])^{C_i} \\
&= \Sigma\hat{\mathrm{H}}^{\cdot}(C_i, \mathrm{THH}(\mathbb{F}_p)),
\end{aligned}$$

and similarly for the spectra $\mathrm{THH}(\mathbb{F}_p) \otimes S^{\lambda_d}$. Furthermore, we have an equivalence of spectra

$$\hat{\mathrm{H}}^{\cdot}(C_i, \mathrm{THH}(\mathbb{F}_p) \otimes S^{\lambda_d}) \simeq \hat{\mathrm{H}}^{\cdot}(C_{p^{v_p(i)}}, \mathrm{THH}(\mathbb{F}_p) \otimes S^{\lambda_d}),$$

where v_p denotes the p-adic valuation.

Hesselholt and Madsen have calculated the homotopy groups of the above spectra [**12**, §9],

$$\pi_* \hat{\mathrm{H}}^{\cdot}(C_{p^n}, \mathrm{THH}(\mathbb{F}_p) \otimes S^{\lambda_d}) \cong S_{\mathbb{Z}/p^n\mathbb{Z}}\{t, t^{-1}\},$$

where t is the divided Bott element. More precisely, $\pi_* \hat{\mathrm{H}}^{\cdot}(C_{p^n}, \mathrm{THH}(\mathbb{F}_p) \otimes S^{\lambda_d})$ is a free module of rank 1 over $\mathbb{Z}/p^n\mathbb{Z}[t, t^{-1}]$ on a generator of degree $2d$. A preferred generator is specified in [**11**, Proposition 2.5]. Combining these and (2), we obtain for $i \notin k\mathbb{N}$ a canonical isomorphism

$$\pi_j \hat{\mathrm{H}}^{\cdot}(\mathbb{T}, \mathrm{THH}(\mathbb{F}_p) \otimes \mathrm{N}^{\mathrm{cy}}(\Pi_k, i)) \cong \begin{cases} \mathbb{Z}/p^{v_p(i)}\mathbb{Z}, & j \text{ odd}, \\ 0, & j \text{ even}. \end{cases}$$

Hesselholt and Madsen have similarly showed that for $i \in k\mathbb{N}$, there is a canonical isomorphism

$$\pi_j \hat{\mathrm{H}}^{\cdot}(\mathbb{T}, \mathrm{THH}(\mathbb{F}_p) \otimes \mathrm{N}^{\mathrm{cy}}(\Pi_k, i)) \cong \begin{cases} \mathbb{Z}/p^{v_p(k)}\mathbb{Z}, & j \text{ odd}, \\ 0, & j \text{ even}. \end{cases}$$

From these, we obtain the following.

Theorem 3.3. *If j is an odd integer, then there is a canonical isomorphism*

$$\mathrm{TP}_j(\mathbb{F}_p[x]/(x^k), (x)) \cong \prod_{i \geqslant 1, i \in k\mathbb{N}} \mathbb{Z}/p^{v_p(k)}\mathbb{Z} \ \times \prod_{i \geqslant 1, i \notin k\mathbb{N}} \mathbb{Z}/p^{v_p(i)}\mathbb{Z}.$$

If j is an even integer, then

$$\mathrm{TP}_j(\mathbb{F}_p[x]/(x^k), (x)) = 0.$$

Note that the order of torsion in $\mathrm{TP}_j(\mathbb{F}_p[x]/(x^k), (x))$ is bounded if k is a p-power. More precisely, if $k = p^r$ with a natural number r, for any $i \notin k\mathbb{N}$, $v_p(i) \leqslant r - 1$. This observation gives us the following.

Corollary 3.4. *If $k = p^r$ with a natural number $r \in \mathbb{N}$, the canonical map*

$$\mathrm{TP}_*(\mathbb{F}_p[x]/(x^{p^r}))[1/p] \to \mathrm{TP}_*(\mathbb{F}_p)[1/p]$$

is an isomorphism.

Thus, in this specific case, the analogue of Goodwillie's theorem for TP holds. In addition, by [**15**, Corollary 1.5] and [**13**, Theorem 4.2.10], we get the following.

Corollary 3.5. *Topological negative cyclic homology is not nil-invariant.*

Since the Hochschild homology defined over $\mathbb{F}_p$ of $\mathbb{F}_p$ is $\mathbb{F}_p$, the same argument with the one explained above gives, for all $j \in \mathbb{Z}$,

$$\mathrm{HP}_j^{\mathbb{F}_p}(\mathbb{F}_p[x]/(x^k), (x)) \cong \prod_{i=1}^{\infty} \mathbb{F}_p,$$

where $\mathrm{HP}^{\mathbb{F}_p}$ denotes periodic cyclic homology defined over $\mathbb{F}_p$. Thus, in particular,

$\mathrm{HP}^{\mathbb{F}_p}$ is not nil-invariant. Moreover, for any odd integer j, the canonical map induces the canonical projection

$$\prod_{i\geqslant 1, i\in k\mathbb{N}} \mathbb{Z}/p^{v_p(k)}\mathbb{Z} \times \prod_{i\geqslant 1, i\notin k\mathbb{N}} \mathbb{Z}/p^{v_p(i)}\mathbb{Z} \longrightarrow \prod_{i=1}^{\infty} \mathbb{F}_p,$$

which shows that periodic cyclic homology $\mathrm{HP}^{\mathbb{Z}}$ defined over $\mathbb{Z}$ is also not nil-invariant. See [**4**, Proposition 3.1] for more calculations on more general cases.

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