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THE HOMOLOGY OF PRINCIPALLY DIRECTED ORDERED GROUPOIDS

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(*communicated by Graham Ellis*)

Abstract

We present some homological properties of a relation β on ordered groupoids that generalises the minimum group congruence for inverse semigroups. When β is a transitive relation on an ordered groupoid G, the quotient G/β is again an ordered groupoid, and we construct a pair of adjoint functors between the module categories of G and of G/β. As a consequence, we show that the homology of G is completely determined by that of G/β, generalising a result of Loganathan for inverse semigroups.

1. Introduction

This paper studies some homological properties of a quotient construction for ordered groupoids determined by a certain relation β that generalises the minimal group congruence σ on an inverse semigroup. Modules for inverse semigroups, and the cohomology of an inverse semigroup, were first defined by Lausch in [**9**], and the cohomology used to classify extensions. An approach based on the cohomology of categories was then given by Loganathan [**13**], who showed that Lausch's cohomology of an inverse semigroup S was equal to the cohomology of a left-cancellative category $\mathfrak{L}(S)$ naturally associated to S. Loganathan proves a number of results relating the cohomology of S with that of its semilattice of idempotents $E(S)$ and of its maximum group image S/σ. He also considers the homology of S, but the treatment is brief since [**13**, Proposition 3.5] shows that the homology of S is completely determined by the homology of the group S/σ.

Ordered groupoids and inverse semigroups are closely related, since any inverse semigroup can be considered as a particular kind of ordered groupoid – an *inductive* groupoid – and this correspondence, in fact, gives rise to an isomorphism between the category of inverse semigroups and the category of inductive groupoids. This is the Ehresmann-Schein-Nambooripad Theorem (see [**11**, Theorem 4.1.8]). This close relationship has been exploited in the use of ordered groupoid techniques to prove

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results about inverse semigroups (see [**7, 11, 12, 16**]) and has been the motivation behind various generalisations of results about inverse semigroups to the wider class of ordered groupoids (see [**1, 5, 10**]).

In this paper we revisit Loganathan's results on the homology of inverse semigroups, and we are led to consider the relation β on an ordered groupoid G defined as follows: two elements of G are β-related if and only if they have a lower bound in G. This relation is trivially reflexive and symmetric but need not be transitive: when it is, we say that G is a principally directed ordered groupoid, a choice of terminology justified in Lemma 3.1 below. The β-relation and the class of principally directed ordered groupoids featured in [**5**] (but there called β-transitive ordered groupoids), in the study of the structure of inverse semigroups S with zero. In this setting, $S^* = S \setminus \{0\}$ can be considered as an ordered groupoid, and S^* is then principally directed if and only if S is *categorical at zero*: that is, whenever $a, b, c \in S$ and $abc = 0$ then either $ab = 0$ or $bc = 0$. The structure theorem of Gomes and Howie [**6**] for strongly categorical inverse semigroups with zero can then be deduced from a more general result on principally directed ordered groupoids [**5**, section 4.1]. In this paper, the significance of the transitivity of β is that it permits the construction of a pair of adjoint functors between the module categories of G and of G/β. The left adjoint is simply the colimit over $E(G)$. The right adjoint expands a G/β-module to a G-module. These constructions are discussed in section 4, and generalise the key ingredients of Loganathan's treatment of the homology of inverse semigroups in [**13**]. The fact that the homology of a principally directed ordered groupoid G is determined by the homology of the quotient G/β then follows readily in section 5.

2. Ordered groupoids

A groupoid G is a small category in which every morphism is invertible. The set of identities of G is denoted $E(G)$, following the customary notation for the set of idempotents in an inverse semigroup. We write $g \in G(e, f)$ when g is a morphism starting at e and ending at f. We regard a groupoid as an algebraic structure comprising its morphisms, and compositions of morphisms as a partially defined binary operation (see [**8, 11**]). The identities are then written as $e = g\mathbf{d} = gg^{-1}$ and $f = g\mathbf{r} = g^{-1}g$ respectively. A groupoid map $\theta \colon G \to H$ is just a functor.

Definition 2.1. An ordered groupoid is a pair $(G, \leqslant)$ where G is a groupoid and $\leqslant$ is a partial order defined on G, satisfying the following axioms:

- OG1 $x \leqslant y \Rightarrow x^{-1} \leqslant y^{-1}$, for all $x, y \in G$.
- OG2 Let $x, y, u, v \in G$ such that $x \leqslant y$ and $u \leqslant v$. Then $xu \leqslant yv$ whenever the compositions xu and yv exist.
- OG3 Suppose $x \in G$ and $e \in E(G)$ such that $e \leqslant x\mathbf{d}$, then there is a unique element $(e|x)$ called the *restriction* of x to e such that $(e|x)\mathbf{d} = e$ and $(e|x) \leqslant x$.
- OG4 If $x \in G$ and $e \in E(G)$ such that $e \leqslant x\mathbf{r}$, then there exist a unique element $(x|e)$ called the *corestriction* of x to e such that $(x|e)\mathbf{r} = e$ and $(x|e) \leqslant x$.

It is easy to see that OG3 and OG4 are equivalent: if OG3 holds then we may define a corestriction $(x|e)$ by $(x|e) = (e|x^{-1})^{-1}$.

An ordered functor $\phi\colon G \to H$ of ordered groupoids is an order preserving groupoid-map, that is $g\phi \leqslant h\phi$ if $g \leqslant h$. Ordered groupoids together with ordered functors constitute the category of ordered groupoids, **OGpd**.

Suppose $g, h \in G$ and that the greatest lower bound ℓ of $g\mathbf{r}$ and $h\mathbf{d}$ exist, then we define the *pseudoproduct* of g and h by $g * h = (g|\ell)(\ell|h)$. An ordered groupoid is called *inductive* if the pair $(E(G), \leqslant)$ is a meet semilattice. In an inductive groupoid G, the pseudoproduct is everywhere defined and $(G, *)$ is then an inverse semigroup: see [**11**, Theorem 4.1.8].

Example 2.2. For any set X, consider the set $\mathfrak{I}(X)$ of all bijections between subsets of X. Then $\mathfrak{I}(X)$ is an ordered groupoid, with set of identities $E(\mathfrak{I}(X))$ equal to the set of identity maps on subsets on X. Two bijections $\alpha\colon U \to V$ and $\beta\colon W \to Y$ are composable if and only if $V = W$, resulting in the usual composition $\alpha\beta\colon U \to Y$. The partial order is given by restriction, so that given $\alpha\colon U \to V$ and $\beta\colon W \to Y$, we have $\alpha \leqslant \beta$ if and only if $U \subseteq W$ and, for all $x \in U$, $x\alpha = x\beta$. If we allow the empty map as an element of $\mathfrak{I}(X)$, then the pseudoproduct $*$ is everywhere defined and the inverse semigroup $(\mathfrak{I}(X), *)$ is the *symmetric inverse monoid* on X, see [**11**, Chapter 1].

To any ordered groupoid G we associate a category $\mathfrak{L}(G)$ as follows. The objects of $\mathfrak{L}(G)$ are the identities of G and morphisms are given by pairs $(e, g) \in E(G) \times G$ where $g\mathbf{d} \leqslant e$, with $(e, g)\mathbf{d} = e$ and $(e, g)\mathbf{r} = g\mathbf{r}$. The composition of morphisms is defined by the partial product $(e, g)(f, h) = (e, g * h) = (e, (g|h\mathbf{d})h)$ whenever $g\mathbf{r} = f$. It is easy to see that $\mathfrak{L}(G)$ is left cancellative. This construction originates in the work of Loganathan [**13**], and forms the basis of the treatment in [**13**] of the cohomology of inverse semigroups.

3. Principally directed ordered groupoids

Let G be an ordered groupoid. The relation β on G is defined by

$$g \,\beta\, h \iff \text{there exists } k \in G \text{ with } k \leqslant g \text{ and } k \leqslant h\,.$$

β is evidently reflexive and symmetric but need not be transitive: we shall be concerned with the class of ordered groupoids for which β is indeed transitive, and thus an equivalence relation. We shall denote the β-class of $g \in G$ by $g\beta$. A *principal order ideal* is a subset of G of the form $\{g \in G\colon g \leqslant t\}$ for some $t \in G$, and will be denoted by $(t)^{\downarrow}$.

Lemma 3.1 ([**5**, section 2.2])**.** *The β-relation on an ordered groupoid G is transitive if and only if every principal order ideal in G is a directed set.*

Proof. Suppose that β is transitive, and that $g, h \in (t)^{\downarrow}$. Then $g \,\beta\, t \,\beta\, h$ and so $g \,\beta\, h$, and there exists $k \in G$ with $k \leqslant g$ and $k \leqslant h$: hence $k \in (t)^{\downarrow}$ and $(t)^{\downarrow}$ is a directed set. Conversely, suppose that $g \,\beta\, t \,\beta\, h$: then there exist $k, l \in G$ with $k \leqslant g$, $k \leqslant t$, $l \leqslant t$ and $l \leqslant h$. In particular, $k, l \in (t)^{\downarrow}$, and if $(t)^{\downarrow}$ is a directed set then there exists $c \leqslant t$ with $c \leqslant k$ and $c \leqslant l$. Then $c \leqslant g$ and $c \leqslant h$, and so $g \,\beta\, h$. □

Definition 3.2. An ordered groupoid in which every principal order ideal is a directed set will be called *principally directed*. This terminology is consistent with that of [**10**].

It is clear that if G is principally directed then so is its poset of identities $E(G)$. However, the converse is false. Let A and B be groups with a common subgroup C and let $i\colon C \hookrightarrow A$ and $j\colon C \hookrightarrow B$ be the inclusions. Consider the semilattice $\{0, e, f, 1\}$ with e, f incomparable, and define a semilattice of groups G by $G_1 = C, G_e = A, G_f = B$ and $E(G) = A \times B$ and with the obvious structure maps. Then $ci\,\beta\,c\,\beta\,cj$ for all $c \in C$, but ci and cj are not β-related.

Proposition 3.3 ([**5**, Proposition 2.2])**.** *If G is a principally directed ordered groupoid then the quotient set G/β is a groupoid.*

The groupoid structure on G/β is inherited from G in the following way. If $g, h \in G$ and $g^{-1}g\,\beta\,hh^{-1}$ then there exists $f \in E(G)$ with $f \leqslant g^{-1}g$ and $f \leqslant hh^{-1}$, and the composition of the β-classes of g and h is then defined by

$$(g\beta)(h\beta) = [(g|f)(f|h)]\beta\,.$$

This is easily seen to be independent of any choices made for f and for representatives of $g\beta$ and $h\beta$: see [**5**, section 2.2] for further details. However, there is no natural ordering inherited by G/β, and so we regard G/β as trivially ordered. Lawson [**10**, Theorem 20] states Proposition 3.3 for the special case of *principally inductive* ordered groupoids.

Example 3.4. An *action* of an ordered groupoid G on a set X is given by an ordered functor $G \to \mathfrak{I}(X)$ (see Example 2.2). There is an associated *action groupoid* $X \rtimes G$, which is an ordered groupoid defined as follows:

$$X \rtimes G = \{(x, g) \in X \times G : x \in (g\alpha)\mathbf{d}\},$$

with set of identities $E(X \rtimes G) = \{(x, e) \in X \rtimes G : e \in E(G)\}$. We have $(x, g)\mathbf{d} = (x, gg^{-1})$ and $(x, g)\mathbf{r} = (x(g\alpha), g^{-1}g)$, and the composition of (x, g) and (y, h) is defined, when $y = x(g\alpha)$ and $g^{-1}g = hh^{-1}$, by $(x, g)(y, h) = (x, gh)$. In the case that $X \rtimes G$ is principally directed, then $(X \rtimes G)/\beta$ is the *groupoid of germs* of the action. This holds when G is inductive, giving the construction of [**3**, Example 3.5]: see also [**14**, Chapter 1, p. 15] and [**10**, section 8] for actions of G by partial homeomorphisms of a topological space.

4. Expansion and colimits of modules

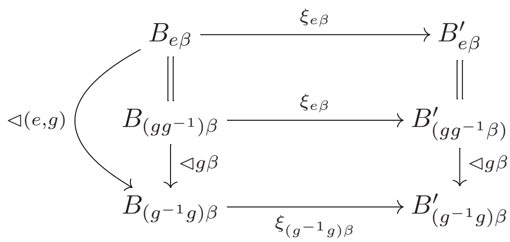
Let G be an ordered groupoid, and $\mathfrak{L}(G)$ its associated left-cancellative category. A G-module is defined to be an $\mathfrak{L}(G)$-module, that is, a functor $\mathcal{A}$ from $\mathfrak{L}(G)$ to the category of abelian groups. A G-module $\mathcal{A}$ is thus comprised of a family of abelian groups $\{A_e : e \in E(G)\}$ together with a group homomorphism $\alpha_{(e,g)}\colon A_e \to A_{g^{-1}g}$ for each arrow (e, g) of $\mathfrak{L}(G)$. We shall often denote $a\alpha_{(e,g)}$ by $a \lhd (e, g)$. Morphisms of G-modules (called G-*maps*) are natural transformations of functors, and so we obtain a category Mod_G of G-modules and G-maps.

Suppose that G is principally directed. No ordering is prescribed for the quotient groupoid G/β and so $\mathfrak{L}(G/\beta) = G/\beta$. If $\mathcal{B}$ is a (G/β)-module then we can *expand* $\mathcal{B}$ to obtain an $\mathfrak{L}(G)$-module $\mathcal{B}^{\uparrow}_{\beta}$ with homomorphisms $\mu_{(e,g)}$ as follows:

- for $e \in E(G)$ we have $(\mathcal{B}^{\uparrow}_{\beta})_e = B_{e\beta}$,

- if $e \geqslant f$ then $e\beta = f\beta$ and $\mu_{(e,f)} = \mathrm{id}$,
- for $x, y \in E(G)$ and for each $g \in G(x,y)$, the map $\mu_{(x,g)} \colon B_{x\beta} \to B_{y\beta}$ is just the map $\mu_{g\beta} \colon B_{x\beta} \to B_{y\beta}$ determined by $\mathcal{B}$.

This defines the *expansion functor* $\mathrm{Mod}_{G/\beta} \to \mathrm{Mod}_{\mathfrak{L}(G)}$ since, if $\xi \colon \mathcal{B} \to \mathcal{B}'$ is a G/β-map then we have a commutative diagram



and so we obtain an $\mathfrak{L}(G)$-map $\xi_\beta^\uparrow \colon \mathcal{B}_\beta^\uparrow \to (\mathcal{B}')_\beta^\uparrow$ with $(\xi_\beta^\uparrow)_e = \xi_{e\beta}$.

Lemma 4.1. *The expansion functor* $\mathrm{Mod}_{G/\beta} \to \mathrm{Mod}_{\mathfrak{L}(G)}$ *preserves epimorphisms.*

Proof. Epimorphisms in Mod are given by families of surjections, and so if ξ is an epimorphism in $\mathrm{Mod}_{G/\beta}$ then so is $\xi_\beta^\uparrow$ in $\mathrm{Mod}_{\mathfrak{L}(G)}$. □

The expansion functor is implicit in [**13**] for the case in which β is replaced by the minimal group congruence σ on an inverse semigroup. We now generalise [**13**, Lemma 3.4] and show that the expansion functor $\mathrm{Mod}_{G/\beta} \to \mathrm{Mod}_{\mathfrak{L}(G)}$ for a principally directed ordered groupoid G admits a left adjoint.

Suppose that $\mathcal{A}$ is an $\mathfrak{L}(G)$-module. We consider the restriction of $\mathcal{A}$ to an $E(G)$-module, involving the same abelian groups $A_e, (e \in E(G))$ but using only the maps $\alpha_{(e,f)} \colon A_e \to A_f$ from $\mathcal{A}$. The colimit $\mathrm{colim}^{E(G)} \mathcal{A}$ is then a direct sum

$$\mathrm{colim}^{E(G)} \mathcal{A} = \bigoplus_{x \in E(G)/\beta} L_x$$

indexed by the β-classes in $E(G)$, and so determines an $E(G/\beta)$-module $\mathcal{L}$ with $C_{e\beta} = L_{e\beta}$ and with trivial action, since $E(G/\beta)$ is a trivially ordered poset. We shall allow ourselves a small abuse of notation, and denote $\mathcal{L}$ by $\mathrm{colim}^{E(G)} \mathcal{A}$.

Proposition 4.2. *If G is principally directed and $\mathcal{A}$ is a G-module then* $\mathrm{colim}^{E(G)} \mathcal{A}$ *is a G/β-module.*

Proof. Let $\mathrm{colim}^{E(G)} \mathcal{A} = \bigoplus L_x$ as above, let $\alpha_e \colon A_e \to L_{e\beta}$ be the canonical map. Suppose that $\overline{a} \in L_{e\beta}$ with $\overline{a} = a\alpha_e$ for some in A_e, and $g \in G$ with $gg^{-1} \,\beta\, e$. Then gg^{-1} and e have a lower bound ℓ, and we define an action of $g\beta$ on $\overline{a}$ by

$$\overline{a} \lhd g\beta = (a\alpha_{(e,\ell)} \lhd (\ell|g))\alpha_z, \tag{1}$$

where $z = (\ell|g)\mathbf{r}$. We have to check that this definition is independent of the choices made for ℓ, a and g.

If we choose a different lower bound ℓ' of gg^{-1} and e, then ℓ and ℓ' are β-related (using the transitivity of β) and so have a lower bound ℓ''. It is sufficient to show, for independence from the choice of ℓ, that the outcome of (1) is unchanged by descent in the partial order, in the following sense.

Suppose that $a \in A_e$, $gg^{-1} = e$ and that $f \leqslant e$. Let $y = g^{-1}g$ and $z = (f|g)\mathbf{r}$. Then (1) gives $a\alpha_e \lhd g\beta = (a \lhd g)\alpha_y$. If we base the calculation at f we obtain $(a\alpha_{(e,f)} \lhd (f|g))\alpha_z$. But in $\mathfrak{L}(G)$,

$$(e,f)(f,(f|g)) = (e,(f|g)) = (e,(e|g))(y,z)$$

and so $a\alpha_{(e,f)} \lhd (f|g) = (a \lhd g)\alpha_{(y,z)}$. Hence

$$(a\alpha_{(e,f)} \lhd (f|g))\alpha_z = (a \lhd g)\alpha_{(y,z)}\alpha_z = (a \lhd g)\alpha_y \,.$$

Therefore the outcome of (1) is independent of the choice of ℓ.

We now consider the choice of a preimage for $\overline{a}$. Suppose that $a\alpha_e = b\alpha_x$. Then $e\beta x$ and so e and x have a lower bound u with $\overline{a} = a\alpha_{(e,u)}\alpha_u = b\alpha_{(x,u)}\alpha_u$. So again it suffices to check what happens if we apply (1) at u. We have

$$\overline{a} \lhd g\beta = (a \lhd g)\alpha_y = (a\alpha_{(e,u)} \lhd (u|g))\alpha_z,$$

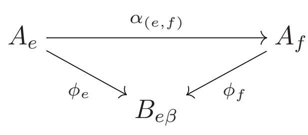
where now $z = (u|g)\mathbf{r}$. But as before, $a\alpha_{(e,u)} \lhd (u|g) = (a \lhd g)\alpha_{(y,z)}$ and $\alpha_{(y,z)}\alpha_z = \alpha_y$. Hence the definition in (1) is independent of the choice of a.

Finally, suppose that $g\,\beta\,h$. Then $gg^{-1}\,\beta\,hh^{-1}$ and so gg^{-1} and hh^{-1} have a lower bound $v \in E(G)$. Then $g\beta = (v|g)\beta = (v|h)\beta = h\beta$, and acting with $(v|g)$ in (1) we obtain

$$\begin{aligned} \overline{a} \lhd (v|g)\beta &= (a\alpha_v^e \lhd (v|g))\alpha_z \\ &= (a \lhd g)\alpha_{(y,z)}\alpha_z \\ &= (a \lhd g)\alpha_y \,. \end{aligned}$$

Hence the definition in (1) is independent of the choice of g, and we have a well-defined action of G/β on $\operatorname{colim}^{E(G)} \mathcal{A}$. □

Let $\mathcal{B}$ be a G/β-module, let $\mathcal{A}$ be a G-module, and suppose that we are given a map $\phi\colon \mathcal{A} \to \mathcal{B}_\beta^\uparrow$, with components $\phi_e\colon A_e \to B_{e\beta}, (e \in E(G))$. Whenever $e \geqslant f$ we have a commutative triangle



(in which $B_{e\beta} = B_{f\beta}$) and so the ϕ_e induce a family of maps ψ with $\psi_{e\beta}\colon L_{e\beta} \to \mathcal{B}_{e\beta}$ and, if $\alpha_e\colon A_e \to \operatorname{colim}^{E(G)} \mathcal{A}$ is the canonical map, then $\phi_e = \alpha_e\psi_{e\beta}$. Therefore ψ determines ϕ, and we have the following Corollary of Proposition 4.2.

Corollary 4.3. *If G is principally directed then $\psi\colon \operatorname{colim}^{E(G)} \mathcal{A} \to \mathcal{B}$ is a G/β-map, and $\phi \mapsto \psi$ is an injection*

$$\rho\colon \operatorname{Mod}_{\mathfrak{L}(G)}(\mathcal{A}, \mathcal{B}_\beta^\uparrow) \to \operatorname{Mod}_{G/\beta}(\operatorname{colim}^{E(G)} \mathcal{A}, \mathcal{B}) \,. \qquad (2)$$

Theorem 4.4. *Let G be a principally directed ordered groupoid. Then the functor $\operatorname{colim}^{E(G)}\colon \operatorname{Mod}_G \to \operatorname{Mod}_{G/\beta}$ is left adjoint to the expansion functor.*

Proof. We wish to construct a function

$$\tau\colon \operatorname{Mod}_{G/\beta}(\operatorname{colim}^{E(G)} \mathcal{A}, \mathcal{B}) \to \operatorname{Mod}_G(\mathcal{A}, \mathcal{B}_\beta^{\uparrow}) . \tag{3}$$

that will be inverse to ρ in (2). For $e \in E(G)$ and $\psi\colon \operatorname{colim}^{E(G)} \mathcal{A} \to \mathcal{B}$, consider the composition

$$A_e \xrightarrow[\alpha_e]{} L_{e\beta} \xrightarrow[\psi_{e\beta}]{} B_{e\beta} = (\mathcal{B}_\beta^{\uparrow})_e .$$

This composition is a G-map since, for $a \in A_e$,

$$(a\alpha_e\psi)\mu_{g\beta} = (a\alpha_e \lhd g\beta)\psi_{(g^{-1}g)\beta}$$

and, evaluating the $g\beta$ action using (1) with $\ell = gg^{-1}$,

$$= (a \lhd (e,g))\alpha_{(g^{-1}g)\beta}\psi_{(g^{-1}g)\beta}$$

and so the diagram

$$\begin{array}{ccccc}
A_e & \xrightarrow{\alpha_e} & L_{e\beta} & \xrightarrow{\psi_{e\beta}} & B_{e\beta} \\
{\scriptstyle \lhd(e,g)}\downarrow & & {\scriptstyle \lhd g\beta}\downarrow & & \downarrow{\scriptstyle \lhd g\beta} \\
A_{g^{-1}g} & \xrightarrow[\alpha_{g^{-1}g}]{} & L_{(g^{-1}g)\beta} & \xrightarrow[\psi_{(g^{-1}g)\beta}]{} & B_{(g^{-1}g)\beta}
\end{array}$$

commutes. Now the injection ρ in (2) carries $(\alpha_e\psi_{e\beta})$ to ψ and so $\tau\rho$ is the identity. A G-map $\phi\colon \mathcal{A} \to \mathcal{B}_\beta^{\uparrow}$ is carried by ρ to the induced map $\psi\colon \mathcal{L} \to \mathcal{B}$, where $\phi_e = \alpha_e\psi_{e\beta}$. But τ carries ψ precisely to this composition, and so $\rho\sigma$ is also the identity, and so in the principally directed case, (2) and (3) exhibit a natural bijection and its inverse. □

4.1. Composition of colimits

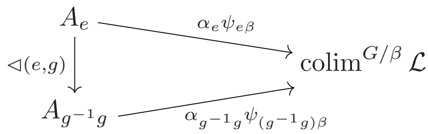
If G is principally directed, then we have seen in Proposition 4.2 that, for every G-module $\mathcal{A}$, the colimit $\mathcal{L} = \operatorname{colim}^{E(G)} \mathcal{A}$ can be considered as a G/β-module. Since G/β need not be connected, $\operatorname{colim}^{G/\beta} \mathcal{L}$ decomposes in general into a direct sum $\operatorname{colim}^{G/\beta} \mathcal{L} = \bigoplus_{p \in \pi_0(G/\beta)} C_p$ indexed by the connected components of G/β. We can therefore form $\operatorname{colim}^{G/\beta} \mathcal{L}$, with canonical maps $\psi_{e\beta}\colon L_{e\beta} \to C_{e\beta}$, where $e\beta$ is the connected component of $e \in G$ in the quotient groupoid G/β.

Proposition 4.5. *The colimit* $\operatorname{colim}^{G/\beta}(\operatorname{colim}^{E(G)} \mathcal{A})$ *is naturally isomorphic to* $\operatorname{colim}^{\mathfrak{L}(G)} \mathcal{A}$.

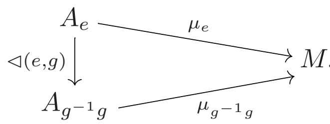
Proof. We show that $\operatorname{colim}^{G/\beta} \mathcal{L}$ has the universal property required of $\operatorname{colim}^{\mathfrak{L}(G)} \mathcal{A}$. As above, we have $\alpha_e\colon A_e \to L_{e\beta}$ and a commutative diagram

$$\begin{array}{ccccc}
A_e & \xrightarrow{\alpha_e} & L_{e\beta} & \xrightarrow{\psi_{e\beta}} & \\
{\scriptstyle \lhd(e,g)}\downarrow & & {\scriptstyle \lhd g\lambda}\downarrow & & \operatorname{colim}^{G/\beta} \mathcal{L} \\
A_{g^{-1}g} & \xrightarrow[\alpha_{g^{-1}g}]{} & L_{(g^{-1}g)\beta} & \xrightarrow[\psi_{(g^{-1}g)\beta}]{} &
\end{array}$$

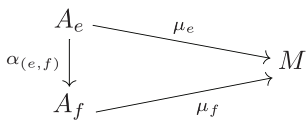
from which we extract the commutative triangles



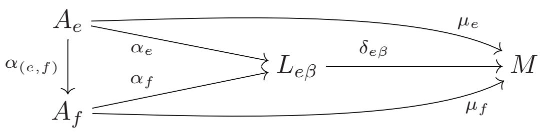
Suppose we are given a family of maps $\mu_e\colon A_e \to M$ to some abelian group M making commutative triangles



In particular, for $f \leqslant e$ we have



and hence a unique family of maps $\delta_{e\beta}\colon L_{e\beta} \to M$ making the diagrams

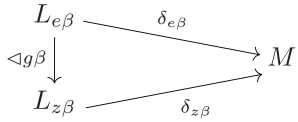


commute.

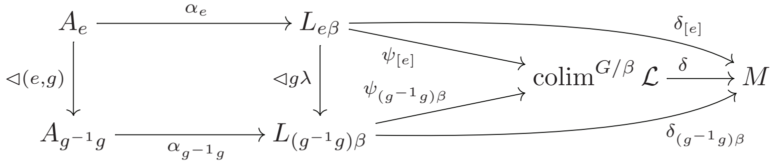
Now consider the action of $g\beta$ on $\overline{a} = a\alpha_e \in L_{e\beta}$. From (1)

$$\begin{aligned}(\overline{a} \lhd g\beta)\delta_{(g^{-1}g)\beta} &= (a\alpha_\ell^e \lhd (\ell|g))\alpha_z\delta_{z\beta} \\ &= (a\alpha_\ell^e \lhd (\ell|g))\mu_z \\ &= a\mu_e \qquad (\text{since } \mu_e = \alpha_{(e,(\ell,g))}\mu_{g^{-1}g}) \\ &= a\alpha_e\delta_{e\beta} \\ &= \overline{a}\delta_{e\beta}.\end{aligned}$$

Hence the triangles



commute and induce a unique map $\delta\colon \operatorname{colim}^{G/\beta}\mathcal{L} \to M$ making the diagram



commute, since $L_{z\beta} = L_{(g^{-1}g)\beta}$. □

5. The homology of principally directed ordered groupoids

The functors $H_n(G,-), n \geqslant 0$, for a fixed ordered groupoid G (or equivalently, for the left-cancellative category $\mathfrak{L}(G)$), may be characterized as functors $\mathrm{Mod}_G \to \mathbf{Ab}$ by the following properties:

(a) $H_n(G,-), n \geqslant 0$ is a homological extension of the colimit $\mathrm{colim}^{\mathfrak{L}(G)}$, so that

- $H_0(G,\mathcal{A}) = \mathrm{colim}^{\mathfrak{L}(G)}(\mathcal{A})$,
- for any short exact sequence $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ of G-modules and for each $n \geqslant 0$, there exists a natural homomorphism $d_n \colon H_{n+1}(G,\mathcal{C}) \to H_n(G\mathcal{A})$ inducing an exact sequence

$$\cdots \to H_{n+1}(G,\mathcal{C}) \to H_n(G,\mathcal{A}) \to H_n(G,\mathcal{B}) \to H_n(G,\mathcal{C}) \to H_{n-1}(G,\mathcal{A}) \to \cdots$$

(b) $H_n(G,\mathcal{P}) = 0$ for all $n > 0$ and all projective modules $\mathcal{P}$.

Theorem 5.1. *For any principally directed ordered groupoid G and G-module $\mathcal{A}$, and any $n \geqslant 0$, the homology groups $H_n(G,\mathcal{A})$ and $H_n(G/\beta, \mathrm{colim}^{E(G)}\,\mathcal{A})$ are isomorphic.*

Proof. We consider the functor $\mathrm{Mod}_{\mathfrak{L}(G)} \to \mathbf{Ab}$ given by

$$\mathcal{A} \mapsto H_n(G/\beta, \mathrm{colim}^{E(G)}\,\mathcal{A}).$$

For $n = 0$ we have

$$H_0(G/\beta, \mathrm{colim}^{E(G)}\,\mathcal{A}) = \mathrm{colim}^{G/\beta}(\mathrm{colim}^{E(G)}\,\mathcal{A}) \cong \mathrm{colim}^{\mathfrak{L}(G)}\,\mathcal{A} = H_0(G,\mathcal{A})$$

by Proposition 4.5. The transitivity of β on $E(G)$ is sufficient to ensure that $\mathcal{A} \mapsto \mathrm{colim}^{E(G)}\,\mathcal{A}$ is exact, (see, for example, [**15**, tag 04AX]). It follows that the sequence of functors $H_n(G/\beta, \mathrm{colim}^{E(G)} -)$ induces, from a short exact sequence $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ of G-modules an exact sequence

$$\cdots \to H_{n+1}(G/\beta, \mathrm{colim}^{E(G)}\,\mathcal{C}) \to H_n(G/\beta, \mathrm{colim}^{E(G)}\,\mathcal{A}) \to H_n(G/\beta, \mathrm{colim}^{E(G)}\,\mathcal{B})$$
$$\to H_n(G/\beta, \mathrm{colim}^{E(G)}\,\mathcal{C}) \to H_{n-1}(G/\beta, \mathrm{colim}^{E(G)}\,\mathcal{A}) \to \cdots .$$

Now suppose that $\mathcal{P}$ is a projective $\mathfrak{L}(G)$-module. By Lemma 4.1 the expansion functor $\mathrm{Mod}_{G/\beta} \to \mathrm{Mod}_{\mathfrak{L}(G)}$ preserves epimorphisms, and so its left adjoint $\mathrm{colim}^{E(G)}$ preserves projectives. Therefore $\mathrm{colim}^{E(G)}\,\mathcal{P}$ is projective, and for $n > 0$ we have $H_n(G/\beta, \mathrm{colim}^{E(G)}\,\mathcal{P}) = 0$. □

We note that the homology of groupoids is discussed in [**8**, chapter 16] (mostly for integer coefficients), and for the important class of étale groupoids in [**4**].

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