

A NOTE ON A HOLSTEIN CONSTRUCTION

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Abstract

We clarify details and fill certain gaps in the construction of a canonical Reedy fibrant resolution for a constant simplicial DG-category due to Holstein.

1. Introduction

The present paper grew out of attempts to understand technical details of a proof in [Hol]. Thus, from the very start, we do not claim that our work contains original insights.

We begin by describing our interest. In the papers [BHW, AØ] homotopy gluing of DG-categories was studied.

The standard example is given by Abelian categories of sheaves on open sets for a Čech covering of a topological space. One seeks a lift for gluing of Abelian categories to DG-level. Unlike with ordinary categories, one requires coherence data on multiple intersections in the covering to be given by weak equivalences, not by isomorphisms. The answer is spelled out naturally in the language of homotopy limits for cosimplicial diagrams of DG-categories. In [AØ] Sebastian Ørsted and the first author provided an explicit model for such a homotopy limit.

The construction relies on an explicit model for *powering* by simplicial sets in the model category of DG-categories due to Holstein (see [Hol, Proposition 3.6]). The key ingredient in the latter is a canonical simplicial resolution of a DG-category introduced in the same paper (see [Hol, Propositions 3.9 and 3.10]). Our goal in the present paper is to add details to the sketch of the proofs of those statements in Holstein's work.

The author's strategy in that paper was to generalize a proof of Tabuada that a certain explicit DG-category provides a *path object* construction (see [Tab2, Proposition 3.3]). However, the original proof of Tabuada contained a minor flaw which was inherited in Holstein's approach. We fill both gaps, and this, together with certain explicit calculations, is the main content of the present note.

Let us outline the structure of the paper. In the second section we recall the construction of Dwyer-Kan model structure on the category of DG-categories. Then,

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following Lefèvre-Hasegawa [**Lef**] and Faonte [**Fao**], we discuss close relatives of DG-functors called A_∞ -functors. We describe the category of A_∞ -functors between two DG-categories playing the role of internal Hom in the category of DG-categories. We conclude the section by recalling the Reedy model structure on a diagram category with values in a model category. In particular, this includes our main object of interest – the category of simplicial DG-categories.

In the third section we provide a detailed proof of Holstein’s theorem filling the gap in his original approach. In particular, the proof of fibrancy of matching maps is given by explicit lifts.

Our proof is based on direct calculations of lifts and on the use of an elegant description for homotopy equivalences of A_∞ -functors suggested to us by Efimov. In the appendix, we provide an alternative approach to the proof developing the ideas of Tabuada and Holstein and correcting their inaccuracies. The main strategy there is to reduce the statement to the case of pretriangulated DG-categories via the construction of pretriangulated envelope.

Acknowledgments

The gaps in the last part of the paper [**Hol**] were noticed by several people, in particular, by Boris Shoikhet. We thank Boris Shoikhet for sharing his concerns at an early stage of the present work. We also thank Julian Holstein for stimulating discussions.

The idea to study pointwise homotopy equivalences of A-infinity functors is due to Alexander Efimov, the context and the exact reference were kindly provided by Sebastian Ørsted. This improvement clarified and simplified the exposition greatly, thus the final form of the paper owes a lot to Efimov and Ørsted. We thank Edouard Balzin for careful proofreading of the text, and Timothy Logvinenko for useful comments.

2. Homotopy theory of DG-categories

Below we collect a few constructions and statements to be used in the next section and necessary to formulate the theorem of Holstein. We work over a base field k . Recall that a DG-category is a category enriched over the monoidal category $\mathbf{Com}(k\text{-Mod})$. The homotopy category for a DG-category \mathcal{A} is denoted by $H^0(\mathcal{A})$. We denote the category of small DG-categories and DG-functors by $\mathbf{DGCat}(k)$.

2.1. Dwyer-Kan model structure for DG-categories

Recall that a DG-functor is called a quasiequivalence, if it induces quasiisomorphisms on all Hom complexes and becomes an equivalence of the homotopy categories. Quasiequivalences are a part of *Dwyer-Kan* model structure on $\mathbf{DGCat}(k)$ constructed in [**Tab**]. Recall the description of the three standard classes of morphisms.

We say that a DG-functor $F: \mathcal{A} \rightarrow \mathcal{D}$ is

- a weak equivalence, if it is a quasiequivalence;
- a fibration, if it is surjective on all Hom complexes and is an isofibration at the level of H^0 , i.e. for a homotopy equivalence $F(x) \xrightarrow{u} y$ in \mathcal{D} there exists a

homotopy equivalence $x \xrightarrow{u'} y'$ such that $F(u') = u$:

$$\begin{array}{ccc} \mathcal{A} & & x \xrightarrow{u'} y' \\ \downarrow F & & \downarrow \\ \mathcal{D} & & F(x) \xrightarrow{u} y; \end{array}$$

- a cofibration, if it admits the left lifting property with respect to all trivial fibrations.

Theorem 2.1. *The category $\text{DGCat}(k)$ is equipped with cofibrantly generated model structure with weak equivalences, fibrations and cofibrations defined as above.*

2.2. A_∞ functors as inner Hom

In $\text{DGCat}(k)$, one can take the naive tensor product $\mathcal{A} \otimes \mathcal{D}$ and the naive inner Hom $\text{DGFun}(\mathcal{A}, \mathcal{D})$ which make $\text{DGCat}(k)$ into a closed monoidal category. However, these notions are not consistent with the model structure discussed above, and thus do not make $\text{HoDGCat}(k)$ into a closed monoidal category. This can be amended by considering derived versions, \otimes^L and RHom (see [Toë]), which are defined up to quasiequivalence but which make $\text{HoDGCat}(k)$ into a closed monoidal category.

Of existing models for RHom , we make use of the one given by the DG-category of A_∞ -functors.

Definition 2.2. For two DG-categories \mathcal{A}, \mathcal{B} , a strictly unital A_∞ functor $F: \mathcal{A} \rightarrow \mathcal{B}$ consists of the following data:

- $F_0: \text{Ob}\mathcal{A} \rightarrow \text{Ob}\mathcal{B}$;
- for all $n \geq 1$ and $x_0, \dots, x_n \in \text{Ob}\mathcal{A}$,

$$F_n: \mathcal{A}(x_{n-1}, x_n) \otimes \dots \otimes \mathcal{A}(x_0, x_1) \rightarrow B(F_0(x_0), F_0(x_n))$$

of degree $1 - n$, subject to

$$\begin{aligned} \sum_{s=0}^{n-2} (-1)^{n-s} F_{n-1}(Id^{\otimes s} \otimes m \otimes Id^{\otimes(n-s-2)}) + \sum_{s=0}^{n-1} (-1)^{n-1} F_n(Id^{\otimes s} \otimes d \otimes Id^{\otimes(n-s-1)}) \\ = dF_n + \sum_{s=1}^{n-1} (-1)^{ns} m(F_s \otimes F_{n-s}), \end{aligned}$$

where d is the differential and m is the composition.

Definition 2.3. For two DG-categories \mathcal{A}, \mathcal{B} , the DG-category $A_\infty \text{Fun}(\mathcal{A}, \mathcal{B})$ has strictly unital A_∞ functors as objects. For F, G such functors, the complex $A_\infty \text{Fun}(\mathcal{A}, \mathcal{B})(F, G)$ is, in degree l ,

$$\prod_{\substack{n \geq 0 \\ x_0, \dots, x_n \in \text{Ob}(\mathcal{A})}} \text{Hom}(\mathcal{A}(x_{n-1}, x_n) \otimes \dots \otimes \mathcal{A}(x_0, x_1), \mathcal{B}(F_0(x_0), G_0(x_n)))[l - n]).$$

For $a \in A_\infty \text{Fun}^l(\mathcal{A}, \mathcal{B})(F, G)$, its differential $d_{A_\infty}(a)$ has its component at (x_0, \dots, x_n)

equal to

$$\begin{aligned} & \pm d(a_{x_0, \dots, x_n}) + \sum_{i=1}^n \pm m(a_{x_i, \dots, x_n} \otimes F_{x_0, \dots, x_n}) + \sum_{i=0}^{n-1} \pm m(G_{x_i, \dots, x_n} \otimes a_{x_0, \dots, x_i}) \\ & + \sum_{i=0}^{n-1} \pm a_{x_0, \dots, x_n} (Id^{\otimes i} \otimes d \otimes Id^{\otimes (n-i-1)}) + \sum_{i=0}^{n-2} \pm a_{x_0, \dots, \hat{x}_i, \dots, x_n} (Id^{\otimes i} \otimes m \otimes Id^{\otimes (n-i-2)}) \end{aligned}$$

The definitions above are a special case of the general theory of A_∞ categories and their morphisms. The discussion in full generality and including sign conventions can be found e.g. in [Lef].

In [Fao], the following theorem is proved.

Theorem 2.4. *The DG-category $A_\infty \text{Fun}(\mathcal{A}, \mathcal{B})$ is a model for $\text{RHom}(\mathcal{A}, \mathcal{B})$.*

2.3. Reedy model structure for diagrams

To talk about (co)simplicial DG-categories, we need the following technique (see [Hir] or [Hov]).

Definition 2.5. A Reedy category is a category \mathcal{I} together with a degree function $d: \text{Ob}(\mathcal{I}) \rightarrow \lambda$ (where λ is an ordinal, typically \mathbb{N}) and with two full subcategories \mathcal{I}^+ and \mathcal{I}^- , subject to the following conditions:

- every non-identity map in \mathcal{I}^+ increases the degree;
- every non-identity map in \mathcal{I}^- decreases the degree;
- every map f in \mathcal{I} admits a unique factorization $f = f^+ \circ f^-$, where $f^- \in \mathcal{I}^-$ and $f^+ \in \mathcal{I}^+$.

The *simplicial category* Δ of finite ordinals and order preserving maps is an example of a Reedy category – in its case, $d([n]) = n$, Δ^+ consists of injections and Δ^- consists of surjections. Also, for \mathcal{I} a Reedy category, \mathcal{I}^{op} is also a Reedy category with the same degree function, with $(\mathcal{I}^{\text{op}})^+ = (\mathcal{I}^-)^{\text{op}}$ and with $(\mathcal{I}^{\text{op}})^- = (\mathcal{I}^+)^{\text{op}}$.

For a Reedy category \mathcal{I} and an arbitrary model category \mathcal{M} , the diagram category $\mathcal{M}^{\mathcal{I}}$ is equipped with Reedy model structure. We need the following definitions to describe it.

- Definition 2.6.**
1. For $i \in \mathcal{I}$, the latching category $\delta(I^+ \downarrow i)$ is a full subcategory of the overcategory $(\mathcal{I}^+ \downarrow i)$ consisting of all arrows except for id_i .
 2. For $i \in \mathcal{I}$ and $D \in \mathcal{M}^{\mathcal{I}}$, the corresponding latching object is

$$L_i D = \text{colim}_{j \rightarrow i \in \delta(I^+ \downarrow i)} D(j).$$

3. Dually, for $i \in \mathcal{I}$, the matching category $\delta(i \downarrow \mathcal{I}^-)$ is a full subcategory of the undercategory $(i \downarrow \mathcal{I}^-)$ consisting of all arrows except for id_i .
4. For $i \in \mathcal{I}$ and $D \in \mathcal{M}^{\mathcal{I}}$, the corresponding matching object is

$$M_i D = \lim_{i \rightarrow j \in \delta(i \downarrow \mathcal{I}^-)} D(j).$$

Note that there are natural maps $L_i D \xrightarrow{l_i D} D(i) \xrightarrow{m_i D} M_i D$, and, for a map of diagrams $f: D \rightarrow D'$, maps $L_i(f): L_i D \rightarrow L_i D'$ and $M_i(f): M_i D \rightarrow M_i D'$. Let us say that a map of diagrams $f: D \rightarrow D'$ is

- a Reedy weak equivalence, if $\forall i \in \mathcal{I}$ the map $f_i: D(i) \rightarrow D'(i)$ is a weak equivalence in \mathcal{M} ;
- a Reedy cofibration, if $\forall i \in \mathcal{I}$, the arrow

$$l_i f: L_i(D') \coprod_{L_i D} D(i) \rightarrow D'(i)$$

is a cofibration in \mathcal{M} :

$$\begin{array}{ccc}
 L_i(D) & \xrightarrow{L_i(f)} & L_i(D') \\
 l_i D \downarrow & & \downarrow \\
 D(i) & \longrightarrow & L_i(D') \coprod_{L_i D} D(i) \\
 & \searrow f_i & \downarrow l_i D' \\
 & & D'(i)
 \end{array}$$

- a Reedy fibration, if $\forall i \in \mathcal{I}$ the arrow

$$m_i f: D(i) \rightarrow M_i(D) \times_{M_i(D')} D'(i)$$

is a fibration in \mathcal{M} :

$$\begin{array}{ccc}
 D(i) & \xrightarrow{f_i} & D'(i) \\
 \downarrow m_i D & \dashrightarrow m_i f & \downarrow m_i D' \\
 M_i(D) \times_{M_i(D')} D'(i) & \longrightarrow & D'(i) \\
 \downarrow M_i(f) & & \downarrow \\
 M_i(D) & \xrightarrow{M_i(f)} & M_i(D')
 \end{array}$$

Theorem 2.7. *The three classes of morphisms define a model structure on the category $\mathcal{M}^{\mathcal{I}}$.*

One notices that for $0 \rightarrow D$ the Reedy cofibrancy condition boils down to the cofibrancy of $L_i(D) \rightarrow D(i)$ for every i , and, dually, for $D \rightarrow 1$ the Reedy fibrancy condition boils down to the fibrancy of $D(i) \rightarrow M_i(D)$ for every i . Thus a diagram is Reedy cofibrant if all its latching maps are cofibrations, and a diagram is Reedy fibrant if all its matching maps are fibrations.

In this note, our source category is Δ^{op} , and our target category is $\text{DGCat}(k)$ with Dwyer-Kan model structure.

3. Reedy fibrant replacement for simplicial DG-categories

3.1. Holstein construction

Denote the DG-category obtained by the k -linearization of the category for the totally ordered set $\{0, \dots, n\}$ by $k[n]$.

For a DG-category \mathcal{A} , the DG-category $A_\infty \text{Fun}^\circ(k[n], \mathcal{A})$ has A_∞ functors $k[n] \rightarrow \mathcal{A}$ sending arrows to homotopy equivalences as objects and the complexes of A_∞ natural transformations as morphisms. We spell out the formulas for our case. An

object $(X, f) \in A_\infty \text{Fun}^\circ(k[n], \mathcal{A})$ is the data of $(n + 1)$ objects X_0, \dots, X_n in \mathcal{A} and the morphisms $\{f_I\}$ where I runs over all subsets of $\{0, \dots, n\}$ of cardinalities at least 2, with $f_{i_0, i_1, \dots, i_k} \in \mathcal{A}^{1-k}(X_{i_0}, X_{i_k})$, subject to the following conditions:

- $d(f_{i_0, \dots, i_k}) = \sum_{s=1}^{k-1} (-1)^s f_{i_0, \dots, \widehat{i_s}, \dots, i_k} - \sum_{s=1}^{k-1} (-1)^s f_{i_s, \dots, i_k} \circ f_{i_0, \dots, i_s}$;
- all $f_{i,j}$ are homotopy equivalences.

Following Holstein, we use the following notation:

- $d(\phi)_{i_0, \dots, i_k} = d(\phi_{i_0, \dots, i_k})$;
- $(\Delta\phi)_{i_0, \dots, i_k} = (-1)^{|\phi|} \sum_{s=1}^{k-1} (-1)^s \phi_{i_0, \dots, \widehat{i_s}, \dots, i_k}$;
- $(\phi \circ \psi)_{i_0, \dots, i_k} = \sum_{s=0}^k (-1)^{|\phi|s} \phi_{i_s, \dots, i_k} \circ \psi_{i_0, \dots, i_s}$, where one should read 0 if indexing subset is impossible.

In this notation, upon fixing $|f| = 1$, the first of the conditions above becomes Maurer-Cartan equation:

$$d(f) + \Delta f + f \circ f = 0.$$

The Hom complexes in $A_\infty \text{Fun}^\circ(k[\bullet], \mathcal{A})$ are the complexes of A_∞ natural transformations, namely

$$A_\infty \text{Fun}^\circ(k[\bullet], \mathcal{A})((X, f), (Y, g)) = \bigoplus_{\{i_0, \dots, i_k\} \subset \{0, \dots, n\}} \mathcal{A}(X_{i_0}, Y_{i_k})[-k]$$

with differential

$$d_{A_\infty}(a) = d(a) + \Delta a + a \circ f - (-1)^{|a|} g \circ a.$$

Explicitly, a degree m morphism $a: (X, f) \rightarrow (Y, g)$ consists of components $\{a_I\}$ where I runs over all non-empty subsets of $\{0, \dots, n\}$, with $a_{i_1, \dots, i_k} \in \mathcal{A}^{1-k}(X_{i_1}, Y_{i_k})$.

As $k[\bullet]$ is a cosimplicial DG-category, $A_\infty \text{Fun}^\circ(k[\bullet], \mathcal{A})$ becomes a simplicial DG-category, with structure maps obtained by precompositions with structure maps of $k[\bullet]$.

One of the main results in the paper [Hol] is the following theorem (see Propositions 3.9 and 3.10 in that paper).

Theorem 3.1. *The simplicial DG-category $A_\infty \text{Fun}^\circ(k[\bullet], \mathcal{A})$, considered as an object of $\text{DGCat}(k)^{\Delta^{\text{op}}}$, is a Reedy fibrant replacement of $c\mathcal{A}$, the constant simplicial DG-category for a DG-category \mathcal{A} , with respect to Dwyer-Kan model structure on the target model category $\text{DGCat}(k)$.*

For convenience, we denote $A_\infty \text{Fun}^\circ(k[\bullet], \mathcal{A}) =: F_\bullet(\mathcal{A})$.

The proof naturally consists of two parts. Firstly, one has to show that for every n , the natural (constant functor) inclusion $\mathcal{A} \rightarrow F_n(\mathcal{A})$ is a quasiequivalence. Secondly, one has to show that $F_\bullet(\mathcal{A})$ is Reedy fibrant.

3.2. Quasiequivalences

In both parts of the proof, we rely on the following general fact from the homotopy theory of A_∞ -functors, due to Lefèvre-Hasegawa [Lef, Proposition 8.2.2.3]. We reduce the generality by considering DG-categories instead of A_∞ -categories.

Lemma 3.2. *Let \mathcal{A}, \mathcal{B} be two DG-categories, F, G two A_∞ -functors $\mathcal{A} \rightarrow \mathcal{B}$ and $a: F \rightarrow G$ a closed A_∞ natural transformation of degree 0. Then a is a homotopy equivalence in $A_\infty \text{Fun}(\mathcal{A}, \mathcal{B})$ if and only if for every $X \in \mathcal{A}$ the component $a_X: F(X) \rightarrow G(X)$ is a homotopy equivalence in \mathcal{B} .*

Note if the DG-category $A_\infty \text{Fun}(\mathcal{A}, \mathcal{B})$ is replaced by the “naive version of inner Hom” $\text{DGFun}(\mathcal{A}, \mathcal{B})$, then the statement of the lemma above would not hold.

We can now prove the following theorem.

Theorem 3.3. *For every n , the constant functor inclusion $c: \mathcal{A} \rightarrow F_n(\mathcal{A})$ is a quasiequivalence.*

Proof. We first check that c induces quasiisomorphism on all Hom complexes. It is injective on cohomology – if for $f: X \rightarrow Y$ we have $cf = d_{A_\infty}(g)$, then, in particular, $f = (cf)_0 = d(g_0)$. To show that c is surjective on cohomology, let a be a closed map $cX \rightarrow cY$ for $X, Y \in \mathcal{A}$. Let us check that a is in the same cohomology class as $c(a_0)$, i.e. that $a - c(a_0)$ is exact. The fact that $d_{A_\infty}(a) = 0$ corresponds to the following formulas:

$$\begin{cases} d(a_i) = 0, \\ d(a_{i_0 \dots i_k}) = a_{i_1 \dots i_k} - a_{i_0 \dots i_{k-1}} + \sum_{s=1}^{k-1} (-1)^s a_{i_0 \dots \widehat{i_s} \dots i_k}. \end{cases}$$

Then $a - c(a_0) = d(b)$, where

$$b_{i_0 \dots i_k} = \begin{cases} 0 & i_0 = 0, \\ a_{0i_0 \dots i_k} & i_0 \neq 0. \end{cases}$$

We then check that c is essentially fully faithful at the level of H^0 , namely that any object $(X, f) \in F_n(\mathcal{A})$ is homotopy equivalent to an object in the image of c . Indeed, consider the object cX_0 . The A_∞ -natural transformation $a: cX_0 \rightarrow (X, f)$ is given by

$$a_i = \begin{cases} 1_{X_0} & i = 0, \\ f_{0i} & \text{otherwise,} \end{cases} \quad \text{and} \quad a_{i_0 \dots i_k} = \begin{cases} 0 & i_0 = 0, \\ f_{0i_0 \dots i_k} & \text{otherwise.} \end{cases}$$

The fact that $d_{A_\infty}(a) = 0$ follows from Maurer-Cartan condition for f .

Note that 1_{X_0} and f_{0i} are all homotopy equivalences in \mathcal{A} . Then, by Lemma 3.2, a is a homotopy equivalence. □

Remark 3.4. In [Hol], it was first shown that every $(X, f) \in F_n(\mathcal{A})$ can be strictified, i.e. it is homotopy equivalent to an (\tilde{X}, \tilde{f}) where all compositions are strict and $\tilde{f}_{i_0 \dots i_k} = 0$ for $k > 1$. However, Lemma 3.2 does not become elementary even in this generality, and once we have this lemma, strictification becomes unnecessary.

3.3. Reedy fibrancy

We now prove Reedy fibrancy of $F_\bullet(\mathcal{A})$ by showing that the matching maps are Dwyer-Kan fibrations – namely, that they are surjective on all the Hom complexes and that they are isofibrations at the level of H^0 . We begin from explicitly describing these matching maps.

By definition of a matching object, we have

$$M_n F(\mathcal{A}) = \lim_{\delta([n] \downarrow (\Delta^{\text{op}})_-)} F_\bullet(\mathcal{A}) = \lim_{[m] \hookrightarrow [n]} F_m(\mathcal{A}).$$

This is the data of A_∞ functors without highest homotopies. Namely, an object $(X, f) \in M_n F(\mathcal{A})$ is the data of $(n+1)$ objects X_0, \dots, X_n in \mathcal{A} and the morphisms $\{f_I\}$ where I runs over all subsets of $\{0, \dots, n\}$ of cardinalities from 2 to n (that is, the subset $\{0, \dots, n\}$ is not included) with $f_{i_0, i_1, \dots, i_k} \in \mathcal{A}^{1-k}(X_{i_0}, X_{i_k})$, satisfying the following conditions:

- $d(f) + \Delta f + f \circ f = 0$;
- all $f_{i,j}$ are homotopy equivalences.

Similarly, the morphisms are given by complexes of A_∞ natural transformations without highest homotopies. Namely, a degree m morphism $a: (X, f) \rightarrow (Y, g)$ is the set of morphisms $\{a_I\}$ where I runs over all non-empty subsets of $\{0, \dots, n\}$ except for $\{0, \dots, n\}$ itself, with $a_{i_1, \dots, i_k} \in \mathcal{A}^{1-k}(X_{i_1}, Y_{i_k})$, and with differential given by

$$d_{A_\infty}(a) = d(a) + \Delta a + a \circ f - (-1)^{|a|} g \circ a.$$

The matching map $m_n: F_n(\mathcal{A}) \rightarrow M_n F(\mathcal{A})$ is the natural forgetful functor that, on objects, forgets $f_{0,1,\dots,n}$, and, on morphisms, forgets $a_{0,1,\dots,n}$. We write $(X, f) \mapsto (X, f_{\leq n})$.

The first part of Reedy fibrancy for $F_\bullet(\mathcal{A})$ is the following elementary proposition.

Proposition 3.5. *The forgetful functor m_n is surjective on Hom complexes.*

Proof. A preimage of a truncated A_∞ transformation a between $(X, f_{\leq n})$ and $(Y, g_{\leq n})$ can be obtained by simply assigning any value (e.g. 0) to $a_{0,1,\dots,n}$, as there are no conditions on the components. \square

Showing that m_n is a homotopy isofibration requires more work. In our computations, we use the following lemma, from [Kon, Section 5, Theorem 1] (see also [Sho, Lemma 3.6]).

Lemma 3.6. *For any DG-category \mathcal{A} and a homotopy equivalence $f \in \mathcal{A}^0(X, Y)$ it is always possible to find $f \in \mathcal{A}^0(Y, X)$, $r_X \in \mathcal{A}^{-1}(X, X)$, $r_Y \in \mathcal{A}^{-1}(Y, Y)$ and $r_{XY} \in \mathcal{A}^{-2}(X, Y)$ such that:*

- $gf = 1_X + d(r_X)$;
- $fg = 1_Y + d(r_Y)$;
- $fr_X - r_Y f = d(r_{XY})$.

Now suppose that we have an object $(Y, g) \in M_n F(\mathcal{A})$ and a homotopy equivalence $a: (X, f_{\leq n}) \rightarrow (Y, g)$ (with homotopy inverse \bar{a}). To show that m_n is an isofibration on H^0 , we need to lift a to a homotopy equivalence in $F_n(\mathcal{A})$.

Remark 3.7. In [Hol], the lift of the object is constructed – namely, $g_{0,\dots,n}$ is given with $d(g_{0,\dots,n}) = (\Delta g + g \circ g)_{0,\dots,n}$. We insignificantly modify the lift and provide the computation for the sake of reader's convenience. In what follows, let $\alpha \circ' \beta$ denote $\alpha \circ \beta$ without the term $\alpha_{0,\dots,n} \circ \beta_0$. Let r_{Y_0} be such that $a_0 \bar{a}_0 = 1_{Y_0} + d(r_{Y_0})$. The indexing subset is always $\{0, 1, \dots, n\}$ and is omitted.

Proposition 3.8. *Setting*

$$g_{0,\dots,n} := (\Delta a + a \circ f - g \circ' a)\bar{a}_0 - (\Delta g + g \circ g)r_{Y_0}$$

indeed gives $d(g_{0,\dots,n}) = \Delta g + g \circ g$, thus this lifts the object.

Proof. One first checks that $d(\Delta g + g \circ g) = 0$. Then

$$d((\Delta g + g \circ g)r_{Y_0}) = (\Delta g + g \circ g)d(r_{Y_0}) = (\Delta g + g \circ g)(a_0\bar{a}_0 - 1).$$

So we are left to see that

$$d((\Delta a + a \circ f - g \circ' a)\bar{a}_0') = (\Delta g + g \circ g)a_0\bar{a}_0$$

– or that

$$d(\Delta a + a \circ f - g \circ' a) = (\Delta g + g \circ g)a_0,$$

which is an explicit computation. \square

We now construct the *closed* lift of the morphism a – namely, we give a formula for $a_{0,\dots,n}$ with $d(a_{0,\dots,n}) = (\Delta a + a \circ f - g \circ a)_{0,\dots,n}$. Let r_{X_0} be such that $\bar{a}_0 a_0 = 1_{X_0} + d(r_{X_0})$, let r_{Y_0} be such that $a_0\bar{a}_0 = 1_{Y_0} + d(r_{Y_0})$, and let $r_{X_0 Y_0}$ be such that $a_0 r_{X_0} - r_{Y_0} a_0 = d(r_{X_0 Y_0})$ (such r_{X_0} , r_{Y_0} and $r_{X_0 Y_0}$ can always be found due to Lemma 3.6). The indexing subset is again $\{0, 1, \dots, n\}$ and is omitted.

Proposition 3.9. *Setting*

$$a_{0,\dots,n} := (\Delta a + a \circ f - g \circ' a)r_{X_0} + (\Delta g + g \circ g)r_{X_0 Y_0}$$

indeed gives $d(a_{0,\dots,n}) = \Delta a + a \circ f + g \circ a$, thus this lifts the morphism.

Proof. We start from observing that $g \circ a = g \circ' a + g_{0,\dots,n} a_0$ and we can insert our value of $g_{0,\dots,n}$. This gives

$$\Delta a + a \circ f - g \circ a = \Delta a + a \circ f - g \circ' a - (\Delta a + a \circ f - g \circ' a)\bar{a}_0 a_0 + (\Delta g + g \circ g)r_{Y_0} a_0.$$

We know that $d(\Delta a + a \circ f - g \circ' a) = (\Delta g + g \circ g)a_0$, so

$$d((\Delta a + a \circ f - g \circ' a)r_{X_0}) = (\Delta g + g \circ g)a_0 r_{X_0} + (\Delta a + a \circ f - g \circ' a)(\bar{a}_0 a_0 - 1).$$

Then we are left to notice that indeed

$$(\Delta g + g \circ g)(a_0 r_{X_0} - r_{Y_0} a_0) = d((\Delta g + g \circ g)r_{X_0 Y_0})$$

and thus we have constructed the lift. \square

Having Lemma 3.2 in our possession, we are left to notice that the degree 0 components of the lift are a_i , which are homotopy equivalences in \mathcal{A} as a was a homotopy equivalence in $M_n F(\mathcal{A})$. Thus, we have proved the following theorem.

Theorem 3.10. *For all n , matching maps $F_n(\mathcal{A}) \rightarrow M_n F(\mathcal{A})$ are Dwyer-Kan fibrations. Consequently, $F_\bullet(\mathcal{A})$ is Reedy fibrant.*

Remark 3.11. In [Hol], the Dwyer-Kan fibrancy of the matching maps was proved for the case when \mathcal{A} is pretriangulated, by a strategy involving contraction of the cones. This strategy can, in fact, be performed in the case of arbitrary \mathcal{A} , which we demonstrate in Appendix A.

Remark 3.12. In the framework of ∞ -local systems, the meaning of Reedy fibrancy is the following: if a is a homotopy equivalence between two ∞ -local systems on the simplex boundary, one of which was restricted from the simplex, then this homotopy equivalence can be lifted to a homotopy equivalence between two ∞ -local systems on the simplex.

Appendix A. An alternative proof of Reedy fibrancy

We now present a proof of Theorem 3.10 that does not rely on Lemma 3.2.

A.1. Contraction of cones and pretriangulated envelopes

We have to verify that lift of Proposition 3.9 is a homotopy equivalence in $F_n(\mathcal{A})$. While an explicit computation might be possible, it appears to be very cumbersome even in the case $n = 1$ (see [Sho, Lemma 3.5]). There exists, however, a strategy involving contractions of cones (see [Tab2]).

Definition A.1. For an object X in some DG-category \mathcal{A} , its contraction is $b_X \in \mathcal{A}^{-1}(X, X)$ with $d(b_X) = 1_X$.

Lemma 3.6 precisely states that for any homotopy equivalence \mathcal{A} , you can construct a contraction of its cone in $\text{Mod } \mathcal{A}$. However, one does not have to go as far as the whole category of DG-modules. Following [Dri], recall the construction of the pretriangulated envelope.

Definition A.2. For a DG-category \mathcal{A} , its pretriangulated envelope $\text{Pretr}(\mathcal{A})$ has one-sided twisted complexes as objects – these are formal expressions $(\bigoplus_{i=1}^n C_i[r_i], q)$, where C_i are objects of \mathcal{A} , r_i are integers and q is a set of morphisms $q_{ij} \in (\mathcal{A}(C_j, C_i)[r_i - r_j])^1$ subject to $q_{ij} = 0$ for $i \geq j$ and $dq + q \circ q = 0$. The morphisms are given by

$$\text{Pretr}(\mathcal{A})\left(\left(\bigoplus_{i=1}^n C_i[r_i], q\right), \left(\bigoplus_{j=1}^m C'_j[r'_j], q'\right)\right) = \bigoplus_{i,j} \mathcal{A}(C_j, C'_i)[r'_i - r_j].$$

That is, a degree k morphism $f: (\bigoplus_{i=1}^n C_i[r_i], q) \rightarrow (\bigoplus_{i=1}^m C'_i[r'_i], q')$ is a set of components $f_{ij} \in (\mathcal{A}(C_j, C'_i)[r'_i - r_j])^k$, with matrix multiplication for composition and with differential given by

$$d_{TC}(f) = d(f) + q' \circ f - (-1)^k f \circ q.$$

There are natural fully faithful embeddings $\mathcal{A} \hookrightarrow \text{Pretr}(\mathcal{A}) \hookrightarrow \text{Mod } \mathcal{A}$. For any $f \in Z^0(\mathcal{A}(X, Y))$, its cone is an object of $\text{Pretr}(\mathcal{A})$ defined as $\text{Cone}(f) := (Y \oplus X[1], q)$ with $q_{12} = f$ (this is compatible with the embedding $\text{Pretr}(\mathcal{A}) \hookrightarrow \text{Mod } \mathcal{A}$). We say that \mathcal{A} already has all the cones if $\text{Cone}(f)$ is always isomorphic to some object in the image of the embedding $\mathcal{A} \hookrightarrow \text{Pretr}(\mathcal{A})$. It can be checked that $\text{Pretr}(\mathcal{A})$ has all the cones.

Note that for DG-categories that have all the cones, we can now prove the following lemma.

Lemma A.3. *If \mathcal{A} has all the cones, then the matching maps $m_n: F_n(\mathcal{A}) \rightarrow M_n F(\mathcal{A})$ are fibrations.*

Proof. We are left to check that if \tilde{a} is a closed lift of a homotopy equivalence $a: (X, f_{\leq n}) \rightarrow (Y, g)$, then \tilde{a} is a homotopy equivalence in $F_n(\mathcal{A})$. We notice that if \mathcal{A} has all the cones, then $F_n(\mathcal{A})$ and $M_nF(\mathcal{A})$ also have all the cones. So $\text{Cone}(a)$ is an object of $M_nF(\mathcal{A})$ which (by Lemma 3.6) has a contraction b . Note that for any functor, the induced functor on pretriangulated envelopes respects cones, so $m_n(\text{Cone}(\tilde{a})) = \text{Cone}(a)$. Lifting b to a contraction of $\text{Cone}(\tilde{a})$ will then show that \tilde{a} is a homotopy equivalence. And indeed, any contraction can be lifted along m_n . Let b be a contraction of $(X, f_{\leq n})$. Then the lift, as shown in [Hol], is obtained by setting

$$b_{0, \dots, n} = b_0(\Delta b + b \circ f + f \circ b). \quad \square$$

We now show how the assumption of \mathcal{A} having all the cones can be omitted. In [Tab2], this was done for the case $n = 1$ via a quasiequivalence $\text{Pretr}(F_1(\mathcal{A})) \simeq F_1(\text{Pretr}(\mathcal{A}))$.

A.2. Proof of Theorem 3.10

Consider the following commutative square, where the horizontal arrows are fully faithful embeddings given by compositions of F_n (respectively M_nF) with natural embeddings $\mathcal{A} \hookrightarrow \text{Pretr}(\mathcal{A})$:

$$\begin{CD} F_n(\mathcal{A}) @<<< F_n(\text{Pretr}(\mathcal{A})) \\ @VVV @VVV \\ M_nF(\mathcal{A}) @<<< M_nF(\text{Pretr}(\mathcal{A})) \end{CD}$$

For a homotopy equivalence $a: (X, f_{\leq n}) \rightarrow (Y, g)$ in $M_nF(\mathcal{A})$, we have constructed in Proposition 3.9 its closed lift along the left vertical arrow. Under embeddings, this is also a legitimate lift along the right vertical arrow. As the category $\text{Pretr}(\mathcal{A})$ has all the cones, we know from Lemma A.3 that any closed lift of a homotopy equivalence is a homotopy equivalence. So we are left to observe that embeddings respect homotopy equivalences, and that if a morphism is a homotopy equivalence in the larger category then it is also a homotopy equivalence in the smaller category. This concludes the proof.

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