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# AN ALGEBRAIC REPRESENTATION OF GLOBULAR SETS

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(*communicated by Ronald Brown*)

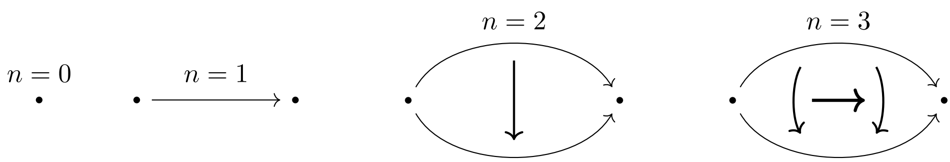
*Abstract*

We describe a fully faithful embedding of the category of (reflexive) globular sets into the category of counital cosymmetric $R$-coalgebras when $R$ is an integral domain. This embedding is a lift of the usual functor of $R$-chains and the extra structure consists of a derived form of cup coproduct. Additionally, we construct a functor from group-like counital cosymmetric $R$-coalgebras to $\omega$-categories and use it to connect two fundamental constructions associated to oriented simplices: Steenrod's cup-$i$ coproducts and Street's orientals. The first defines the square operations in the cohomology of spaces, the second, the nerve of higher-dimensional categories.

## 1. Introduction

Globular sets are presheaves over a category $\mathbb{G}$ whose objects are non-negative integers. They generalize directed graphs and constitute one of the major geometric shapes for higher category theory, providing models for strict and non-strict higher-dimensional categories when enriched with further structure.

We depict the representable globular set $\mathbb{G}_n$ for small values of $n$:




The globular set $\partial\,\mathbb{G}_{n+1}$ obtained by removing the identity from $\mathbb{G}_{n+1}$ models the $n$-sphere together with its antipodal map. We are interested in the functor $\mathrm{C}_\bullet$ of chains from globular sets to differential graded $R$-modules. Let $W$ be defined as the colimit of the diagram

$$\mathrm{C}_\bullet(\partial\,\mathbb{G}_0) \longrightarrow \mathrm{C}_\bullet(\partial\,\mathbb{G}_1) \longrightarrow \cdots$$

induced from a standard set of inclusions $\mathbb{G}_n \to \mathbb{G}_{n+1}$. We notice that the antipodal map makes $W$ into a free differential graded $R[\Sigma_2]$-module. For any globular set $X$

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we will construct a natural $R[\Sigma_2]$-module chain map

$$\Delta\colon W \otimes \mathrm{C}_\bullet(X) \longrightarrow \mathrm{C}_\bullet(X) \otimes \mathrm{C}_\bullet(X)$$

together with a natural chain map $\varepsilon : \mathrm{C}_\bullet(X) \to R$ satisfying appropriate counitality relations. We can think of this structure as a lift to the chain level of the counital cocommutative $R$-coalgebra on the homology of $X$ (a structure pre-dual to the usual cup product in cohomology).

We will show that when $R$ is an integral domain, this lift of the functor of chains is a fully faithful embedding of the category of globular sets into the category of counital cosymmetric $R$-coalgebras. We can think of this result as a non-linear globular form of the Dold-Kan Theorem. In more diagramatic language, our map fits into the following commutative diagram

$$\begin{array}{ccc} \mathrm{Set}^{\mathbb{G}^{\mathrm{op}}} & \dashrightarrow & \mathrm{coAlg}_{\mathrm{R}} \\ \downarrow & \searrow^{\mathrm{C}_\bullet} & \downarrow \\ \mathcal{M}\mathrm{od}_R^{\mathbb{G}^{\mathrm{op}}} & \longrightarrow & \mathrm{Ch}_R, \end{array}$$

where the lower triangle consists of a free functor followed by a fully faithful embedding and the upper triangle consists of a fully faithful embedding followed by a forgetful functor.

We will then focus on the full subcategory $\mathrm{coAlg}_R^{gl}$ of group-like counital cosymmetric $R$-coalgebras and on a model for strict higher-dimensional categories known as $\omega$-categories. We will describe a functor, similar to those used by Street, Brown, and Steiner in their respective studies of parity complexes, linear $\omega$-categories, and augmented directed complexes, from $\mathrm{coAlg}_R^{gl}$ to $\omega\mathrm{Cat}$ behaving like a free functor on pasting diagrams. We will use our version to relate two fundamental constructions on oriented simplices: Steenrod's cup-$i$ coproducts and Street's orientals. The first defines the square operations on the cohomology of spaces, the second, the nerve of $\omega$-categories.

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## 2. Globular sets and counital cosymmetric $R$-coalgebras

In this section we will describe how to represent, when $R$ is an integral domain, the category of globular sets algebraically as a full subcategory of the category of counital cosymmetric $R$-coalgebras. These are models for counital $R$-coalgebras commutative up to coherent homotopies ($E_\infty$-coalgebras are examples). We will also review an important construction of Steenrod providing concrete examples of such $R$-coalgebra when $R = \mathbb{F}_2$ and used to define his square operations.

## 2.1. Globular sets

The **globe category** $\mathbb{G}$ has set of objects the non-negative integers and its morphisms are generated by

$$\sigma_n, \tau_n \colon n \to n+1, \qquad\qquad \iota_n \colon n \to n-1$$

subject to the relations

$$\begin{aligned} \tau_n\,\tau_{n-1} &= \sigma_n\,\tau_{n-1}, & \qquad \sigma_n\,\sigma_{n-1} &= \tau_n\,\sigma_{n-1}, \\ \iota_{n+1}\,\tau_n &= \mathrm{id}_n, & \qquad \iota_{n+1}\,\sigma_n &= \mathrm{id}_n. \end{aligned} \tag{1}$$

Let Set be the category of small sets. We denote the category of contravariant functors from $\mathbb{G}$ to Set by $\mathrm{Set}^{\mathbb{G}^{\mathrm{op}}}$ and refer to it as the category of **globular sets**. For a globular set $X$ we use the notation

$$X_n = X(n), \qquad t_n = X(\tau_n), \qquad s_n = X(\sigma_n), \qquad i_n = X(\iota_n).$$

Furthermore, abusing notation, we let $t_n \colon X(k) \to X(n)$ stand for any composition of the form $t_n r$ where $r \colon X(k) \to X(n+1)$ is induced from an arbitrary morphism. Thanks to (1) this map is independent of $r$ and determined by the integer $k$. We follow a similar convention for $s_n$.

## 2.2. Augmented differential graded $R$-modules

Let $R$ be a commutative and unital ring. The category of differential (homologically) graded $R$-modules concentrated in non-negative degrees is denoted $\mathrm{Ch}_R$. We reserve the word chain complex for when $R$ equals $\mathbb{Z}$.

Let $C$ be a differential graded $R$-module and $n$ a non-negative integer; we denote

$$C_{\leqslant n} = C_0 \oplus C_1 \oplus \cdots \oplus C_n.$$

A pair $(C, \varepsilon)$ with $C$ and $\varepsilon \colon C \to R$ in $\mathrm{Ch}_R$ is called an **augmented differential graded $R$-module** and a morphism between two of them is a morphism of underlying differential graded $R$-modules making the diagram

$$\begin{array}{ccc} C' & \longrightarrow & C \\ & {\scriptstyle \varepsilon'}\searrow \quad \swarrow{\scriptstyle \varepsilon} & \\ & R & \end{array}$$

commutative.

The functor $\mathrm{C}_\bullet \colon \mathrm{Set}^{\mathbb{G}^{\mathrm{op}}} \to \mathrm{Ch}_R$ is defined for $X \in \mathrm{Set}^{\mathbb{G}^{\mathrm{op}}}$ by

$$\mathrm{C}_n(X) = R\{X_n\} \,/\, R\{i_n(X_{n-1})\}, \qquad \partial_n = t_{n-1} - s_{n-1}\,.$$

It admits a natural lift to the category of augmented differential graded $R$-modules by defining for $x \in X_n$

$$\varepsilon(x) = \begin{cases} 1 & n = 0, \\ 0 & n \neq 0. \end{cases}$$

## 2.3. Counital cosymmetric $R$-coalgebras

Let $\Sigma_2$ be the group with one non-identity element $T$. Let us consider the following resolution of $R$ by free $R[\Sigma_2]$-modules:

$$W = \qquad R[\Sigma_2] \xleftarrow[1-T]{} R[\Sigma_2] \xleftarrow[1+T]{} R[\Sigma_2] \xleftarrow[1-T]{} \cdots$$

and let $\varepsilon_W : W \to R$ be the unique $R[\Sigma_2]$-linear map extending the identity $R \to R$.

Given any differential graded $R$-module $C$ we make $C \otimes C$ into a differential graded $R[\Sigma_2]$-module using the transposition of factors $T(x \otimes y) = (-1)^{rs} y \otimes x$ where $r$ and $s$ are the degrees of $x$ and $y$.

A **counital cosymmetric** $R$**-coalgebra** is an augmented differential graded $R$-module $(C, \varepsilon)$ together with

$$\Delta \colon W \otimes C \to C \otimes C$$

an $R[\Sigma_2]$-linear chain map making the following diagrams commute:

$$\begin{array}{ccc} W \otimes C & \xrightarrow{\Delta} & C \otimes C \\ & \searrow^{\varepsilon_W \otimes \mathrm{id}} & \downarrow{\scriptstyle 1 \otimes \varepsilon} \\ & & C, \end{array} \qquad \begin{array}{ccc} W \otimes C & \xrightarrow{\Delta} & C \otimes C \\ & \searrow^{\varepsilon_W \otimes \mathrm{id}} & \downarrow{\scriptstyle \varepsilon \otimes 1} \\ & & C\ . \end{array}$$

A **coalgebra map** between counital cosymmetric $R$-coalgebras is a map $f$ of underlying augmented differential graded $R$-modules making the following diagram commute:

$$\begin{array}{ccc} W \otimes C' & \xrightarrow{\mathrm{id} \otimes f} & W \otimes C \\ {\scriptstyle \Delta'}\downarrow & & \downarrow{\scriptstyle \Delta} \\ C' \otimes C' & \xrightarrow{f \otimes f} & C \otimes C\, . \end{array}$$

We denote the category of counital cosymmetric $R$-coalgebras with coalgebra maps by $\mathrm{coAlg}_R$.

We use the adjunction isomorphism

$$\mathrm{Hom}_{R[\Sigma_2]}(W \otimes C, C \otimes C) \longrightarrow \mathrm{Hom}_{R[\Sigma_2]}\big(W, \mathrm{Hom}(C, C \otimes C)\big)$$

to represent $\Delta$ by a collection of maps $\Delta_k \colon C \to C \otimes C$ satisfying

$$\partial \Delta_k - (-1)^k \Delta_k \partial = \big(1 + (-1)^k T\big) \Delta_{k-1} \tag{2}$$

with the convention that $\Delta_{-1} = 0$.

## 2.4. Steenrod cup-$i$ coalgebras

Alexander-Whitney's approximation to the diagonal map

$$\Delta_0 \colon \mathrm{C}_\bullet \longrightarrow \mathrm{C}_\bullet \otimes \mathrm{C}_\bullet$$

defines a natural non-commutative coproduct on the integral chains of any simplicial set whose linear dual descends to the commutative cup product on its cohomology.

In [Ste47], Steenrod constructed a cosymmetric $\mathbb{Z}$-coalgebra

$$\Delta\colon W \otimes \mathrm{C}_\bullet \longrightarrow \mathrm{C}_\bullet \otimes \mathrm{C}_\bullet$$

extending the Alexander-Whitney coproduct, which when considered with $\mathbb{F}_2$-coefficients defines the square operations

$$Sq^k\colon H^\bullet(-;\mathbb{F}_2) \longrightarrow H^{\bullet+k}(-;\mathbb{F}_2).$$

Since these operations are homological in nature, any pair of natural homotopy equivalent cosymmetric $\mathbb{F}_2$-coalgebra structures give rise to isomorphic square operations. Yet, Steenrod's original construction appears ubiquitously in the literature in various equivalent forms. For example, in [MM18c], the author finds it in the action of a finitely presented prop arising from just three maps: Alexander-Whitney's diagonal, the augmentation, and the join map. In [MM18b], it is induced from the action of a cellular $E_\infty$-operad on the geometric realization of cubical sets. And in [MS03] and [BF04], McClure-Smith and Berger-Fresse find it in the action of their respective Sequence and Barratt-Eccles operads.

The universality of this cosymmetric $\mathbb{F}_2$-coalgebra is formalized via an axiomatic characterization in [MM18a]. In this note, we provide further evidence for its fundamental nature by deriving from it in Theorem 3.12 another fundamental construction: the nerve of higher-dimensional categories.

Let us review its description as presented in [MM18a]. Let $P\binom{n}{k}$ be the set of all $U = \{0 \leqslant u_1 < \cdots < u_k \leqslant n\}$. For any such $U$ define the composition of face maps

$$d_U = d_{u_1} \cdots d_{u_k}$$

and the pair

$$U^- = \{u_i \in U \,:\, u_i \not\equiv i \bmod 2\},$$
$$U^+ = \{u_i \in U \,:\, u_i \equiv i \bmod 2\}.$$

**Definition 2.1** ([MM18a])**.** For any simplicial set $X$ its **Steenrod cup-$i$ coalgebra** $\big(\mathrm{C}_\bullet(X;\mathbb{F}_2), \Delta, \varepsilon\big)$ is defined by

$$\Delta_i(x) \;=\; \sum_{U \in P\binom{n}{n-i}} d_{U^-}\, x \otimes d_{U^+}\, x \tag{3}$$

and

$$\varepsilon(x) = \begin{cases} 1 & n = 0, \\ 0 & n \neq 0, \end{cases}$$

where $x \in X_n$.

*Remark 2.2.* The cup product and Steenrod squares in cohomology are obtained from the Steenrod cup-$i$ coalgebra by defining

$$[\alpha] \smile [\beta] = \big[(\alpha \otimes \beta)\Delta_0\big]$$

and

$$Sq^k[\alpha] = \big[(\alpha \otimes \alpha)\Delta_{|\alpha|-k}\big].$$

These two cohomological structures are related via the Cartan Formula. In [MM19], using the definitions above, the author gave an effective chain level proof of the Cartan

Formula. In [**MM18d**], based on (2.1) and the definitions above, a novel algorithm for the computation of Steenrod squares of finite simplicial complexes was developed and added to the toolkit of topological data analysis.

### 2.5. Globular $R$-coalgebras

We now describe a counital cosymmetric $R$-coalgebra naturally associated to a globular set and state our main theorem.

**Definition 2.3.** For any globular set $X$ its **globular $R$-coalgebra** $\big(\mathrm{C}_\bullet(X;R), \Delta, \varepsilon\big)$ is defined by

$$\Delta_k(x) = \begin{cases} 0 & n < k, \\ x \otimes x & n = k, \\ t_k x \otimes x \, + (-1)^{(n+1)k} \, x \otimes s_k x & k < n \end{cases}$$

and

$$\varepsilon(x) = \begin{cases} 1 & n = 0, \\ 0 & n > 0, \end{cases}$$

where $x \in X_n$.

**Theorem 2.4.** *Let $R$ be an integral domain. The assignment*

$$X \longrightarrow (\mathrm{C}_\bullet(X;R), \Delta, \varepsilon)$$

*induces a full and faithful embedding of* $\mathrm{Set}^{\mathbb{G}^{\mathrm{op}}}$ *into* $\mathrm{coAlg}_R$.

The proof of this theorem occupies Section 4.

*Remark 2.5.* We can think of this statement as a non-linear globular form of the Dold-Kan Theorem. A conjecture, verified in the author's thesis [**Med15**] for special cases, is that including the higher arity parts of an $E_\infty$-coalgebra structure on the chains of simplicial sets results in a similar non-linear (simplicial) Dold-Kan Theorem.

## 3. Group-like coalgebras and higher-dimensional categories

### 3.1. $\omega$-categories and the functor $\mu$

In this subsection we recall the definition of $\omega$-categories, which are a globular model of strict higher-dimensional categories. We also review a natural construction associating an $\omega$-category to any differential graded $R$-module.

**Definition 3.1.** An **$\omega$-category** is a globular set $X$ together with maps

$$\circ_m \colon X_n \times_{X_m} X_n \longrightarrow X_m,$$

where

$$X_n \times_{X_m} X_n = \big\{(y,x) \in X \times X \mid s(y) = t(x)\big\}$$

satisfying relations of associativity, unitality and interchange. For the complete list of relations we refer the reader to Definition 1.4.8 in [**Lei04**]. When $t_m(x) = s_m(y) = z$ we write $y \circ_z x$ for $y \circ_m x$.

The next definition appears in [**Ste04**] where it is credited to [**BH03**] and [**Str91**].

**Definition 3.2** (Street, Brown-Higgins, Steiner)**.** The functor

$$\mu\colon \mathrm{Ch}_R \longrightarrow \omega\mathrm{Cat}$$

is defined as follows: for $C$ a differential graded $R$-module let $\mu(C)$ be the $R$-submodule of the infinite product of $C$ with itself generated by all sequences

$$c = (c_0^-, c_0^+, c_1^-, c_1^+, \dots)$$

satisfying

i) $c_n^-, c_n^+ \in C_n$,

ii) $c_n^-, c_n^+ = 0$ for $n >> 0$,

iii) $\partial c_{n+1}^- = \partial c_{n+1}^+ = c_n^+ - c_n^-$.

We can make this $R$-module into a globular set by defining

$$\mu(C)_n = \{c \in \mu(C) \ : \ \forall k > n,\ c_k^- = c_k^+ = 0\}$$

and

$$\begin{aligned} s_k(c) &= (c_0^-, c_0^+, \dots, c_{k-1}^-, c_{k-1}^+, c_k^-, c_k^-, 0, 0, \dots),\\ t_k(c) &= (c_0^-, c_0^+, \dots, c_{k-1}^-, c_{k-1}^+, c_k^+, c_k^+, 0, 0, \dots),\\ i_k(c) &= c. \end{aligned}$$

We can make this globular set into an $\omega$-category by defining

$$\begin{aligned} b \ \circ_c \ a &= b + a - c \\ &= \big(b_0^- + a_0^- - c_0^-,\ b_0^+ + a_0^+ - c_0^+, \dots\big). \end{aligned}$$

## 3.2. Group-like coalgebras and the functor $\xi$

In this subsection we define group-like elements in counital cosymmetric $R$-coalgebras and consider $\mathrm{coAlg}_R^{gl}$, the full subcategory of counital cosymmetric $R$-coalgebras admitting a basis of group-like elements. We then introduce a functor from $\mathrm{coAlg}_R^{gl}$ to $\omega\mathrm{Cat}$ using the notion of atom associated to a group-like element.

**Definition 3.3.** Let $\big(C, \Delta, \varepsilon\big)$ be a counital cosymmetric coalgebra. We call $c \in C_n$ a **group-like element** if for any integer $k$ we have

$$\begin{aligned} &\Delta_k(c) \in C_{\leqslant n} \otimes C_{\leqslant n},\\ &\Delta_n(c) = c \otimes c \end{aligned}$$

and, when $n = 0$,

$$\varepsilon(c) = 1.$$

We say that $C$ is **group-like** if it admits a basis of group-like elements and denote the full subcategory of group-like counital cosymmetric $R$-coalgebras as $\mathrm{coAlg}_R^{gl}$.

A consequence of the following lemma applied to the identity map is that if a counital cosymmetric coalgebra admits a basis of group-like elements, then that basis is unique.

**Lemma 3.4.** *Let $R$ be an integral domain. If $f\colon R[A] \to R[B]$ is a coalgebra map between counital cosymmetric $R$-coalgebras with bases of group-like elements $A$ and $B$. Then, for any $a \in A$ either $f(a) = 0$ or there exists $b \in B$ such that $f(a) = b$.*

*Proof.* For $a \in A_n$ there is a collection of elements $b_i \in B_n$ and coefficients $\beta_i \in R$ such that

$$f(a) = \sum_i \beta_i \, b_i. \tag{4}$$

Applying $\Delta_n$ to (4) gives

$$\sum_i \beta_i \, b_i \otimes b_i = \Delta_n f(a) = (f \otimes f)\Delta_n(a) = \sum_{i,\,j} \beta_i \, \beta_j \; b_i \otimes b_j.$$

The equations $0 = \beta_i \, \beta_j$ for $i \neq j$ together with $\beta_i = \beta_i^2$ imply, since $R$ is an integral domain, that each coefficient $\beta_i$ equals 0 except possibly one of them that must equal 1. □

*Example 3.5.* Steenrod cup-$i$ coalgebras as well as globular $R$-coalgebras are group-like.

**Definition 3.6.** Let $C$ be a differential graded $R$-module with a basis $B$. For $b \in B$ let $\pi_b\colon C \to R$ be the $R$-linear map sending $b$ to 1 and the other basis elements to 0. Define the maps

$$\pi_b^+, \, \pi_b^- : C \otimes C \longrightarrow C$$

by

$$\pi_b^+ = \mathrm{id} \otimes \pi_b \qquad \text{and} \qquad \pi_b^- = \pi_b^+ T,$$

where $T$ is the transposition of factors.

We make a note of the following straightforward observation for later use:

**Lemma 3.7.** *Let $C$ be a differential graded $R$-module with a basis $B$. If $b \in B_n$ and $\eta \in \{+, -\}$ then*

$$\partial \pi_b^\eta = \pi_b^\eta \partial$$

*on $C_{\leqslant n} \otimes C_{\leqslant n}$ and*

$$\pi_b^\eta = 0$$

*on $C_{\leqslant n-1} \otimes C_{\leqslant n-1}$.*

**Definition 3.8.** Let $(C, \Delta, \varepsilon)$ be a counital cosymmetric coalgebra. For every group-like element $b \in C$ define its **atom** as

$$\langle b \rangle = \big( \langle b \rangle_0^-, \langle b \rangle_0^+, \langle b \rangle_1^-, \langle b \rangle_1^+, \dots \big)$$

with

$$\langle b \rangle_k^\eta = \begin{cases} (-1)^k \, \pi_b^- \Delta_k b & \eta = -, \\ \pi_b^+ \Delta_k b & \eta = +. \end{cases}$$

**Lemma 3.9.** *Let $C$ be a counital cosymmetric coalgebra. For any group-like element $b \in C$ the atom $\langle b \rangle$ is in $\mu(C)$.*

*Proof.* We need to prove that for any group-like element $b$ of degree $n$ the sequence $\langle b\rangle$ satisfies conditions i), ii), and iii) in Definition 3.2. Since the first two conditions are immediate, we are left with showing that for any non-negative integer $k$

$$\partial\langle b\rangle^{+}_{k+1} = \partial\langle b\rangle^{-}_{k+1} = \langle b\rangle^{+}_{k} - \langle b\rangle^{-}_{k.} \tag{5}$$

For $k > n$ and $\eta \in \{+,-\}$ we have $\langle b\rangle^{\eta}_{k} = 0$ so (5) holds. For $k = n$ we notice that $\langle b\rangle^{+}_{k} = \langle b\rangle^{-}_{k} = b$ and (5) follows. For $k < n$, using Lemma 3.7, we have

$$\begin{aligned}\langle b\rangle^{+}_{k} - \langle b\rangle^{-}_{k} &= \pi^{+}_{b}\Delta_k b - (-1)^k\pi^{-}_{b}\Delta_k b = \pi^{+}_{b}\big(1 + (-1)^{k+1}T\big)\Delta_k b \\ &= \pi^{+}_{b}\big(\partial\Delta_{k+1}b - (-1)^k\Delta_{k+1}\partial b\big) = \partial\pi^{+}_{b}\Delta_{k+1}b \\ &= \partial\langle b\rangle^{+}_{k+1}\end{aligned}$$

and

$$\begin{aligned}\langle b\rangle^{+}_{k} - \langle b\rangle^{-}_{k} &= \pi^{+}_{b}\Delta_k b - (-1)^k\pi^{-}_{b}\Delta_k b = \pi^{-}_{b}\big(T + (-1)^{k+1}\big)\Delta_k b \\ &= (-1)^{k+1}\pi^{-}_{b}\big(\partial\Delta_{k+1}b - (-1)^k\Delta_{k+1}\partial b\big) \\ &= (-1)^{k+1}\partial\pi^{-}_{b}(\Delta_{k+1}b) = \partial\langle b\rangle^{-}_{k+1}\end{aligned}$$

as desired. □

**Lemma 3.10.** *Let $R$ be an integral domain. The assignment sending a group-like counital cosymmetric $R$-coalgebra $C$ to the sub-$\omega$-category of $\mu(C)$ generated by its atoms is functorial.*

*Proof.* The statement follows from the fact, proven in Lemma 3.4, that when $R$ is an integral domain a coalgebra map between group-like $R$-coalgebras sends group-like elements to either group-like elements or to 0. □

**Definition 3.11.** Let

$$\xi\colon \mathrm{coAlg}^{gl}_{R} \longrightarrow \omega\mathrm{Cat}$$

be the functor described in Lemma 3.10.

### 3.3. Street's orientals

In this subsection we state the second main result of this note: the functor $\xi$ sends the Steenrod coalgebra of a standard simplex to the free $\omega$-category generated by that simplex.

Historically, Roberts [**Rob77**] pioneered the idea of using higher-dimensional categories as the coefficient objects for non-abelian cohomology. A key ingredient for this enterprise is the construction of a nerve functor from $\omega$-categories to simplicial sets. Such a functor $N$ can be obtained from the construction of a natural cosimplicial $\omega$-category

$$\begin{aligned}\mathcal{O}\colon \mathbf{\Delta} &\longrightarrow \omega\mathrm{Cat}, \\ [n] &\mapsto \mathcal{O}_n\end{aligned}$$

by setting

$$N(\mathcal{C})_n = \mathrm{Hom}_{\omega\mathrm{Cat}}\big(\mathcal{O}_n, \mathcal{C}\big).$$

This was accomplished by Street in [**Str87**] where he says the following about the

$\omega$-categories $\mathcal{O}_n$ : "[t]hese objects seem to be fundamental structures of nature so I decided they should have a short descriptive name. I settled on oriental."

We will not use the original definition of Street but an equivalent one given by Steiner in [**Ste04**] and further explored in [**Ste07**]. It is presented as Definition 3.15 after a review of Steiner's theory of augmented directed complexes.

We are ready to state the second main result of this work.

**Theorem 3.12.** *Let $\big(\mathrm{C}_\bullet(\mathbf{\Delta}^n;\mathbb{F}_2),\Delta,\varepsilon\big)$ be the Steenrod cup-$i$ coalgebra associated to the $n$-th representable simplicial set $\mathbf{\Delta}^n$. Then,*

$$\xi\big(\mathrm{C}_\bullet(\mathbf{\Delta}^n;\mathbb{F}_2),\Delta,\varepsilon\big)=\mathcal{O}_n.$$

The proof of this theorem occupies Subsection 3.5.

### 3.4. Steiner's augmented directed complexes

In this subsection we give an extremely abridged exposition of Steiner's rich theory of augmented directed complexes with the aim of proving Theorem 3.12. The original source is [**Ste04**].

We refer to the objects of $\mathrm{Ch}_\mathbb{Z}$ simply as chain complexes.

Let $C$ be a chain complex together with a basis. We write $C^+$ for the submonoid containing all elements written as linear combinations of basis elements with only non-negative coefficients. We use the following notation for the induced canonical decomposition:

$$c=c^+-c^-$$

with $c^+$ and $c^-$ in $C^+$.

Let $(C,\varepsilon)$ be an augmented chain complex with a basis $B$. For $b\in B_n$ define recursively

$$b_i^+=\begin{cases}0 & i>n,\\ b & i=n,\\ (\partial b_{i+1}^+)^+ & i<n\end{cases}\qquad\text{and}\qquad b_i^-=\begin{cases}0 & i>n,\\ b & i=n,\\ (\partial b_{i+1}^-)^- & i<n.\end{cases}$$

The basis is said to be **unital** if $\varepsilon(b_0^+)=\varepsilon(b_0^-)=1$ for every $b\in B$.

**Definition 3.13.** A **strong augmented directed complex** or simply a **SADC** is an augmented chain complex $C$ with a unital basis such that the transitive closure of the reflexive relation $\leqslant$ defined by

$$c_1\leqslant c_2$$

if and only if

$$(\partial c_2)^- - c_1\in C^+$$

or

$$(\partial c_1)^+ - c_2\in C^+$$

is anti-symmetric, i.e., it defines a partial order on $C$.

A morphism between two SADCs is an augmented chain map $f\colon C_1\to C_2$ such that

$$f(C_1^+)\subset f(C_2^+).$$

**Definition 3.14.** Let $(C, B)$ be a SADC. For $b \in B$ define its **Steiner atom** to be

$$\big(b_0^-, b_0^+, b_1^-, b_1^+, \dots\big) \in \mu(C).$$

Steiner showed that assigning to a SADC, let us call it $(C, B)$, the sub-$\omega$-category generated inside $\mu(C)$ by its Steiner atoms defines a full and faithful embedding

$$\nu\colon \mathrm{SADC} \longrightarrow \omega\mathrm{Cat}.$$

We refer the reader to Sections 5.6, 6.1, and 6.2 in [Ste04] for these statements.

Additionally, Steiner gives the following definition of Street's orientals in Section 3.8 loc. cit.:

**Definition 3.15** ([Ste04])**.** Let $\boldsymbol{\Delta}^n$ denote the $n$-th representable simplicial set. The chain complex $\mathrm{C}_\bullet(\boldsymbol{\Delta}^n;\mathbb{Z})$ together with the canonical basis

$$B = \big\{[m] \to [n] \ :\ \text{injective}\big\}$$

define a SADC and

$$\mathcal{O}_n = \nu\big(\mathrm{C}_\bullet(\boldsymbol{\Delta}^n;\mathbb{Z}),\ B\big).$$

### 3.5. Proof of Theorem 3.12

We will exhibit a bijection between the set of atoms of $\xi\big(\mathrm{C}_\bullet(\boldsymbol{\Delta}^n;\mathbb{F}_2), \Delta, \varepsilon\big)$ and of Steiner atoms of $\nu\big(\mathrm{C}_\bullet(\boldsymbol{\Delta}^n;\mathbb{Z}), B\big)$ which, since these are generators, will establish the theorem.

We will verify that for every non-degenerate simplex $\sigma\colon [m] \to [n]$ and $\eta \in \{-,+\}$ we have

$$\sigma_i^\eta = \pi_\sigma^\eta \Delta_i \sigma, \tag{6}$$

where this equality holds with $\mathbb{Z}$-coefficients using the canonical set lift $\mathbb{F}_2 \to \mathbb{Z}$ with $0 \mapsto 0$ and $1 \mapsto 1$.

For $i > m$, both sides of (6) are equal to 0.

For $i \leqslant m$, let $r = m - i$. Then, by (3), we have

$$\pi_\sigma^- \Delta_i \sigma = \sum_{\substack{U \in P\binom{m}{r} \\ U^- = \emptyset}} d_{U^+}\sigma \qquad \text{and} \qquad \pi_\sigma^+ \Delta_i \sigma = \sum_{\substack{U \in P\binom{m}{r} \\ U^+ = \emptyset}} d_{U^-}\sigma.$$

Then, in the case $r = 0$, (6) holds because $\pi_\sigma^- \Delta_i \sigma = \sigma$ and $\pi_\sigma^+ \Delta_i \sigma = \sigma$, so $\pi_\sigma^- \Delta_i \sigma = \sigma_i^-$ and $\pi_\sigma^+ \Delta_i \sigma = \sigma_i^+$. Assuming the identity for $r$ we compute

$$\partial \sigma_i^- = \sum_j (-1)^j d_j \sigma_i^- = \sum_j (-1)^j \sum_{\substack{U \in P\binom{m}{r} \\ U^- = \emptyset}} d_j d_{U^+}\sigma.$$

We will prove the identity for $r+1$ by rewriting the above identity as

$$\partial \sigma_i^- = \sum_{\substack{U \in P\binom{m}{r+1} \\ U^- = \emptyset}} d_{U^+}\sigma \quad - \sum_{\substack{U \in P\binom{m}{r+1} \\ U^+ = \emptyset}} d_{U^-}\sigma.$$

For $U = \{u_1 < \cdots < u_r\} \in P\binom{n}{r}$ with $U^- = \emptyset$ and $0 \leqslant j \leqslant i$ we can use the simplicial identities to write

$$d_j d_{U^+} = d_j d_{u_1} \dots d_{u_r} = d_{u_1} \dots d_{u_l}\, d_{j+l}\, d_{u_l+1} \dots d_{u_r}$$

with $u_l < j+l < u_{l+1}$. Notice that if $j \equiv 1 \bmod 2$ and $l < r$ then

$$V = \{u_1 < \cdots < u_l < j+l < \widehat{u}_{l+1} < \cdots < u_r\} \in P\binom{m}{r}$$

with $V^- = \emptyset$ and, calling $k = u_{l+1} - l - 1$,

$$(-1)^j d_j d_U + (-1)^k d_k d_V = 0.$$

If $j \equiv 0 \bmod 2$ and $1 < l$ then

$$W = \{u_1 < \cdots < \widehat{u}_l < j+l < u_{l+1} < \cdots < u_r\} \in P\binom{m}{r}$$

with $W^- = \emptyset$ and, calling $k = u_l - l$,

$$(-1)^j d_j d_U + (-1)^k d_k d_W = 0.$$

This implies that the only non-zero terms are of the form

$$\begin{cases} d_{u_1} \dots d_{u_r} d_{j+r} & j \text{ odd,} \\ d_j d_{u_1} \dots d_{u_r} & j \text{ even} \end{cases}$$

for $U = \{u_1 < \cdots < u_r\} \in P\binom{n}{r}$ with $U^- = \emptyset$. Therefore,

$$\partial \sigma_i = \sum_{\substack{U \in P\binom{m}{r+1} \\ U^- = \emptyset}} d_{U^+}\sigma \; - \sum_{\substack{U \in P\binom{m}{r+1} \\ U^+ = \emptyset}} d_{U^-}\sigma$$

as claimed.

## 4. Proof of Theorem 2.4

We will prove Theorem 2.4 by establishing a sequence of lemmas. Unless stated otherwise, all algebraic constructions are taken over a general commutative and unital ring $R$.

**Lemma 4.1.** *For any globular set $X$ the triple $\big(\mathrm{C}_\bullet(X), \Delta, \varepsilon\big)$ is a counital cosymmetric $R$-coalgebras.*

*Proof.* Showing that $\Delta \colon W \otimes \mathrm{C}_\bullet(X) \to \mathrm{C}_\bullet(X) \otimes \mathrm{C}_\bullet(X)$ is a $R[\Sigma_2]$-linear chain map is equivalent to establishing (2) for all $k \geqslant 0$. We will split the verification into six cases. For the remainder of this proof let us consider $x \in X_n$.

If $k = n = 0$:

$$\begin{aligned} \partial \Delta_k x - (-1)^k \Delta_k \partial x &= \partial(x \otimes x) \\ &= 0. \end{aligned}$$

If $k = 0 < n$:

$$\begin{aligned} \partial \Delta_k x - (-1)^k \Delta_k \partial x = \; & t_0 x \otimes (t_{n-1} - s_{n-1})x \; + \; (t_{n-1} - s_{n-1})x \otimes s_0 x \\ & - t_0 t_{n-1} x \otimes t_{n-1} x \; - \; t_{n-1} x \otimes s_0 t_{n-1} x \\ & + t_0 s_{n-1} x \otimes s_{n-1} x \; + \; t_{n-1} x \otimes s_0 s_{n-1} x \\ = \; & 0. \end{aligned}$$

If $0 < k = n + 1$:

$$\begin{aligned}\partial\Delta_k x - (-1)^k \Delta_k \partial x &= 0 \\ &= x \otimes x - x \otimes x \\ &= \big(1 + (-1)^{n+1} T\big)(x \otimes x) \\ &= \big(1 + (-1)^k T\big)\Delta_{k-1}(x).\end{aligned}$$

If $0 < k = n$:

$$\begin{aligned}\partial\Delta_k x - (-1)^k \Delta_k \partial x &= (t_{n-1} - s_{n-1})x \otimes x + (-1)^n x \otimes (t_{n-1} - s_{n-1})x \\ &= \big(t_{k-1}x \otimes x - (-1)^n x \otimes s_{k-1}x\big) \\ &\quad + (-1)^n \big(x \otimes t_{k-1}x - (-1)^n s_{k-1}x \otimes x\big) \\ &= \big(1 + (-1)^n T\big)\big(t_{k-1}x \otimes x - (-1)^n x \otimes s_{k-1}x\big) \\ &= \big(1 + (-1)^n T\big)\big(t_{k-1}x \otimes x + (-1)^{(n+1)(k-1)} x \otimes s_{k-1}x\big) \\ &= \big(1 + (-1)^n T\big)\Delta_{k-1}(x).\end{aligned}$$

If $0 < k = n - 1$:

$$\begin{aligned}\partial\Delta_k x - (-1)^k \Delta_k \partial x &= (t_{k-1} - s_{k-1})t_k x \otimes x + (-1)^{(n-1)} t_k x \otimes (t_{n-1} - s_{n-1})x \\ &\quad + (-1)^{n+1}(t_{n-1} - s_{n-1})x \otimes s_k x - x \otimes (t_{k-1} - s_{k-1})s_k x \\ &\quad + (-1)^{k+1}\big(t_k x \otimes t_k x - s_k x \otimes s_k x\big) \\ &= \big(t_{k-1}x \otimes x + x \otimes s_{k-1}x\big) - \big(x \otimes t_{k-1}x + s_{k-1}x \otimes x\big) \\ &= (1 + (-1)^k T)\big(t_{k-1}x \otimes x + x \otimes s_{k-1}x\big) \\ &= (1 + (-1)^k T)\Delta_{k-1}(x).\end{aligned}$$

If $0 < k < n - 1$:

$$\begin{aligned}\partial\Delta_k x - (-1)^k \Delta_k \partial x &= (t_{k-1} - s_{k-1})t_k x \otimes x + (-1)^k t_k x \otimes (t_{n-1} - s_{n-1})x \\ &\quad + (-1)^{(n+1)k}\big((t_{n-1} - s_{n-1})x \otimes s_k x \\ &\quad + (-1)^n x \otimes (t_{k-1} - s_{k-1})s_k x\big) \\ &\quad + (-1)^{k+1}\big(t_k t_{n-1}x \otimes t_{n-1}x + (-1)^{nk} t_{n-1}x \otimes s_k t_{n-1}x\big) \\ &\quad + (-1)^k\big(t_k s_{n-1}x \otimes s_{n-1}x + (-1)^{nk} s_{n-1}x \otimes s_k s_{n-1}x\big) \\ &= \big(t_{k-1}x \otimes x + (-1)^{(n+1)(k+1)} x \otimes s_{k-1}x\big) \\ &\quad + \big((-1)^{(n(k+1)+k)} x \otimes t_{k-1}x - s_{k-1}x \otimes x\big) \\ &= (1 + (-1)^k T)\big(t_{k-1}x \otimes x + (-1)^{(n+1)(k-1)} x \otimes s_{k-1}x\big) \\ &= (1 + (-1)^k T)\Delta_{k-1}(x).\end{aligned}$$

Showing that $\varepsilon$ is a counit for $\Delta$ follows from the fact that for any $x \in X_n$

$$\Delta_0 x = \begin{cases} t_0 x \otimes x + x \otimes s_0 x & n \neq 0, \\ x \otimes x & n = 0 \end{cases}$$

and $\varepsilon(x') = 1$ for any $x' \in X_0$. □

**Lemma 4.2.** *For any morphism $F\colon X \to Y$ of globular sets, the chain map*

$$\mathrm{C}_\bullet(F)\colon \mathrm{C}_\bullet(X) \longrightarrow \mathrm{C}_\bullet(Y)$$

*is a coalgebra map.*

*Proof.* Denote $\mathrm{C}_\bullet(F)$ by $f$. Since $F(X_n) \subseteq Y_n$ we have $\varepsilon f = f\varepsilon$ and since $Ft_n = t_nF$ and $Fs_n = s_nF$ we have

$$\begin{aligned}\big(f \otimes f\big)\Delta_k(x) &= F(t_kx) \otimes F(x) + (-1)^{(n+1)k} F(x) \otimes F(s_kx) \\ &= t_kF(x) \otimes F(x) + (-1)^{(n+1)k} F(x) \otimes s_kF(x) \\ &= \Delta_k f(x)\end{aligned}$$

for any $x \in X$ and $k \geqslant 0$. □

**Lemma 4.3.** *If $R$ is an integral domain, the function*

$$\mathrm{Set}^{\mathbb{G}^{\mathrm{op}}}(X,Y) \longrightarrow \mathrm{coAlg}_R\big(\mathrm{C}_\bullet(X), \mathrm{C}_\bullet(Y)\big)$$

*is a bijection.*

*Proof.* Injectivity is immediate. For establishing surjectivity, let us consider $f \in \mathrm{coAlg}_R\big(\mathrm{C}_\bullet(X), \mathrm{C}_\bullet(Y)\big)$. We will construct $F \in \mathrm{Set}^{\mathbb{G}^{\mathrm{op}}}(X,Y)$ such that $\mathrm{C}_\bullet(F) = f$. From Lemma 3.4 we know that for any $x \in X$ either $f(x) = y$ for some $y \in Y$ or $f(x) = 0$. Let $x \in X_n$ not in the image of $i_n$. Define

$$F(x) = \begin{cases} f(x) & f(x) \neq 0 \\ i_n\big(F(t_{n-1}x)\big) & f(x) = 0. \end{cases}$$

This recursive definition is well defined because of the augmentation preserving property of $f$. For $x = i_n(y)$ we recursively define $F(x) = i_nF(y)$. We will prove next that $F\colon X \to Y$ is a map of globular sets.

Let $x \in X_n$ and without loss of generality assume it is not in the image of $i_n$. Let us first assume $f(x) = 0$, then

$$t_{n-1}F(x) = t_{n-1}i_nF(t_{n-1}x) = F(t_{n-1}x)$$

and

$$s_{n-1}F(x) = s_{n-1}i_nF(t_{n-1}x) = F(t_{n-1}x) \overset{?}{=} F(s_{n-1}x).$$

We claim that $F(t_{n-1}x)$ must equal $F(s_{n-1}x)$. Observe that since $f$ is a chain map

$$\boxed{f(x) = 0} \Longrightarrow \boxed{f(t_{n-1}x) = f(s_{n-1}x)} \tag{7}$$

If $f(t_{n-1}x) \neq 0$ or $f(s_{n-1}x) \neq 0$ implication (7) establishes the claims. If $f(t_{n-1}x) = f(s_{n-1}x) = 0$ we have

$$F(t_{n-1}x) = i_{n-1}F(t_{n-2}t_{n-1}x) = i_{n-1}F(t_{n-2}s_{n-1}x) = F(s_{n-1}x).$$

Let us now assume $f(x) \neq 0$. For any $0 < k \leqslant n$ the identity

$$\Delta_{n-k}f(x) = (f \otimes f)\Delta_{n-k}(x)$$

reads

$$\begin{aligned} t_{n-k}F(x) \otimes F(x) \ &+ (-1)^{(n+1)k} F(x) \otimes s_{n-k}F(x) \\ &= f(t_{n-k}x) \otimes F(x) \ + \ (-1)^{(n+1)k} F(x) \otimes f(s_{n-k}x). \end{aligned} \tag{8}$$

Therefore, for $0 < k \leqslant n$

$$\begin{array}{c}\boxed{f(t_{n-k}x) \neq 0} \Longrightarrow \boxed{t_{n-k}F(x) = F(t_{n-k}x)} \\ \boxed{f(s_{n-k}x) \neq 0} \Longrightarrow \boxed{s_{n-k}F(x) = F(s_{n-k}x)}\end{array} \tag{9}$$

Therefore, if both $f(t_{n-1}x) \neq 0$ and $f(s_{n-1}x) \neq 0$ we are done.

Let us assume $f(t_{n-1}x) = 0$ and notice that $n-1$ must be greater than 0. It follows from (8) that $t_{n-1}F(x)$ is in the image of $i_{n-1}$. Writing $t_{n-1}F(x) = i_{n-1}y$ and applying $t_{n-2}$ to this identity gives $t_{n-2}F(x) = y$. Hence,

$$F(t_{n-1}x) \overset{\text{def}}{=} i_{n-1}F(t_{n-2}t_{n-1}x) = i_{n-1}F(t_{n-2}x) \overset{?}{=} i_{n-1}t_{n-2}F(x) = i_{n-1}y = t_{n-1}F(x).$$

Similarly, when $f(s_{n-1}x) = 0$ we have

$$F(s_{n-1}x) \overset{\text{def}}{=} i_{n-1}F(t_{n-2}s_{n-1}x) = i_{n-1}F(t_{n-2}x) \overset{?}{=} i_{n-1}t_{n-2}F(x) = s_{n-1}F(x).$$

Therefore, we have reduced both claims: $F(t_{n-1}x) = t_{n-1}F(x)$ when $f(t_{n-1}x) = 0$ and $s_{n-1}F(x) = F(s_{n-1}x)$ when $f(s_{n-1}x) = 0$ to showing $F(t_{n-2}x) = t_{n-2}F(x)$. If $f(t_{n-2}x) \neq 0$ then (9) finishes the proof. If not, we repeat the argument and reduce it to $F(t_{n-3}x) = t_{n-3}F(x)$. Because of the augmentation preserving property of $f$ this regression has to end. □

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