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CROSSED MODULES AND SYMMETRIC COHOMOLOGY OF GROUPS

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Abstract

This paper links the third symmetric cohomology (introduced by Staic [**10**] and Zarelua [**12**]) to crossed modules with certain properties. The equivalent result in the language of 2-groups states that an extension of 2-groups corresponds to an element of HS^3 iff it possesses a section which preserves inverses in the 2-categorical sense. This ties in with Staic's (and Zarelua's) result regarding HS^2 and abelian extensions of groups.

1. Introduction

Let G be a group and M a G-module. Symmetric cohomology $HS^*(G,M)$ was introduced by Staic [**9**] as a variant of classical group cohomology $H^*(G,M)$. There has been interest in the mathematical community in exploring symmetric cohomology. Several papers have already been published on this topic, e.g. [**2**], [**3**], [**8**], [**11**]. Zarelua's prior definition [**12**] of exterior cohomology $H^*_\lambda(G,M)$ is very closely related to this, as shown in [**6**]. Namely, we proved in [**6**] that symmetric cohomology has a functorial decomposition

$$HS^*(G,M) = H^*_\lambda(G,M) \oplus H^*_\delta(G,M),$$

where $H^i_\delta(G,M) = 0$ when $0 \leqslant i \leqslant 4$ or M has no elements of order two.

There are natural homomorphisms $\alpha^n\colon HS^n(G,M) \to H^n(G,M)$ (and $\beta^n\colon H^n_\lambda(G,M) \to H^n(G,M)$), which are isomorphisms for $n = 0, 1$ and a monomorphism for $n = 2$. This was shown by Staic in [**10**]. According to our results in [**6**], the map α^n is an isomorphism for $n = 2$ if G has no elements of order two. In this case, α^3 is a monomorphism. More generally, α^n is an isomorphism if G is torsion free.

Classical $H^2(G,M)$ classifies extensions of G by M. Staic showed in [**10**] that $HS^2(G,M)$ classifies a subclass of extensions of G by M, namely those that have a section-preserving inversion.

It is also well known that $H^3(G,M)$ classifies the so-called crossed extensions of G by M [**4**] or, equivalently, $H^3(G,M)$ classifies 2-groups Γ, with $\pi_0(\Gamma) = G$ and $\pi_1(\Gamma) = M$.

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The main result of this paper is that for groups G with no elements of order 2 the group $HS^3(G,M)$ classifies crossed extensions of G by M with a certain condition on the section.

2. Preliminaries

2.1. Preliminaries on symmetric cohomology of groups

Let G be a group and M a G-module. Recall that the group cohomology $H^*(G,M)$ is defined as the cohomology of the cochain complex $C^*(G,M)$, where the group of n-cochains of G with coefficients in M is the set of functions from G^n to M: $C^n(G,M) = \{\phi\colon G^n \to M\}$ and the coboundary map $d^n\colon C^n(G,M) \to C^{n+1}(G,M)$ is defined by

$$d^n(\phi)(g_0,g_1,\dots,g_n) = g_0\cdot\phi(g_1,\dots,g_n) + \sum_{i=1}^{n}(-1)^i\phi(g_0,\dots,g_{i-2},g_{i-1}g_i,g_{i+1},\dots,g_n) + (-1)^{n+1}\phi(g_0,\dots,g_{n-1}).$$

A cochain ϕ is called *normalised* if $\phi(g_1,\dots,g_n)=0$ whenever some $g_j=1$. The collection of normalised cochains is denoted by $C^*_N(G,M)$ and the classical normalisation theorem claims that the inclusion $C^*_N(G,M)\to C^*(G,M)$ induces an isomorphism on cohomology.

In [**9**] Staic introduced a subcomplex $CS^*(G,M)\subset C^*(G,M)$, whose homology is known as the symmetric cohomology of G with coefficients in M and is denoted by $HS^*(G,M)$. The definition is based on an action of Σ_{n+1} on $C^n(G,M)$ (for every n) compatible with the differential. In order to define this action, it is enough to define how the transpositions $\tau_i=(i,i+1)$, $1\leqslant i\leqslant n$ act. For $\phi\in C^n(G,M)$ one defines:

$$(\tau_i\phi)(g_1,g_2,g_3,\dots,g_n) = \begin{cases} -g_1\phi(g_1^{-1},g_1g_2,g_3,\dots,g_n), & \text{if } i=1,\\ -\phi(g_1,\dots,g_{i-2},g_{i-1}g_i,g_i^{-1},g_ig_{i+1},\dots,g_n), & 1<i<n,\\ -\phi(g_1,g_2,g_3,\dots,g_{n-1}g_n,g_n^{-1}), & \text{if } i=n.\end{cases}$$

Denote by $CS^n(G,M)$ the subgroup of the invariants of this action. That is, $CS^n(G,M)=C^n(G,M)^{\Sigma_{n+1}}$. Staic proved that $CS^*(G,M)$ is a subcomplex of $C^*(G,M)$ [**9**], [**10**] and hence the groups $HS^*(G,M)$ are well defined. There is an obvious natural transformation

$$\alpha^n\colon HS^n(G,M)\to H^n(G,M),\quad n\geqslant 0.$$

According to [**9**], [**10**], α^n is an isomorphism if $n=0,1$ and is a monomorphism for $n=2$. For extensive study of the homomorphism α^n for $n\geqslant 2$ we refer to [**6**].

Denote by $CS^*_N(G,M)$ the intersection $CS^*(G,M)\cap C^*_N(G,M)$. Unlike classical cohomology the inclusion $CS^*_N(G,M)\to CS^*(G,M)$ does not always induce an isomorphism on cohomology. The groups $H^*_\lambda(G,M)=H^*(CS^*_N(G,M))$ are isomorphic to the so called *exterior cohomology of groups* introduced by Zarelua in [**12**]. According to [**6**] the canonical map

$$\gamma_n\colon H^n_\lambda(G,M)\to HS^n(G,M)$$

induced by the inclusion $CS^*_N(G,M)\to CS^*(G,M)$, is an isomorphism if $n\leqslant 4$, or M has no elements of order two.

2.2. Symmetric extensions and HS^2

It is a classical fact that $H^2(G,M)$ classifies the extension of G by M and $H^3(G,M)$ classifies the crossed extensions of G by M. One can ask what objects classify the symmetric cohomology groups $HS^2(G,M)$ and $HS^3(G,M)$. The answer to this question in dimension two was given in [**10**]. The aim of this work is to prove a similar result in dimension three.

Let G be a group and M a G-module. Recall that an extension of G by M is a short exact sequence of groups

$$0 \to M \xrightarrow{i} K \xrightarrow{p} G \to 0$$

such that for any $k \in K$ and $m \in M$, one has $ki(m)k^{-1} = i(p(k)m)$. An s-section to this extension is a map $s\colon G \to K$ such that $p \circ s(x) = x$ for all $x \in G$. Let $\mathbf{Extgr}(G,M)$ be the category whose objects are extensions of G and M and morphisms are commutative diagrams

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{i} & K & \xrightarrow{p} & G & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle id} & & \downarrow & & \downarrow{\scriptstyle id} & & \\ 0 & \longrightarrow & M & \xrightarrow{i'} & K' & \xrightarrow{p'} & G & \longrightarrow & 0. \end{array}$$

The set of connected components of the category $\mathbf{Extgr}(G,M)$ is denoted by $\mathsf{Extgr}(G,M)$. It is well known that there exists a natural map $\mathsf{Extgr}(G,M) \to H^2(G,M)$, which is a bijection. To construct this map, one needs to choose an s-section s and then define $f \in C^2(G,M)$ by

$$s(x)s(y) = i(f(x,y))s(xy).$$

One checks that f is a 2-cocycle and its class in H^2 is independent of the chosen s-section s.

Let $0 \to M \xrightarrow{i} K \xrightarrow{p} G \to 0$ be an extension and s an s-section. Then s is called *symmetric* if $s(x^{-1}) = s(x)^{-1}$ holds for all $x \in G$. An extension is called *symmetric* if it possesses a symmetric s-section. The symmetric extensions form a full subcategory $\mathbf{ExS}(G,M)$ of the category $\mathbf{Extgr}(G,M)$. The set of connected components of $\mathbf{ExS}(G,M)$ is denoted by $\mathsf{ExS}(G,M)$. The main result of [**10**] claims that the restriction of the bijection $\mathsf{Extgr}(G,M) \to H^2(G,M)$ on $\mathsf{ExS}(G,M)$ yields a bijection $\mathsf{ExS}(G,M) \to HS^2(G,M)$.

2.3. Crossed modules

Recall the classical relationship between third cohomology of groups and crossed modules. A crossed module is a group homomorphism $\partial\colon T \to R$ together with an action of R on T satisfying:

$$\partial({}^r t) = r\partial(t)r^{-1} \qquad \text{and} \qquad {}^{\partial t}s = tst^{-1},\ r \in R, t, s \in T.$$

It follows from the definition that the image $\mathrm{Im}(\partial)$ is a normal subgroup of R, and the kernel $M = \mathrm{Ker}(\partial)$ is in the centre of T. Moreover, the action of R on T induces an action of G on $\mathrm{Ker}(\partial)$, where $G = \mathrm{Coker}(\partial)$.

A *morphism* from a crossed module $\partial\colon T \to R$ to a crossed module $\partial'\colon T' \to R'$ is a pair of group homomorphisms $(\phi\colon T \to T', \psi\colon R \to R')$ such that

$$\psi \circ \partial = \partial' \circ \phi, \quad \phi({}^r t) = {}^{\psi(r)}\phi(t), \; r \in R, t \in T.$$

For a group G and for a G-module M one denotes by $\mathbf{Xext}(G,M)$ the category of exact sequences

$$0 \to M \to T \xrightarrow{\partial} R \to G \to 0,$$

where $\partial\colon T \to R$ is a crossed module and the action of G on M induced from the crossed module structure coincides with the prescribed one. The morphisms in $\mathbf{Xext}(G,M)$ are commutative diagrams

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & T & \xrightarrow{\partial} & R & \xrightarrow{p} & G & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle id} & & \\ 0 & \longrightarrow & M & \longrightarrow & T' & \xrightarrow{\partial'} & R' & \xrightarrow{p'} & G & \longrightarrow & 0, \end{array}$$

where (ϕ, f) is a morphism of crossed modules $(T, G, \partial) \to (T', G', \partial')$. We let $\mathsf{Xext}(G,M)$ be the class of the connected components of the category $\mathbf{Xext}(G,M)$. Objects of the category $\mathbf{Xext}(G,M)$ are called *crossed extensions* of G by M.

It is a classical fact (see for example [**4**]) that there is a canonical bijection

$$\xi\colon \mathsf{Xext}(G,M) \to H^3(G,M).$$

The map ξ has the following description. Let $0 \to M \to T \xrightarrow{\partial} R \xrightarrow{p} G \to 0$ be a crossed extension. An *s-section* of it is a pair of maps $(s\colon G \to R, \sigma\colon G \times G \to T)$ for which the following hold

$$ps(x) = x, \quad s(x)s(y) = \partial(\sigma(x,y))s(xy), \; x, y \in G.$$

An s-section is called *normalised* if $s(1) = 1$ and $\sigma(1,x) = 1 = \sigma(x,1)$ for all $x \in G$. It is clear that every crossed extension has a normalised s-section. Any (normalised) s-section (s,σ) gives rise to a (normalised) 3-cocycle $f \in Z^3(G,M)$ defined by

$$f(x,y,z) = {}^{s(x)}\sigma(y,z)\sigma(x,yz)\sigma(xy,z)^{-1}\sigma(x,y)^{-1}. \tag{1}$$

To show the dependence of f on σ and s, we sometimes write f_σ or even $f_{s,\sigma}$ instead of f. Then the map ξ assigns the class of f in $H^3(G,M)$ to the class of $0 \to M \to T \xrightarrow{\partial} R \xrightarrow{p} G \to 0$ in $\mathsf{Xext}(G,M)$.

3. A characterisation of symmetric cocycles

In this section we prove the following auxiliary results, which will be used in the next section.

Lemma 3.1. *If $\phi \in CS^n_N(G,M)$, $n \geqslant 2$ then $\phi(g_1, \dots, g_n) = 0$, whenever $g_{i+1} = g_i^{-1}$ for some $1 \leqslant i \leqslant n-1$.*

Proof. By definition we have $(\tau_i(\phi) - \phi)(g_1, \dots, g_n) = 0$ for any $g_1, \dots, g_n \in G$. If $g_{i+1} = g_i^{-1}$ for some i, then $\tau_i(\phi)(g_1, \dots, g_n) = 0$ by the normalisation condition. Hence $\phi(g_1, \dots, g_i, g_i^{-1}, \dots, g_n) = 0$. □

The converse in general is not true, however we have the following important fact.

Lemma 3.2. *If $\phi \in C_N^n(G,M)$ is a cocycle, $n \geqslant 2$, then $\phi \in CS_N^n(G,M)$ iff $\phi(g_1,\dots,g_n)=0$, whenever $g_{i+1}=g_i^{-1}$ for some $1 \leqslant i \leqslant n-1$.*

Proof. Thanks to Lemma 3.1 we need to prove that $\tau_i(\phi)-\phi=0$, if $\phi(g_1,\dots,g_n)=0$, whenever $g_{k+1}=g_k^{-1}$ for some $1 \leqslant k \leqslant n-1$.

$$x_1\phi(x_2,\dots,x_{n+1})+\sum_{k=1}^{n}(-1)^k\phi(x_1,\dots,x_kx_{k+1},\dots,x_{n+1})+(-1)^{n+1}\phi(x_1,\dots,x_n)=0,$$

for any $x_1,\dots,x_{n+1}\in G$. First we take

$$x_k=\begin{cases} g_1, & k=1,\\ g_1^{-1}, & k=2,\\ g_1g_2, & k=3,\\ g_{k-1}, & k\geqslant 4.\end{cases}$$

to obtain

$$(\tau_1\phi-\phi)(g_1,\dots,g_n)=0.$$

Next, fix $1<i<n$ and put

$$x_k=\begin{cases} g_k, & k\leqslant i,\\ g_i^{-1}, & k=i+1,\\ g_ig_{i+1}, & k=i+2,\\ g_{k-1}, & k\geqslant i+3,\end{cases}$$

to obtain

$$((-1)^i\tau_i\phi+(-1)^{i+1}\phi)(g_1,\dots,g_n)=0.$$

Thus, $\tau_i\phi-\phi=0$, $1<i<n$.

Finally, we take

$$x_k=\begin{cases} g_k, & k\leqslant n,\\ g_n^{-1}, & k=n+1,\end{cases}$$

to obtain

$$((-1)^n\tau_n\phi+(-1)^{n+1}\phi)(g_1,\dots,g_n)=0.$$

Thus, $\tau_n\phi-\phi=0$ and the lemma follows. □

In particular, a cocycle $\phi\in C_N^3(G,M)$ is symmetric (and hence defines a class in $HS^3(G,M)=H_\lambda^3(G,M)$) iff

$$\phi(x,x^{-1},y)=0=\phi(x,y,y^{-1})$$

for all $x,y\in G$.

The next lemma helps us to distinguish boundary elements in $CS_N^3(G,M)$.

Lemma 3.3. *Suppose $\phi(x,y,z)$ is a normalised symmetric cocycle. Also suppose that it is a coboundary: so there exists a $g(x,y)\in C_N^2(G,M)$ such that*

$$\phi(x,y,z)=xg(y,z)-g(xy,z)+g(x,yz)-g(x,y).$$

Then g is symmetric iff $g(x,x^{-1})=0$ for all $x\in G$.

Proof. Recall that g is symmetric iff
$$g(x,y) = -xg(x^{-1},xy) = -g(xy,y^{-1}).$$
If these conditions hold, we can take $y = x^{-1}$ to obtain
$$g(x,x^{-1}) = -g(1,x) = 0,$$
because g is normalised. Conversely, assume $g(x,x^{-1}) = 0$ for all $x \in G$. Since ϕ is symmetric, we have
$$0 = \phi(x,y,y^{-1}) = xg(y,y^{-1}) - g(xy,y^{-1}) + g(x,1) - g(x,y)$$
and
$$0 = \phi(x,x^{-1},z) = xg(x^{-1},z) - g(1,z) + g(x,x^{-1}z) + g(x,x^{-1}).$$
Since g is normalized we have $g(x,1) = g(1,z) = 0$. By assumption, we also have $g(y,y^{-1}) = 0 = g(x,x^{-1})$. Hence,
$$g(x,y) + g(xy,y^{-1}) = 0 \qquad \text{and} \qquad xg(x^{-1},z) + g(x,x^{-1}z) = 0.$$
Replacing z by xy in the last equality, one gets symmetric conditions on g. □

4. Third symmetric cohomology and crossed modules

We start with proving the following result, which links symmetric cocycles and crossed modules. Our notation is the same as at the end of Section 2.3.

Proposition 4.1. *The class of* $0 \to M \to T \xrightarrow{\partial} R \xrightarrow{p} G \to 0$ *in* $\mathsf{Xext}(G,M)$ *lies in the image of the composite map*
$$HS^3(G,M) \xrightarrow{\alpha^3} H^3(G,M) \xrightarrow{\xi^{-1}} \mathsf{Xext}(G,M)$$
iff the crossed extension $0 \to M \to T \xrightarrow{\partial} R \xrightarrow{p} G \to 0$ *has a normalised s-section* (s,σ) *for which the following two identities hold:*
$$^{s(x)}\sigma(x^{-1},y)\sigma(x,x^{-1}y) = \sigma(x,x^{-1}),$$
$$\sigma(x,y)\sigma(xy,y^{-1}) = {}^{s(x)}\sigma(y,y^{-1}).$$

Proof. In fact, $f = f_{\sigma,\tau}$ is symmetric iff
$$f(x,x^{-1},y) = 0 = f(x,y,y^{-1}),$$
thanks to Lemma 3.2. By definition of f, these conditions are equivalent to
$$^{s(x)}\sigma(x^{-1},y)\sigma(x,x^{-1}y)\sigma(1,y)^{-1}\sigma(x,x^{-1})^{-1} = 1$$
and
$$^{s(x)}\sigma(y,y^{-1})\sigma(x,1)\sigma(xy,y^{-1})^{-1}\sigma(x,y)^{-1} = 1.$$
Since $\sigma(1,-) = 1 = \sigma(-,1)$, we obtain
$$^{s(x)}\sigma(x^{-1},y)\sigma(x,x^{-1}y) = \sigma(x,x^{-1})$$
and
$$^{s(x)}\sigma(y,y^{-1}) = \sigma(x,y)\sigma(xy,y^{-1})$$
and we are done. □

Definition 4.2. A normalised s-section (s, σ) of a crossed extension

$$0 \to M \to T \xrightarrow{\partial} R \xrightarrow{p} G \to 0$$

is called weakly symmetric if the following identities hold:

i) $s(x^{-1}) = s(x)^{-1}$,

ii) $\sigma(x, x^{-1}) = 1$, $x, y \in G$.

We have the following easy fact.

Lemma 4.3. *Let G be a group which has no elements of order two. Then any crossed extension $0 \to M \to T \xrightarrow{\partial} R \xrightarrow{p} G \to 0$ has a weakly symmetric s-section.*

Proof. In this case $G \setminus \{1\}$ is a disjoint union of two element subsets of the form $\{x, x^{-1}\}$, $x \neq 1$. Let us choose a representative in each class. If x is a representative, we set $s(x)$ to be an element in $p^{-1}(x)$. We then extend s to whole G by $s(1) = 1$ and $s(x^{-1}) = s(x)^{-1}$, where x is a representative. We see that for $y = x^{-1}$, one has $s(x)s(y)s(xy)^{-1} = 1$. Thus one can choose σ with the property $\sigma(x, x^{-1}) = 1$ and the lemma follows. □

Definition 4.4. A weakly symmetric s-section (s, σ) of a crossed extension

$$0 \to M \to T \xrightarrow{\partial} R \xrightarrow{p} G \to 0$$

is called symmetric if the following identities hold:

i) $\sigma(x, y) \cdot {}^{s(x)}\sigma(x^{-1}, xy) = 1$, $x, y \in G$,

ii) $\sigma(x, y) \cdot \sigma(xy, y^{-1}) = 1$, $x, y \in G$.

A crossed extension $0 \to M \to T \xrightarrow{\partial} R \xrightarrow{p} G \to 0$ is called symmetric if it has a symmetric s-section.

Symmetric crossed extensions of G by M form a subset $\mathsf{XextS}(G, M)$ of the set of $\mathsf{Xext}(G, M)$.

Example. Denote by $C_n(x)$ the multiplicatively written cyclic group of order $n \in \mathbb{N} \cup \infty$ with generator x.

Let $G = C_9(t)$. We take $M = \mathbb{Z}/9\mathbb{Z}$ considered as a trivial G-module. We let $R = C_\infty(x)$ and $T = M \times C_\infty(y)$. Define

$$\partial \colon T \to R$$

by

$$\partial(m, y^k) = x^{9k}.$$

Here $m \in \mathbb{Z}/9\mathbb{Z}$ and $k \in \mathbb{Z}$. Define an action of R on T as follows. Since R is the infinite cyclic group, it suffices to define the action of $x \in R = C_\infty(x)$ on T. We put

$$^x(m,0) = (m,0) \qquad ^x(1,y) = (m_0, y),$$

where $m_0 \in \mathbb{Z}/9\mathbb{Z}$ is fixed. This forces

$$^{x^i}(m, y^k) = (m + kim_0, y^k).$$

Then

$$0 \to \mathbb{Z}/9\mathbb{Z} \xrightarrow{i} \mathbb{Z}/9\mathbb{Z} \times C_\infty(y) \xrightarrow{\partial} C_\infty(x) \xrightarrow{p} C_9(t) \to 0$$

is a crossed extension, whose class in $H^3(C_9(t), \mathbb{Z}/9\mathbb{Z}) = \mathbb{Z}/9\mathbb{Z}$ corresponds to the element $m_0 \in \mathbb{Z}/9\mathbb{Z}$. Here $i(m) = (m,1)$ and $p(x) = t$.

Now we will define maps s, σ as follows. It will be convenient to write elements of $G = C_9(t)$ as powers of t with exponent i, where $|i| \leqslant 4$. So,

$$G = \{t^i, \text{ where } |i| \leqslant 4\}.$$

For $|i|, |j| \leqslant 4$ we set

$$s(t^i) = x^i$$

and

$$\sigma(t^i, t^j) = \begin{cases} (0,1), & \text{if } |i+j| \leqslant 4, \\ (0,y), & \text{if } i+j > 4, \\ (0,y^{-1}), & \text{if } i+j < -4. \end{cases}$$

Then the pair (s, σ) is a weakly symmetric normalised s-section. However, it is symmetric iff $3|m_0$. In fact, the equality $\sigma(t^i, t^j)\sigma(t^{i+j}, t^{-j}) = 1$ holds for all values $|i|, |j| \leqslant 4$, while the equation

$$\sigma(t^i, t^j) \cdot {}^{x^i}\sigma(t^{-i}, t^{i+j}) = 1$$

does not always hold. For example, for $i = 3, j = 2$ it takes the form $(-3m_0, 1) = (0, 1)$. Hence we have the condition $3|m_0$.

Theorem 4.5. *Let G be a group which has no elements of order two. Then there is a natural bijection*

$$HS^3(G, M) \cong \mathsf{XextS}(G, M).$$

Proof. By Lemma 4.3, any crossed extension has a weakly symmetric s-section (s, σ). By Proposition 4.1, the corresponding 3-cocycle is symmetric if

$$^{s(x)}\sigma(x^{-1}, y)\sigma(x, x^{-1}y) = 1,$$
$$\sigma(x, y)\sigma(xy, y^{-1}) = 1.$$

Now, if we replace x by x^{-1} in the first identity and then act by $s(x)$, we see that these conditions are exactly the ones in Definition 4.4. Hence by Proposition 4.1, the image of the composite map

$$HS^3(G, M) \xrightarrow{\alpha^3} H^3(G, M) \cong \mathsf{Xext}(G, M)$$

is exactly $\mathsf{XextS}(G, M)$. On the other hand, since G has no elements of order two, the map α^3 is injective. This follows from the part ii) of Corollary 4.4 [**6**], because $HS^3 = H^4_\lambda$, thanks to Theorem 3.9 [**6**]. It follows that the induced map $HS^3(G, M) \to \mathsf{Xext}(G, M)$ is a bijection. □

5. Interpretation in terms of 2-groups

For us (strict) 2-groups are group objects in the category of small categories. Thus we have a category C (in fact a groupoid) equipped with a bifunctor $\cdot\colon \mathsf{C} \times \mathsf{C} \to \mathsf{C}$, $(a, b) \mapsto a \cdot b$ which is strictly associative and satisfies the group object axioms. These objects are also known under the name categorical groups and 1-cat-groups, see [**5**]. Recall the relationship between crossed modules and 2-groups [**5**]. Let $\partial\colon T \to R$ be a crossed module. It defines a 2-group $\mathsf{Ca}_{T\to R}$. Objects of $\mathsf{Ca}_{T\to R}$ are elements of R. A morphism from $r \in R$ to $r' \in R$ is an element $t \in T$ such that

$$r' = \partial(t)r.$$

In this situation we use the notation $r \xrightarrow{t} r'$. The composite of arrows $r \xrightarrow{t} r' \xrightarrow{t'} r''$ is $r \xrightarrow{t't} r''$. It is clear that $r \xrightarrow{1} r$ is the identity arrow Id_r in the category $\mathsf{Ca}_{T\to R}$. Any morphism in $\mathsf{Ca}_{T\to R}$ is an isomorphism. The inverse of $r \xrightarrow{t} r'$ is $r' \xrightarrow{t^{-1}} r$. As usual we set $M = \mathrm{Ker}(\partial)$. Observe that any $m \in M$ defines an endomorphism $r \xrightarrow{m} r$ of $r \in R$ and, conversely, any endomorphism of r has this form.

The bifunctor

$$\mathsf{Ca}_{T\to R} \times \mathsf{Ca}_{T\to R} \xrightarrow{\cdot} \mathsf{Ca}_{T\to R}$$

is given on objects by the multiplication rule in the group R, while on morphisms it is given by

$$(r \xrightarrow{t} z) \cdot (x \xrightarrow{s} y) = rx \xrightarrow{t({}^{z}s)} zy, \quad r, x, y, z \in R, s, t \in T.$$

In particular, we have

$$(x \xrightarrow{t} y) \cdot \mathrm{Id}_z = xz \xrightarrow{t} yz,$$
$$\mathrm{Id}_r \cdot (x \xrightarrow{t} y) = rx \xrightarrow{{}^{r}t} ry.$$

It is well known that any 2-group is isomorphic to the 2-group of the form $\mathsf{Ca}_{T\to R}$. For a uniquely defined (up to isomorphism) crossed module $\partial\colon T \to R$, see for example [**5**].

In particular, crossed modules give rise to monoidal categories. So we can consider monoidal functors. We recall the corresponding definition. Let C and D be 2-groups. An *s-functor* $\mathsf{C} \to \mathsf{D}$ is a pair (F, ξ), where $F\colon \mathsf{C} \to \mathsf{D}$ is a functor and ξ is a natural transformation from the composite functor $\mathsf{C} \times \mathsf{C} \xrightarrow{\cdot} \mathsf{C} \xrightarrow{F} \mathsf{D}$ to the composite functor $\mathsf{C} \times \mathsf{C} \xrightarrow{F\times F} \mathsf{D} \times \mathsf{D} \xrightarrow{\cdot} \mathsf{D}$. Thus for any objects x and y of C we have a morphism $\xi_{x,y}\colon F(x \cdot y) \to F(x) \cdot F(y)$, which is natural in x and y.

In what follows, we will assume that (F, ξ) is normalised, meaning that $F(1) = 1$ and $\xi_{x,y} = \mathrm{Id}$, if $x = 1$ or $y = 1$. Thus, for any object x we have a morphism $\xi(x, x^{-1})\colon 1 \to F(x) \cdot F(x^{-1})$, which will play an important role later.

An s-functor is *monoidal* if for any objects x, y, z of the category C the diagram

$$\begin{array}{ccc} F(x\cdot y\cdot z) & \xrightarrow{\xi_{x\cdot y,z}} & F(x\cdot y)\cdot F(z) \\ \downarrow{\scriptstyle \xi_{x,y\cdot z}} & & \downarrow{\scriptstyle \xi_{x,y}\cdot \mathrm{Id}_{F(z)}} \\ F(x)\cdot F(y\cdot z) & \xrightarrow[\mathrm{Id}_{F(x)}\cdot \xi_{y,z}]{} & F(x)\cdot F(y)\cdot F(z) \end{array} \tag{2}$$

commutes.

Let $0 \to M \to T \xrightarrow{\partial} R \xrightarrow{p} G \to 0$ be a crossed extension. In this situation we have three 2-groups $\mathsf{Ca}_{M\to 0}$, $\mathsf{Ca}_{T\to R}$ and Ca_G. The third one is G considered as a discrete category (equivalently, the 2-group, corresponding to the crossed module $1 \to G$). Without giving too much detail, we mention that these three 2-groups form a short exact sequence in the 2-mathematical sense (see [**7**, Définition 10.1.] for the exact mathematical meaning of exact sequences of 2-groups). The homomorphism p yields the strict monoidal functor $\mathsf{Ca}_{T\to R} \to \mathsf{Ca}_G$, which is still denoted by p.

One can consider sections of p. We will consider different levels of compatibility of sections with monoidal structures. The weakest condition to ask of such a section is to be a functor. Since Ca is a discrete category, we see that such a section of the functor p is nothing but a set section of the map $p\colon R \to G$. Next is to ask the functor $\mathsf{Ca}_G \to \mathsf{Ca}_{T\to R}$ to be an s-functor. Call them s-sections of p. One easily observes that there is a one-to-one correspondence between s-sections (F, ξ) of the functor p and s-sections of a crossed extension $0 \to M \to T \xrightarrow{\partial} R \xrightarrow{p} G \to 0$. In fact, if (s, σ) is an s-section, then (F, ξ) is an s-functor $\mathsf{Ca}_G \to \mathsf{Ca}_{T\to R}$, for which $F \circ p = \mathrm{Id}_{\mathsf{Ca}_G}$. Here the functor F and natural transformation ξ are defined as follows. Since Ca_G is a discrete category, the functor F is uniquely determined by the rule:

$$F(g) = s(g),\ \ g \in G.$$

Next, the natural transformation ξ is uniquely determined by the family of morphisms

$$\xi_{g,h} = \left(s(gh) \xrightarrow{\sigma(g,h)} s(g)s(h) \right),\ \ g, h \in G.$$

An even stronger assumption is to ask the pair (F, ξ) to be a monoidal functor. As the following well-known result shows this condition is a 2-dimensional analogue of splitting a short exact sequence.

Proposition 5.1. *The class of a crossed extension*

$$0 \to M \to T \xrightarrow{\partial} R \xrightarrow{p} G \to 0$$

is zero in $H^3(G, M)$ *iff the strict monoidal functor* $p\colon \mathsf{Ca}_{T\to R} \to \mathsf{Ca}_G$ *has a section* $\mathsf{Ca}_G \to \mathsf{Ca}_{T\to R}$*, which is monoidal. That is, there exists an* s*-section* (s, σ) *of* $0 \to M \to T \xrightarrow{\partial} R \xrightarrow{p} G \to 0$*, for which the corresponding* s*-functor* (F, ξ) *is monoidal.*

Since we did not find an appropriate reference, we give the proof.

Proof. Let (s, σ) be an s-section of $0 \to M \to T \xrightarrow{\partial} R \xrightarrow{p} G \to 0$. Then Diagram 2 in the definition of the monoidal functor for (F,ξ) has the form

$$\begin{array}{ccc} s(x\cdot y\cdot z) & \xrightarrow{\sigma(x\cdot y,z)} & s(x\cdot y)\cdot s(z) \\ \downarrow{\scriptstyle \sigma(x,y\cdot z)} & & \downarrow{\scriptstyle \sigma(x,y)} \\ s(x)\cdot s(y\cdot z) & \xrightarrow[{}^{s(x)}\sigma(y,z)]{} & s(x)\cdot s(y)\cdot s(z). \end{array}$$

Thus commutativity of this diagram is equivalent to the vanishing of the 3-cocycle f in (1). This finishes one direction of the proof. Conversely, let f_σ defined in (1) be a coboundary and

$$f_\sigma(x,y,z) = xk(y,z) - k(xy,z) + k(x,yz) - k(x,y)$$

for a function $k\colon G^2 \to M$. Denote by $\bar{k}$ the corresponding map from G^2 to T. Since M is commutative but T does not have to be commutative, we use the additive notation for k and the multiplicative notation for $\bar{k}$. Take $\tau(g,h) = \sigma(g,h)\bar{k}(g,h)^{-1}$. Since $M = ker(\partial)$, we have that (s,τ) is an s-section. Moreover, because $M \subseteq Z(T)$, we have:

$$\begin{aligned} f_\tau(x,y,z) &= {}^{s(x)}\tau(y,z)\tau(x,yz)\tau(xy,z)^{-1}\tau(x,y)^{-1} \\ &= {}^{s(x)}(\sigma(y,z)\bar{k}(y,z)^{-1})\sigma(x,yz)\bar{k}(x,yz)^{-1}(\sigma(xy,z)\bar{k}(xy,z)^{-1})^{-1}(\sigma(x,y)\bar{k}(x,y)^{-1})^{-1} \\ &= {}^{s(x)}\sigma(y,z)\sigma(x,yz)\sigma(xy,z)^{-1}\sigma(x,y)^{-1}\,{}^{s(x)}\bar{k}(y,z)^{-1}\bar{k}(x,yz)^{-1}\bar{k}(xy,z)\bar{k}(x,y) \\ &= f_\sigma(x,y,z)f_\sigma(x,y,z)^{-1} \\ &= 1. \end{aligned}$$

And so we have that (s,τ) is an s-section for which $f_\tau = 0$. □

We now introduce another condition on an s-functor (F,ξ), which is weaker than the monoidal functor.

Definition 5.2. An s-functor (F,ξ) is called symmetric if the diagrams

$$\begin{array}{ccc} F(y) & \xrightarrow{\xi(x,x^{-1}y)} & F(x)\cdot F(x^{-1}y) \\ & \searrow{\scriptstyle \xi(x,x^{-1})\cdot \mathrm{Id}_{F(y)}} & \downarrow{\scriptstyle \mathrm{Id}_{F(x)}\cdot\xi(x^{-1},y)} \\ & & F(x)\cdot F(x^{-1})\cdot F(y) \end{array}$$

and

$$\begin{array}{ccc} F(x) & \xrightarrow{\xi(xy,y^{-1})} & F(xy)\cdot F(y^{-1}) \\ & \searrow{\scriptstyle \mathrm{Id}_{F(x)}\cdot\xi(y,y^{-1})} & \downarrow{\scriptstyle \xi(x,y)\cdot \mathrm{Id}_{F(y^{-1})}} \\ & & F(x)\cdot F(y)\cdot F(y^{-1}) \end{array}$$

commute.

Then the following holds:

Lemma 5.3. *1. Any monoidal functor is a symmetric s-functor.*

2. The s-functor corresponding to an s-section (s,σ) is symmetric if (s,σ) satisfies the conditions listed in Proposition 4.1.

The first can be seen by taking $z = y^{-1}$ in Diagram 2. To see the second remark, note that the first diagram of Definition 5.2 follows from the first condition of Proposition 4.1, and the second diagram follows from the second condition.

Thus 2-groups, for which the corresponding class in $H^3(G, M)$ lies in the image of $HS^3(G, M)$ can be characterised as those for which there exists a symmetric s-functor $\mathsf{Ca}_G \to \mathsf{Ca}_{T\to R}$, which is a section of $p\colon \mathsf{Ca}_{T\to R} \to \mathsf{Ca}_G$.

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