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TIME-REVERSAL HOMOTOPICAL PROPERTIES OF CONCURRENT SYSTEMS

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(*communicated by Ronald Brown*)

Abstract

Directed topology was introduced as a model of concurrent programs, where the flow of time is described by distinguishing certain paths in the topological space representing such a program. Algebraic invariants which reflect this directedness have been introduced to classify directed spaces. In this work we study the properties of such invariants with respect to the reversal of the flow of time in directed spaces. Known invariants, natural homotopy and homology, have been shown to be unchanged under this time-reversal. We show that these can be equipped with additional algebraic structure witnessing this reversal. Specifically, when applied to a directed space and to its reversal, we show that these enhanced invariants yield dual objects. We further refine natural homotopy by introducing a notion of relative directed homotopy and showing the existence of a long exact sequence of natural homotopy systems.

1. Introduction

1.1. Time-reversal properties of concurrent systems

Directed topology was originally introduced as a model, and a tool, for studying and classifying concurrent systems in computer science [**18, 9**]. In this approach, the possible states of several processes running concurrently are modeled as points in a topological space of configurations, in which executions are described by paths. Restricted areas appear when these processes have to synchronize, to perform a joint task, or to use a shared object that cannot be shared by more than a certain number of processes. It is natural to study the homotopical properties of this configuration space in order to deduce some interesting properties of the parallel programs involved, for verification purposes, or for classifying synchronization primitives. A usual model for concurrent processes is actually that of higher-dimensional automata, which are based on (pre-)cubical sets, and are the most expressive known models in concurrency theory [**22**]. But in contrast to ordinary algebraic topology, the invariants of interest are invariants only under continuous deformations which have to respect the flow of time. In short, the only relevant homotopies are the ones which never invert the flow

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of time. For mathematical developments and some applications we refer the reader to the two books [**11, 5**].

Directed spaces and concurrent programs. Directed topological invariants, most notably the computationally tractable ones such as homology, have been long in the making (starting again with [**9**]). Most directed homology theories have proven too weak to classify essential features of directed topology, until the proposal [**4, 3**]. Let us review quickly the main idea from [**3**]. Recall from [**11**] that a *directed space*, or a *dispace* for short, is a pair $\mathcal{X} = (X, dX)$, where X is a topological space and dX is a set of *paths* in X, i.e., continuous maps from $[0, 1]$ to X, called *directed paths*, of *dipaths* for short, such that every constant path is directed, and such that dX is closed under monotonic reparametrization and concatenation.

Partially-ordered spaces, or pospaces, form particular dispaces: these are topological spaces X equipped with a partial order $\leqslant$ on X which is closed under the product topology. The directed structure is thus given by paths $p\colon [0, 1] \to X$ such that $p(s) \leqslant p(t)$, for all $s \leqslant t$ in $[0, 1]$. Another useful class of dispaces is given by the directed geometric realization of finite precubical sets, see, e.g., [**6**]. These are made of gluings of cubical cells, on which the dispace structure is locally that of a particular partially-ordered space: each n-dimensional cell is identified with $[0, 1]^n$ ordered componentwise. This last class is in particular very useful in applications to concurrency and distributed systems theory, see, e.g., [**5**].

As an example (see [**3**]), we have depicted two dispaces in Figure 1, which are built as the gluing of squares (the white ones), each of which is equipped with the product ordering of $\mathbb{R}^2$. They are the directed geometric realization of certain precubical sets as

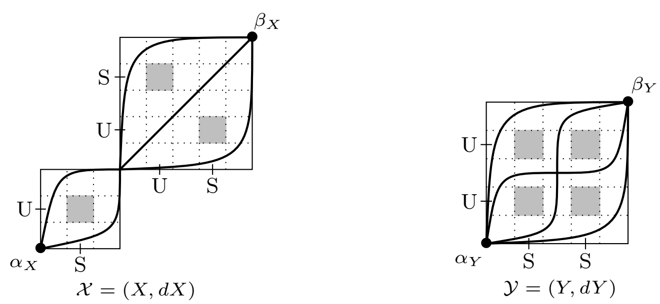

Figure 1: Examples of pospaces coming from non-equivalent concurrent programs.

we mentioned above, *i.e.* are higher-dimensional automata in the sense of [**18**]. They are not dihomeomorphic spaces since they are already not homotopy equivalent: the fundamental group of the leftmost one, that we call X, is the free group on three generators, whereas the fundamental group of the rightmost one, that we call Y, is the free group on four generators. Consider now the topological space of dipaths, with the compact-open topology, from the lowest point of X (resp. Y), which we denote by α_X (resp. α_Y), to the highest point of X (resp. Y), which we denote by β_X (resp. β_Y). The topological space $\overrightarrow{Di}(\mathcal{X})(\alpha_X, \beta_X)$ of directed paths from α_X to β_X, is homotopy equivalent to a six point space, corresponding to the six dihomotopy classes of dipaths depicted in Figure 1. The topological space $\overrightarrow{Di}(\mathcal{Y})(\alpha_Y, \beta_Y)$ is also homotopy equivalent to a six point space, corresponding again to the six dihomotopy classes of dipaths depicted in Figure 1. However, these two dispaces should not be considered as equivalent, in the sense that they correspond to distinct concurrent programs. Therefore comparing spaces of dipaths exclusively between two particular points in each space is not sufficient for distinguishing these dispaces.

Natural homotopy. The main idea of [**3**] is to encode how the homotopy types of the spaces of directed paths vary when we move the end points. With the possibility of considering all the directed path spaces, we can distinguish the two former pospaces. Indeed, the space of directed paths between α_X and β'_X, as in Figure 2, has the homotopy type of a discrete space with four points. Furthermore, we can show that

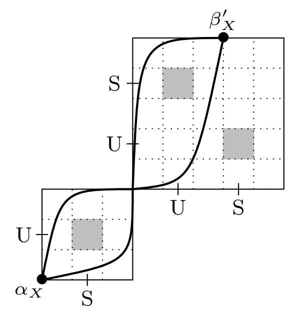

Figure 2: Changing the base points to exhibit a particular space of directed paths.

in Y, there is no pair of points between which we have a directed path space with that homotopy type. The algebraic structure which logs all of the homotopy types of the directed path spaces between each pair of points is a functor called a *natural system*, see Subsection 2.4.

Time reversal invariance. Now, let us consider the concurrent program semantics, and its model as a directed space $\mathcal{X} = (X, dX)$, and invert the flow of time. If we orient the time flow from left to right and from bottom to top, we must rotate its representation as a dispace, as illustrated in Figure 3. As concurrent processes,

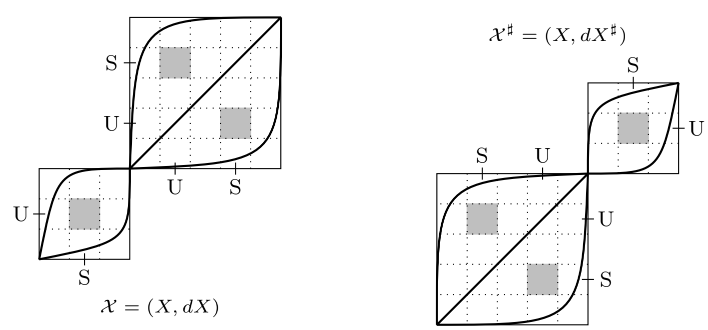

Figure 3: The dispace of a concurrent program and its time-reversed dispace.

these two programs should not be considered as equivalent under any form of well accepted equivalence. These two concurrent programs actually have equivalent prime event structure representations, see [10], that are not bisimulation equivalent [23] under any kind of sensible bisimulation. Fajstrup and Hess noted that natural homotopy and homology theories do not distinguish between these two cases, but produce isomorphic natural systems.

It is one purpose of this paper to show that natural homotopy and homology theories lack an algebraic ingredient, a form of composition, that will make the invariants associated to these two dispaces non-isomorphic, but rather dual to each other. This composition was introduced by Porter [17] in order to link natural systems seen as coefficients of generalized cohomology theories [1] to coefficients for Quillen cohomology theories [19]. This additional structure on natural systems allows us to interpret natural homotopy not as a functor, but as a category. With this point of view, reversal of time in a directed space is witnessed as an action of the opposite functor on the associated category.

1.2. Main results and organisation of the article

In the following section we recall categorical preliminaries used in our constructions. In Subsections 2.1 and 2.2, we recall the notion of internal group over a category and the category Act of actions as introduced by Grandis in [12]. We also recall the embeddings of the categories Gp of groups and Set_* of pointed sets into the category Act. These embeddings preserve exactness of sequences when Act is endowed with the structure of a homological category presented in [12]. In Subsection 2.4, we recall the notion of natural system, central in this work. These were introduced in [20] and used as coefficients in the cohomology of small categories in [1] and monoids in [15], as well as to define homological finiteness invariants for convergent rewriting systems in [13, 14]. A *natural system on a category* $\mathcal{C}$ with values in a category $\mathbf{V}$ is a functor $D\colon F\mathcal{C} \to \mathbf{V}$, where $F\mathcal{C}$ is the *category of factorizations* of $\mathcal{C}$, whose 0-cells are the 1-cells of $\mathcal{C}$ and the 1-cells correspond to factorizations of 1-cells in $\mathcal{C}$. We denote by $\mathsf{opNat}(\mathbf{V})$ the category of pairs $(\mathcal{C}, D)$ where $\mathcal{C}$ is a category and D is a natural system on $\mathcal{C}$ with values in $\mathbf{V}$. The category of natural systems on a category $\mathcal{C}$ with values in the category Ab of Abelian groups is equivalent to the category of internal groups in the category $\mathsf{Cat}_{\mathcal{C}_0}/\mathcal{C}$ of categories over $\mathcal{C}$. In order to extend such an equivalence to natural systems with values in the category Gp of groups, Porter in [17] considers natural systems enriched with composition pairings. Specifically, given a natural system $D\colon F\mathcal{C} \to \mathbf{V}$, a *composition pairing* associated to D consists of families

$$\nu_{f,g}\colon D_f \times D_g \to D_{fg} \qquad \nu_x\colon T \to D_{1_x},$$

of morphisms of $\mathbf{V}$ indexed by composable 1-cells f, g *resp.* 0-cells x of $\mathcal{C}$, satisfying coherence conditions as recalled in Subsection 2.5. Porter showed that the category of natural systems on a category $\mathcal{C}$ with values in the category of groups and with composition pairings is equivalent to the category of internal groups in the category of categories over $\mathcal{C}$. We recall this equivalence in Subsection 2.7 and explain in Subsection 2.8 that such an equivalence can equally be established for natural systems with values in the category Set_* by considering split objects in the category of categories over $\mathcal{C}$.

The aim of Section 3 is to relate the notion of directed homotopy of directed spaces to certain internal groups, refining this invariant of directed spaces. We recall notions from directed algebraic topology in Subsection 3.1. In particular, we define the functor $\overrightarrow{\mathbf{P}}\colon \mathsf{dTop} \to \mathsf{Cat}$ which associates to a dispace $\mathcal{X}$ the *trace category of* $\mathcal{X}$, whose 0-cells are points of X, 1-cells are traces of $\mathcal{X}$, *i.e.* classes of dipaths of $\mathcal{X}$ modulo reparametrization, in which composition is given by concatenation of traces. We are interested in the properties of the natural homotopy and natural homology functors as introduced in [**4, 2**], see Subsection 3.1.5. The functors $\overrightarrow{P}_n(\mathcal{X})$ and $\overrightarrow{H}_n(\mathcal{X})$, for a dispace $\mathcal{X}$, are natural systems extending the homotopy and homology functors on topological spaces to directed spaces. They extend to functors

$$\overrightarrow{P}_n\colon \mathsf{dTop} \to \mathsf{opNat}(\mathsf{Act}), \quad \text{and} \quad \overrightarrow{H}_n\colon \mathsf{dTop} \to \mathsf{opNat}(\mathsf{Ab}),$$

sending a dispace $\mathcal{X}$ to $(\overrightarrow{\mathbf{P}}(\mathcal{X}), \overrightarrow{P}_n(\mathcal{X}))$ and $(\overrightarrow{\mathbf{P}}(\mathcal{X}), \overrightarrow{H}_n(\mathcal{X}))$ respectively. In Subsection 3.2, we show that the natural systems $\overrightarrow{P}_n(\mathcal{X})$ and $\overrightarrow{H}_n(\mathcal{X})$ admit composition pairings. This additional structure allows us to relate the natural systems $\overrightarrow{P}_n(\mathcal{X})$ and $\overrightarrow{H}_n(\mathcal{X})$ to internal groups or split objects in the category $\mathsf{Cat}_X/\overrightarrow{\mathbf{P}}(\mathcal{X})$, giving the main result of this section:

Theorem 3.5. *Let $\mathcal{X} = (X, dX)$ be a dispace. For each $n \leqslant 1$ (resp. $n \geqslant 2$) there exists a split object $\mathcal{C}^n_{\mathcal{X}}$ (resp. internal group $\mathcal{C}^n_{\mathcal{X}}$) in $\mathsf{Cat}_X/\overrightarrow{\mathbf{P}}(\mathcal{X})$ such that*

$$\overrightarrow{P}_n(\mathcal{X})_f = (\mathcal{C}^n_{\mathcal{X}})_f,$$

for all traces f of $\mathcal{X}$, and this assignment is functorial in $\mathcal{X}$.

Using this result, we define a functor $\mathcal{C}^n_-\colon \mathsf{dTop} \to \mathsf{Cat}$. We explain in Subsection 3.2 that for all $n \geqslant 1$, the natural system $\overrightarrow{H}_n(\mathcal{X})$ is automatically equipped with a composition pairing, and thus can be interpreted as an internal abelian group $\mathcal{A}^n_{\mathcal{X}}$ in the category $\mathsf{Cat}_X/\overrightarrow{\mathbf{P}}(\mathcal{X})$. Moreover, the assignment $\mathcal{A}^n_-\colon \mathsf{dTop} \to \mathsf{Cat}$ is functorial for all $n \geqslant 1$. Finally, in Subsection 3.3 we recall the notion of fundamental category from [**11**] and provide a result relating it to natural homotopy.

In Section 4 we study the behaviour of natural homology and natural homotopy with respect to reversal of time in dispaces. First, we define the *time-reversed*, or *opposite*, dispace of a dispace $\mathcal{X}$ as the dispace $\mathcal{X}^\sharp = (X, dX^\sharp)$, where $dX^\sharp$ is the set of paths $t \mapsto f(1-t)$ with $f \in dX$. For every $n \geqslant 0$, we describe an explicit isomorphism

$$I_n(\mathcal{X})\colon \mathcal{C}^n_{\mathcal{X}^\sharp} \Longrightarrow (\mathcal{C}^n_{\mathcal{X}})^o,$$

which is natural in $\mathcal{X}$, showing the main result of this paper:

Theorem 4.2. *For any $n \geqslant 0$, the functor $\mathcal{C}^n_-\colon \mathsf{dTop} \to \mathsf{Cat}$ is strongly time-reversal.*

Finally, in Section 5, we introduce a notion of relative homotopy for dispaces, and establish a long exact sequence, as in the case of ordinary topological spaces, using the homological category structure on Act as introduced by Grandis in [**12**]:

Theorem 5.1. *Let $\mathcal{X}$ be a dispace and $\mathcal{A}$ be a directed subspace of $\mathcal{X}$. There is an exact sequence in $\mathsf{NatSys}(\overrightarrow{\mathbf{P}}(\mathcal{A}), \mathsf{Act})$:*

$$\cdots \longrightarrow \overrightarrow{P}_n(\mathcal{A}) \longrightarrow \overrightarrow{P}_n(\mathcal{X}) \longrightarrow \overrightarrow{P}_n(\mathcal{X},\mathcal{A}) \xrightarrow{\partial_n} \overrightarrow{P}_{n-1}(\mathcal{A}) \longrightarrow \cdots$$
$$\cdots \to \overrightarrow{P}_2(\mathcal{A}) \xrightarrow{v} \overrightarrow{P}_2(\mathcal{X}) \xrightarrow{f} (\overrightarrow{P}_2(\mathcal{X},\mathcal{A}), \overrightarrow{P}_2(\mathcal{X})) \xrightarrow{g} \overrightarrow{P}_1(\mathcal{A}) \xrightarrow{h} \overrightarrow{P}_1(\mathcal{X}) \to \overrightarrow{P}_1(\mathcal{X},\mathcal{A}) \to 0.$$

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2. Categorical preliminaries

In this section we recall categorical constructions used in this article. First we recall the notion of an internal group over a category. Then, in Subsection 2.2 we recall the embeddings of the categories Gp of groups and Set_* of pointed sets into the category Act of actions, and their exactness properties as shown in [**12**]. In Subsection 2.5, we recall the notion of natural system on a category as well as the notion of composition pairing associated to a natural system, allowing the description of natural systems of groups in terms of internal groups over a category [**17**].

2.1. Internal groups

We denote by Cat the category of (small) categories. For a category $\mathcal{C}$, we will denote by $\mathcal{C}_0$ its set of 0-cells (*i.e.* objects) and by $\mathcal{C}_1$ its set of 1-cells (*i.e.* arrows). Given a set X, we denote by Cat_X the subcategory of Cat consisting of those categories with X as their set of 0-cells, and in which we take only the functors which are the identity on 0-cells. Given a category $\mathcal{B}$, we denote by $\mathsf{Cat}_{\mathcal{B}_0}/\mathcal{B}$ the category whose objects are pairs $(\mathcal{C}, p)$, with $\mathcal{C}$ in $\mathsf{Cat}_{\mathcal{B}_0}$, and where $p\colon \mathcal{C} \to \mathcal{B}$ is a functor which is the identity on 0-cells. A morphism in $\mathsf{Cat}_{\mathcal{B}_0}/\mathcal{B}$ from $(\mathcal{C}, p)$ to $(\mathcal{C}', p')$ is a functor $f\colon \mathcal{C} \to \mathcal{C}'$ such that $p = p' \circ f$. Note that $\mathsf{Cat}_{\mathcal{B}_0}/\mathcal{B}$ has arbitrary limits, and that its terminal object is the pair $(\mathcal{B}, id_{\mathcal{B}})$. Given an object $(\mathcal{C}, p)$ in $\mathsf{Cat}_{\mathcal{B}_0}/\mathcal{B}$ and a 1-cell $f\colon x \to y$ of $\mathcal{B}$, the *fibre of f in $\mathcal{C}$*, denoted by $\mathcal{C}_f$, is the pre-image of f in $\mathcal{C}$ by p, that is

$$\mathcal{C}_f = \{c\colon x \to y \text{ in } \mathcal{C}_1 \mid p(c) = f\}.$$

Let $\mathcal{A}$ be a category with finite products and denote by T its terminal object. Recall from [**16**, III.6] that an *(internal) group* in $\mathcal{A}$ is a tuple $\mathcal{G} = (G, \mu, \eta, (-)^{-1})$, where G is an object of $\mathcal{A}$, and $\mu\colon G \times G \to G$, $\eta\colon T \to G$, and $(-)^{-1}\colon G \to G$ are morphisms of $\mathcal{A}$, respectively called the *multiplication*, *identity*, and *inverse* maps, which must satisfy the group axioms. A *morphism of internal groups* from $\mathcal{G}$ to $\mathcal{G}'$ is a morphism $f\colon G \to G'$ of $\mathcal{A}$ that commutes with the associated multiplication and identity morphisms. The category of internal groups in $\mathcal{A}$ and their morphisms is denoted by $\mathsf{Gp}(\mathcal{A})$. The groups which additionally satisfy the commutativity condition $\mu = \mu \circ \tau$, where τ exchanges the factors of the product, constitute a full subcategory of $\mathsf{Gp}(\mathcal{A})$ called the *category of abelian groups* in $\mathcal{A}$, denoted by $\mathsf{Ab}(\mathcal{A})$.

We now turn to the case of groups in the category $\mathsf{Cat}_{\mathcal{B}_0}/\mathcal{B}$. Since $(\mathcal{B}, id_{\mathcal{B}})$ is its terminal object, given a group $((\mathcal{C}, p), \mu, \eta, (-)^{-1})$ of $\mathsf{Cat}_{\mathcal{B}_0}/\mathcal{B}$, we have $id_{\mathcal{B}} = p \circ \eta$. This implies that every fibre $\mathcal{C}_f$ is non-empty, and that η splits p in $\mathsf{Cat}_{\mathcal{B}_0}$. Therefore each hom-set $\mathcal{C}(x,y)$ is a coproduct of the fibres:

$$\mathcal{C}(x,y) = \coprod_{f\in\mathcal{B}(x,y)} \mathcal{C}_f,$$

whose elements are denoted by (c,f), with $c\in\mathcal{C}_f$. The product in $\mathsf{Cat}_{\mathcal{B}_0}/\mathcal{B}$ of an object $(\mathcal{C},p)$ with itself is given by pullback over $\mathcal{B}$ in $\mathsf{Cat}_{\mathcal{B}_0}$, and is denoted by $(\mathcal{C}\times_{\mathcal{B}}\mathcal{C},\widetilde{p})$, where the category $\mathcal{C}\times_{\mathcal{B}}\mathcal{C}$ has $\mathcal{B}_0$ as its set of 0-cells and for x,y in $\mathcal{B}_0$,

$$(\mathcal{C}\times_{\mathcal{B}}\mathcal{C})(x,y) = \{(c,d)\in\mathcal{C}(x,y)^2 \mid p(c)=p(d)\}.$$

The functor $\widetilde{p}$ is the identity on 0-cells, and assigns to each pair (c,d) of 1-cells in $\mathcal{C}\times_{\mathcal{B}}\mathcal{C}$ their common image under p. The hom-sets of this product thus admit the following decomposition:

$$(\mathcal{C}\times_{\mathcal{B}}\mathcal{C})(x,y) = \coprod_{f\in\mathcal{B}(x,y)} \mathcal{C}_f\times\mathcal{C}_f.$$

Furthermore, by definition of μ, $\widetilde{p}=p\circ\mu$ holds. Thus, for all $c,d\in\mathcal{C}_f$, we have $f=\widetilde{p}(c,d)=p(\mu(c,d))$, and therefore $\mu(c,d)\in\mathcal{C}_f$. As a consequence we obtain induced maps $\mu_f\colon\mathcal{C}_f\times\mathcal{C}_f\to\mathcal{C}_f$ for each 1-cell f of $\mathcal{B}$. This endows each fibre with a group structure.

The *opposite group* of an internal group $((\mathcal{C},p),\mu,\eta,(-)^{-1})$ in $\mathsf{Cat}_{\mathcal{B}_0}/\mathcal{B}$ is the internal group $(\mathcal{C}^o,p^o)$ in $\mathsf{Cat}_{\mathcal{B}_0}/\mathcal{B}^o$, for which the multiplication, identity and inverse maps, denoted respectively by μ^o, η^o and $(-)_o^{-1}$, are the induced opposite maps of μ, η and $(-)^{-1}$. Note that the fibre group $\mathcal{C}_f$ in $\mathcal{C}$ associated to a 1-cell f of $\mathcal{B}$ is equal to the fibre group $\mathcal{C}^o_{f^o}$ associated to its opposite f^0.

2.2. The category of actions

In order to formulate results in Section 5 concerning long exact sequences induced by relative directed homotopy functors, we recall from [**12**] the definition of the category of actions of groups on pointed sets, denoted by Act. Objects of Act are *actions*, defined as pairs (X,G) where X is a pointed set, whose base point we shall denote by 0_X, and G is a group with identity element 1_G, equipped with a right action of G on X. The base point of X is not assumed to be fixed by the action, and we will write

$$G_0 = Fix_G(0_X) = \{g\in G \mid 0_X\cdot g = 0_X\}$$

to denote the subgroup of G fixing the base point 0_X. A morphism in Act is a pair $f=(f',f'')\colon(X,G)\to(Y,H)$ where $f'\colon X\to Y$ is a morphism of pointed sets, and $f''\colon G\to H$ is a morphism of groups compatible with the action, in the sense that for all $g\in G$ and all $x\in X$,

$$f'(x\cdot g) = f'(x)\cdot f''(g).$$

We consider Act as a homological category as introduced by Grandis in [**12**, Section 6.4]. With this structure, the kernel of a morphism $f\colon(X,G)\to(Y,H)$ is the inclusion

$$(Ker(f'),{f''}^{-1}(H_0))\to(X,G),$$

where $Ker(f') := {f'}^{-1}(\{0_Y\})$. Observe that ${f''}^{-1}(H_0)$ is the subset of G consisting of elements g such that $x=x'\cdot g$ for some x,x' in $Ker(f')$. Dually, the cokernel of a morphism $f\colon (X,G)\to(Y,H)$ is the projection

$$(Y,H)\to(Y/R,H),$$

where R is an equivalence relation on Y defined by $y\equiv_R y'$ if and only if either y or y' is an element of $f(X)$ and there exists some h in H with $y = y'\cdot h$.

2.3. Embeddings of Gp and Set_* in Act

There are embeddings of the categories Gp and Set_* into the category Act that preserve exactness of sequences and morphisms. In the case of Set_*, there are adjoint functors,

$$J\colon \mathsf{Set}_*\to\mathsf{Act},\qquad V\colon \mathsf{Act}\to\mathsf{Set}_*,$$

defined by $J(X)=(X,\{1\})$ and $V(X,G)=X/G$ for all pointed sets X and groups G with a right action on X, where $(X,\{1\})$ is the action of the trivial group on X, and X/G is the quotient of X by the G-orbits of the action, pointed at the class of 0_X. The functor J induces an equivalence of categories between Set_* and the full homological subcategory of Act consisting of actions of the trivial group. This, along with the fact that J preserves null morphisms, means that it preserves exactness of sequences.

On the other hand, the category Gp can be realized as a retract of the category Act, via the functors

$$K\colon \mathsf{Gp}\to\mathsf{Act},\qquad R\colon \mathsf{Act}\to\mathsf{Gp},$$

defined by $K(G)=(|G|,G)$ and $R(X,G)=G/\overline{G_0}$, where $(|G|,G)$ is the usual right action of G on the underlying set $|G|$, pointed at 1_G, and $\overline{G_0}$ is the normal closure in G of G_0. These show that Gp is a retract of Act in the sense that $R\circ K = id_{\mathsf{Gp}}$ since the action of G on itself is transitive. As a consequence, a sequence of groups viewed in Act is exact if and only if the sequence is exact in the usual sense.

2.4. Natural systems

The *category of factorizations* of a category $\mathcal{C}$, denoted by $F\mathcal{C}$, is the category whose 0-cells are the 1-cells of $\mathcal{C}$, and a 1-cell from f to f' is a pair (u,v) of 1-cells of $\mathcal{C}$ such that $ufv=f'$ holds in $\mathcal{C}$. Composition is given by

$$(u,v)(u',v')=(u'u,vv'),$$

whenever u' and v are composable with u and v' respectively, and the identity on $f\colon x\to y$ is the pair $(1_x,1_y)$. A *natural system on a category* $\mathcal{C}$ *with values in a category* $\mathbf{V}$ is a functor $D\colon F\mathcal{C}\to\mathbf{V}$. We will denote by D_f (resp. $D(u,v)$) the image of a 0-cell f (resp. 1-cell (u,v)) of $F\mathcal{C}$. In most cases, we will consider natural systems with values in the category Set_* of pointed sets, the category Gp of groups, the subcategory Ab of abelian groups, or the category Act, then called *natural systems of pointed sets*, *of groups*, *of abelian groups*, or *of actions* respectively.

We denote by $\mathsf{NatSys}(\mathcal{C},\mathbf{V})$ the category whose objects are natural systems on $\mathcal{C}$ with values in $\mathbf{V}$ and in which morphisms are natural transformations between functors. The category of natural systems with values in $\mathbf{V}$, denoted by $\mathsf{opNat}(\mathbf{V})$, is defined as follows:

1. its objects are the pairs $(\mathcal{C}, D)$ where $\mathcal{C}$ is a category and D is a natural system on $\mathcal{C}$ with values in $\mathbf{V}$,
2. its morphisms are pairs
$$(\Phi, \tau)\colon (\mathcal{C}, D) \to (\mathcal{C}', D')$$
consisting of a functor $\Phi\colon \mathcal{C} \to \mathcal{C}'$ and a natural transformation $\tau\colon D \to \Phi^* D'$, where the natural system $\Phi^* D'\colon F\mathcal{C} \to \mathbf{V}$ is defined by
$$(\Phi^* D')(f) = D'(\Phi f),$$
for every 1-cell f in $\mathcal{C}$, and $\Phi^* D'(u, v) = D'(\Phi(u), \Phi(v))$ for 1-cells u and v in $\mathcal{C}$,
3. composition of morphisms $(\Psi, \sigma)\colon (\mathcal{C}', D') \to (\mathcal{C}'', D'')$ and $(\Phi, \tau)\colon (\mathcal{C}, D) \to (\mathcal{C}', D')$ is defined by
$$(\Psi, \sigma) \circ (\Phi, \tau) := (\Psi \circ \Phi, (\Phi^* \sigma) \circ \tau),$$
where $\Psi \circ \Phi$ denotes composition of functors and where the component of the natural transformation $(\Phi^* \sigma) \circ \tau$ at a 1-cell f of $\mathcal{C}$ is $\tau_f \sigma_{\Phi(f)}$.

2.5. Natural systems and composition pairings

Let $\mathbf{V}$ be a category with finite products. Given a natural system D on a category $\mathcal{C}$ with values in $\mathbf{V}$, recall from [**17**] that a *composition pairing* associated to D consists of two families of morphisms of $\mathbf{V}$
$$\big(\, \nu_{f,g}\colon D_f \times D_g \to D_{fg} \,\big)_{f,g \in \mathcal{C}_1} \qquad \text{and} \qquad \big(\, \nu_x\colon T \to D_{1_x} \,\big)_{x \in \mathcal{C}_0},$$
where T is the terminal object in $\mathbf{V}$, the indexing 1-cells f and g are composable, and such that the three following coherence conditions are satisfied:

1. *naturality condition*: the diagram
$$\begin{array}{ccc} D_f \times D_g & \xrightarrow{\nu_{f,g}} & D_{fg} \\ {\scriptstyle D(u,1)\times D(1,v)}\downarrow & & \downarrow{\scriptstyle D(u,v)} \\ D_{uf} \times D_{gv} & \xrightarrow[\nu_{uf,gv}]{} & D_{ufgv} \end{array}$$
commutes in $\mathbf{V}$ for all 1-cells f, g, u, v in $\mathcal{C}_1$ such that the composites are defined.
2. *The cocycle condition:* the diagram
$$\begin{array}{ccc} D_f \times D_g \times D_h & \xrightarrow{\nu_{f,g} \times id_{D_h}} & D_{fg} \times D_h \\ {\scriptstyle id_{D_f} \times \nu_{g,h}}\downarrow & & \downarrow{\scriptstyle \nu_{fg,h}} \\ D_f \times D_{gh} & \xrightarrow[\nu_{f,gh}]{} & D_{fgh} \end{array}$$
commutes for all 1-cells f, g and h of $\mathcal{C}$ such that the composite fgh is defined.
3. *The unit conditions:* the diagrams
$$\begin{array}{ccc} D_f & \xleftarrow{\nu_{f,1_y}} & D_f \times D_{1_y} \\ & {\scriptstyle \cong}\nwarrow & \uparrow{\scriptstyle 1_{D_f} \times \nu_y} \\ & & D_f \times T \end{array} \qquad \begin{array}{ccc} D_{1_x} \times D_f & \xrightarrow{\nu_{1_x,f}} & D_f \\ {\scriptstyle \nu_x \times 1_{D_f}}\uparrow & \nearrow{\scriptstyle \cong} & \\ T \times D_f & & \end{array}$$
commute for every 1-cell $f\colon x \to y$ of $\mathcal{C}$.

The category of natural systems on $\mathcal{C}$ with values in $\mathbf{V}$ which admit a composition pairing is the category whose objects are pairs (D, ν), with D a natural system on $\mathcal{C}$ and ν a composition pairing associated to D. The morphisms are natural transformations $\alpha\colon D \to D'$ *compatible with the composition pairings* ν and ν', in the sense that the following diagram commutes in $\mathbf{V}$

$$\begin{array}{ccc} D_f \times D_g & \xrightarrow{\nu_{f,g}} & D_{fg} \\ {\scriptstyle \alpha_f \times \alpha_g}\downarrow & & \downarrow{\scriptstyle \alpha_{fg}} \\ D'_f \times D'_g & \xrightarrow[\nu'_{f,g}]{} & D'_{fg} \end{array}$$

for all composable 1-cells f and g in $\mathcal{C}$. We will denote this category of natural systems admitting a composition pairing by $\mathsf{NatSys}_\nu(\mathcal{C}, \mathbf{V})$. We denote by $\mathsf{opNat}_\nu(\mathbf{V})$ the subcategory of $\mathsf{opNat}(\mathbf{V})$ consisting of natural systems with values in $\mathbf{V}$ which admit a composition pairing, in which we take only those morphisms $(\Phi, \tau)\colon (\mathcal{C}, (D, \nu)) \to (\mathcal{C}', (D', \nu'))$ such that τ is compatible with the composition pairings ν and $\Phi^*\nu'$.

2.6. Commutator condition

Consider a natural system of groups $D\colon F\mathcal{B} \to \mathsf{Gp}$. For all composable 1-cells f and g of $\mathcal{B}$, define a homomorphism $\nu_{f,g}\colon D_f \times D_g \to D_{fg}$, by setting

$$\nu_{f,g}(d, d') = D(f, 1)(d').D(1, g)(d),$$

for all $d \in D_f$ and $d' \in D_g$, where the right hand side is a product in D_{fg}. Porter proved in [17] that a natural system of groups D on a category $\mathcal{B}$ admits a composition pairing if, and only if, the condition

$$[D(f, 1)(d'), D(1, g)(d)] = 1,$$

holds for all $d \in D_f$, and $d' \in D_g$, where the bracket denotes the commutator in D_{fg}. In this case, the composition pairing is uniquely given by

$$\nu_{f,g}(d, d') = D(f, 1)(d').D(1, g)(d) = D(1, g)(d).D(f, 1)(d'),$$

for all 1-cells f, g and $d \in D_f$ and $d' \in D_g$ such that the composition fg is defined. Note that as a consequence of this characterization, every natural system of *abelian* groups admits a composition pairing [17].

Remark 2.1. The compatibility condition for natural transformations is always satisfied in the case of natural systems of groups with composition pairings. Indeed, if $\alpha\colon D \to D'$ is a transformation of natural systems, we have

$$D'(1, g)(\alpha_f(d)) = \alpha_{fg}(D(1, g)(d)) \text{ and } D'(f, 1)(\alpha_g(d')) = \alpha_{fg}(D(f, 1)(d'))$$

for all $d \in D_f$ and $d' \in D_g$. Thus

$$\alpha_{fg}(\nu_{f,g}(d, d')) = \nu'_{f,g}(\alpha_f(d), \alpha_g(d')).$$

We thereby deduce that $\mathsf{NatSys}_\nu(\mathcal{B}, \mathsf{Gp})$ (resp. $\mathsf{opNat}_\nu(\mathsf{Gp})$) is a full subcategory of $\mathsf{NatSys}(\mathcal{B}, \mathsf{Gp})$ (resp. $\mathsf{opNat}(\mathsf{Gp})$), and that the categories $\mathsf{NatSys}(\mathcal{B}, \mathsf{Ab})$ and $\mathsf{NatSys}_\nu(\mathcal{B}, \mathsf{Ab})$ are equal.

2.7. Natural systems and internal groups

Given a natural system $D\colon F\mathcal{B} \to \mathsf{Gp}$ with composition pairing ν, we construct an internal group in the category $\mathbf{Cat}_{\mathcal{B}_0}/\mathcal{B}$. First, we construct a category $\mathcal{C}$ of $\mathbf{Cat}_{\mathcal{B}_0}$, whose hom-sets are defined as

$$\mathcal{C}(x,y) := \coprod_{f\in\mathcal{B}(x,y)} D_f,$$

for all x and y in $\mathcal{B}_0$. The 1-cells of $\mathcal{C}$ are denoted by pairs (c,f) where $c \in D_f$. For all 0-cells x,y,z of $\mathcal{B}$, the 0-composition maps

$$\star_0^{x,y,z}\colon \mathcal{C}(x,y)\times\mathcal{C}(y,z) \to \mathcal{C}(x,z)$$

are defined fibre by fibre using the decomposition

$$\mathcal{C}(x,y)\times\mathcal{C}(y,z) \quad = \coprod_{f\in B(x,y),g\in B(y,z)} D_f \times D_g,$$

and the homomorphisms $\nu_{f,g}\colon D_f\times D_g \to D_{fg}$, by setting

$$(c,f)\star_0(d,g) := (\nu_{f,g}(c,d), fg),$$

for all c in D_f and d in D_g. The associativity of composition $\star_0$ is a consequence of the cocycle condition, and the identity on a 0-cell x is the pair $(1_{D_{1_x^{\mathcal{B}}}}, 1_x^{\mathcal{B}})$, where $1_x^{\mathcal{B}}$ denotes the identity on x in $\mathcal{B}$.

Let $p\colon \mathcal{C}\to\mathcal{B}$ denote the functor which is the identity on 0-cells, and which maps the pair (c,f) to f. Then the pair $(\mathcal{C},p)$ is an object of $\mathbf{Cat}_{\mathcal{B}_0}/\mathcal{B}$. Now let us see that it is an internal group. The unit functor $\eta\colon(\mathcal{B},id_{\mathcal{B}})\to(\mathcal{C},p)$ is induced by the functor $\eta\colon\mathcal{B}\to\mathcal{C}$ defined by $\eta(f) := (1_{D_f}, f)$, for every 1-cell f in $\mathcal{B}$. The multiplication map $\mu\colon(\mathcal{C},p)\times(\mathcal{C},p)\to(\mathcal{C},p)$ is defined by $\mu((c,f),(d,f)) := (c.d,f)$, for all c,d in D_f and where $c.d$ denotes the product of c and d in D_f. The functoriality of μ is a consequence of $\nu_{f,g}$ being a homomorphism of groups. Finally, the inverse map $(-)^{-1}\colon(\mathcal{C},p)\to(\mathcal{C},p)$ is induced by the functor $(-)^{-1}\colon\mathcal{C}\to\mathcal{C}$ defined by $(c,f)^{-1} = (c^{-1},f)$, for all c in D_f. This construction induces a functor

$$\mathsf{NatSys}_\nu(\mathcal{B},\mathsf{Gp}) \to \mathsf{Gp}(\mathbf{Cat}_{\mathcal{B}_0}/\mathcal{B}),$$

which assigns an internal group in $\mathbf{Cat}_{\mathcal{B}_0}/\mathcal{B}$ to each natural system of groups on $\mathcal{B}$. Porter proves in [**17**] that this functor induces an equivalence of categories

$$\mathsf{NatSys}_\nu(\mathcal{B},\mathsf{Gp}) \simeq \mathsf{Gp}(\mathbf{Cat}_{\mathcal{B}_0}/\mathcal{B}).$$

2.8. Natural systems and split objects

Given a category $\mathcal{B}$, we define *the category of split objects in* $\mathsf{Cat}_{\mathcal{B}_0}/\mathcal{B}$, denoted by $\mathsf{Split}(\mathsf{Cat}_{\mathcal{B}_0}/\mathcal{B})$, as the full subcategory of $\mathsf{Cat}_{\mathcal{B}_0}/\mathcal{B}$ whose objects are pairs $((\mathcal{C},p),\epsilon)$, where $(\mathcal{C},p)$ is an object of $\mathsf{Cat}_{\mathcal{B}_0}/\mathcal{B}$ and ϵ is a morphism of $\mathsf{Cat}_{\mathcal{B}_0}$ such that $id_{\mathcal{B}} = p\circ\epsilon$.

Note that internal groups are split objects. The equivalence of categories stated in Subsection 2.7 from [**17**] can be adapted to show that there is an equivalence of categories

$$\mathsf{NatSys}_\nu(\mathcal{B},\mathsf{Set}_*) \simeq \mathsf{Split}(\mathsf{Cat}_{\mathcal{B}_0}/\mathcal{B}).$$

3. Directed homotopy as an internal group

In Section 3.1 we recall the notion of dispace from [11] and the definition of natural homotopy and natural homology as introduced in [4, 2]. These are natural systems extending the classical algebraic invariants to dispaces. In Section 3.2, we show that these natural systems have an associated composition pairing, and relate them to certain internal groups or split objects. Finally, in Section 3.3 we recall the notion of fundamental category from [11] and relate it to natural homotopy.

3.1. Directed homology and homotopy

In this subsection we recall the notion of dispaces from [11], and define algebraic invariants for these spaces, natural homotopy and natural homology, as introduced in [4, 2].

3.1.1. Directed spaces

Recall from [11] that a *directed space*, or *dispace*, is a pair $\mathcal{X} = (X, dX)$, where X is a topological space and dX is a set of paths in X, *i.e.* continuous maps from $[0,1]$ to X, called *directed paths*, or *dipaths* for short, satisfying the three following conditions:

1. Every constant path is directed,
2. dX is closed under monotonic reparametrization,
3. dX is closed under concatenation.

We will denote by $f \star g$ the concatenation of dipaths f and g, defined via monotonic reparametrization. A morphism $\varphi\colon (X, dX) \to (Y, dY)$ of dispaces is a continuous function $\varphi\colon X \to Y$ that preserves directed paths, *i.e.*, for every path $p\colon [0,1] \to X$ in dX, the path $\varphi_* p\colon [0,1] \to Y$ belongs to dY. The category of dispaces is denoted dTop. An isomorphism in dTop from (X, dX) to (Y, dY) is a homeomorphism from X to Y that induces a bijection between the sets dX and dY.

Note that the forgetful functor $U\colon \mathsf{dTop} \to \mathsf{Top}$ admits left and right adjoint functors. The left adjoint functor sends a topological space X to the dispace (X, X_d), where X_d is the set of constant directed paths. The right adjoint sends X to the dispace $(X, X^{[0,1]})$, where $X^{[0,1]}$ is the set of all paths in X.

For a dispace $\mathcal{X} = (X, dX)$ and x, y in X, we denote by $\overrightarrow{Di}(\mathcal{X})(x,y)$ the space of dipaths f in X with source $x = f(0)$ and target $y = f(1)$, equipped with the compact-open topology.

3.1.2. The trace category

The *trace space* of a dispace $\mathcal{X}$ from x to y, denoted by $\overrightarrow{\mathfrak{T}}(\mathcal{X})(x,y)$, is the quotient of $\overrightarrow{Di}(\mathcal{X})(x,y)$ by monotonic reparametrization, equipped with the quotient topology. The *trace* of a dipath f in $\mathcal{X}$, denoted by $\overline{f}$ or f if no confusion is possible, is the equivalence class of f modulo monotonic reparametrization. The concatenation of dipaths of $\mathcal{X}$ is compatible with this quotient, inducing a concatenation of traces defined by $\overline{f} \star \overline{g} := \overline{f \star g}$, for all dipaths f and g of $\mathcal{X}$. We will denote by

$$\overrightarrow{\mathbf{P}}\colon \mathsf{dTop} \to \mathsf{Cat}$$

the functor which associates to a dispace $\mathcal{X}$ the *trace category of* $\mathcal{X}$, whose 0-cells are points of X, 1-cells are traces of $\mathcal{X}$, and composition is given by concatenation of traces.

3.1.3. Dihomotopies

The *directed unit interval*, denoted by $\uparrow I$, is the dispace with underlying topological space $[0,1]$ and in which dipaths are non-decreasing maps from $[0,1]$ to $[0,1]$. The *directed cylinder* of a dispace $\mathcal{X}$, denoted by $\mathcal{X}\times\uparrow I$, is the dispace $(X\times[0,1], d(\mathcal{X}\times\uparrow I))$, where

$$d(\mathcal{X}\times\uparrow I) = \{c = (c_1, c_2)\colon [0,1] \to X\times[0,1] \mid c_1 \in dX \text{ and } c_2 \text{ monotonic}\}.$$

Recall from [**11**] that an *elementary dihomotopy between morphisms* $\varphi, \psi\colon \mathcal{X} \to \mathcal{Y}$ is a morphism of dispaces $h\colon \mathcal{X}\times\uparrow I \to \mathcal{Y}$, such that $h(x,0) = \varphi(x)$ and $h(x,1) = \psi(x)$ for all x in X. Dihomotopy between morphisms is defined as the symmetric and transitive closure of elementary dihomotopies. In particular, given a dispace $\mathcal{X}$ and dipaths f and g of $\mathcal{X}$, an *elementary dihomotopy of dipaths* is an elementary dihomotopy between the morphisms $f, g\colon \uparrow I \to \mathcal{X}$. Two dipaths are thus dihomotopic if there exists a zig-zag of elementary dihomotopies connecting them.

3.1.4. Trace diagrams

The *pointed trace diagram in* Top_* of a dispace $\mathcal{X}$ is the functor

$$\overrightarrow{T}_*(\mathcal{X})\colon F\overrightarrow{\mathbf{P}}(\mathcal{X}) \to \mathsf{Top}_*$$

sending a trace $\overline{f}\colon x \to y$ to the pointed topological space $(\overrightarrow{\mathfrak{T}}(\mathcal{X})(x,y), \overline{f})$, and a 1-cell $(\overline{u}, \overline{v})$ of $F\overrightarrow{\mathbf{P}}(\mathcal{X})$ to the continuous map

$$\overline{u} \star \_ \star \overline{v}\colon \overrightarrow{\mathfrak{T}}(\mathcal{X})(x,y) \to \overrightarrow{\mathfrak{T}}(\mathcal{X})(x',y')$$

which sends a trace $\overline{f}$ to $\overline{u} \star \overline{f} \star \overline{v}$. The functor $\overrightarrow{T}_*(\mathcal{X})$ extends to a functor

$$\overrightarrow{T}_*\colon \mathsf{dTop} \to \mathsf{opNat}(\mathsf{Top}_*)$$

whose codomain is the category of natural systems with values in Top_*, defined in Section 2.4, and sends a dispace $\mathcal{X}$ to the pair $(\overrightarrow{\mathbf{P}}(\mathcal{X}), \overrightarrow{T}_*(\mathcal{X}))$. Observe that a morphism of dispaces $\varphi\colon \mathcal{X} \to \mathcal{Y}$ induces continuous maps

$$\varphi_{x,y}\colon \overrightarrow{\mathfrak{T}}(\mathcal{X})(x,y) \to \overrightarrow{\mathfrak{T}}(\mathcal{Y})(\varphi(x), \varphi(y))$$

for all points x, y of X. Thus we obtain natural transformations between the corresponding trace diagrams:

$$\overrightarrow{\varphi}_*\colon \overrightarrow{T}_*(\mathcal{X}) \Rightarrow \overrightarrow{T}_*(\mathcal{Y}).$$

3.1.5. Natural homotopy and natural homology

Recall from [**4, 2**] that the 1^{st} *natural homotopy functor of* $\mathcal{X}$ is the natural system denoted by $\overrightarrow{P}_1(\mathcal{X})\colon F\overrightarrow{\mathbf{P}}(\mathcal{X}) \to \mathsf{Set}$, and defined as the composite

$$F\overrightarrow{\mathbf{P}}(\mathcal{X}) \xrightarrow{\overrightarrow{T}_*(\mathcal{X})} \mathsf{Top}_* \xrightarrow{\pi_0} \mathsf{Set}_*,$$

where π_0 is the 0^{th} homotopy functor with values in Set_*. That is, for a trace $\overline{f}$ on $\mathcal{X}$ from x to y,

$$\overrightarrow{P}_1(\mathcal{X})_{\overline{f}} = (\pi_0(\overrightarrow{\mathfrak{T}}(\mathcal{X})(x,y)), [\overline{f}]),$$

where $[\overline{f}]$ denotes the path-connected component of $\overline{f}$ in $\overrightarrow{\mathfrak{T}}(\mathcal{X})(x,y)$. For $n \geqslant 2$, the n^{th} *natural homotopy functor of* $\mathcal{X}$, denoted by $\overrightarrow{P}_n(\mathcal{X})\colon F\overrightarrow{\mathbf{P}}(\mathcal{X}) \to \mathsf{Gp}$, is defined as the composite

$$F\overrightarrow{\mathbf{P}}(\mathcal{X}) \xrightarrow{\overrightarrow{T}_*(\mathcal{X})} \mathsf{Top}_* \xrightarrow{\pi_{n-1}} \mathsf{Gp},$$

where π_{n-1} is the $(n-1)^{th}$ homotopy functor. Note that for $n \geqslant 3$, the functor $\overrightarrow{P}_n(\mathcal{X})$ has values in Ab. Finally, for $n = 0$, we define $\overrightarrow{P}_0(\mathcal{X})\colon F\overrightarrow{\mathbf{P}}(\mathcal{X}) \to \mathsf{Set}_*$ as the functor sending a trace $\overline{f}$ to the pointed singleton $(\{\overline{f}\}, \overline{f})$.

Using the inclusion functors $J\colon \mathsf{Set}_* \to \mathsf{Act}$ and $K\colon \mathsf{Gp} \to \mathsf{Act}$ defined in Subsection 2.3, the classical homotopy functors can be realized as functors $\pi_n\colon \mathsf{Top}_* \to \mathsf{Act}$, for all $n \geqslant 0$. With this interpretation, we define natural homotopy as functors

$$\overrightarrow{P}_n(\mathcal{X})\colon F\overrightarrow{\mathbf{P}}(\mathcal{X}) \to \mathsf{Act},$$

for all $n \geqslant 0$.

Recall from [4], that for $n \geqslant 1$, the n^{th} *natural homology functor of* $\mathcal{X}$ is the functor denoted by $\overrightarrow{H}_n(\mathcal{X})\colon F\overrightarrow{\mathbf{P}}(\mathcal{X}) \to \mathsf{Ab}$, and defined as the composite

$$F\overrightarrow{\mathbf{P}}(\mathcal{X}) \xrightarrow{\overrightarrow{T}(\mathcal{X})} \mathsf{Top} \xrightarrow{H_{n-1}} \mathsf{Ab},$$

where H_{n-1} is the $(n-1)^{th}$ singular homology functor.

The functors $\overrightarrow{P}_n(\mathcal{X})$ and $\overrightarrow{H}_n(\mathcal{X})$, for $\mathcal{X}$ in dTop, extend to functors

$$\overrightarrow{P}_n\colon \mathsf{dTop} \longrightarrow \mathsf{opNat}(\mathsf{Act}), \quad \text{and} \quad \overrightarrow{H}_n\colon \mathsf{dTop} \longrightarrow \mathsf{opNat}(\mathsf{Ab}),$$

sending a dispace $\mathcal{X}$ to $(\overrightarrow{\mathbf{P}}(\mathcal{X}), \overrightarrow{P}_n(\mathcal{X}))$ and $(\overrightarrow{\mathbf{P}}(\mathcal{X}), \overrightarrow{H}_n(\mathcal{X}))$ respectively.

These definitions are coherent with the classical notions of homotopy and homology as illustrated by the following two results.

Proposition 3.1. *Given a topological space X, the dispace $\mathcal{X} = (X, X^{[0,1]})$ is such that for every x in X,*

$$\overrightarrow{P}_n(\mathcal{X})_{c_x} \cong \pi_n(X, x),$$

where c_x denotes the trace of the constant dipath equal to x.

As a consequence, given a dispace $\mathcal{X} = (X, X^{[0,1]})$, if X is n-connected, then for every $x \in X$, the space $\overrightarrow{\mathfrak{T}}(\mathcal{X})(x,x)$ is also $(n-1)$-connected. Applying the Hurewicz theorem, Proposition 3.1 yields the following result.

Corollary 3.2. *For $n \geqslant 1$ an $(n-1)$-connected topological space X, the dispace $\mathcal{X} = (X, X^{[0,1]})$ is such that for every x in X*

$$\overrightarrow{H}_i(\mathcal{X})_{c_x} \cong H_i(X),$$

for all $i \leqslant n$.

3.2. Directed homotopy as an internal group or a split object

In this subsection we show that for any dispace $\mathcal{X}$, the natural systems $\overrightarrow{P}_n(\mathcal{X})$ and $\overrightarrow{H}_n(\mathcal{X})$ admit composition pairings. We treat the case $\overrightarrow{P}_1(\mathcal{X})$ in Lemma 3.3

separately from the case $\overrightarrow{P}_n(\mathcal{X})$ for $n \geqslant 2$ in Lemma 3.4. Finally, using the equivalence of categories stated in Subsection 2.7, we describe the natural homotopy functor $\overrightarrow{P}_n(\mathcal{X})$ as split objects, or internal group when $n \geqslant 2$, in the category $\mathsf{Cat}_X/\overrightarrow{\mathbf{P}}(\mathcal{X})$. We also treat the case of the natural homology functors $\overrightarrow{H}_n(\mathcal{X})$, for $n \geqslant 1$, which we describe as internal abelian groups in the category $\mathsf{Cat}_X/\overrightarrow{\mathbf{P}}(\mathcal{X})$.

Lemma 3.3. *The natural system of pointed sets $\overrightarrow{P}_1(\mathcal{X})$ admits a composition pairing ν given, for all composable traces $f\colon x \to y, g\colon y \to z$ of $\mathcal{X}$, by*

$$\nu_{f,g}([f'],[g']) = [f' \star g']$$

for any $[f']$ in $\pi_0(\overrightarrow{\mathfrak{T}}(\mathcal{X})(x,y), f)$ and $[g']$ in $\pi_0(\overrightarrow{\mathfrak{T}}(\mathcal{X})(y,z), g)$.

Proof. Observe that the maps $\nu_x\colon \{*\} \to \overrightarrow{P}_1(\mathcal{X})$ for x in X are uniquely determined since the singleton is the initial object in Set_*. For composable traces f and g of $\mathcal{X}$, the maps $\nu_{f,g}$ are well defined and are morphisms of Set_*. Thus, we only have to check the cocycle, unit, and naturality conditions. The cocycle condition is a consequence of the fact that the composition is associative. The right unit condition is verified, since for $f\colon x \to y$, the following diagram

$$\begin{array}{ccc}
\overrightarrow{P}_1(\mathcal{X})_f & \xleftarrow{\;\nu_{f,1_y}\;} & \overrightarrow{P}_1(\mathcal{X})_f \times \overrightarrow{P}_1(\mathcal{X})_{1_y} \\
 & \nwarrow{\scriptstyle \cong} & \uparrow{\scriptstyle id_{\overrightarrow{P}_1(\mathcal{X})_f} \times \nu_y} \\
 & & \overrightarrow{P}_1(\mathcal{X})_f \times \{*\}
\end{array}$$

commutes. Indeed, if c_y denotes the constant path equal to y, we have $[f] = [f \star c_y] = [f \circ \nu_y(*)]$, since $[c_y]$ is the pointed element of $\overrightarrow{P}_1(\mathcal{X})_{1_y}$. The left unit condition is similarly verified. Finally, the naturality condition follows from the associativity of concatenation of traces. Indeed, the equality

$$[(u \star f) \star (g \star v)] = [u \star (f \star g) \star v]$$

holds for any traces u, v, f, g of $\mathcal{X}$ such that the composites are defined. ☐

Lemma 3.4. *For every $n \geqslant 2$, the natural system of groups $\overrightarrow{P}_n(\mathcal{X})$ admits a composition pairing ν defined by*

$$\nu_{f,g}(\sigma, \tau) = \sigma \star \tau,$$

for all composable traces $f\colon x \to y$ and $g\colon y \to z$ of $\mathcal{X}$ and homotopy classes σ in $\pi_{n-1}(\overrightarrow{\mathfrak{T}}(\mathcal{X})(x,y), f)$ and τ in $\pi_{n-1}(\overrightarrow{\mathfrak{T}}(\mathcal{X})(y,z), g)$, where $\sigma \star \tau$ denotes the homotopy class in $\overrightarrow{\mathfrak{T}}(\mathcal{X})(x,z)$ of the map $t \mapsto \sigma(t) \star \tau(t)$.

Proof. First observe that the maps ν_x, for x in X, are uniquely determined since the trivial group is the initial object in Gp. Let us prove that $\overrightarrow{P}_n(\mathcal{X})$ verifies the commutator condition recalled in Subsection 2.6. Given composable 1-cells f and g of $\overrightarrow{\mathbf{P}}(\mathcal{X})$, the 1-cell $(1, g)$ of $F\overrightarrow{\mathbf{P}}(\mathcal{X})$ induces a map

$$\overrightarrow{P}_n(\mathcal{X})(1,g)\colon \pi_{n-1}(\overrightarrow{\mathfrak{T}}(\mathcal{X})(x,y), f) \to \pi_{n-1}(\overrightarrow{\mathfrak{T}}(\mathcal{X})(x,z), f \star g)$$

sending a class σ in $\pi_{n-1}(\overrightarrow{\mathfrak{T}}(\mathcal{X})(x,y), f)$ to the homotopy class of the map $t \mapsto \sigma(t) \star g$, denoted by $\sigma \star g$. We obtain a similar homomorphism from the 1-cell $(f, 1)$, sending τ in $\pi_{n-1}(\overrightarrow{\mathfrak{T}}(\mathcal{X})(y,z),g)$ to the homotopy class of the map $t \mapsto f \star \tau(t)$, denoted by $f \star \tau$.

Let σ, σ' in $\pi_{n-1}(\overrightarrow{\mathfrak{T}}(\mathcal{X})(x,y),f)$ and τ, τ' in $\pi_{n-1}(\overrightarrow{\mathfrak{T}}(\mathcal{X})(y,z),g)$. The following relation

$$(\sigma \star \tau) \cdot (\sigma' \star \tau') = (\sigma \cdot \sigma') \star (\tau \cdot \tau'),$$

where $\cdot$ denotes the product in homotopy groups, holds in $\pi_{n-1}(\overrightarrow{\mathfrak{T}}(\mathcal{X})(x,z), f \star g)$. Indeed, unscrewing the definitions for concatenation of traces and for maps $\mathbb{S}^{n-1} \to \overrightarrow{\mathfrak{T}}(\mathcal{X})(x,z)$ this equality holds even before quotienting by the homotopy relation. Using this relation, we have

$$(\sigma \star g) \cdot (f \star \tau) = (\sigma \cdot f) \star (g \cdot \tau) = \sigma \star \tau = (f \cdot \sigma) \star (\tau \cdot g) = (f \star \tau) \cdot (\sigma \star g)$$

for all σ in $\pi_{n-1}(\overrightarrow{\mathfrak{T}}(\mathcal{X})(x,y),f)$ and τ in $\pi_{n-1}(\overrightarrow{\mathfrak{T}}(\mathcal{X})(y,z),g)$. We conclude via the commutator condition from Section 2.6 that $\overrightarrow{P}_n(\mathcal{X})$ admits a composition pairing, given by $\nu_{f,g}(\sigma,\tau) = \sigma \star \tau$. □

Theorem 3.5. *Let $\mathcal{X} = (X, dX)$ be a dispace. For each $n \leqslant 1$ (resp. $n \geqslant 2$) there exists a split object $\mathcal{C}^n_{\mathcal{X}}$ (resp. internal group $\mathcal{C}^n_{\mathcal{X}}$) in $\mathsf{Cat}_X/\overrightarrow{\mathbf{P}}(\mathcal{X})$ such that*

$$\overrightarrow{P}_n(\mathcal{X})_f = (\mathcal{C}^n_{\mathcal{X}})_f,$$

for all traces f of $\mathcal{X}$, and this assignment is functorial in $\mathcal{X}$.

Proof. Using the equivalences of categories recalled in Subsection 2.8 (resp. in Subsection 2.7), and Lemma 3.3 (resp. Lemma 3.5) we obtain a split object $\mathcal{C}^1_{\mathcal{X}}$ (resp. an internal group $\mathcal{C}^n_{\mathcal{X}}$) in $\mathsf{Cat}_X/\overrightarrow{\mathbf{P}}(\mathcal{X})$ associated to $\overrightarrow{P}_1(\mathcal{X})$ (resp. $\overrightarrow{P}_n(\mathcal{X})$ for $n \geqslant 2$). Let us prove that this assignment defines a functor

$$\mathcal{C}^n_{-} \colon \mathsf{dTop} \to \mathsf{Cat}.$$

Any morphism $\varphi\colon \mathcal{X} \to \mathcal{Y}$ of dispaces induces continuous maps $\varphi_{x,y}\colon \overrightarrow{\mathfrak{T}}(\mathcal{X})(x,y) \to \overrightarrow{\mathfrak{T}}(\mathcal{Y})(\varphi(x),\varphi(y))$ for all points x, y in X such that $\overrightarrow{\mathfrak{T}}(\mathcal{X})(x,y) \neq \emptyset$. We define a functor $\mathcal{C}^n_{\varphi}\colon \mathcal{C}^n_{\mathcal{X}} \to \mathcal{C}^n_{\mathcal{Y}}$ on a 0-cell x and a 1-cell (σ, f) of $\mathcal{C}^n_{\mathcal{X}}$ by setting $\mathcal{C}^n_{\varphi}(x) = \varphi(x)$, and

$$\mathcal{C}^n_{\varphi}(\sigma, f) = (\pi_{n-1}(\varphi_{x,y})(\sigma), \overrightarrow{\mathbf{P}}(\varphi)(f)).$$

Functoriality follows from that of π_{n-1} and $\overrightarrow{\mathbf{P}}$. □

Let us describe the categories $\mathcal{C}^n_{\mathcal{X}}$ for $n \geqslant 0$. The 0-cells of $\mathcal{C}^n_{\mathcal{X}}$ are the points of X, and the set of 1-cells of $\mathcal{C}^n_{\mathcal{X}}$ with source x and target y is given by

$$\mathcal{C}^n_{\mathcal{X}}(x,y) = \coprod_{f \in \overrightarrow{\mathbf{P}}(\mathcal{X})(x,y)} \overrightarrow{P}_n(\mathcal{X})_f.$$

The projection p onto the second factor extends the category $\mathcal{C}^n_{\mathcal{X}}$ into an object of $\mathsf{Cat}_X/\overrightarrow{\mathbf{P}}(\mathcal{X})$.

For $0 \leqslant n \leqslant 1$, the functor p is split by $\epsilon_n\colon \overrightarrow{\mathbf{P}}(X) \to \mathcal{C}^n_{\mathcal{X}}$ defined on any trace f on $\mathcal{X}$ by $\epsilon_n(f) = ([f], f)$. Note that for any trace f, $\overrightarrow{P}_0(\mathcal{X})_f = \{\, [f] \,\}$, hence $\epsilon_0(\overrightarrow{\mathbf{P}}(X)) = \mathcal{C}^0_{\mathcal{X}}$.

The composition is defined, for $0 \leqslant n \leqslant 1$, by

$$([f'], f)([g'], g) = ([f' \star g'], f \star g),$$

for all $[f'] \in \overrightarrow{P}_n(\mathcal{X})_f$ and $[g'] \in \overrightarrow{P}_n(\mathcal{X})_g$. Note that $\mathcal{C}^0_{\mathcal{X}}$ is isomorphic to $\overrightarrow{\mathbf{P}}(\mathcal{X})$.

For $n \geqslant 2$, the functor p is split by the identity map $\eta\colon \overrightarrow{\mathbf{P}}(X) \to \mathcal{C}^n_{\mathcal{X}}$ defined by $\eta(f) = (1_{D_f}, f)$, where 1_{D_f} is the homotopy class of the constant loop equal to f. The inverse map is given by the inverse in each homotopy group, that is $(\sigma, f)^{-1} = (\sigma^{-1}, f)$. Recall that the product in $\mathsf{Cat}_X/\overrightarrow{\mathbf{P}}(\mathcal{X})$ is the fibred product over $\overrightarrow{\mathbf{P}}(\mathcal{X})$, so we can use the internal multiplication in each homotopy group to define the multiplication map μ by setting $\mu((\sigma, f), (\sigma', f)) = (\sigma \cdot \sigma', f)$. The composition of (σ, f) and (τ, g), for homotopy classes σ and τ above f and g respectively, is given by

$$(\sigma, f) \star_0 (\tau, g) = (\nu_{f,g}(\sigma, \tau), f \star g) = (\sigma \star \tau, f \star g).$$

Recall from Remark 2.1 that as a consequence of the commutation condition and the triviality of the compatibility criterion for natural transformations, the categories $\mathsf{NatSys}(\overrightarrow{\mathbf{P}}(\mathcal{X}), \mathsf{Ab})$ and $\mathsf{NatSys}_\nu(\overrightarrow{\mathbf{P}}(\mathcal{X}), \mathsf{Ab})$ coincide. For all $n \geqslant 1$, the natural system $\overrightarrow{H}_n(\mathcal{X})$ is thus equipped with a composition pairing, and via the equivalence

$$\mathsf{Ab}(\mathsf{Cat}_X/\overrightarrow{\mathbf{P}}(\mathcal{X})) \cong \mathsf{NatSys}_\nu(\overrightarrow{\mathbf{P}}(\mathcal{X}), \mathsf{Ab})$$

we obtain an internal abelian group $\mathcal{A}^n_{\mathcal{X}}$ in the category $\mathsf{Cat}_X/\overrightarrow{\mathbf{P}}(\mathcal{X})$. Moreover, using similar arguments as in the proof of Proposition 3.5, one proves that the assignment $\mathcal{A}^n_{-}\colon \mathsf{dTop} \to \mathsf{Cat}$ is functorial for all $n \geqslant 1$.

3.3. Fundamental category of a dispace

The *fundamental category* of a dispace $\mathcal{X}$, denoted by $\overrightarrow{\mathbf{\Pi}}(\mathcal{X})$, is the homotopy category of $\overrightarrow{\mathbf{P}}(\mathcal{X})$ when interpreted as a 2-category. Explicitly, the trace category $\overrightarrow{\mathbf{P}}(\mathcal{X})$ can be extended into a $(2,1)$-category by adding a 2-cell corresponding to every dihomotopy $h\colon \uparrow I \times \uparrow I \to \mathcal{X}$ between traces f and g such that $f(0) = h(0, s) = g(0)$ and $f(1) = h(1, s) = g(1)$ for all $s \in \uparrow I$. The fundamental category is the quotient of this $(2,1)$-category by the congruence generated by these 2-cells. We refer the reader to [**7, 11**] for a fuller treatment of fundamental categories of dispaces. The operation described above defines a functor

$$\overrightarrow{\mathbf{\Pi}}\colon \mathsf{dTop} \to \mathsf{Cat}$$

sending a dispace to its fundamental category.

Given a dispace $\mathcal{X}$, consider the quotient functor $\pi\colon \overrightarrow{\mathbf{P}}(\mathcal{X}) \to \overrightarrow{\mathbf{\Pi}}(\mathcal{X})$, which is the identity on 0-cells and which associates a trace f to its class $[f]$ modulo path-connectedness. Similarly to [**8**, Theorem 1], we have the following result.

Proposition 3.6. *Given a dispace $\mathcal{X}$, suppose that there exists a functorial section σ of the functor $\pi\colon \overrightarrow{\mathbf{P}}(\mathcal{X}) \to \overrightarrow{\mathbf{\Pi}}(\mathcal{X})$. Then the natural system $\overrightarrow{P}_n(\mathcal{X})$ is trivial for all $n \geqslant 2$.*

Proof. We show that each trace space is contractible. Let $\overrightarrow{t}(\mathcal{X})$ (resp. $\overrightarrow{t}(\mathcal{X}) \times [0,1]$) denote the natural system of topological spaces on $\overrightarrow{\mathbf{\Pi}}(\mathcal{X})$ which maps a class $[f]\colon x \to$

y to the space $[f]$ (resp. $[f] \times [0,1]$), where $[f]$ is viewed as a subspace of $\overrightarrow{\mathfrak{T}}(\mathcal{X})(x,y)$. For a dipath g in $[f]$, denote by $g_{|[s,r]}$ the restriction of g to the interval $[s,r] \subseteq [0,1]$. Now we define a natural transformation $H\colon \overrightarrow{t}(\mathcal{X}) \times [0,1] \Rightarrow \overrightarrow{t}(\mathcal{X})$ such that the component $H_{[f]}$ sends a pair $(g,s) \in [f] \times [0,1]$ to the dipath

$$H_{[f]}(g,s)(t) = \begin{cases} g(t) & t \in [0, \frac{s}{2}], \\ \sigma([g_{|[\frac{s}{2}, 1-\frac{s}{2}]}]) & t \in [\frac{s}{2}, 1-\frac{s}{2}], \\ g(t) & t \in [1-\frac{t}{2}, 1]. \end{cases}$$

Then $H_{[f]}(g,-)$ is a homotopy from g to $\sigma([f])$ for every g in $[f]$. Indeed, when $s=0$, we have $\sigma([g]) = \sigma([f])$ and when $s=1$ we have that $\sigma([g_{|[\frac{s}{2},1-\frac{s}{2}]}])$ is the point $g(\frac{1}{2})$ because the map σ is the identity on points. Thus every connected component of every trace space of $\mathcal{X}$ is contractible. □

Remark 3.7. Recall that the homotopy groups $\pi_n(X,x)$ and $\pi_n(X,y)$ of a topological space X are isomorphic for any path-connected points x and y of X. In the definition of natural homotopy we consider the homotopy groups of trace spaces $\overrightarrow{\mathfrak{T}}(\mathcal{X})(x,y)$ based at each trace f. However, for natural homotopy, choosing a single base-point in each connected component of each trace space of a dispace $\mathcal{X}$ requires a section as described above, and for such a section to define a natural system of groups on the fundamental category, it must be functorial. By the proposition, in this case the only non-trivial homotopy functor is $\overrightarrow{P}_1(\mathcal{X})$. In the case of natural homotopy, choosing a base point in each path-connected component is thus not possible in non-trivial cases.

Finally, note that natural homology decomposes, for any trace $f\colon x \to y$ on $\mathcal{X}$, into

$$\overrightarrow{H}_n(\mathcal{X})_f \cong \bigoplus_{[f] \in \overrightarrow{\Pi}(\mathcal{X})(x,y)} H_{n-1}([f]),$$

where $H_{n-1}([f])$ is the $(n-1)^{th}$ singular homology of the connected space $[f] \subset \overrightarrow{\mathfrak{T}}(\mathcal{X})(x,y)$.

4. Time-reversal invariance

In this section we study the effect of reversal of time on homotopical and homological invariants of dispaces. We show that the natural homotopy and homology functors are *time-symmetric*, that is do not capture the reversal of time. We then show that using composition pairings, the categorical interpretations $\mathcal{C}^n_-\colon \mathsf{dTop} \to \mathsf{Cat}$ and $\mathcal{A}^n_-\colon \mathsf{dTop} \to \mathsf{Cat}$ of natural homotopy and homology capture the reversal of time via duality of categories, and are thus said to be *time-reversal*. First, in Subsection 4.1 we define the notion of time-reversed dispace and show that natural homotopy and homology are time-symmetric. We then prove the main result of this section in Subsection 4.2, which states that the functors $\mathcal{C}^n_-$ and $\mathcal{A}^n_-$ are time-reversal.

4.1. Time-reversal in dispaces

Given a dispace $\mathcal{X} = (X, dX)$, for any dipath f in dX, we denote by $f^\sharp$ the dipath defined by $f^\sharp(t) = f(1-t)$, for all t in $[0,1]$. We define its *time-reversed dispace*, or *opposite dispace*, as the dispace $\mathcal{X}^\sharp = (X, dX^\sharp)$ where $dX^\sharp$ is defined by $dX^\sharp = \{f^\sharp \mid f \in dX\}$.

Note that $dX^\sharp$ is easily verified to be a set of directed paths according to the conditions listed in Section 3.1.1. This defines a functor $(-)^\sharp\colon \mathsf{dTop} \to \mathsf{dTop}$, sending a dispace $\mathcal{X}$ to its opposite. Notice that if $\phi\colon \mathcal{X} \to \mathcal{Y}$ is a morphism of dispaces, this functor leaves the continuous map $\phi\colon X \to Y$ unchanged, since $(\phi_* f)^\sharp = \phi_*(f^\sharp)$.

4.1.1. Time-reversal properties

A dispace $\mathcal{X}$ is called *time-symmetric* if the dispaces $\mathcal{X}$ and $\mathcal{X}^\sharp$ are isomorphic. In that case, by functoriality of $\overrightarrow{\mathbf{P}}$ and $\mathcal{C}^n_-$, there exist covariant isomorphisms

$$\overrightarrow{\mathbf{P}}(\mathcal{X}) \xrightarrow{\sim} \overrightarrow{\mathbf{P}}(\mathcal{X}^\sharp), \qquad \overrightarrow{P}_n(\mathcal{X}) \xrightarrow{\sim} \overrightarrow{P}_n(\mathcal{X}^\sharp), \qquad \text{and} \qquad \mathcal{C}^n_{\mathcal{X}} \xrightarrow{\sim} \mathcal{C}^n_{\mathcal{X}^\sharp}.$$

A dispace $\mathcal{X} = (X, dX)$ is called *time-contractible* when $dX = dX^\sharp$. In that case any dipath is reversible, that is $f \in dX$ implies $f^\sharp \in dX$. Note that for a dispace $\mathcal{X} = (X, dX)$, $dX = X^{[0,1]}$ implies that $\mathcal{X}$ is time-contractible but the converse is not true in general (for example, any directed space in which only the constant paths are directed is time-contractible). Thus the directed homotopy of a time-contractible dispace $\mathcal{X}$ does not necessarily coincide with the homotopy of its underlying space X in the sense of Proposition 3.1.

A functor $F\colon \mathsf{dTop} \to \mathbf{V}$ is *time-symmetric with respect to a category* $\mathbf{V}$ if the following diagram

$$\begin{array}{ccc} \mathsf{dTop} & \xrightarrow{F} & \mathbf{V} \\ {\scriptstyle (-)^\sharp}\downarrow & & \downarrow{\scriptstyle \|} \\ \mathsf{dTop} & \xrightarrow[F]{} & \mathbf{V} \end{array}$$

commutes up to isomorphism. Such a functor is *strongly time-symmetric with respect to* $\mathbf{V}$ if there exists a natural isomorphism $F((-)^\sharp) \Rightarrow F$. A functor $F\colon \mathsf{dTop} \to \mathsf{Cat}$ is *time-reversal* if the following diagram

$$\begin{array}{ccc} \mathsf{dTop} & \xrightarrow{F} & \mathsf{Cat} \\ {\scriptstyle (-)^\sharp}\downarrow & & \downarrow{\scriptstyle (-)^o} \\ \mathsf{dTop} & \xrightarrow[F]{} & \mathsf{Cat} \end{array}$$

commutes up to isomorphism. Such a functor is *strongly time-reversal* if there exists a natural isomorphism $F((-)^\sharp) \Rightarrow F(-)^o$.

4.1.2. Time-symmetry of directed homology and homotopy

For any dispace $\mathcal{X}$ the equalities

$$\overrightarrow{\mathbf{P}}(\mathcal{X}^\sharp) = \overrightarrow{\mathbf{P}}(\mathcal{X})^o \qquad \text{and} \qquad \overrightarrow{\mathbf{\Pi}}(\mathcal{X}^\sharp) = \overrightarrow{\mathbf{\Pi}}(\mathcal{X})^o$$

hold in Cat, and hence the functors $\overrightarrow{\mathbf{P}}$ and $\overrightarrow{\mathbf{\Pi}}$ are strongly time-reversal. The functor which sends a dispace $\mathcal{X}$ to $F\overrightarrow{\mathbf{P}}(\mathcal{X})$ is strongly time-symmetric with respect to Cat.

Indeed, the isomorphism of categories

$$F^\sharp \colon F\overrightarrow{\mathbf{P}}(\mathcal{X}) \to F(\overrightarrow{\mathbf{P}}(\mathcal{X}^\sharp))$$

sending a trace f to its opposite $f^\sharp$ and a 1-cell of (u, v) in $F\overrightarrow{\mathbf{P}}(\mathcal{X})$ to the 1-cell $(v^\sharp, u^\sharp)$, is the component at $\mathcal{X}$ of a natural isomorphism. Note that the functors $\overrightarrow{P}_n$ and $\overrightarrow{H}_n$ are not strongly time-symmetric with respect to $\mathsf{opNat}(\mathsf{Act})$. However, consider the category $\mathsf{Diag}(\mathsf{Act})$, whose objects are the pairs $(\mathcal{C}, F)$, where $\mathcal{C}$ is a category and $F \colon \mathcal{C} \to \mathsf{Act}$ is a functor, and whose morphisms are pairs $(\Phi, \tau) \colon (\mathcal{C}, F) \to (\mathcal{C}', F')$, where $\Phi \colon \mathcal{C} \to \mathcal{C}'$ is a functor and $\tau \colon F \to F'\Phi$ is a natural transformation, with natural composition. By definition, the functors $\overrightarrow{P}_n$ and $\overrightarrow{H}_n$ are strongly time-symmetric with respect to $\mathsf{Diag}(\mathsf{Act})$.

For $n \geqslant 0$, we compare the functors $\overrightarrow{P}_n(\mathcal{X})$ and $\overrightarrow{P}_n(\mathcal{X}^\sharp)$ in $\mathsf{NatSys}(\overrightarrow{\mathbf{P}}(\mathcal{X}), \mathsf{Act})$ by precomposing the latter with the isomorphism $F^\sharp$. Observe that, for all points x, y in X, we have homeomorphisms

$$\alpha_{x,y} \colon \overrightarrow{\mathfrak{T}}(\mathcal{X})(x, y) \to \overrightarrow{\mathfrak{T}}(\mathcal{X}^\sharp)(y, x)$$

sending a trace f to its opposite $f^\sharp$. These induce group isomorphisms $\overrightarrow{P}_n(\mathcal{X})_f \xrightarrow{\sim} \overrightarrow{P}_n(\mathcal{X}^\sharp)_{f^\sharp}$ for all $n \geqslant 2$. By definition, $(F^\sharp)^* \overrightarrow{P}_n(\mathcal{X}^\sharp)_f = \overrightarrow{P}_n(\mathcal{X}^\sharp)_{f^\sharp}$, so we get components of a natural isomorphism

$$\begin{gathered}\alpha_f \colon \overrightarrow{P}_n(\mathcal{X})_f \longrightarrow (F^\sharp)^* \overrightarrow{P}_n(\mathcal{X}^\sharp)_f, \\ [\sigma] = [(s, t) \mapsto \sigma_s(t)] \longmapsto [(s, t) \mapsto \sigma_s(1 - t)] =: [\sigma^\sharp],\end{gathered}$$

where s is the parameter for the map $\mathbb{S}^{n-1} \to \overrightarrow{\mathfrak{T}}(\mathcal{X})(x, y)$, and t is the parameter for the dipath σ_s. Thus the pair $(F^\sharp, \alpha)$ is an isomorphism in the category $\mathsf{Diag}(\mathsf{Gp})$. Such an isomorphism can similarly be established in the category $\mathsf{Diag}(\mathsf{Set}_*)$ for natural homotopy in the case $n = 1$. The functor $F^\sharp$ and the isomorphisms are components at $\mathcal{X}$ of natural isomorphisms, hence $\overrightarrow{P}_n$ is strongly time-symmetric with respect to $\mathsf{Diag}(\mathsf{Act})$ for all $n \geqslant 1$.

A corresponding isomorphism for natural homology, $\overrightarrow{H}_n(\mathcal{X}) \cong \overrightarrow{H}_n(\mathcal{X}^\sharp)$, can be similarly established in $\mathsf{Diag}(\mathsf{Ab})$ using the functor $F^\sharp$ and the homeomorphisms $\alpha_{x,y}$, showing that $\overrightarrow{H}_n$ is strongly time-symmetric with respect to $\mathsf{Diag}(\mathsf{Ab})$ for all $n \geqslant 1$.

4.2. Time-reversibility of natural homotopy

Following Theorem 3.5, the category $\mathcal{C}^n_{\mathcal{X}}$ with the projection $p \colon \mathcal{C}^n_{\mathcal{X}} \to \overrightarrow{\mathbf{P}}(\mathcal{X})$ onto the second factor is an internal group in $\mathsf{Cat}_X/\overrightarrow{\mathbf{P}}(\mathcal{X})$. On the other hand, the category $\mathcal{C}^n_{\mathcal{X}^\sharp}$ obtained from the natural system $\overrightarrow{P}_n(\mathcal{X}^\sharp)$ via the construction given in Section 3.2 has 0-cells $x \in X$, while 1-cells are of the form $(\sigma^\sharp, f^\sharp) \colon y \to x$ where $\sigma^\sharp \in \overrightarrow{P}_n(\mathcal{X}^\sharp)_{f^\sharp}$ and $f^\sharp \colon y \to x$ is a trace in $\mathcal{X}^\sharp$. Composition is given by

$$(\tau^\sharp, g^\sharp) \star_0^{\mathcal{C}^n_{\mathcal{X}^\sharp}} (\sigma^\sharp, f^\sharp) = (\tau^\sharp \star \sigma^\sharp, g^\sharp \star f^\sharp).$$

We denote the associated projection by $p^\sharp$. We define for $n \geqslant 2$

$$I_n(\mathcal{X}) \colon \mathcal{C}^n_{\mathcal{X}^\sharp} \to (\mathcal{C}^n_{\mathcal{X}})^o,$$

the isomorphism of categories which is the identity on 0-cells, and which sends a 1-cell $(\sigma^\sharp, f^\sharp)$ of $\mathcal{C}^n_{\mathcal{X}^\sharp}$ to $(\sigma, f)^o$. The functoriality of $I_n(\mathcal{X})$ follows from the equality

$$(\tau^\sharp, g^\sharp) \star_0^{\mathcal{C}^n_{\mathcal{X}^\sharp}} (\sigma^\sharp, f^\sharp) = (\tau^\sharp \star \sigma^\sharp, g^\sharp \star f^\sharp) = ((\sigma \star \tau)^\sharp, (f \star g)^\sharp).$$

The opposite group $(\mathcal{C}^n_{\mathcal{X}})^o$ can be interpreted as an internal group in $\mathsf{Cat}_X/\overrightarrow{\mathbf{P}}(\mathcal{X}^\sharp)$ by composing the projection $p^o\colon (\mathcal{C}^n_{\mathcal{X}})^o \to \overrightarrow{\mathbf{P}}(\mathcal{X})^o$ with the canonical isomorphism $\overrightarrow{\mathbf{P}}(\mathcal{X})^o \simeq \overrightarrow{\mathbf{P}}(\mathcal{X}^\sharp)$. We denote by $\widetilde{p}^o$ this composition. Then the following diagram commutes

$$\begin{array}{ccc} \mathcal{C}^n_{\mathcal{X}^\sharp} & \xrightarrow{\;I_n(\mathcal{X})\;} & (\mathcal{C}^n_{\mathcal{X}})^o \\ & {\scriptstyle p^\sharp}\searrow \quad \swarrow{\scriptstyle \widetilde{p}^o} & \\ & \overrightarrow{\mathbf{P}}(\mathcal{X}^\sharp). & \end{array}$$

We thereby deduce that $I_n(\mathcal{X})$ is a morphism of $\mathsf{Cat}_X/\overrightarrow{\mathbf{P}}(\mathcal{X}^\sharp)$. Furthermore, it is a group isomorphism, since the fibre groups above a 1-cell $f^\sharp$ of $\overrightarrow{\mathbf{P}}(\mathcal{X}^\sharp)$ are isomorphic:

$$(\mathcal{C}^n_{\mathcal{X}})^o_{f^\sharp} = (\mathcal{C}^n_{\mathcal{X}})_f = \overrightarrow{P}_n(\mathcal{X})_f \cong \overrightarrow{P}_n(\mathcal{X}^\sharp)_{f^\sharp} = (\mathcal{C}^n_{\mathcal{X}^\sharp})_{f^\sharp}.$$

An isomorphism $\mathcal{C}^1_{\mathcal{X}^\sharp} \cong (\mathcal{C}^1_X)^o$ can similarly be established in the category $\mathsf{Split}(\mathsf{Cat}_X/\overrightarrow{\mathbf{P}}(\mathcal{X}^\sharp))$. We have thus proved the following result.

Proposition 4.1. *Given a dispace $\mathcal{X} = (X, dX)$, $\mathcal{C}^n_{\mathcal{X}^\sharp}$ and $(\mathcal{C}^n_{\mathcal{X}})^o$ are isomorphic in $\mathsf{Gp}(\mathsf{Cat}_X/\overrightarrow{\mathbf{P}}(\mathcal{X}^\sharp))$ for all $n \geqslant 2$, and in $\mathsf{Split}(\mathsf{Cat}_X/\overrightarrow{\mathbf{P}}(\mathcal{X}^\sharp))$ for $n = 1$. In particular, the functors $\mathcal{C}^n_-$ are time-reversal for all $n \geqslant 1$.*

For any $n \geqslant 0$, the functors $I_n(\mathcal{X})$ give components of a natural transformation. Indeed, by precomposing (resp. composing) the functor $\mathcal{C}^n_-$ with $(\cdot)^\sharp$ (resp. $(\cdot)^o$), any morphism $\phi\colon \mathcal{X} \to \mathcal{Y}$ of dispaces yields a commuting diagram

$$\begin{array}{ccc} \mathcal{C}^n_{\mathcal{X}^\sharp} & \xrightarrow{\;I_n(\mathcal{X})\;} & (\mathcal{C}^n_{\mathcal{X}})^o \\ {\scriptstyle \mathcal{C}^n_{\phi^\sharp}}\downarrow & & \downarrow{\scriptstyle (\mathcal{C}^n_{\phi})^o} \\ \mathcal{C}^n_{\mathcal{Y}^\sharp} & \xrightarrow[\;I_n(\mathcal{Y})\;]{} & (\mathcal{C}^n_{\mathcal{Y}})^o \end{array}$$

in Cat. Furthermore, as shown above, these components are all isomorphisms, that is there exists a natural isomorphism

$$I_n : \mathcal{C}^n_{(-)^\sharp} \Longrightarrow (\mathcal{C}^n_-)^o.$$

We have thus proved the following result.

Theorem 4.2. *For any $n \geqslant 0$, the functor $\mathcal{C}^n_-\colon \mathsf{dTop} \to \mathsf{Cat}$ is strongly time-reversal.*

A consequence of Theorem 4.2 is that for any dispace $\mathcal{X}$, the category $\mathcal{C}^n_{\mathcal{X}}$ is dual to the category $\mathcal{C}^n_{\mathcal{X}^\sharp}$. It can similarly be shown that the functor $\mathcal{A}^n_-\colon \mathsf{dTop} \to \mathsf{Cat}$ associated to natural homology is strongly time-reversal for all $n \geqslant 1$. In the particular case of a time-symmetric space $\mathcal{X}$, the category $\mathcal{C}^n_{\mathcal{X}}$ is self-dual, *i.e.* there exists a covariant isomorphism of categories

$$\mathcal{C}^n_{\mathcal{X}} \cong (\mathcal{C}^n_{\mathcal{X}})^o.$$

The time-reversibility of a functor with values in Cat is expressed via duality of categories. However, given some category $\mathbf{V}$, we can define a notion of time-reversal with respect to $\mathsf{opNat}(\mathbf{V})$ which is compatible with the interpretation of natural systems with composition pairings as categories when $\mathbf{V} = \mathsf{Act}$. Consider the functor

$$(-)^\flat \colon \mathsf{opNat}(\mathbf{V}) \to \mathsf{opNat}(\mathbf{V})$$

which sends a pair $(\mathcal{C}, D)$ to the pair $(\mathcal{C}^o, (F^o)^* D)$, where $F^o \colon F(\mathcal{C}^o) \to F\mathcal{C}$ is the covariant functor sending a 0-cell f^o of $F\mathcal{C}^o$ to f, and a 1-cell (v^o, u^o) to (u, v). To a morphism

$$(\Phi, \alpha) \colon (\mathcal{C}, D) \to (\mathcal{C}', D')$$

of $\mathsf{opNat}(\mathbf{V})$, the functor $(-)^\flat$ associates the morphism (Φ^o, α^o), where Φ^o is the opposite functor $\mathcal{C}^o \to (\mathcal{C}')^o$, and where the component $\alpha^o_{f^o}$ at f^o a 1-cell of $\mathcal{C}^o$ is the component α_f of α at f.

Then for F a functor $\mathsf{dTop} \to \mathsf{opNat}(\mathbf{V})$, we say that F is *time-reversal with respect to* $\mathsf{opNat}(\mathbf{V})$ if the following diagram

$$\begin{array}{ccc} \mathsf{dTop} & \xrightarrow{F} & \mathsf{opNat}(\mathbf{V}) \\ {\scriptstyle (-)^\sharp}\downarrow & & \downarrow{\scriptstyle (-)^\flat} \\ \mathsf{dTop} & \xrightarrow[F]{} & \mathsf{opNat}(\mathbf{V}) \end{array}$$

commutes up to isomorphisms of the form (id, α). Explicitly, this means that if $F(\mathcal{X}) = (\mathcal{C}, D)$, then $F(\mathcal{X}^\sharp) = (\mathcal{C}^o, D')$ with $(F^o)^* D$ naturally isomorphic to D'.

Given F a functor $\mathsf{dTop} \to \mathsf{opNat}_\nu(\mathsf{Act})$, we can extend to the following diagram

$$\begin{array}{ccccc} \mathsf{dTop} & \xrightarrow{F} & \mathsf{opNat}_\nu(\mathsf{Act}) & \xrightarrow{\mathcal{E}} & \mathsf{Cat} \\ {\scriptstyle (-)^\sharp}\downarrow & & \downarrow{\scriptstyle (-)^\flat} & & \downarrow{\scriptstyle (-)^o} \\ \mathsf{dTop} & \xrightarrow[F]{} & \mathsf{opNat}_\nu(\mathsf{Act}) & \xrightarrow[\mathcal{E}]{} & \mathsf{Cat}, \end{array} \qquad (1)$$

where the functor $\mathcal{E} \colon \mathsf{opNat}(\mathsf{Act}) \to \mathsf{Cat}$ sends a pair $(\mathcal{C}, D, \nu)$ to the category in $\mathsf{Cat}_{\mathcal{C}_0}/\mathcal{C}$ defined using the construction described in Subsection 2.7 and Subsection 2.8. The rightmost square commutes strictly. Indeed, denoting by $\mathcal{E}_{(\mathcal{C},D,\nu)}$ the category obtained from the natural system (D, ν) on the category $\mathcal{C}$, we have that $\mathcal{E}_{(\mathcal{C},D,\nu)^\flat}$ is the category with the same 0-cells as $\mathcal{C}^o$ and in which 1-cells are defined via the hom-sets

$$\mathcal{E}(y, x) = \coprod_{f^o \in \mathcal{C}^o(y,x)} D_f\,,$$

since by definition, $D^\flat_{f^o} = D_f$. On the other hand, $\mathcal{E}_{(\mathcal{C},D,\nu)}$ has the same 0-cells as $\mathcal{C}$ and 1-cells are defined via the hom-sets

$$\mathcal{E}(x, y) = \coprod_{f \in \mathcal{C}(x,y)} D_f\,.$$

Thus $\mathcal{E}_{(\mathcal{C},D,\nu)^\flat}$ coincides with $\mathcal{E}^o_{(\mathcal{C},D,\nu)}$. Hence, if the leftmost square in diagram (1) commutes up to isomorphism, then the outer square commutes up to isomorphism. This proves the following result.

Proposition 4.3. *Any functor* $F\colon \mathsf{dTop} \to \mathsf{opNat}_\nu(\mathsf{Act})$ *which is time-reversal with respect to* $\mathsf{opNat}(\mathsf{Act})$ *can be extended into a time-reversal functor* $\mathcal{E} \circ F\colon \mathsf{dTop} \to \mathsf{Cat}$.

5. Relative directed homotopy

In this section, we introduce a notion of relative homotopy for dispaces, and establish a long exact sequence, as in the case of regular topological spaces, using the homological category structure on Act as introduced by Grandis in [**12**].

Given a pointed pair of topological spaces (X, A), *i.e.* a space X and a subspace $A \subseteq X$ pointed at $x \in A$, for $n \geqslant 1$, we denote $\pi_n(X, A)$ the n^{th} relative homotopy of (X, A). Note that $\pi_1(X, A)$ is considered as a pointed set, the pointed element being the class of paths f such that f is homotopic to a path g with its image contained in A. The assignment of relative homotopy groups to a pointed pair of spaces is functorial. Its domain is the category of pointed pairs of topological spaces, denoted by Top_{*2}, in which a morphism $f\colon (X, A, x) \to (Y, B, y)$ is a continuous map $f\colon X \to Y$ such that $f(A) \subseteq f(B)$ and $f(x) = y$, and its codomain is Set_* for $n = 1$, Gp for $n = 2$ and Ab for $n \geqslant 3$. We will therefore consider these as functors with values in Act for all $n \geqslant 1$.

5.1. Natural relative homotopy sequence

Let us extend relative homotopy to dispaces. A *directed subspace* of a dispace $\mathcal{X} = (X, dX)$ is a dispace $\mathcal{A} = (A, dA)$ such that $A \subseteq X$ is a subspace and $dA \subseteq dX$. We define *the category of pairs of dispaces*, denoted dTop_2, as the category having objects $(\mathcal{X}, \mathcal{A})$ with $\mathcal{A}$ a directed subspace of $\mathcal{X}$, and in which a morphism $\varphi\colon (\mathcal{X}, \mathcal{A}) \to (\mathcal{Y}, \mathcal{B})$ is a morphism $\varphi\colon \mathcal{X} \to \mathcal{Y}$ of dispaces such that $\varphi(A) \subseteq B$ and $\varphi_*(dA) \subseteq dB$.

For $n \geqslant 2$, the n^{th} *natural system of relative directed homotopy* associated to the pair $(\mathcal{X}, \mathcal{A})$ is the natural system on $\overrightarrow{\mathbf{P}}(\mathcal{A})$, denoted by $\overrightarrow{P}_n(\mathcal{X}, \mathcal{A})$, sending a dipath $f\colon x \to y$ in dA to the $(n-1)^{th}$ relative homotopy group of the pointed pair $(\overrightarrow{\mathfrak{T}}(\mathcal{X})(x, y), \overrightarrow{\mathfrak{T}}(\mathcal{A})(x, y), f)$, and whose group homomorphisms induced by extensions (u, v) are defined by concatenation of paths as in Section 3.1.4. Using a notion of relative trace diagrams $\mathsf{dTop}_2 \to \mathsf{Top}_{*2}$ and similar arguments as those in Section 3, it can be shown that, for each $n \geqslant 2$, $\overrightarrow{P}_n(\mathcal{X}, \mathcal{A})$ extends to functors

$$\overrightarrow{P}_n\colon \mathsf{dTop}_2 \to \mathsf{opNat}(\mathsf{Act}).$$

Grandis shows in [**12**, Theorem 6.4.9] that given a pointed pair of topological spaces (X, A), there is a long exact sequence in Act:

$$\cdots \longrightarrow \pi_n(A) \longrightarrow \pi_n(X) \longrightarrow \pi_n(X, A) \xrightarrow{\partial_n} \pi_{n-1}(A) \longrightarrow \cdots$$
$$\cdots \xrightarrow{v} \pi_1(X) \xrightarrow{f} (\pi_1(X, A), \pi_1(X)) \xrightarrow{g} \pi_0(A) \xrightarrow{h} \pi_0(X) \longrightarrow \pi_0(X, A) \longrightarrow 0.$$

Note that this assignment is functorial from Top_{*2} to the category of long exact sequences in Act. All of the morphisms of this sequence are induced by inclusions, except the last non-trivial homomorphism and the homomorphisms ∂_n, which are given by restriction to the distinguished face: $\partial_n([\sigma]) = [\sigma]_{|I^{n-1}}$. Also recall that we have not defined a relative homotopy group for $n = 0$; the object $\pi_0(X, A)$ is defined to be $Cok(h)$. It is thus the quotient of the set of path-connected components of X obtained by identifying the components which intersect A.

All the terms above $\pi_1(X)$ are groups, and the existence of this long exact sequence in Gp is well known. Since exactness is carried into Act, this induces that the sequence is exact in Act up to this object. As observed above, $\pi_1(X, A)$ is not a group, but a pointed set. There is a right action of the group $\pi_1(X)$ on this pointed set given by concatenation; the elements of $\pi_1(X, A)$ have ending point 0_X, and so we can concatenate with elements of $\pi_1(X)$ on the right. The sequence is exact at $\pi_1(X) = (|\pi_1(X)|, \pi_1(X))$ since the image of v is precisely $Ker(f')$; indeed, $Fix_{\pi_1(X)}(0_{\pi_1(X,A)}) = \pi_1(A)$. The map g sends $(\tau, \sigma) \in (\pi_1(X, A), \pi_1(X))$ to the path-connected component of $\tau(0) \in A$. We therefore view it is a pointed set map from $\pi_1(X, A)$ to $\pi_0(A)$. The sequence is exact at $(\pi_1(X, A), \pi_1(X))$ because the antecedents under g of the pointed element of $\pi_0(A)$, namely the component containing the base point x, are elements of the orbit of the pointed element 0 of $\pi_1(X, A)$, *i.e.* $0 \cdot \pi_1(X)$. This coincides with the image of f since it is defined by sending $\sigma \in \pi_1(X)$ to $(0 \cdot \sigma, \sigma)$. Lastly, exactness at $\pi_0(A)$ is a consequence of the inverse image under h of the pointed element $[x]$ in $\pi_0(X)$ being exactly $\{[x]\}$, the pointed element in $\pi_0(A)$, since h is induced by the inclusion $A \hookrightarrow X$. Furthermore, for $\tau \in \pi_1(X, A)$, $g(\tau)$ is necessarily in the same path connected component as x. Thus, the image of g coincides with the kernel of h.

We endow the category $\mathsf{NatSys}(\overrightarrow{\mathbf{P}}(\mathcal{A}), \mathsf{Act})$ with the structure of a homological category by letting null morphisms be those natural transformations which are null component-wise in Act. A sequence of natural systems of actions is then *exact* when it is point-wise exact in Act. As a consequence we obtain the following long exact sequence of natural homotopy systems:

Theorem 5.1. *Let $\mathcal{X}$ be a dispace and $\mathcal{A}$ be a directed subspace of $\mathcal{X}$. There is an exact sequence in* $\mathsf{NatSys}(\overrightarrow{\mathbf{P}}(\mathcal{A}), \mathsf{Act})$*:*

$$\cdots \longrightarrow \overrightarrow{P}_n(\mathcal{A}) \longrightarrow \overrightarrow{P}_n(\mathcal{X}) \longrightarrow \overrightarrow{P}_n(\mathcal{X}, \mathcal{A}) \overset{\partial_n}{\to} \overrightarrow{P}_{n-1}(\mathcal{A}) \longrightarrow \cdots$$
$$\cdots \to \overrightarrow{P}_2(\mathcal{A}) \overset{v}{\to} \overrightarrow{P}_2(\mathcal{X}) \overset{f}{\to} (\overrightarrow{P}_2(\mathcal{X}, \mathcal{A}), \overrightarrow{P}_2(\mathcal{X})) \overset{g}{\to} \overrightarrow{P}_1(\mathcal{A}) \overset{h}{\to} \overrightarrow{P}_1(\mathcal{X}) \to \overrightarrow{P}_1(\mathcal{X}, \mathcal{A}) \to 0.$$

A dispace $\mathcal{X}$ is called *dicontractible* if all its natural homotopy functors $\overrightarrow{P}_n(\mathcal{X})$ are trivial, e.g. are constant functors into a singleton for $n = 1$ or a trivial group for $n \geqslant 2$. Following Theorem 5.1, if $\mathcal{A}$ is a dicontractible directed subspace of $\mathcal{X}$, then we have an isomorphism

$$\overrightarrow{P}_n(\mathcal{X}) \simeq \overrightarrow{P}_n(\mathcal{X}, \mathcal{A}),$$

in $\mathsf{NatSys}(\overrightarrow{\mathbf{P}}(\mathcal{A}), \mathsf{Gp})$ for all $n \geqslant 3$. Note that when (X, dX) is the geometric realization of a non-self-linked precubical set (a large class of precubical sets, in which e.g. the semantics of concurrent systems can be expressed, see [**6**] for more details), the dicontractibility condition is equivalent to asking that all path spaces are contractible, since, by Proposition 3.14 of [**21**], all its trace spaces have the homotopy type of a CW-complex.

5.2. A long exact fibration sequence in directed topology

Recall that a morphism $\varphi\colon \mathcal{X} \to \mathcal{Y}$ of dispaces induces a natural transformation $\overrightarrow{\varphi}\colon \overrightarrow{T}_*(\mathcal{X}) \Rightarrow \overrightarrow{T}_*(\mathcal{Y})$. We consider morphisms $p\colon \mathcal{E} \to \mathcal{B}$ of dispaces such that each component $\overrightarrow{p}_e$ of the induced natural transformation is a fibration $\overrightarrow{\mathfrak{T}}(\mathcal{E})(x,y) \to \overrightarrow{\mathfrak{T}}(\mathcal{B})(p(x),p(y))$, for every e a dipath of $\mathcal{E}$. We define the associated *natural system of fibres*, denoted $\overrightarrow{T}_*(\mathcal{F})$, as the natural system of pointed topological spaces on $\overrightarrow{\mathbf{P}}(\mathcal{E})$ which sends a dipath e to

$$\overrightarrow{T}(\mathcal{F})_e = \left(\overrightarrow{p}_e^{\,-1}(p(e))\ ,\ e\right).$$

Now for each 1-cell e of $\overrightarrow{\mathbf{P}}(\mathcal{E})$, denote by $\overrightarrow{P}_n(\mathcal{F})_e$ (resp. $\overrightarrow{P}_n(\mathcal{E},\mathcal{F})_e$) the homotopy group (resp. relative homotopy group)

$$\pi_{n-1}\left(\overrightarrow{T}(\mathcal{F})_e\right) \qquad (\text{resp. } \pi_{n-1}\left(\overrightarrow{T}(\mathcal{E})_e, \overrightarrow{T}(\mathcal{F})_e\right)).$$

These are natural systems on $\overrightarrow{\mathbf{P}}(\mathcal{E})$. Furthermore, for each e dipath of $\mathcal{E}$, the sequence

$$\overrightarrow{T}(\mathcal{F})_e \to \overrightarrow{T}(\mathcal{E})_e \to \overrightarrow{T}(\mathcal{B})_{p(e)}$$

of topological spaces induces a long exact sequence of homotopy groups. Extending this to lower-dimensional homotopy groups via [**12**, Theorem 6.4.9] yields the following result.

Theorem 5.2. *Let $p\colon \mathcal{E} \to \mathcal{B}$ be a morphism of dispaces inducing (Serre) fibrations $\overrightarrow{p}_e$ for every 1-cell e of $\overrightarrow{\mathbf{P}}(\mathcal{E})$. Then we obtain a long exact sequence in $\mathsf{NatSys}(\overrightarrow{\mathbf{P}}(\mathcal{E}), \mathsf{Act})$:*

$$\cdots \to \overrightarrow{P}_n(\mathcal{F}) \to \overrightarrow{P}_n(\mathcal{E}) \to \overrightarrow{P}_n(\mathcal{E},\mathcal{F}) \to \overrightarrow{P}_{n-1}(\mathcal{F}) \to \cdots$$
$$\cdots \to \overrightarrow{P}_2(\mathcal{F}) \to \overrightarrow{P}_2(\mathcal{E}) \to \left(\overrightarrow{P}_2(\mathcal{E},\mathcal{F})\ ,\ \overrightarrow{P}_2(\mathcal{E})\right) \to \overrightarrow{P}_1(\mathcal{F}) \to \overrightarrow{P}_1(\mathcal{E}) \to \overrightarrow{P}_1(\mathcal{E},\mathcal{F}) \to 0.$$

Furthermore, $\overrightarrow{P}_n(\mathcal{E},\mathcal{F}) \cong p^*(\overrightarrow{P}_n(\mathcal{B}))$ for all $n \geqslant 2$. In particular, when $\overrightarrow{T}_*(\mathcal{B})_{p(e)}$ is path connected for all dipaths e of $\mathcal{E}$, the isomorphism holds for all $n \geqslant 1$.

Example 5.3. Given a dispace $\mathcal{B}$ and a fibration $p\colon E \to B$, we obtain a morphism of dispaces from $\mathcal{E}$ to $\mathcal{B}$ inducing fibrations on trace spaces by setting $\mathcal{E} = (E, dE)$ where $dE = \{e \in E^{[0,1]} \mid p \circ e \in dB\}$. In particular, any morphism of dispaces from $\mathcal{E}'$ to $\mathcal{B}'$ such that the underlying map from E' to B' is a fibration induces fibrations on trace spaces.

References

[**1**] Hans-Joachim Baues and Günther Wirsching. Cohomology of small categories. *J. Pure Appl. Algebra*, 38(2–3):187–211, 1985.

[**2**] Jérémy Dubut. *Directed homotopy and homology theories for geometric models of true concurrency*. PhD thesis, 09 2017.

[**3**] Jérémy Dubut, Eric Goubault, and Jean Goubault-Larrecq. Natural homology. In *Automata, Languages, and Programming – 42nd International Colloquium, ICALP 2015, Kyoto, Japan, July 6–10, 2015, Proceedings, Part II*, pages 171–183, 2015.

[4] Jérémy Dubut, Eric Goubault, and Jean Goubault-Larrecq. Directed homology theories and Eilenberg-Steenrod axioms. *Appl. Categ. Struct.*, pages 1–33, 2016.

[5] Lisbeth Fajstrup, Eric Goubault, Emmanuel Haucourt, Samuel Mimram, and Martin Raussen. *Directed Algebraic Topology and Concurrency.* Springer, 2016.

[6] Lisbeth Fajstrup, Eric Goubault, and Martin Raussen. Algebraic topology and concurrency. *Theor. Comp. Sci.*, 357(1–3):241–278, 2006.

[7] Eric Goubault. Some geometric perspectives in concurrency theory. *Homol. Homotopy Appl.*, 5(2):95–136, 2003.

[8] Eric Goubault. On directed homotopy equivalences and a notion of directed topological complexity. arXiv:1709.05702, 2017.

[9] Eric Goubault and Thomas P. Jensen. Homology of higher dimensional automata. In *CONCUR '92, Third International Conference on Concurrency Theory, Stony Brook, NY, USA, August 24–27, 1992, Proceedings*, pages 254–268, 1992.

[10] Eric Goubault and Samuel Mimram. Directed homotopy in non-positively curved spaces. *Submitted*, 2016.

[11] Marco Grandis. *Directed Algebraic Topology, Models of Non-Reversible Worlds.* Cambridge University Press, 2009.

[12] Marco Grandis. *Homological Algebra: In Strongly Non-Abelian Settings.* World Scientific Publishing Company, 2013.

[13] Yves Guiraud and Philippe Malbos. Higher-dimensional normalisation strategies for acyclicity. *Adv. Math.*, 231(3–4):2294–2351, 2012.

[14] Yves Guiraud and Philippe Malbos. Identities among relations for higher-dimensional rewriting systems. In *OPERADS 2009*, volume 26 of *Sémin. Congr.*, pages 145–161. Soc. Math. France, Paris, 2013.

[15] Jonathan Leech. Cohomology theory for monoid congruences. *Houston J. Math.*, 11(2):207–223, 1985.

[16] Saunders MacLane. *Categories for the Working Mathematician*, volume 5 of *Grad. Texts Math.*, Springer-Verlag, New York, second edition, 1998.

[17] Timothy Porter. Group objects in Cat_{Σ_0}/B, 2012. Preprint.

[18] Vaughan Pratt. Modeling concurrency with geometry. In *Proceedings of the 18th ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages*, POPL '91, pages 311–322. ACM, 1991.

[19] Daniel Quillen. On the (co-) homology of commutative rings. In *Applications of Categorical Algebra (Proc. Sympos. Pure Math., Vol. XVII, New York, 1968)*, pages 65–87. Amer. Math. Soc., Providence, R.I., 1970.

[20] Daniel Quillen. Higher algebraic K-theory. I. *Lect. Notes Math.*, 341:85–147, 1973.

[21] Martin Raussen. Trace spaces in a pre-cubical complex. *Topol. Appl.*, 156(9):1718–1728, 2009.

[22] Rob J. van Glabbeek. On the expressiveness of higher dimensional automata. *Theor. Comput. Sci.*, 356(3):265–290, 2006.

[23] Rob J. van Glabbeek. Bisimulation. In *Encyclopedia of Parallel Computing*, pages 136–139, 2011.

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