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HOMOLOGY PRO STABILITY FOR TOR-UNITAL PRO RINGS

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(*communicated by Charles A. Weibel*)

Abstract

Let $\{A_m\}_m$ be a pro system of associative commutative, not necessarily unital, rings. Assume that the pro systems of Tor-groups $\{\mathrm{Tor}_i^{\mathbb{Z}\ltimes A_m}(\mathbb{Z},\mathbb{Z})\}_m$ vanish for all $i>0$. Then we prove that the pro systems $\{H_l(\mathrm{GL}_n(A_m))\}_m$ stabilize up to pro isomorphisms for n large enough relative to l and the stable range of A_m's.

1. Introduction

Homology stability for general linear groups is a simple but deep question in homological algebra. Let R be an associative unital ring. We consider the general linear groups $\mathrm{GL}_n(R)$ of R and their sequence

$$\mathrm{GL}_n(R) \hookrightarrow \mathrm{GL}_{n+1}(R) \hookrightarrow \mathrm{GL}_{n+2}(R) \hookrightarrow \cdots,$$

where each embedding is given by sending α to $\left(\begin{smallmatrix}\alpha & 0\\ 0 & 1\end{smallmatrix}\right)$. The question is whether the induced sequence of the integral group homology

$$H_l(\mathrm{GL}_n(R)) \to H_l(\mathrm{GL}_{n+1}(R)) \to H_l(\mathrm{GL}_{n+2}(R)) \to \cdots$$

stabilizes for n large enough relative to l. There have been many works on this problem, and the most striking result was obtained by Suslin.

Theorem 1.1 (Suslin [**Su82**]). *Let R be an associative unital ring and $l \geqslant 0$. Then the canonical map*

$$H_l(\mathrm{GL}_n(R)) \to H_l(\mathrm{GL}_{n+1}(R))$$

is surjective for $n \geqslant \max(2l, l+\mathrm{sr}(R)-1)$ and bijective for $n \geqslant \max(2l+1, l+\mathrm{sr}(R))$, where $\mathrm{sr}(R)$ is the stable range of R.

Things become much harder and more interesting if we consider *non-unital* rings. Then homology stability is strongly related to K-theory excision and Tor-unitality.

$$\begin{tikzcd}
 & \text{homology stability} \arrow[dr] & \\
\text{Tor-unitality} \arrow[ur, dotted, "?"] \arrow[rr, leftrightarrow] & & K\text{-theory excision}
\end{tikzcd}$$

Let R be an associative unital ring and A a two-sided ideal of R. We define the n-th

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relative K-group by

$$K_n(R,A) := \pi_n \operatorname{hofib}(B\operatorname{GL}(R)^+ \to B\operatorname{GL}(R/A)^+).$$

We say that A *satisfies K-theory excision* if, for every unital ring R which contains A as a two-sided ideal and for every $n \geqslant 1$, the canonical map

$$K_n(\mathbb{Z} \ltimes A, A) \xrightarrow{\sim} K_n(R,A)$$

is an isomorphism. It is well-known that K-theory excision fails in general. However, if the homology $H_l(\operatorname{GL}_n(A))$ stabilizes for n large enough, then A satisfies K-theory excision[1]. This being the case, homology stability for non-unital rings fails in general, even if the stable range of A is finite.

On the other hand, in [**Su95**], Suslin completely determined the obstruction to K-theory excision: An associative ring A satisfies K-theory excision if and only if A is *Tor-unital*, i.e. $\operatorname{Tor}_i^{\mathbb{Z}\ltimes A}(\mathbb{Z},\mathbb{Z}) = 0$ for all $i > 0$. Hence, we may hope that Tor-unital rings satisfy homology stability. Again, Suslin gave a partial solution.

Theorem 1.2 (Suslin [**Su96**]). *Let A be a Tor-unital $\mathbb{Q}$-algebra, $r = \max(\operatorname{sr}(A), 2)$ and $l \geqslant 0$. Then the canonical map*

$$H_l(\operatorname{GL}_n(A)) \to H_l(\operatorname{GL}_{n+1}(A))$$

is surjective for $n \geqslant 2l + r - 2$ and bijective for $n \geqslant 2l + r - 1$.

Unfortunately, commutative rings rarely happen to be Tor-unital. Instead, a recent trend has been to think about *Tor-unital pro rings*. We say that a pro system $\{A_m\}$ of associative rings is *Tor-unital* if the pro systems $\{\operatorname{Tor}_i^{\mathbb{Z}\ltimes A_m}(\mathbb{Z},\mathbb{Z})\}_m$ vanish for all $i > 0$. A notable result by Morrow [**Mo18**] is that, for any ideal A of a noetherian commutative ring, the pro ring $\{A^m\}_{m\geqslant 1}$ of successive powers of A is Tor-unital. Besides, Geisser and Hesselholt [**GH06**] generalized Suslin's excision theorem to the pro setting: If $\{A_m\}$ is a Tor-unital pro ring then the canonical map

$$\{K_n(\mathbb{Z} \ltimes A_m, A_m)\}_m \xrightarrow{\sim} \{K_n(R_m, A_m)\}_m$$

is a pro isomorphism for any pro system of unital rings $\{R_m\}$ together with a level map $\{A_m\} \to \{R_m\}$ which exhibits each A_m as a two-sided ideal of R_m.

Our main theorem is an integral pro version of Theorem 1.2.

Theorem 1.3 (Theorem 5.13). *Let $\{A_m\}$ be a commutative Tor-unital pro ring*[2], *$r = \max_m(\operatorname{sr}(A_m), 2)$ and $l \geqslant 0$. Then the canonical map*

$$\{H_l(\operatorname{GL}_n(A_m))\}_m \to \{H_l(\operatorname{GL}_{n+1}(A_m))\}_m$$

is a pro epimorphism for $n \geqslant 2l + r - 2$ and a pro isomorphism for $n \geqslant 2l + r - 1$.

It follows from Theorem 1.3 that if $\{A_m\}$ is commutative Tor-unital then the conjugate action of $\operatorname{GL}_n(\mathbb{Z})$ on $\{H_l(\operatorname{GL}_n(A_m))\}_m$ is pro trivial for $n \geqslant 2l + r - 1$, cf. Corollary 5.14. Together with a standard argument this reproves Geisser-Hesselholt's

[1] The stability implies that the conjugate action of $\operatorname{GL}(R)$ on $H_l(\operatorname{GL}(A))$ is trivial for any unital ring R which contains A as a two-sided ideal. Then K-theory excision for A follows by a standard Hochschild-Serre spectral sequence argument.

[2] "commutative" means that each A_m is commutative. However, this condition may not be essential. We expect that the theorem is true without the commutativity assumption.

pro excision theorem for commutative Tor-unital pro rings of finite stable range.

In [**CE16**], Calegari and Emerton independently proved homology stability for $\varprojlim_m H_l(\mathrm{GL}_n((p\mathbb{Z})^m);\mathbb{Z}_p)$ and $\varprojlim_m H_l(\mathrm{GL}_n((p\mathbb{Z})^m);\mathbb{F}_p)$ for a prime number p. See also [**Ca15**] for related works and different backgrounds.

1.1. Outline

In §2, we prove the pro stability for $H_1(\mathrm{GL}_n)$, cf. Theorem 2.5. This essentially follows from Vaseršteĭn's stability for relative K_1.

In §3, we recall some properties of Tor-unital rings. In particular, we review the theory of special morphisms between pseudo-free modules over Tor-unital rings exploited in [**Su95**]. Roughly speaking, special morphisms are non-unital substitutions for multiplications by units.

In §4, which is the technical heart of this paper, we study triangular spaces and prove a pro acyclicity of the union of triangular spaces, cf. Theorem 4.9. This is an integral pro version of [**Su96**, Corollary 5.7]. However, the proof there relies on the Malcev theory, which works only for $\mathbb{Q}$-algebras, and we need a new argument. Our new input is the theory of special morphisms recalled in §3.

In §5, we complete the proof of Theorem 1.3. The building blocks are the pro stability for $H_1(\mathrm{GL}_n)$ in §2 and the pro acyclicity of triangular spaces in §4. Then the drift of the argument follows [**Su96**, §6].

1.2. Notation

1. A ring means an associative, not necessarily unital, ring.
2. $\mathrm{sr}(A)$ is the stable range of a ring A, i.e. the minimum number $r \geqslant 1$ such that the stable range condition [**Va69**, $(2.2)_n$] holds for every $n \geqslant r$.
3. Let A be a ring and $n \geqslant 1$.
   (a) The general linear group $\mathrm{GL}_n(A)$ is the kernel of the canonical map $\mathrm{GL}_n(\mathbb{Z} \ltimes A) \to \mathrm{GL}_n(\mathbb{Z})$.
   (b) The elementary subgroup $E_n(A)$ is the subgroup of $\mathrm{GL}_n(A)$ generated by the elementary matrices $e_{ij}(a)$ with $a \in A$ and $1 \leqslant i \neq j \leqslant n$.

   We regard $\mathrm{GL}_n(A)$ as a subgroup of $\mathrm{GL}_{n+1}(A)$ by sending a matrix α to $\left(\begin{smallmatrix} \alpha & 0 \\ 0 & 1 \end{smallmatrix}\right)$. We write $\mathrm{GL}(A) = \mathrm{GL}_\infty(A) = \bigcup_n \mathrm{GL}_n(A)$ and $E(A) = E_\infty(A) = \bigcup_n E_n(A)$.
4. A pro ring is a pro system of rings indexed by a filtered poset. Typically, we denote a pro ring by a bold letter $\mathbf{A} = \{A_m\}$ and the structure maps $A_m \to A_n$ by $\iota_{m,n}$ or just by ι.
5. A unital (resp. commutative) pro ring is a pro ring which is levelwise unital (resp. commutative). Unless otherwise stated, we use standard operations of rings levelwise for pro rings: E.g. $\mathrm{GL}_n(\mathbf{A}) = \{\mathrm{GL}_n(A_m)\}_m$, $\mathrm{Tor}_*^{\mathbb{Z}\ltimes\mathbf{A}}(\mathbb{Z},\mathbb{Z}) = \{\mathrm{Tor}_*^{\mathbb{Z}\ltimes A_m}(\mathbb{Z},\mathbb{Z})\}_m$, etc.
6. A left ideal of a pro ring $\mathbf{A} = \{A_m\}_{m\in J}$ is a pro ring $\mathbf{B} = \{B_m\}_{m\in J}$ with a level map $\mathbf{B} \to \mathbf{A}$ which exhibits each B_m as a left ideal of A_m.

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2. Pro stability for K_1

In this section, we prove pro stability for $H_1(\mathrm{GL}_n)$ (Theorem 2.5). This follows from Vaseršteĭn's stability for relative K_1 (Theorem 2.1) and Tits' lemma (Lemma 2.2).

2.1. Vaseršteĭn's stability

Let R be a unital ring and A a two-sided ideal of R. The normal elementary subgroup $E_n(R,A)$ is the smallest normal subgroup of $E_n(R)$ which contains $E_n(A)$. We write $E(R,A) = E_\infty(R,A) = \bigcup_n E_n(R,A)$. By Whitehead's lemma, $E(R,A)$ is a normal subgroup of $\mathrm{GL}(A)$. We define the relative K_1-group $K_1(R,A)$ to be the quotient group $\mathrm{GL}(A)/E(R,A)$.

Theorem 2.1 (Vaseršteĭn [Va69])**.** *The canonical map*

$$\mathrm{GL}_n(A) \to K_1(R,A)$$

is surjective for $n \geqslant \mathrm{sr}(A)$, and the kernel is $E_n(R,A)$ for $n \geqslant \mathrm{sr}(A)+1$.

2.2. Tits' lemma

Let R be a unital ring and A a two-sided ideal of R. The following lemma generalizes [Ti76, Proposition 2] to possibly noncommutative rings.

Lemma 2.2. *For $n \geqslant 3$, $E_n(R,A^2) \subset [E_n(A), E_n(A)]$.*

Proof. Note the standard equality of elementary matrices:

$$[e_{ij}(a), e_{kl}(b)] = \begin{cases} 1 & \text{if } j \neq k, i \neq l, \\ e_{il}(ab) & \text{if } j = k, i \neq l, \\ e_{kj}(-ba) & \text{if } j \neq k, i = l, \end{cases}$$

which we use throughout the proof. One immediate consequence is an inclusion relation $E_n(A^2) \subset [E_n(A), E_n(A)]$ for $n \geqslant 3$.

For $r = (r_1, \dots, r_n) \in R^n$ with $r_j = 1$, we write

$$X_j(r) := \prod_{k \neq j} e_{jk}(r_k) \quad \text{and} \quad X^j(r) := \prod_{k \neq j} e_{kj}(r_k).$$

Fix $1 \leqslant j \leqslant n$. It is easy to see that every $x \in E_n(R)$ has the form

$$x_{2m}(U) := X^j(u_{2m})X_j(u_{2m-1}) \cdots X^j(u_2)X_j(u_1)$$

for some $m > 0$ and $U = (u_1, u_2, \dots, u_{2m}) \in (R^n)^{2m}$. We set $x_0(\emptyset) := 1$ and

$$x_{2m-1}(V) := X^j(v_{2m-1})X_j(v_{2m-2}) \cdots X_j(v_2)X^j(v_1)$$

for $m > 0$ and $V = (v_1, v_2, \dots, v_{2m-1}) \in (R^n)^{2m-1}$.

Consider the following assertion.

$(\heartsuit)_N$ For every $U \in (R^n)^N$, $x_N(U)E_n(A^2)x_N(U)^{-1} \subset [E_n(A), E_n(A)]$.

We have seen $(\heartsuit)_0$. Let $N > 0$ and suppose that $(\heartsuit)_l$ holds for $l < N$. We shall prove $(\heartsuit)_N$ in the case N even; the case N odd is proved in the same way.

Let $U = (u_1, \ldots, u_N) \in (R^n)^N$ and $x := x_N(U)$. For $e_{ik}(a)$ with $a \in A^2$, $1 \leqslant i, k \leqslant n$ and $k \neq j$, we have $X_j(u_1)e_{ik}(a)X_j(-u_1) \in E_n(A^2)$ and thus by the induction hypothesis $xe_{ik}(a)x^{-1} \in [E_n(A), E_n(A)]$. For $e_{ij}(a)$ with $a \in A^2$ and $1 \leqslant i \neq j \leqslant n$, we have

$$\begin{aligned} X_j(u_1)e_{ij}(a)X_j(-u_1) &= e_{ji}(u_{1,i})\Big(\prod_{k\neq i,j} e_{ik}(-au_{1,k}) \cdot e_{ij}(a)\Big)e_{ji}(-u_{1,i}) \\ &= \prod_{k\neq i,j} e_{jk}(-u_{1,i}au_{1,k})e_{ik}(-au_{1,k}) \cdot e_{ji}(u_{1,i})e_{ij}(a)e_{ji}(-u_{1,i}). \end{aligned}$$

Hence, it follows from the induction hypothesis that $xE_n(A^2)x^{-1}$ is generated by $y_ie_{ij}(a)y_i^{-1}$, $y_i = X^j(u_N)X_j(u_{N-1})\cdots X^j(u_2)e_{ji}(u_{1,i})$, with $a \in A^2$ and $1 \leqslant i \neq j \leqslant n$ modulo $[E_n(A), E_n(A)]$.

For $U = (u_1, \ldots, u_N) \in (R^n)^N$ and $1 \leqslant p \leqslant N/2$, we set

$$\begin{aligned} y_i^{2p-1}(U) &:= X^j(u_N)X_j(u_{N-1})\cdots X^j(u_{2p})e_{ji}(u_{2p-1,i})\cdots e_{ij}(u_{2,i})e_{ji}(u_{1,i}), \\ y_i^{2p}(U) &:= X^j(u_N)X_j(u_{N-1})\cdots X_j(u_{2p+1})e_{ij}(u_{2p,i})\cdots e_{ij}(u_{2,i})e_{ji}(u_{1,i}). \end{aligned}$$

We claim that:

$(\diamondsuit)_Q$ For $U \in (R^n)^N$, $x_N(U)E_n(A^2)x_N(U)^{-1}$ is generated by $y_i^Q(U)e_{ij}(a)y_i^Q(U)^{-1}$, $a \in A^2$, $1 \leqslant i \neq j \leqslant n$ modulo $[E_n(A), E_n(A)]$.

We have seen $(\diamondsuit)_1$. Let $Q > 1$ and suppose that $(\diamondsuit)_l$ holds for $l < Q$. We prove $(\diamondsuit)_Q$ in the case Q even; the case Q odd is proved in the same way.

Let $U = (u_1, \ldots, u_N) \in (R^n)^N$. According to $(\diamondsuit)_{Q-1}$, $x_N(U)E_n(A^2)x_N(U)^{-1}$ is generated by $y_i^{Q-1}(U)e_{ij}(a)y_i^{Q-1}(U)^{-1}$, $a \in A^2$, $1 \leqslant i \neq j \leqslant n$ modulo $[E_n(A), E_n(A)]$. We fix $1 \leqslant i \neq j \leqslant n$ for a moment. Now,

$$X^j(u_Q)e_{ji}(u_{Q-1,i}) = e_{ij}(u_{Q,i})e_{ji}(u_{Q-1,i}) \prod_{k\neq i,j} e_{kj}(u_{Q,k})e_{ki}(u_{Q,k}u_{Q-1,i}).$$

Hence, by putting $\tilde{y} := \prod_{k\neq i,j} e_{ki}(u_{2p,k}u_{2p-1,i})$, we have

$$\begin{aligned} y_i^{Q-1}(U) = X^j(u_N)&X_j(u_{N-1})\cdots \\ &\cdots X_j(u_{Q+1})e_{ij}(u_{Q,i})e_{ji}(u_{Q-1,i})X^j(u'_{Q-2})\cdots X^j(u'_2)X_j(u'_1)\tilde{y} \end{aligned}$$

for some $u'_1, \ldots, u'_{Q-2} \in R^n$ with $u'_{q,i} = u_{q,i}$. For $Q-1 \leqslant q \leqslant N$, we set

$$u'_q := \begin{cases} u_{q,i}e_i + e_j & \text{if } q = Q-1, Q, \\ u_q & \text{if } q > Q \end{cases}$$

and $U' := (u'_1, \ldots, u'_N)$, so that $y_i^{Q-1}(U) = x_N(U')\tilde{y}$ and $y_i^q(U') = y_i^q(U)$ for $q \geqslant Q$. By applying $(\diamondsuit)_{Q-1}$ to U', we see that $x_N(U')E_n(A^2)x_N(U')^{-1}$ is generated by $y_i^Q(U')e_{ij}(a)y_i^Q(U')^{-1}$, $a \in A^2$ modulo $[E_n(A), E_n(A)]$. Varying i, this proves $(\diamondsuit)_Q$ for the given $U \in (R^n)^N$, and thus for all $U \in (R^n)^N$.

Thanks to $(\diamondsuit)_N$, we are reduced to showing that $ye_{ij}(ab)y^{-1} \in [E_n(A), E_n(A)]$ for $y = e_{ij}(r_N)e_{ji}(r_{N-1}) \cdots e_{ij}(r_2)e_{ji}(r_1)$ with $a, b \in A$, $r_1, \ldots, r_N \in R$ and $1 \leqslant i \neq j \leqslant n$. Observe that we have

$$e_{ij}(r_1)e_{ji}(ab)e_{ij}(-r_1) = e_{ij}(r_1)[e_{jt}(a), e_{ti}(b)]e_{ij}(-r_1) = [e_{it}(r_1a)e_{jt}(a), e_{tj}(-br_1)e_{ti}(b)]$$

for $t \neq i, j$. Now, it is clear that

$$y'[e_{it}(r_1a)e_{jt}(a), e_{tj}(-br_1)e_{ti}(b)](y')^{-1} \in [E_n(A), E_n(A)]$$

for $y' = e_{ij}(r_N)e_{ji}(r_{N-1}) \cdots e_{ij}(r_2)$, and thus we get $(\heartsuit)_N$. $\square$

Corollary 2.3. *Let $\mathbf{R} = \{R_m\}$ be a unital pro ring and $\mathbf{A} = \{A_m\}$ a two-sided ideal of $\mathbf{R}$. Suppose that $\mathbf{A}/\mathbf{A}^2 = \{A_m/A_m^2\} = 0$. Then, for $3 \leqslant n \leqslant \infty$, the canonical maps*

$$\begin{tikzcd}
E_n(\mathbf{A}) \arrow[rr, "\simeq"] & & E_n(\mathbf{R}, \mathbf{A}) \\
{[E_n(\mathbf{A}), E_n(\mathbf{A})]} \arrow[u, "\simeq"] \arrow[rr, "\simeq"] & & {[E_n(\mathbf{R}, \mathbf{A}), E_n(\mathbf{R}, \mathbf{A})]} \arrow[u, "\simeq"']
\end{tikzcd}$$

are pro isomorphisms.

Proof. Since all the indicated maps are injections, it suffices to show that the map $[E_n(\mathbf{A}), E_n(\mathbf{A})] \to E_n(\mathbf{R}, \mathbf{A})$ is a pro epimorphism. By the assumption $\mathbf{A}/\mathbf{A}^2 = 0$, there exists $s \geqslant m$ for each m such that $\iota_{s,m}(A_s) \subset A_m^2$. Therefore,

$$\iota_{s,m}E_n(R_s, A_s) \subset E_n(R_m, A_m^2) \subset [E_n(R_m, A_m), E_n(R_m, A_m)],$$

where the last inclusion is by Lemma 2.2. This proves that the map $[E_n(\mathbf{A}), E_n(\mathbf{A})] \to E_n(\mathbf{R}, \mathbf{A})$ is a pro epimorphism. $\square$

2.3. Pro excision and pro stability

Let $\mathbf{R} = \{R_m\}$ be a unital pro ring and $\mathbf{A} = \{A_m\}$ a two-sided ideal of $\mathbf{R}$. We define $\mathrm{sr}(\mathbf{A}) := \max_m(\mathrm{sr}(A_m))$.

Theorem 2.4 (Pro excision)**.** *Suppose that $\mathbf{A}/\mathbf{A}^2 = 0$. Then the canonical map*

$$H_1(\mathrm{GL}(\mathbf{A})) \xrightarrow{\sim} K_1(\mathbf{R}, \mathbf{A})$$

is a pro isomorphism.

Proof. Since $K_1(\mathbf{R}, \mathbf{A})$ is levelwise abelian, we have a levelwise exact sequence

$$H_1(E(\mathbf{R}, \mathbf{A})) \longrightarrow H_1(\mathrm{GL}(\mathbf{A})) \longrightarrow K_1(\mathbf{R}, \mathbf{A}) \longrightarrow 0.$$

It follows from Corollary 2.3 that $H_1(E(\mathbf{R}, \mathbf{A})) = 0$, and thus we get the desired isomorphism. $\square$

Theorem 2.5 (Pro stability)**.** *Suppose that $\mathbf{A}/\mathbf{A}^2 = 0$. Then the canonical map*

$$H_1(\mathrm{GL}_n(\mathbf{A})) \to H_1(\mathrm{GL}(\mathbf{A}))$$

is a pro epimorphism for $n \geqslant \mathrm{sr}(\mathbf{A})$ and a pro isomorphism for $n \geqslant \max(3, \mathrm{sr}(\mathbf{A}) + 1)$.

Proof. The composite

$$H_1(\mathrm{GL}_n(\mathbf{A})) \to H_1(\mathrm{GL}(\mathbf{A})) \xrightarrow{\sim} K_1(\mathbf{R}, \mathbf{A})$$

is a levelwise surjection for $n \geqslant \mathrm{sr}(\mathbf{A})$ by Theorem 2.1. Since the last map is a pro isomorphism by Theorem 2.4, the first map is a pro epimorphism for $n \geqslant \mathrm{sr}(\mathbf{A})$.

Consider the commutative diagram

$$\begin{tikzcd}
H_1(E_n(\mathbf{R},\mathbf{A})) \arrow[r] \arrow[d] & H_1(\mathrm{GL}_n(\mathbf{A})) \arrow[r] \arrow[d] & H_1(\mathrm{GL}_n(\mathbf{R},\mathbf{A})/E_n(\mathbf{R},\mathbf{A})) \arrow[r] \arrow[d] & 0 \\
H_1(E(\mathbf{R},\mathbf{A})) \arrow[r] & H_1(\mathrm{GL}(\mathbf{A})) \arrow[r] & H_1(K_1(\mathbf{R},\mathbf{A})) \arrow[r] & 0
\end{tikzcd}$$

with levelwise exact rows. The left terms are zero for $n \geqslant 3$ by Corollary 2.3. According to Theorem 2.1, the right vertical map is a levelwise bijection for $n \geqslant \mathrm{sr}(\mathbf{A})+1$. Hence, the middle term is a pro isomorphism for $n \geqslant \max(3, \mathrm{sr}(\mathbf{A})+1)$. □

Theorem 2.6. *Set $\bar{E}_n(\mathbf{A}) := \mathrm{GL}_n(\mathbf{A}) \cap E(\mathbf{A})$. Suppose that $\mathbf{A}/\mathbf{A}^2 = 0$. Then the canonical map*

$$E_n(\mathbf{A}) \to \bar{E}_n(\mathbf{A})$$

is a pro isomorphism for $n \geqslant \max(3, \mathrm{sr}(\mathbf{A})+1)$.

Proof. Let $\bar{E}_n(\mathbf{R},\mathbf{A}) := \mathrm{GL}_n(\mathbf{A}) \cap E(\mathbf{R},\mathbf{A})$. According to Theorem 2.1, the canonical map $E_n(\mathbf{R},\mathbf{A}) \to \bar{E}_n(\mathbf{R},\mathbf{A})$ is a levelwise bijection for $n \geqslant \mathrm{sr}(\mathbf{A})+1$. Hence, the theorem follows from Corollary 2.3. □

3. Tor-unital pro rings

The treatment of this section closely follows Suslin [**Su95**] and Geisser–Hesselholt [**GH06**].

Definition 3.1. A pro ring $\mathbf{A} = \{A_m\}$ is *Tor-unital* if

$$\mathrm{Tor}_i^{\mathbb{Z}\ltimes\mathbf{A}}(\mathbb{Z},\mathbb{Z}) = \{\mathrm{Tor}_i^{\mathbb{Z}\ltimes A_m}(\mathbb{Z},\mathbb{Z})\}_m = 0$$

as pro abelian groups for all $i > 0$.

Example 3.2.

(i) A unital pro ring is Tor-unital.

(ii) (Morrow [**Mo18**]) Let A be an ideal of a noetherian commutative ring, then the pro ring $\{A^m\}_{m\geqslant 1}$ of the successive powers of A is Tor-unital.

Definition 3.3. Let $\mathbf{A} = \{A_m\}_{m\in J}$ be a pro ring.

(i) A *left* $\mathbf{A}$*-module* is a pro abelian group $\mathbf{M} = \{M_m\}_{m\in J}$ with a level map $\mathbf{A} \times \mathbf{M} \to \mathbf{M}$ which exhibits each M_m as a left A_m-module. A *morphism between left* $\mathbf{A}$*-modules* $\mathbf{M} = \{M_m\}$ *and* $\mathbf{N} = \{N_m\}$ is a level map $f\colon \mathbf{M} \to \mathbf{N}$ such that each $f_m\colon M_m \to N_m$ is a morphism of left A_m-modules.

(ii) A left $\mathbf{A}$-module $\mathbf{P}$ is *pseudo-free* if there is an isomorphism of left $\mathbf{A}$-modules $\mathbf{A} \otimes \mathbf{L} \xrightarrow{\sim} \mathbf{P}$ for some pro system $\mathbf{L}$ of free abelian groups. We call such an $\mathbf{L}$ a *free basis of* $\mathbf{P}$.

(iii) A morphism $f\colon \mathbf{P} \to \mathbf{M}$ of left $\mathbf{A}$-modules is *special* if $\mathbf{P}$ is pseudo-free with a free basis $\mathbf{L}$ and f is induced from a level morphism of pro abelian groups $\mathbf{L} \to \mathbf{M}$.

Proposition 3.4 (Suslin [**Su95**], Geisser–Hesselholt [**GH06**]). *Let $\mathbf{A} = \{A_m\}$ be a Tor-unital pro ring. Suppose we are given an augmented complex*

$$\cdots \to \mathbf{C}_1 \to \mathbf{C}_0 \xrightarrow{\epsilon} \mathbf{C}_{-1}$$

of left $\mathbf{A}$-modules such that:[3]

(i) *Each $\mathbf{C}_k$ with $k \geqslant -1$ is pseudo-free.*

(ii) *The homology $H_k(C_{\bullet,m})$ is annihilated by A_m for every m and $k \geqslant -1$.*

Then

$$H_k(\mathbf{C}_\bullet) = \{H_k(C_{\bullet,m})\}_m = 0$$

for all $k \geqslant -1$.

In fact, a finer result holds.

Proposition 3.5. *Let $\mathbf{A} = \{A_m\}_{m\in J}$ be a Tor-unital pro ring and $k \geqslant -1$. Then there exists $s(m) \geqslant m$ for each $m \in J$ such that the map*

$$\iota_{s(m),m} \colon H_k(C_{\bullet,s(m)}) \to H_k(C_{\bullet,m})$$

is zero for all augmented complexes of left $\mathbf{A}$-modules which satisfy conditions (i) *and* (ii)*.*

Proof. Let $\mathbf{C}$ be a pseudo-free left $\mathbf{A}$-module with a free basis $\mathbf{L}$. Then we have levelwise isomorphisms

$$\mathrm{Tor}_q^{\mathbb{Z}\ltimes\mathbf{A}}(\mathbb{Z}, \mathbf{C}) \simeq \mathrm{Tor}_q^{\mathbb{Z}\ltimes\mathbf{A}}(\mathbb{Z}, \mathbf{A}\otimes\mathbf{L}) \simeq \mathrm{Tor}_q^{\mathbb{Z}\ltimes\mathbf{A}}(\mathbb{Z}, \mathbf{A})\otimes\mathbf{L} \simeq \mathrm{Tor}_{q+1}^{\mathbb{Z}\ltimes\mathbf{A}}(\mathbb{Z}, \mathbb{Z})\otimes\mathbf{L}.$$

Since $\mathbf{A}$ is Tor-unital, we see that

$$\mathrm{Tor}_q^{\mathbb{Z}\ltimes\mathbf{A}}(\mathbb{Z}, \mathbf{C}) = 0$$

for every $q \geqslant 0$.

Let $\mathbf{Z}_k$ and $\mathbf{B}_{k-1}$ be the kernel and the image of $\mathbf{C}_k \to \mathbf{C}_{k-1}$ respectively. By assumption (ii), we have a levelwise inclusion $\mathbf{AC}_{-1} \subset \mathbf{B}_{-1}$, and thus there is a levelwise surjection $\mathbf{C}_{-1}/\mathbf{AC}_{-1} \twoheadrightarrow H_{-1}(\mathbf{C}_\bullet)$. Since $\mathbf{C}_{-1}$ is pseudo-free, $\mathbf{C}_{-1}/\mathbf{AC}_{-1} = \mathrm{Tor}_0^{\mathbb{Z}\ltimes\mathbf{A}}(\mathbb{Z}, \mathbf{C}_{-1}) = 0$. Therefore, $H_{-1}(\mathbf{C}_\bullet) = 0$.

Let $k \geqslant 0$ and suppose that $H_l(\mathbf{C}_\bullet) = 0$ for $l < k$. Consider the levelwise spectral sequence

$$\mathbf{E}^1_{pq} = \begin{cases} \mathrm{Tor}_q^{\mathbb{Z}\ltimes\mathbf{A}}(\mathbb{Z}, \mathbf{C}_p) & \text{if } 0 \leqslant p \leqslant k, \\ \mathrm{Tor}_q^{\mathbb{Z}\ltimes\mathbf{A}}(\mathbb{Z}, \mathbf{Z}_k) & \text{if } p = k+1, \\ 0 & \text{otherwise,} \end{cases}$$

which arises from the brutal truncation of the complex

$$\mathbf{Z}_k \to \mathbf{C}_k \to \mathbf{C}_{k-1} \to \cdots \to \mathbf{C}_0.$$

By the induction hypothesis, the complex is pro quasi-isomorphic to $\mathbf{C}_{-1}$ and thus

$$\mathbf{E}^\infty_q \simeq \mathrm{Tor}_q^{\mathbb{Z}\ltimes\mathbf{A}}(\mathbb{Z}, \mathbf{C}_{-1}) = 0$$

---

[3] We thank Takeshi Saito for pointing out an unnecessary condition, the augmentation ϵ is special, which was in the first draft and in [**Su95, GH06**] too.

for $q \geqslant 0$. Since $\mathbf{C}_p$ is pseudo-free, we also have $\mathbf{E}^1_{pq} = 0$ for $0 \leqslant p \leqslant k$. Hence,

$$\mathbf{Z}_k/\mathbf{A}\mathbf{Z}_k = \mathrm{Tor}_0^{\mathbb{Z}\ltimes\mathbf{A}}(\mathbb{Z}, \mathbf{Z}_k) = \mathbf{E}^\infty_{k+1} = 0.$$

On the other hand, by the assumption (ii), we have $\mathbf{A}\mathbf{Z}_k \subset \mathbf{B}_k$. Therefore, $H_k(\mathbf{C}_\bullet) = 0$. This proves Proposition 3.4. The finer assertion (Proposition 3.5) is also clear from this proof. □

Lemma 3.6. *Let $\mathbf{A}$ be a pro ring and $\mathbf{P}$ a pseudo-free left $\mathbf{A}$-module. Then there exists an augmented complex $\mathbf{P}_\bullet$ of left $\mathbf{A}$-modules with $\mathbf{P}_{-1} = \mathbf{P}$ which satisfies conditions* (i), (ii) *and*

(iii) *The augmentation $\epsilon\colon \mathbf{P}_0 \to \mathbf{P}_{-1}$ is special*

We call $\mathbf{P}_\bullet$ a *pro resolution* of $\mathbf{P}$.

Proof. Write $\mathbf{P} = \{P_m\}$ and let $\mathbb{Z}[\mathbf{P}] = \{\mathbb{Z}[P_m]\}$ be the pro system of the free abelian groups generated by the sets P_m. Then $\mathbf{P}_0 := \mathbf{A} \otimes \mathbb{Z}[\mathbf{P}]$ is a pseudo-free $\mathbf{A}$-module and the canonical map $\mathbb{Z}[\mathbf{P}] \to \mathbf{P}$ induces a special morphism $\epsilon\colon \mathbf{P}_0 \to \mathbf{P}$.

Let $\mathbf{R} = \{R_m\}$ be the kernel of ϵ, and $\mathbb{Z}[\mathbf{R}] = \{\mathbb{Z}[R_m]\}$ the pro system of the free abelian group generated by R_m. Then $\mathbf{P}_1 := \mathbf{A} \otimes \mathbb{Z}[\mathbf{R}]$ is a pseudo-free $\mathbf{A}$-module. Repeating this procedure, we obtain an augmented complex $\mathbf{P}_\bullet$ with $\mathbf{P}_{-1} = \mathbf{P}$ which satisfies the desired conditions. □

4. Pro acyclicity of triangular spaces

The goal of this section is to prove Theorem 4.9.

4.1. Preliminaries on homology

For a simplicial set X, we denote by $C_*(X)$ the complex freely generated by X_* with the differential being the alternating sum of the faces. We write $H_n(X) = H_n(C_*(X))$. Also, we write $\tilde{H}_n(X)$ for the reduced homology.

Let G be a group. We write EG for the simplicial set whose degree n-part is $G^{\times(n+1)}$ and whose i-th face (resp. the i-th degeneracy) omits the i-th entry (resp. repeats the i-th entry). We give a right G-action on EG by letting $(g_0, \dots, g_n) \cdot g := (g_0 g, \dots, g_n g)$. The classifying space BG is defined to be EG/G.

By a pro object or pro system, we mean a pro object whose index category is a left filtered small category.

Lemma 4.1. *Let $f\colon X \to Y$ be a morphism between pro systems of pointed simplicial sets. Suppose that f induces pro isomorphisms*

$$\pi_n(X) \xrightarrow{\sim} \pi_n(Y)$$

for all $n \geqslant 0$. Then f induces pro isomorphisms

$$H_n(X) \xrightarrow{\sim} H_n(Y)$$

for all $n \geqslant 0$.

Proof. Since $\mathbb{Z}\pi_0(X) \simeq H_0(X)$, the assertion is clear for $n = 0$. Hence, by taking the connected components (i.e. the levelwise homotopy fiber of $X \to \tau_{\leqslant 0}X$), we may assume that X and Y are connected.

Then, according to [**Is01**], the induced map $P_n(X) \to P_n(Y)$ on the n-th Postnikov sections for $n \geqslant 1$ is a strict weak equivalence, i.e. isomorphic to a levelwise weak equivalence. Hence, the induced map $C_*(P_n(X)) \to C_*(P_n(Y))$ is isomorphic to a levelwise quasi-isomorphism.

On the other hand, by Hurewicz theorem and Serre spectral sequence, we have levelwise isomorphisms $H_k(X) \simeq H_k(P_n(X))$ for $k \leqslant n$. Now, in the commutative diagram

$$\begin{tikzcd} H_k(X) \arrow[r] \arrow[d, "\simeq"] & H_k(Y) \arrow[d, "\simeq"] \\ H_k(P_n(X)) \arrow[r, "\simeq"] & H_k(P_n(Y)), \end{tikzcd}$$

the vertical maps and the bottom map are pro isomorphisms for $k \leqslant n$, and so is the top map. This completes the proof. $\square$

For a simplicial group G, we consider the bi-simplicial set BG constructed degree-wise. For a bi-simplicial set X, we denote by $C_*(X)$ the double-complex freely generated by X_* with the differential being the alternating sum of the faces.

Corollary 4.2. *Let $f\colon P \to Q$ be a morphism between pro systems of simplicial abelian groups. Suppose that f induces pro isomorphisms*

$$\pi_n(P) \xrightarrow{\sim} \pi_n(Q)$$

for all $n \geqslant 0$. Then f induces pro isomorphisms

$$H_n(\operatorname{Tot} C_*(BP)) \xrightarrow{\sim} H_n(\operatorname{Tot} C_*(BQ))$$

for all $n \geqslant 0$.

Proof. Now, the map $B_kP \to B_kQ$ induces pro isomorphisms $\pi_n(B_kP) \to \pi_n(B_kQ)$ for all $n \geqslant 0$. Hence, by Lemma 4.1, the induced maps

$$H_n(C_*(B_kP)) \to H_n(C_*(B_kQ))$$

are pro isomorphisms for all $n \geqslant 0$. By a standard spectral sequence argument, we obtain the corollary. $\square$

Let us quote a lemma from [**Su95**, §2].

Lemma 4.3. *Let G be a group and H a group with a left G-action. Then there exists a natural quasi-isomorphism*

$$C_*(B(G \ltimes H)) \simeq C_*(EG) \otimes_G C_*(BH).$$

Let $G = \{G_m\}$ be a pro group (= a pro system of groups). A *left G-module M* is a pro abelian group $M = \{M_m\}$ with a level map $G \times M \to M$ which exhibits each M_m as a left G_m-module. A *morphism between left G-modules $M = \{M_m\}$ and* $N = \{N_m\}$ is a level map $f\colon M \to N$ such that each $f_m\colon M_m \to N_m$ is a morphism of left G_m-modules. These form the category of left G-modules, and we consider simplicial objects in this category; simplicial left G-modules and morphisms between them.

Corollary 4.4. *Let G be a pro group. Let P and Q be simplicial left G-modules and $f\colon P \to Q$ a morphism between them. Suppose that f induces pro isomorphisms*

$$\pi_n(P) \xrightarrow{\sim} \pi_n(Q)$$

for all $n \geqslant 0$. Then f induces pro isomorphisms

$$H_n(\operatorname{Tot} C_*(B(G \ltimes P))) \xrightarrow{\sim} H_n(\operatorname{Tot} C_*(B(G \ltimes Q)))$$

for all $n \geqslant 0$, where the semi-direct products are taken levelwise and degreewise.

Proof. This follows from Corollary 4.2 and Lemma 4.3. □

4.2. The key lemma

Let A be a ring and P a left A-module. Let σ be a finite poset. We define a group $T^\sigma(A,P)$ by

$$T^\sigma(A,P) := \prod_{i<_\sigma j,\, j\notin \max\sigma} A_{(i,j)} \times \prod_{i<_\sigma j,\, j\in\max\sigma} P_{(i,j)},$$

where $A_{(i,j)}$ and $P_{(i,j)}$ are just the copies of A and P respectively. For $\alpha \in T^\sigma(A,P)$, we denote its (i,j)-th component by $\alpha_{i,j}$; thus $\alpha_{i,j} \in A$ if $j \notin \max\sigma$, and $\alpha_{i,j} \in P$ if $j \in \max\sigma$. For $\alpha, \beta \in T^\sigma(A,P)$, we define the composition $\alpha \cdot \beta$ by

$$(\alpha\cdot\beta)_{i,j} = \alpha_{i,j} + \beta_{i,j} + \sum_{i<_\sigma k <_\sigma j} \alpha_{i,k}\beta_{k,j}$$

for $i <_\sigma j$. We set $T^\sigma(A) := T^\sigma(A,A)$.

Set $\sigma_0 := \sigma \setminus \max\sigma$ and $M^\sigma(P) := \prod_{i<_\sigma j,\, j\in\max\sigma} P_{(i,j)}$. Then we have an identification

$$T^\sigma(A,P) = T^{\sigma_0}(A) \ltimes M^\sigma(P)$$

and canonical inclusion and projection

$$T^{\sigma_0}(A) \hookrightarrow T^\sigma(A,P) \twoheadrightarrow T^{\sigma_0}(A).$$

Let $\theta\colon \sigma \to \tau$ be an embedding of finite posets. Then it induces a morphism of groups

$$T^\theta\colon T^\sigma(A) \to T^\tau(A).$$

If θ sends maximal elements to maximal elements, then it also induces a morphism $T^\sigma(A,P) \to T^\tau(A,P)$ for any left A-module P, which we also denote by T^θ.

Let $f\colon P \to Q$ be a morphism of A-modules. Then it induces a morphism of groups

$$T^f\colon T^\sigma(A,P) \to T^\sigma(A,Q).$$

If $\theta\colon \sigma \to \tau$ sends maximal elements to maximal elements, then we define

$$T^{f,\theta}\colon T^\sigma(A,P) \to T^\tau(A,Q)$$

to be the composite $T^f \circ T^\theta = T^\theta \circ T^f$.

For a finite poset σ and $p \geqslant 0$, let $[p]$ be the poset $0 < 1 < 2 < \cdots < p$ and endow $\sigma \times [p]$ with the lexicographical order. We define

$$\sigma\bigstar[p] := \sigma \times [p] \setminus \max\sigma \times \{1, \dots, p\}.$$

We denote by ϕ (resp. φ) the embedding $\sigma \to \sigma \times [p]$ (resp. $\sigma \to \sigma\bigstar[p]$), $a \mapsto (a, 0)$. Note that $\varphi^{-1}(\max(\sigma\bigstar[p])) = \max\sigma$ and that $(\sigma\bigstar[p])_0 = \sigma_0 \times [p]$.

The following lemma is a variant of Lemma 7.4 in [**Su82**].

Lemma 4.5. *Let $\{A_m\}_{m\in\Xi}$ be a commutative Tor-unital pro ring and $l \geqslant 0$. Then there exist $p_l \geqslant 0$ and $s_l(m) \geqslant m$ for each $m \in \Xi$ such that:*

(i) *For all finite posets σ and all pseudo-free $\{A_m\}$-modules $\{P_m\}$, the map*

$$\iota_{s_l(m),m} H_l(T^{\varphi})\colon \tilde{H}_l(T^{\sigma}(A_{s_l(m)}, P_{s_l(m)})) \to \tilde{H}_l(T^{\sigma\bigstar[p_l]}(A_m, P_m))$$

is equal to zero.

(ii) *For all finite posets σ and all special morphisms $f\colon \{P_m\} \to \{Q_m\}$ between pseudo-free $\{A_m\}$-modules, the map*

$$\iota_{s_l(m),m} H_l(T^{f,\varphi})\colon \tilde{H}_l(T^{\sigma}(A_{s_l(m)}, P_{s_l(m)})) \to \tilde{H}_l(T^{\sigma\bigstar[p_l]}(A_m, Q_m))$$

is equal to zero.

The lemma fails for general non-unital rings and the use of Tor-unitality is essential here. We also remark that (i) is not a special case of (ii) because the identity morphism is not special unless the ring is unital. Even if one is only interested in (i), the proof requires (ii) in its induction step.

Proof. We prove the lemma by induction on $l \geqslant 0$. The case $l = 0$ is clear, here we can take $p_0 = 0$ and $s_0(m) = m$. Let $L > 0$ and suppose that we have constructed $p_0 \leqslant p_1 \leqslant \cdots \leqslant p_{L-1}$ and $s_0(m) \leqslant s_1(m) \leqslant \cdots \leqslant s_{L-1}(m)$ which satisfy the conditions of the lemma.

Set $q := p_{L-1} + 1$ and $t(m) := s_{L-1}(m)$. First, we prove the following.

Sublemma 4.6. *For all finite posets σ and all special morphisms $f\colon \{P_m\} \to \{Q_m\}$ between pseudo-free $\{A_m\}$-modules, the diagram*

$$\begin{tikzcd} H_L(T^{\sigma}(A_{t(m)}, P_{t(m)})) \arrow[r, "\iota_{t(m),m} H_L(T^{f,\varphi})"] \arrow[d, two heads] & H_L(T^{\sigma\bigstar[q]}(A_m, Q_m)) \\ H_L(T^{\sigma_0}(A_{t(m)})) \arrow[r, "\iota_{t(m),m} H_L(T^{\phi})"] & H_L(T^{\sigma_0 \times [q]}(A_m)) \arrow[u, hook] \end{tikzcd}$$

commutes, where the vertical maps are the canonical projection and inclusion.

Proof. Let $f\colon \{P_m\} \to \{Q_m\}$ be a special morphism between pseudo-free $\{A_m\}$-modules and $\{L_m\}$ a free basis of $\{P_m\}$ such that f is induced from a map $\{L_m\} \to \{Q_m\}$. Note that we have an equality $\{\varinjlim_i L_m^{(i)}\} = \{L_m\}$, where $\{L_m^{(i)}\}$ is a subsystem of $\{L_m\}$ such that each $L_m^{(i)}$ is finitely generated and the limit runs over all such systems. Hence, we have

$$\varinjlim_i C_*(BM^\sigma(A_m \otimes L_m^{(i)})) \simeq C_*(BM^\sigma(P_m))$$

for every m. It follows that

$$C_*(BT^\sigma(A_m, P_m)) \simeq \varinjlim_i C_*(BT^\sigma(A_m, A_m \otimes L_m^{(i)}))$$

and

$$H_*(T^\sigma(A_m, P_m)) \simeq \varinjlim_i H_*(T^\sigma(A_m, A_m \otimes L_m^{(i)})).$$

Consequently, to show the sublemma, we may assume that $\{P_m\} = \{A_m \otimes_{\mathbb{Z}} L_m\}$ with L_m a free abelian group of finite rank. We may also assume that $\{Q_m\} = \{A_m \otimes_{\mathbb{Z}} M_m\}$ with M_m a free abelian group of finite rank.

Fix $m \in \Xi$. Let $e_1, \dots, e_I$ be a basis of $L_{t(m)}$ and $f_1, \dots, f_J$ a basis of M_m. Since f is special, the map $\iota_{t(m),m} f\colon P_{t(m)} \to Q_m$ is induced by a map $\alpha\colon L_{t(m)} \to Q_m$, which sends e_i to $\sum_j \alpha_{i,j} f_j$ with $\alpha_{i,j} \in A_m$. We denote $\iota_{t(m),m} f$ also by α.

If $\alpha = 0$, then the diagram

$$\begin{tikzcd}
H_L(T^\sigma(A_{t(m)}, P_{t(m)})) \arrow[r, "\iota_{t(m),m} H_L(T^{f,\varphi})"] \arrow[d] & H_L(T^{\sigma\bigstar[q]}(A_m, Q_m)) \\
H_L(T^{\sigma_0}(A_{t(m)})) \arrow[r, "\iota_{t(m),m} H_L(T^{\phi})"] & H_L(T^{\sigma_0 \times [q]}(A_m)) \arrow[u, hook]
\end{tikzcd}$$

commutes, and thus the sublemma holds in this case. Let $(u, v) \in [1, I] \times [1, J]$ and suppose that the sublemma holds if $\alpha_{i,j} = 0$ for $(i, j) \geqslant (u, v)$ with respect to the lexicographical order. We prove the sublemma in case $\alpha_{i,j} = 0$ for $(i, j) > (u, v)$. We define a map $\beta\colon P_{t(m)} \to Q_m$ by sending e_i to $\delta_{i,u} f_v$.

We define an embedding $\psi\colon \sigma \to \sigma\bigstar[q]$ by

$$\psi(x) = \begin{cases} (x, 0) & \text{if } x \in \max \sigma, \\ (x, q) & \text{if } x \notin \max \sigma. \end{cases}$$

Then the image τ of ψ intersects with $\sigma\bigstar[q-1] \subset \sigma\bigstar[q]$ only at $\max \sigma \times \{0\}$, and thus the composite

$$T^\sigma \xrightarrow{\text{diag}} T^\sigma \times T^\sigma \xrightarrow{T^\varphi \times T^\psi} T^{\sigma\bigstar[q-1]} \times T^\tau \xrightarrow{\text{prod}} T^{\sigma\bigstar[q]}$$

is a morphism of groups. By applying this construction to the product

$$T^{\alpha,\varphi} \times T^{\beta,\psi}\colon T^\sigma(A_{t(m)}, P_{t(m)})^{\times 2} \to T^{\sigma\bigstar[q-1]}(A_m, Q_m) \times T^\tau(A_m, Q_m),$$

we get a morphism of groups

$$T^{\alpha,\varphi} \cdot T^{\beta,\psi}\colon T^\sigma(A_{t(m)}, P_{t(m)}) \to T^{\sigma\bigstar[q]}(A_m, Q_m).$$

Since $q - 1 = p_{L-1}$ and $t(m) = s_{L-1}(m)$, by the induction hypothesis and by the

Künneth formula, we obtain

$$H_L(T^{\alpha,\varphi} \cdot T^{\beta,\psi}) = H_L(T^{\alpha,\varphi}) + H_L(T^{\beta,\psi}). \tag{4.1}$$

We set

$$\omega := \prod_{x \in \sigma_0} e_{\varphi(x),\psi(x)}(\alpha_{u,v}) \in T^{\sigma \bigstar [q]}(A_m).$$

We define $(\alpha'_{i,j}) \in M_{I,J}(A^m)$ by $\alpha'_{i,j} = \alpha_{i,j}$ unless $(i,j) = (u,v)$ and $\alpha'_{u,v} = 0$, which induces a map $\alpha' \colon Q_{t(m)} \to P_m$ by sending e_i to $\sum_j \alpha'_{i,j} f_j$.

Claim 4.7. *We have an equality*[4]

$$\mathrm{Ad}(\omega) \circ (T^{\alpha',\varphi} \cdot T^{\beta,\psi}) = T^{\alpha,\varphi} \cdot T^{\beta,\psi}. \tag{4.2}$$

We calculate the (k,l)-entry of (4.2) at $U \in T^\sigma(A_{t(m)}, P_{t(m)})$. It suffices to do this for;

(1) $(k,l) = (\varphi(x), \varphi(y))$ with $x \in \sigma_0$ and $y \in \sigma$.
(2) $(k,l) = (\varphi(x), \psi(y))$ with $x \in \sigma_0$ and $y \in \sigma_0$.
(3) $(k,l) = (\psi(x), \psi(y))$ with $x \in \sigma_0$ and $y \in \sigma_0$.
(4) $(k,l) = (\psi(x), \varphi(y))$ with $x \in \sigma_0$ and $y \in \sigma$.

Case (1):

$$\begin{aligned}
&\big(\mathrm{Ad}(\omega) \circ (T^{\alpha',\varphi} \cdot T^{\beta,\psi})(U)\big)_{\varphi(x),\varphi(y)} \\
&\quad = \big((T^{\alpha',\varphi} \cdot T^{\beta,\psi})(U)\big)_{\varphi(x),\varphi(y)} + \alpha_{u,v} \cdot \big((T^{\alpha',\varphi} \cdot T^{\beta,\psi})(U)\big)_{\psi(x),\varphi(y)} \\
&\quad = T^{\alpha',\varphi}(U)_{\varphi(x),\varphi(y)} + \alpha_{u,v} \cdot T^{\beta,\psi}(U)_{\psi(x),\varphi(y)} \\
&\quad = \begin{cases} U_{x,y} & \text{if } y \in \sigma_0 \\ \alpha'(U_{x,y}) + \beta(U_{x,y})\alpha_{u,v} = \alpha(U_{x,y}) & \text{if } y \in \max \sigma \end{cases} \\
&\quad = \big((T^{\alpha,\varphi} \cdot T^{\beta,\psi})(U)\big)_{\varphi(x),\varphi(y)}.
\end{aligned}$$

Case (2):

$$\begin{aligned}
&\big(\mathrm{Ad}(\omega) \circ (T^{\alpha',\varphi} \cdot T^{\beta,\psi})(U)\big)_{\varphi(x),\psi(y)} \\
&\quad = \alpha_{u,v} \cdot \big((T^{\alpha',\varphi} \cdot T^{\beta,\psi})(U)\big)_{\varphi(x),\varphi(y)} - \big((T^{\alpha',\varphi} \cdot T^{\beta,\psi})(U)\big)_{\psi(x),\psi(y)} \cdot \alpha_{u,v} \\
&\quad = \alpha_{u,v} U_{x,y} - U_{x,y}\alpha_{u,v} \\
&\quad = 0 \\
&\quad = \big((T^{\alpha,\varphi} \cdot T^{\beta,\psi})(U)\big)_{\varphi(x),\psi(y)}.
\end{aligned}$$

Case (3):

$$\begin{aligned}
&\big(\mathrm{Ad}(\omega) \circ (T^{\alpha',\varphi} \cdot T^{\beta,\psi})(U)\big)_{\psi(x),\psi(y)} \\
&\quad = \big((T^{\alpha',\varphi} \cdot T^{\beta,\psi})(U)\big)_{\psi(x),\psi(y)} - \big((T^{\alpha',\varphi} \cdot T^{\beta,\psi})(U)\big)_{\psi(x),\phi(y)} \cdot \alpha_{u,v} \\
&\quad = \big((T^{\alpha,\varphi} \cdot T^{\beta,\psi})(U)\big)_{\psi(x),\psi(y)}.
\end{aligned}$$

Case (4):

$$\begin{aligned}
\big(\mathrm{Ad}(\omega) \circ (T^{\alpha',\varphi} \cdot T^{\beta,\psi})(U)\big)_{\psi(x),\varphi(y)} &= \big((T^{\alpha',\varphi} \cdot T^{\beta,\psi})(U)\big)_{\psi(x),\varphi(y)} \\
&= \big((T^{\alpha,\varphi} \cdot T^{\beta,\psi})(U)\big)_{\psi(x),\varphi(y)}.
\end{aligned}$$

[4]Here is the only place we need the commutativity of pro rings

Consequently, we obtain the equality (4.2).

Again, by the induction hypothesis and by the Künneth formula, we obtain

$$H_L(T^{\alpha,\varphi} \cdot T^{\beta,\psi}) = H_L(T^{\alpha',\varphi} \cdot T^{\beta,\psi}) = H_L(T^{\alpha',\varphi}) + H_L(T^{\beta,\psi}). \tag{4.3}$$

It follows from (4.1, 4.3) that

$$H_L(T^{\alpha,\varphi}) = H_L(T^{\alpha',\varphi}).$$

Therefore, by induction, we get the sublemma. □

We return to the proof of Lemma 4.5. We prove (i) for $l = L$. Let $\{P_m\}$ be a pseudo-free $\{A_m\}$-module. Let $\{P_m[-]\}$ be a pro resolution of $\{P_m\}$ as in Lemma 3.6. Then, by Proposition 3.4 and Corollary 4.4, $\{P_m[-_{\geqslant 0}]\} \to \{P_m\}$ induces a pro isomorphism

$$\Theta\colon \{H_L(T^\sigma(A_m, P_m[-_{\geqslant 0}]))\}_m \xrightarrow{\sim} \{H_L(T^\sigma(A_m, P_m))\}_m.$$

In fact, by Proposition 3.5, there exists $r(m) \geqslant m$ for each $m \in \Xi$, which does not depend on $\{P_m\}$, $\{P_m[-]\}$ and σ, such that the canonical maps $\ker \Theta_{r(m)} \to \ker \Theta_m$ and $\operatorname{coker} \Theta_{r(m)} \to \operatorname{coker} \Theta_m$ are equal to zero. We set

$$p := p_L := \Big(\prod_{l=1}^{L-1} (p_l + 1)\Big)(q+1) - 1,$$
$$s(m) := s_L(m) := r(s_1(\cdots(s_{L-1}(t(m)))\cdots)).$$

We claim that (i) for $l = L$ holds with these definitions. We prove it by induction on $n := \#\sigma \geqslant 1$. The case $n = 1$ is clear, and so let $n > 1$.

By Lemma 4.3, we have

$$C_*(BT^\sigma(A_m, P_m[-_{\geqslant 0}])) = C_*(ET^{\sigma_0}(A_m)) \otimes_{T^{\sigma_0}(A_m)} C_*(BM^\sigma(P_m[-_{\geqslant 0}]))$$

and thus we have a first quadrant spectral sequence

$$(E^1_{s,t})^\sigma_m = H_t(T^\sigma(A_m, P_m[s])) \Rightarrow H_{s+t}(T^\sigma(A_m, P_m[-_{\geqslant 0}])).$$

It is clear that $(E^2_{s,0})^\sigma_m = 0$ for $s > 0$. Hence, the spectral sequence induces a filtration

$$0 = F^\sigma_{-1,m} \subset F^\sigma_{0,m} \subset \cdots \subset F^\sigma_{L-1,m} = H_L(T^\sigma(A_m, P_m[-_{\geqslant 0}]))$$

with $F^\sigma_{i,m}/F^\sigma_{i-1,m}$ a subquotient of $H_{L-i}(T^\sigma(A_m, P_m[i]))$.

Note that the map $\varphi\colon \sigma \to \sigma\bigstar[p]$ induces a morphism of spectral sequences

$$\begin{array}{ccc}
(E^1_{s,t})^\sigma_m = H_t(T^\sigma(A_m, P_m[s])) & \Longrightarrow & H_{s+t}(T^\sigma(A_m, P_m[-_{\geqslant 0}])) \\
\downarrow & & \downarrow \\
(E^1_{s,t})^{\sigma\bigstar[p]}_m = H_t(T^{\sigma\bigstar[p]}(A_m, P_m[s])) & \Longrightarrow & H_{s+t}(T^{\sigma\bigstar[p]}(A_m, P_m[-_{\geqslant 0}])).
\end{array}$$

By the induction hypothesis, the induced map

$$F^\sigma_{i,s_{L-i}(m)}/F^\sigma_{i-1,s_{L-i}(m)} \to F^{\sigma\bigstar[p_{L-i}]}_{i,m}/F^{\sigma\bigstar[p_{L-i}]}_{i-1,m}$$

is zero for $1 \leqslant i \leqslant L-1$. Also, observe that $(\sigma\bigstar[a])\bigstar[b] = \sigma\bigstar[(a+1)(b+1)-1]$. It follows that, by putting $s'(m) := s_1(\cdots(s_{L-1}(t(m)))\cdots)$ and $p' := \prod_{l=1}^{L-1}(p_l+1) - 1$, the canonical map

$$\iota_{s'(m),t(m)}H_l(T^{\varphi})\colon H_L(T^{\sigma}(A_{s'(m)},P_{s'(m)}[-_{\geqslant 0}]))\to H_L(T^{\sigma\bigstar[p']}(A_{t(m)},P_{t(m)}[-_{\geqslant 0}]))$$

factors through $F^{\sigma\bigstar[p']}_{0,t(m)}$.

Now, we have lifts (maps of sets) in the commutative diagram

$$\begin{tikzcd}
& & H_L(T^{\sigma}(A_{s(m)},P_{s(m)})) \arrow[dl, dotted] \arrow[d, "\iota_{s(m),s'(m)}"] \\
F^{\sigma}_{0,s'(m)} \arrow[r, hook] \arrow[d] & H_L(T^{\sigma}(A_{s'(m)},P_{s'(m)}[-_{\geqslant 0}])) \arrow[r, "\Theta"] \arrow[d] \arrow[dl, dotted] & H_L(T^{\sigma}(A_{s'(m)},P_{s'(m)})) \arrow[d, "\iota_{s'(m),t(m)}H_L(T^{\varphi})"'] \\
F^{\sigma\bigstar[p']}_{0,t(m)} \arrow[r, hook] \arrow[d, dotted, bend right] & H_L(T^{\sigma\bigstar[p']}(A_{t(m)},P_{t(m)}[-_{\geqslant 0}])) \arrow[r, two heads, "\Theta"] & H_L(T^{\sigma\bigstar[p']}(A_{t(m)},P_{t(m)})) \arrow[d, equal] \\
H_L(T^{\sigma\bigstar[p']}(A_{t(m)},P_{t(m)}[0])) \arrow[u, two heads] \arrow[rr, "H_L(T^{\epsilon,\varphi})"] & & H_L(T^{\sigma\bigstar[p']}(A_{t(m)},P_{t(m)})).
\end{tikzcd}$$

Consider the following diagram

$$\begin{tikzcd}
& H_L(T^{\sigma\bigstar[p']}(A_{t(m)},P_{t(m)}[0])) \arrow[r, "H_L(T^{\epsilon,\varphi})"] \arrow[d] & H_L(T^{\sigma\bigstar[p]}(A_m,P_m)) \arrow[d, equal] \\
H_L(T^{\sigma}(A_{s(m)},P_{s(m)})) \arrow[r, "H_L(T^{\varphi})"'] \arrow[d] \arrow[ur, dotted] & H_L(T^{\sigma\bigstar[p']}(A_{t(m)},P_{t(m)})) \arrow[r, "H_L(T^{\varphi})"'] \arrow[d] & H_L(T^{\sigma\bigstar[p]}(A_m,P_m)) \\
H_L(T^{\sigma_0}(A_{s(m)})) \arrow[r, "H_L(T^{\phi})"'] & H_L(T^{\sigma_0\times[p']}(A_{t(m)})) \arrow[r, "H_L(T^{\phi})"'] & H_L(T^{\sigma_0\times[p]}(A_m)), \arrow[u]
\end{tikzcd}$$

where we omit the structure maps $\iota_{*,*}$. The right rectangle commutes by Sublemma 4.6, though the lower right square may not commute. It follows from a simple diagram chase that the bottom rectangle commutes. Therefore, by the induction hypothesis for $n=\#\sigma$, the middle composite $\iota_{s(m),m}H_L(T^{\varphi})$ equals zero.

Finally, (ii) for $l=L$ follows immediately from (i) for $l=L$ and Sublemma 4.6. $\square$

Let σ be a partial ordering on $\{1,\dots,n\}$. Then we can naturally regard $T^{\sigma}(A)$ as a subgroup of $\mathrm{GL}_n(A)$. For $k\geqslant 0$, we define ${}^k\tilde{\sigma}$ to be the partial ordering on $\{1,\dots,n+k\}$ obtained from σ by adding the relations $i<n+j$ for $i\in\{1,\dots,n\}$ and $1\leqslant j\leqslant k$. We set

$${}^k\tilde{T}^{\sigma}(A,P):=\begin{cases}T^{\sigma}(A) & \text{if } k=0\\ T^{{}^k\tilde{\sigma}}(A,P) & \text{if } k\geqslant 1\end{cases}=T^{\sigma}(A)\ltimes M_{n,k}(P).$$

We write Π_n for the set of all partial orderings on $\{1,\dots,n\}$.

Corollary 4.8. *Let* $\mathbf{A}$ *be a commutative Tor-unital pro ring. Let* $\sigma_1,\dots,\sigma_t\in\Pi_n$ *and* $l\geqslant 0$. *Then there exists* $p\geqslant 0$ *such that the canonical map*

$$\tilde{H}_l\Big(\bigcup_{i=1}^{t}B^k\tilde{T}^{\sigma_i}(\mathbf{A},\mathbf{P})\Big)\to\tilde{H}_l\Big(\bigcup_{i=1}^{t}B^k\tilde{T}^{\sigma_i\times[p]}(\mathbf{A},\mathbf{P})\Big)$$

is equal to zero as a pro morphism for all $k\geqslant 0$ *and all pseudo-free* $\mathbf{A}$*-modules* $\mathbf{P}$*, where the unions are taken in* $B(\mathrm{GL}_*(\mathbf{A})\ltimes M_{*,k}(\mathbf{P}))$.

In particular, there exists $N \geqslant n$ such that the canonical map

$$\tilde{H}_l\Big(\bigcup_{\sigma\in\Pi_n} B^k\tilde{T}^\sigma(\mathbf{A},\mathbf{P})\Big) \to \tilde{H}_l\Big(\bigcup_{\sigma\in\Pi_N} B^k\tilde{T}^\sigma(\mathbf{A},\mathbf{P})\Big)$$

is equal to zero for all $k \geqslant 0$ and all pseudo-free $\mathbf{A}$-modules $\mathbf{P}$.

Proof. We follow the proof of [**Su82**, Lemma 7.5]. Note that

$${}^k\tilde{T}^{\sigma\times[p]}(A,P) = \begin{cases} T^{\sigma\times[p]}(A) & \text{if } k=0,\\ T^{{}^k\tilde{\sigma}\bigstar[p]}(A,P) & \text{if } k\geqslant 1.\end{cases}$$

Hence, the case $t=1$ is true by Lemma 4.5. Let $t>1$ and suppose that the corollary holds for $s<t$.

We abbreviate ${}^k\tilde{T}^\sigma(\mathbf{A},\mathbf{P})$ as $\tilde{T}^\sigma$. Set $\sigma_{i,t} := \sigma_i \cap \sigma_t$. Then we have a commutative diagram

$$\begin{tikzcd}
\tilde{H}_l\big(\bigcup_{i=1}^{t-1} B\tilde{T}^{\sigma_i}\big) \oplus \tilde{H}_l(B\tilde{T}^{\sigma_t}) \arrow[r] \arrow[d] & \tilde{H}_l\big(\bigcup_{i=1}^{t} B\tilde{T}^{\sigma_i}\big) \arrow[r] \arrow[d] \arrow[dl, dotted] & \tilde{H}_{l-1}\big(\bigcup_{i=1}^{t-1} B\tilde{T}^{\sigma_{i,t}}\big) \arrow[d] \\
\tilde{H}_l\big(\bigcup_{i=1}^{t-1} B\tilde{T}^{\sigma_i\times[q]}\big) \oplus \tilde{H}_l(B\tilde{T}^{\sigma_t\times[q]}) \arrow[r] & \tilde{H}_l\big(\bigcup_{i=1}^{t} B\tilde{T}^{\sigma_i\times[q]}\big) \arrow[r] & \tilde{H}_{l-1}\big(\bigcup_{i=1}^{t-1} B\tilde{T}^{\sigma_{i,t}\times[q]}\big)
\end{tikzcd}$$

with exact rows. We regard the diagram as a diagram of modules by the Freyd–Mitchell embedding. By the induction hypothesis, the right vertical map is zero for some $q\geqslant 0$. Thus, there exists a lift (a map of sets) as indicated above. Again, by the induction hypothesis, there exists $q'\geqslant 0$ such that the map

$$\tilde{H}_l\Big(\bigcup_{i=1}^{t-1} B\tilde{T}^{\sigma_i}\Big) \oplus \tilde{H}_l(B\tilde{T}^{\sigma_t}) \to \tilde{H}_l\Big(\bigcup_{i=1}^{t-1} B\tilde{T}^{\sigma_i\times[q']}\Big) \oplus \tilde{H}_l(B\tilde{T}^{\sigma_t\times[q']})$$

is zero. It follows from $(\sigma_i\times[q])\times[q'] = \sigma_i\times[q'']$ with $q'' := (q+1)(q'+1)-1$ that the map

$$\tilde{H}_l\Big(\bigcup_{i=1}^{t} B\tilde{T}^{\sigma_i}\Big) \to \tilde{H}_l\Big(\bigcup_{i=1}^{t} B\tilde{T}^{\sigma_i\times[q'']}\Big)$$

is zero. This completes the proof. □

4.3. The pro acyclicity theorem

We set ${}^k\tilde{T}^\sigma(A) := {}^k\tilde{T}^\sigma(A,A) = T^\sigma(A)\ltimes M_{n,k}(A)$ for $\sigma\in\Pi_n$.

Theorem 4.9. *Let $\mathbf{A}$ be a commutative Tor-unital pro ring and $l\geqslant 0$. Then:*

(i) *For $n\geqslant 2l+1$ and for any $k\geqslant 0$,*

$$\tilde{H}_l\Big(\bigcup_{\sigma\in\Pi_n} B^k\tilde{T}^\sigma(\mathbf{A})\Big) = 0,$$

where the union is taken in $B(\mathrm{GL}_n(\mathbf{A})\ltimes M_{n,k}(\mathbf{A}))$.

(ii) *For $n\geqslant 2l$ and for any $k\geqslant 0$, the canonical map*

$$H_l\Big(\bigcup_{\sigma\in\Pi_n} BT^\sigma(\mathbf{A})\Big) \to H_l\Big(\bigcup_{\sigma\in\Pi_n} B^k\tilde{T}^\sigma(\mathbf{A})\Big)$$

is a pro isomorphism.

Proof. We write ${}^k\tilde{X}_n(\mathbf{A}) = \bigcup_{\sigma\in\Pi_n} B^k\tilde{T}^\sigma(\mathbf{A})$ and $X_n(\mathbf{A}) = {}^0\tilde{X}_n(\mathbf{A})$.

We prove the theorem by induction on l. The case $l = 0$ is trivial. Let $L > 0$ and suppose that the theorem holds for $l < L$.

Sublemma 4.10. *Let $k \geqslant 0$. The canonical map*

$$H_L({}^k\tilde{X}_n(\mathbf{A})) \to H_L({}^k\tilde{X}_{n+1}(\mathbf{A}))$$

is a pro epimorphism for $n \geqslant 2L$ and a pro isomorphism for $n \geqslant 2L+1$.

Proof. Let us introduce some notation. Let A be a ring, $\sigma \in \Pi_n$ and $1 \leqslant i \leqslant n$. We define $T_n^{\sigma,i}(A)$ to be the subgroup of $T_n^\sigma(A)$ consisting of all α with $\alpha_{i,j} = \alpha_{j,i} = 0$ for $i \neq j$. For $k \geqslant 0$, we set

$${}^k\tilde{X}_n^i(A) := \bigcup_{\sigma\in\Pi_n} BT^{{}^k\tilde{\sigma},i}(A)$$

and write ${}^k\tilde{X}_n^{i_1,\dots,i_p}(A)$ for the intersection of ${}^k\tilde{X}_n^{i_1}(A), \dots, {}^k\tilde{X}_n^{i_p}(A)$. Then it is easy to see that ${}^k\tilde{X}_n^{i_1,\dots,i_p}(A) \simeq {}^k\tilde{X}_{n-p}(A)$.

Consider the spectral sequence

$${}^k\tilde{E}^1_{p,q} = \coprod_{i_0,\dots,i_p} H_q\big({}^k\tilde{X}_{n+1}^{i_0,\dots,i_p}(\mathbf{A})\big) \Rightarrow H_{p+q}\Big(\bigcup_{1\leqslant i\leqslant n+1} {}^k\tilde{X}_{n+1}^i(\mathbf{A})\Big). \tag{4.4}$$

Since ${}^k\tilde{X}_{n+1}^{i_0,\dots,i_p}(\mathbf{A}) \simeq {}^k\tilde{X}_{n-p}(\mathbf{A})$, it follows from the induction hypothesis that

$${}^k\tilde{E}^2_{0,L} \simeq H_L({}^k\tilde{X}_n(\mathbf{A}))$$

for $n \geqslant 2L$. Hence, the canonical map

$$H_L({}^k\tilde{X}_n(\mathbf{A})) \to H_L\Big(\bigcup_{1\leqslant i\leqslant n+1} {}^k\tilde{X}_{n+1}^i(\mathbf{A})\Big)$$

is a pro epimorphism for $n \geqslant 2L$ and a pro isomorphism for $n \geqslant 2L+1$. According to [**Su82**, Corollary 6.6, see also the remark before Theorem 7.1][5], the canonical map

$$H_L\Big(\bigcup_{1\leqslant i\leqslant n+1} {}^k\tilde{X}_{n+1}^i(\mathbf{A})\Big) \to H_L({}^k\tilde{X}_{n+1}(\mathbf{A}))$$

is a levelwise surjection for $n \geqslant 2L$ and a levelwise bijection for $n \geqslant 2L+1$. Bringing these together, we obtain the sublemma. □

We show (i) for $l = L$. Suppose that $n \geqslant 2L+1$. According to Corollary 4.8, the canonical map

$$H_L({}^k\tilde{X}_n(\mathbf{A})) \to H_L({}^k\tilde{X}_N(\mathbf{A}))$$

is zero for some $N \geqslant n$. On the other hand, by Sublemma 4.10, this map is a pro isomorphism, and thus $H_L({}^k\tilde{X}_n(\mathbf{A})) = 0$.

[5]The proof works for non-unital rings as it is.

To get (ii) for $l = L$, it remains to show that the canonical map

$$H_L(X_{2L}(\mathbf{A})) \to H_L({}^k\tilde{X}_{2L}(\mathbf{A}))$$

is a pro isomorphism. By the spectral sequence (4.4), we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & E^2_{2,L-1} \arrow[r] \arrow[d] & H_L(X_{2L}(\mathbf{A})) \arrow[r] \arrow[d] & H_L(X_{2L+1}(\mathbf{A})) \arrow[r] \arrow[d, "\simeq"] & 0 \\
0 \arrow[r] & {}^k\tilde{E}^2_{2,L-1} \arrow[r] & H_L({}^k\tilde{X}_{2L}(\mathbf{A})) \arrow[r] & H_L({}^k\tilde{X}_{2L+1}(\mathbf{A})) \arrow[r] & 0
\end{tikzcd}$$

with exact rows. Hence, it is enough to show that $E^2_{2,L-1} \to {}^k\tilde{E}^2_{2,L-1}$ is a pro isomorphism; equivalently it is a pro epimorphism. This follows from the diagram

$$\begin{tikzcd}
E^1_{2,L-1} = \bigoplus H_{L-1}(X_{2L-2}(\mathbf{A})) \arrow[r] \arrow[d, "\simeq"] & E^2_{2,L-1} \arrow[r] \arrow[d] & 0 \\
{}^k\tilde{E}^1_{2,L-1} = \bigoplus H_{L-1}({}^k\tilde{X}_{2L-2}(\mathbf{A})) \arrow[r] & {}^k\tilde{E}^2_{2,L-1} \arrow[r] & 0
\end{tikzcd}$$

with exact rows. □

5. Homology pro stability

In this section, we prove homology pro stability for E_n (Theorem 5.6) and for GL_n (Theorem 5.13). We follow Suslin [**Su96**], generalizing his argument to the pro setting.

We say that a levelwise action of a pro group $\{G_m\}$ on a pro object $\{M_m\}$ is *pro trivial* if there exists $s \geqslant m$ for each m such that $\iota_{s,m}(gx) = \iota_{s,m}(x)$ for all $g \in G_s$ and $x \in M_s$.

5.1. Volodin spaces

Let G be a group and $\{G_i\}_{i\in I}$ a family of subgroups of G. We define the *Volodin space* $V(G, \{G_i\}_{i\in I})$ to be the simplicial subset of EG formed by simplices $(g_0, \dots, g_p)$ for which there exists $i \in I$ such that $g_j g_k^{-1} \in G_i$ for all $0 \leqslant j, k \leqslant p$.

The simplicial subset $V(G, \{G_i\}_{i\in I}) \subset EG$ is stable under the right action of G, and $V(G, \{G_i\})/G = \bigcup_{i\in I} BG_i$. Hence, we have a spectral sequence

$$E^2_{p,q} = H_p(G, H_q(V(G, \{G_i\}_{i\in I}))) \Rightarrow H_{p+q}\Big(\bigcup_{i\in I} BG_i\Big). \tag{5.1}$$

Let A be a ring. We consider the Volodin space

$$V_n(A) := V(E_n(A), \{T^\sigma(A)\}_{\sigma\in\Pi_n}).$$

The permutation group Σ_n acts on $V_n(A)$ by conjugation, and $E_n(A)$ acts on $V_n(A)$ by right multiplication.

Here are some properties of Volodin spaces we need.

Lemma 5.1 (Suslin–Wodzicki [**Su96**, Lemma 5.4])**.** *Let G be a subgroup of* $\mathrm{GL}_n(A)$

containing $E_n(A)$ and let $k \geqslant 0$. Then the canonical projection and the inclusion

$$V(G, \{T^\sigma(A)\}_{\sigma\in\Pi_n}) \rightleftarrows V\left(\begin{pmatrix} G & * \\ 0 & 1_k \end{pmatrix}, \left\{\begin{pmatrix} T^\sigma(A) & * \\ 0 & 1_k \end{pmatrix}\right\}_{\sigma\in\Pi_n}\right)$$

are mutually inverse homotopy equivalences.

Lemma 5.2 (Suslin–Wodzicki [**SW92**, Lemma 2.8])**.** *For every $n, l \geqslant 0$, the action of $E_{n+1}(A^2)$ on the image of the canonical map*

$$H_l(V_n(A)) \to H_l(V_{n+1}(A))$$

is trivial.

Corollary 5.3. *Let $\mathbf{A}$ be a pro ring such that $\mathbf{A}/\mathbf{A}^2 = 0$. Then, for every $n, l \geqslant 0$, the action of $E_{n+1}(\mathbf{A})$ on the image of the canonical map*

$$H_l(V_n(\mathbf{A})) \to H_l(V_{n+1}(\mathbf{A}))$$

is pro trivial.

Proof. Write $\mathbf{A} = \{A_m\}$. By the assumption, there exists $s \geqslant m$ for each m such that $\iota_{s,m}A_s \subset A_m^2$. Hence, given x in the image of $H_l(V_n(A_s)) \to H_l(V_{n+1}(A_s))$ and $g \in E_n(A_s)$, we have $\iota_{s,m}(gx) = \iota_{s,m}(x)$. $\square$

5.2. Van der Kallen's acyclicity

Let A be a ring and $n \geqslant 1$. Fix a unital ring R which contains A as a two sided ideal. Let I be a finite subset of $\{1, \ldots, n\}$ and R^n the free right R-module with basis $e_1, \ldots, e_n$. A map $f\colon I \to R^n$ is called an *A-unimodular function* if $\{f(i)\}_{i\in I}$ forms a basis of a free direct summand of R^n and $f(i) \equiv e_i$ modulo A. We denote by $\mathsf{Uni}^I_{A,n} = \mathsf{Uni}^I_{A,n}(R)$ the set of all A-unimodular functions $f\colon I \to R^n$. As the notation suggests, $\mathsf{Uni}^I_{A,n}$ does not depend on R.

We define the associated semi-simplicial set $\mathsf{Uni}_{A,n}$ as follows: A p-simplex is an A-unimodular function $f \in \mathsf{Uni}^I_{A,n}$ for some I with $|I| = p+1$. The i-th face $d_i\colon (\mathsf{Uni}_{A,n})_p \to (\mathsf{Uni}_{A,n})_{p-1}$, $0 \leqslant i \leqslant p$, is defined by

$$(f, \operatorname{dom} f = \{i_0, \ldots, i_p\}) \mapsto f|_{\{i_0,\ldots,\hat{i}_k,\ldots,i_p\}}.$$

As in the preceding section, for a semi-simplicial set X, we denote by $C_*(X)$ the complex freely generated by X_* with the differential being the alternating sum of the faces.

The following result is proved by van der Kallen [**vdK80**] in the case A is unital, and the proof can be easily modified for non-unital rings. We can also find a complete proof in [**Su96**, §2].

Theorem 5.4. *$\tilde{H}_l(C_*(\mathsf{Uni}_{A,n})) = 0$ for $n \geqslant l + \mathrm{sr}(A) + 1$.*

Let $\mathsf{SUni}^I_{A,n}$ (resp. $\overline{\mathsf{SUni}}^I_{A,n}(R)$) be the set of all unimodular functions $f \in \mathsf{Uni}^I_{A,n}(R)$ for which there exists $\alpha \in E_n(A)$ (resp. $\alpha \in E_n(R, A)$) such that $f(i) = e_i\alpha$ for all $i \in I$. These yield semi-simplicial subsets $\mathsf{SUni}_{A,n}$ and $\overline{\mathsf{SUni}}_{A,n}(R)$ of $\mathsf{Uni}_{A,n}(R)$ in an obvious way.

Corollary 5.5.

(i) $\tilde{H}_l(C_*(\overline{\mathsf{SUni}}_{A,n}(R))) = 0$ *for* $n \geqslant l + \mathrm{sr}(A) + 1$.

(ii) *Let* $\mathbf{A}$ *be a pro ring such that* $\mathbf{A}/\mathbf{A}^2 = 0$. *Then*

$$\tilde{H}_l(C_*(\mathsf{SUni}_{\mathbf{A},n})) = 0$$

as pro abelian groups for $n \geqslant l + \mathrm{sr}(\mathbf{A}) + 1$.

Proof. (i) This is [**Su96**, Corollary 2.8].

(ii) Let $\mathbf{R}$ be a unital pro ring which contains $\mathbf{A}$ as a two-sided ideal. By Corollary 2.3, the canonical map $\mathsf{SUni}_{\mathbf{A},n} \to \overline{\mathsf{SUni}}_{\mathbf{A},n}(\mathbf{R})$ is a pro isomorphism. Hence, (ii) follows from (i). □

5.3. Homology pro stability for V_n and E_n

The following is a pro version of [**Su96**, Theorem 6.1].

Theorem 5.6. *Let* $\mathbf{A}$ *be a commutative Tor-unital pro ring. Let* $r = \max(\mathrm{sr}(\mathbf{A}), 2)$ *and* $l \geqslant 0$. *Then:*

(i) *The canonical map*

$$H_l(V_n(\mathbf{A})) \to H_l(V_{n+1}(\mathbf{A}))$$

is a pro epimorphism for $n \geqslant 2l + r + 1$ *and a pro isomorphism for* $n \geqslant 2l + r + 2$.

(ii) *The conjugate action of* Σ_n *on* $H_l(V_n(\mathbf{A}))$ *is pro trivial for* $n \geqslant 2l + r + 2$.

(iii) *The action of* $E_n(\mathbf{A})$ *on* $H_l(V_n(\mathbf{A}))$ *is pro trivial for* $n \geqslant 2l + r + 2$.

(iv) *The canonical map*

$$H_l(E_n(\mathbf{A})) \to H_l\left(\begin{pmatrix} E_n(\mathbf{A}) & * \\ 0 & 1_k \end{pmatrix}\right)$$

is a pro isomorphism for $n \geqslant 2l + r - 2$ *and for any* $k \geqslant 0$.

(v) *The conjugate action of* Σ_n *on* $H_l(E_n(\mathbf{A}))$ *is pro trivial for* $n \geqslant 2l + r - 1$.

(vi) *The canonical map*

$$H_l(E_n(\mathbf{A})) \to H_l(E_{n+1}(\mathbf{A}))$$

is a pro epimorphism for $n \geqslant 2l + r - 2$ *and a pro isomorphism for* $n \geqslant 2l + r - 1$.

We prove Theorem 5.6 by induction on l. The case $l = 0$ is clear. Also, (iv, v, vi) for $l = 1$ holds by the obvious reasons: (v, vi) follows from the fact $H_1(E_n(\mathbf{A})) = 0$ for $n \geqslant 3$. For (iv), note that we have a levelwise exact sequence

$$M_{n,k}(\mathbf{A})_{E_n(\mathbf{A})} \longrightarrow H_1(E_n(\mathbf{A}) \ltimes M_{n,k}(\mathbf{A})) \longrightarrow H_1(E_n(\mathbf{A})) \longrightarrow 0,$$

and it is easy to see that $M_{n,k}(\mathbf{A})_{E_n(\mathbf{A})} = 0$ for $n \geqslant 2$.

Let $L > 0$. The proof is divided into the four steps. We write $(?)_{<N}$ (resp. $(?)_{\leqslant N}$, resp. $(?)_N$) for Theorem 5.6 (?) with $l < N$ (resp. $l \leqslant N$, resp. $l = N$).

Step 1: $(\text{i, ii, iii})_{<L-1} \Rightarrow (\text{iii})_{L-1}$.

Step 2: $(\text{iii})_{\leqslant L-1}$, $(\text{iv})_{<L+1} \Rightarrow (\text{iv})_{L+1}$.

Step 3: $(\text{iv})_{\leqslant L+1}$, $(\text{v, vi})_{<L+1} \Rightarrow (\text{v, vi})_{L+1}$.

Step 4: (i, ii)$_{<L-1}$, (iii)$_{\leqslant L-1}$ (vi)$_{\leqslant L+1}$ $\Rightarrow$ (i, ii)$_{L-1}$.

Remark 5.7. Let us explain how our argument below compares to Suslin's in [**Su96**]. First we remark that, in (i)–(iii) of Theorem 5.6, the range of stability or triviality is different from Suslin's (ours is weaker). We think it was just an error there. Accordingly there is a minor difference in induction systems between ours and Suslin's, but all the essential ideas below are due to Suslin and the arguments are roughly compared as follows:

— Step 1 corresponds to 6.2–6.4 in [**Su96**].

— Step 2 corresponds to Corollary 5.8 and its proof in [**Su96**].

— Step 3 corresponds to 6.5–6.7 in [**Su96**].

— Step 4 corresponds to 6.8–6.10 in [**Su96**].

5.4. Step 1: Covering argument I

Suppose that (i, ii, iii)$_{<L-1}$ hold. We show (iii)$_{L-1}$.[6]

5.4.1. Covering spectral sequence

Let A be a ring. For $I \subset \{1, \ldots, n\}$, let Π_n^I be the set of all partial orderings of $\{1, \ldots, n\}$ for which every $i \in I$ is maximal. Set $V_n(A)^I := V_n(E_n(A), \{T^\sigma(A)\}_{\sigma \in \Pi_n^I})$. Then $V_n(A) = \bigcup_{i=1}^n V_n(A)^i$, and there is a spectral sequence

$$E^1_{p,q}(A) = \coprod_{|I|=p+1} H_q(V_n(A)^I) \Rightarrow H_{p+q}(V_n(A)).$$

We define a map $\phi\colon V_n(A)^I \to \mathsf{SUni}^I_{A,n}$ by $\phi(\alpha_0, \ldots, \alpha_q)(i) = e_i\alpha_0$, $i \in I$. Then ϕ is a morphism of simplicial sets regarding $\mathsf{SUni}^I_{A,n}$ as a constant simplicial set, and the inverse image of the unimodular function $f_0\colon i \mapsto e_i$ is $V(E_n(A)^I, \{T^\sigma(A)\}_{\sigma \in \Pi_n^I})$, where $E_n(A)^I$ is the subgroup of $E_n(A)$ generated by elementary matrices α such that $e_i\alpha = e_i$ for all $i \in I$. For each $f \in \mathsf{SUni}^I_{A,n}$, choose $\Lambda(f) \in E_n(A)$ with $f(i) = e_i\Lambda(f)$, $i \in I$. Since the map ϕ is $E_n(A)$-equivariant, $\Lambda(f)$ gives an isomorphism $\phi^{-1}(f_0) \simeq \phi^{-1}(f)$ and

$$\mathsf{SUni}^I_{A,n} \times V(E_n(A)^I, \{T^\sigma(A)\}_{\sigma \in \Pi_n^I}) \xrightarrow{\sim} V_n(A)^I, \quad (f, u) \mapsto u\Lambda(f).$$

Also, the conjugation by the shuffle permutation σ_I, $\sigma_I\{n-p, \ldots, n\} = I$, gives an isomorphism

$$V(E_n(A)^{n-p,\ldots,n}, \{T^\sigma(A)\}_{\sigma \in \Pi_n^{n-p,\ldots,n}}) \xrightarrow{\sim} V(E_n(A)^I, \{T^\sigma(A)\}_{\sigma \in \Pi_n^I}).$$

Hence, we get an isomorphism

$$\Phi_\Lambda\colon C_p(\mathsf{SUni}_{A,n}) \otimes H_q(V(E_n(A)^{n-p,\ldots,n}, \{T^\sigma(A)\}_{\sigma \in \Pi_n^{n-p,\ldots,n}})) \xrightarrow{\sim} E^1_{p,q}(A).$$

For another choice of Λ', there exists $\gamma(f) \in E_n(A)^{n-p,\ldots,n}$ for each $f \in \mathsf{SUni}_{A,n}$ such that $\Phi_{\Lambda'}(f, u) = \Phi_\Lambda(f, u\gamma(f))$.

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[6] In this step, we only need $\mathrm{Tor}_1^{\mathbb{Z} \ltimes \mathbf{A}}(\mathbb{Z}, \mathbb{Z}) = \mathbf{A}/\mathbf{A}^2 = 0$.

Under the isomorphism Φ_Λ, the differential $d^1\colon E_{p,q} \to E_{p-1,q}$ is given by, for $f \in \mathsf{SUni}^I_{A,n}$ and $u \in H_q(V(E_n(A)^{n-p,\dots,n}, \{T^\sigma(A)\}_{\sigma\in\Pi_n^{n-p,\dots,n}}))$,

$$d^1(f \otimes u) = \sum_{k=0}^{p} (-1)^k d_k f \otimes \tau_{I,k}(\delta u)\tau_{I,k}^{-1}\alpha_k. \tag{5.2}$$

Here, α_k is a certain element in $E_n(A)^{n-p+1,\dots,n}$, $\tau_{I,k} := \sigma^{-1}_{I\setminus\{i_k\}}\sigma_I$, and δ is the map induced from the canonical embedding $E_n(A)^{n-p,\dots,n} \to E_n(A)^{n-p+1,\dots,n}$.

5.4.2. Pro arguments

We write $\mathbf{A} = \{A_m\}_{m\in\Xi}$. Set $\bar{E}_n(\mathbf{A}) := \mathrm{GL}_n(\mathbf{A}) \cap E(\mathbf{A})$. Then the canonical maps

$$\begin{gathered} H_q(V(\bar{E}_{n-p-1}(\mathbf{A}), \{T^\sigma(\mathbf{A})\}_{\sigma\in\Pi_{n-p-1}})) \\ \downarrow\simeq \\ H_q(V(\bar{E}_n(\mathbf{A})^{n-p,\dots,n}, \{T^\sigma(\mathbf{A})\}_{\sigma\in\Pi_n^{n-p,\dots,n}})) \\ \downarrow\simeq \\ H_q\left(V\left(\begin{pmatrix}\bar{E}_{n-p-1}(\mathbf{A}) & * \\ 0 & 1_{p+1}\end{pmatrix}, \left\{\begin{pmatrix}T^\sigma(\mathbf{A}) & * \\ 0 & 1_{p+1}\end{pmatrix}\right\}_{\sigma\in\Pi_{n-p-1}}\right)\right) \end{gathered}$$

are levelwise isomorphisms. Indeed, the second map is an isomorphism by definition and the composite is an isomorphism by Lemma 5.1. Hence, by Theorem 2.6, the canonical map

$$\lambda\colon H_q(V_{n-p-1}(\mathbf{A})) \to H_q(V(E_n(\mathbf{A})^{n-p,\dots,n}, \{T^\sigma(\mathbf{A})\}_{\sigma\in\Pi_n^{n-p,\dots,n}}))$$

is a pro isomorphism for $n-p-1 \geqslant r+1$.

Suppose that $q < L-1$ and $n-p-1 \geqslant 2q+r+2$. Then, by (iii)$_{<L-1}$, the action of $E_{n-p-1}(\mathbf{A})$ on $H_q(V_{n-p-1}(\mathbf{A}))$ is pro trivial. Hence, there exists $s(m) \geqslant m$ for each $m \in \Xi$ such that the composite Ψ_m in the diagram below does not depend on the choice of Λ:

$$\begin{tikzcd} C_p(\mathsf{SUni}_{A_{s(m)},n}) \otimes H_q(V_{n-p-1}(A_{s(m)})) \arrow[d, "\mathrm{id}\otimes\lambda"'] \arrow[ddr, dotted, bend left, "\Psi_m"] & \\ C_p(\mathsf{SUni}_{A_{s(m)},n}) \otimes H_q(V(E_n(A_{s(m)})^{n-p,\dots,n}, \{T^\sigma(A_{s(m)})\}_{\sigma\in\Pi_n^{n-p,\dots,n}})) \arrow[d, "\Phi_\Lambda"', "\simeq"] & \\ E^1_{p,q}(A_{s(m)}) \arrow[r, "\iota_{s(m),m}"] & E^1_{p,q}(A_m). \end{tikzcd}$$

We may assume $s(m+1) > s(m)$ for every m, so that we obtain a morphism of pro abelian groups

$$\Psi\colon C_p(\mathsf{SUni}_{\mathbf{A},n}) \otimes H_q(V_{n-p-1}(\mathbf{A})) \to E^1_{p,q}(\mathbf{A}).$$

Since λ is a pro isomorphism and Φ_Λ is an isomorphism, we see that Ψ is a pro isomorphism.

Now, by (ii)$_{<L-1}$, the action of Σ_{n-p-1} on $H_q(V_{n-p-1}(\mathbf{A}))$ is also pro trivial.

Hence, by modifying $s(m) \geqslant m$ if necessary, we see that the diagram

$$\begin{tikzcd} C_{p+1}(\mathsf{SUni}_{A_{s(m)},n}) \otimes H_q(V_{n-p-2}(A_{s(m)})) \arrow[r, "\iota_{s(m),m}\Phi_\Lambda(\mathrm{id}\otimes\lambda)"] \arrow[d, "\sum(-1)^k d_k \otimes \delta"] & E^1_{p+1,q}(A_m) \arrow[d, "d^1"] \\ C_p(\mathsf{SUni}_{A_{s(m)},n}) \otimes H_q(V_{n-p-1}(A_{s(m)})) \arrow[r, "\iota_{s(m),m}\Phi_{\Lambda'}(\mathrm{id}\otimes\lambda)"] & E^1_{p,q}(A_m) \end{tikzcd}$$

commutes, cf. the formula (5.2). The horizontal maps are the maps Ψ_m unless $n - p - 1 = 2q + r + 2$; in the last case only the bottom horizontal map can be identified with Ψ_m. Consequently, for $q < L - 1$, we obtain a morphism of pro complexes

$$\Psi\colon \sigma_{\leqslant n-2q-r-3}(C_\bullet(\mathsf{SUni}_{\mathbf{A},n}) \otimes H_q(V_{n-1-\bullet}(\mathbf{A}))) \to \sigma_{\leqslant n-2q-r-3} E^1_{\bullet,q}(\mathbf{A}) \tag{5.3}$$

and it is a pro isomorphism.

Claim 5.8. *For $q < L - 1$ and $0 < p \leqslant n - 2q - r - 3$,*

$$E^2_{p,q}(\mathbf{A}) = 0.$$

Proof. Suppose that $q < L - 1$ and $0 < p \leqslant n - 2q - r - 3$. We set

$$F_{p,q}(\mathbf{A}) := C_p(\mathsf{SUni}_{\mathbf{A},n}) \otimes H_q(V_{n-p-1}(\mathbf{A})),$$

which we regard as a complex in p with differential $\partial := \sum(-1)^k d_k \otimes \delta$. First, we show that $H_p(F_{\bullet,q}(\mathbf{A})) = 0$.

By (i)$_{<L-1}$, the canonical map $H_q(V_{n-p-1}(\mathbf{A})) \to H_q(V_{n-p}(\mathbf{A}))$ is a pro isomorphism, and thus

$$\ker(F_{p,q}(\mathbf{A}) \to F_{p-1,q}(\mathbf{A})) \simeq Z_p(\mathsf{SUni}_{\mathbf{A},n}) \otimes H_q(V_{n-p-1}(\mathbf{A})),$$

where $Z_p(\mathsf{SUni}_{\mathbf{A},n}) := \ker(C_p(\mathsf{SUni}_{\mathbf{A},n}) \to C_{p-1}(\mathsf{SUni}_{\mathbf{A},n}))$. By Corollary 5.5, the differential

$$C_{p+1}(\mathsf{SUni}_{\mathbf{A},n}) \to Z_p(\mathsf{SUni}_{\mathbf{A},n})$$

is a pro epimorphism. Also, by (i)$_{<L-1}$, the canonical map

$$H_q(V_{n-p-2}(\mathbf{A})) \to H_q(V_{n-p-1}(\mathbf{A}))$$

is a pro epimorphism. These imply that $\partial\colon F_{p+1}(\mathbf{A}) \to \ker(F_{p,q}(\mathbf{A}) \to F_{p-1,q}(\mathbf{A}))$ is a pro epimorphism, hence $H_p(F_{\bullet,q}(\mathbf{A})) = 0$.

If $p < n - 2L - r - 3$, then Ψ (5.3) induces a pro isomorphism

$$H_p F_{\bullet,q}(\mathbf{A}) \simeq E^2_{p,q}(\mathbf{A}).$$

Hence, in this case, the vanishing of $E^2_{p,q}(\mathbf{A})$ follows from that of $H_p(F_{\bullet,q}(\mathbf{A}))$.

Finally, let $p = n - 2q - r - 3$. Then we have a commutative diagram

$$\begin{tikzcd} F_{p+1,q}(A_{s(m)}) \arrow[r, "\iota_{s(m),m}\Phi_\Lambda(\mathrm{id}\otimes\lambda)"] \arrow[d, "\partial"] & E^1_{p+1,q}(A_m) \arrow[d, "d^1"] \\ F_{p,q}(A_{s(m)}) \arrow[r, "\Psi_m"] & E^1_{p,q}(A_m). \end{tikzcd}$$

Since Ψ is a pro isomorphism, there exists $m' \geqslant m$ such that, for $x \in \ker(E^1_{p,q}(A_{m'}) \to$

$E^1_{p-1,q}(A_{m'}))$, $\iota_{m',m}x$ lifts to $y \in \ker(F_{p,q}(A_{s(m)}) \to F_{p-1,q}(A_{s(m)}))$ along Ψ_m. Further, since $H_{p,q}(F_{\bullet,q}(\mathbf{A})) = 0$, we may assume that $y = \partial z$ for some $z \in F_{p+1,q}(A_{s(m)})$. Hence, $\iota_{m',m}x$ is in the image of the differential d^1. This proves $E^2_{p,q}(\mathbf{A}) = 0$. □

5.4.3. Conclusion

Suppose that $n \geqslant 2L + r$. If $p + q = L - 1$ and $p > 0$, then $q < L - 1$ and $0 < p \leqslant n - 2q - r - 3$. Hence, by Claim 5.8, the $E^2_{p,q}$-terms with $p + q = L - 1$ are zero unless $E^2_{0,L-1}$, and the edge map

$$E^1_{0,L-1}(\mathbf{A}) \to H_{L-1}(V_n(\mathbf{A}))$$

is a pro epimorphism.

Now, the composite

$$\begin{tikzcd}
C_0(\mathsf{SUni}_{A_m,n}) \otimes H_{L-1}(V_{n-1}(A_m)) \arrow[d, "\mathrm{id}\otimes\lambda"'] \arrow[ddr, dotted, bend left] & \\
C_0(\mathsf{SUni}_{A_m,n}) \otimes H_{L-1}(V(E_n(A_m)^{\{n\}}, \{T^\sigma(A_m)\}_{\sigma\in\Pi_n^{\{n\}}})) \arrow[d, "\Phi_\Lambda"'] & \\
E^1_{0,L-1}(A_m) \arrow[r, "\text{edge}"] & H_{L-1}(V_n(A_m))
\end{tikzcd}$$

is given by $f \otimes u \mapsto \sigma_{\{i\}}(\delta u)\sigma_{\{i\}}^{-1}\Lambda(f)$, where f is an A_m-unimodular function with $\operatorname{dom} f = \{i\}$ and $u \in H_{L-1}(V_{n-1}(A_m))$. Since the action of $E_n(\mathbf{A})$ on the image of $\delta\colon H_{L-1}(V_{n-1}(\mathbf{A})) \to H_{L-1}(V_n(\mathbf{A}))$ is pro trivial by Corollary 5.3, the above composite yields a pro morphism

$$C_0(\mathsf{SUni}_{\mathbf{A},n}) \otimes H_{L-1}(V_{n-1}(\mathbf{A})) \to H_{L-1}(V_n(\mathbf{A})), \quad f \otimes u \mapsto \sigma_{\{i\}}(\delta u)\sigma_{\{i\}}^{-1}. \tag{5.4}$$

Furthermore, since the edge map is a pro epimorphism and Φ_Λ is an isomorphism and $\mathrm{id} \otimes \lambda$ is a pro isomorphism, we see that (5.4) is a pro epimorphism.

By Corollary 5.3 again, we conclude that the action of $E_n(\mathbf{A})$ on $H_{L-1}(V_n(\mathbf{A}))$ is pro trivial. This proves (iii)$_{L-1}$.

5.5. Step 2: V to E

Suppose that (iii)$_{\leqslant L-1}$ and (iv)$_{<L+1}$ hold. We show (iv)$_{L+1}$.

Let $n \geqslant 2L + r$ and fix $k \geqslant 0$. We set

$$\tilde{E}_n(\mathbf{A}) := \begin{pmatrix} E_n(\mathbf{A}) & * \\ 0 & 1_k \end{pmatrix}, \quad \tilde{T}^\sigma(\mathbf{A}) := \begin{pmatrix} T^\sigma(\mathbf{A}) & * \\ 0 & 1_k \end{pmatrix}$$

and $\tilde{V}_n(\mathbf{A}) := V(\tilde{E}_n(\mathbf{A}), \{\tilde{T}^\sigma(\mathbf{A})\}_{\sigma\in\Pi_n})$.

By Lemma 5.1, the canonical inclusion and projection $V_n(\mathbf{A}) \rightleftarrows \tilde{V}_n(\mathbf{A})$ are mutually inverse homotopy equivalences. It follows that the action of $\left(\begin{smallmatrix} 1_n & * \\ 0 & 1_k \end{smallmatrix}\right)$ on $H_*(\tilde{V}_n(\mathbf{A}))$ is trivial. By (iii)$_{\leqslant L-1}$, the action of $E_n(\mathbf{A})$ on $H_q(V_n(\mathbf{A})) \simeq H_q(\tilde{V}_n(\mathbf{A}))$ is pro trivial for $q \leqslant L-1$. Hence, the action of $\tilde{E}_n(\mathbf{A})$ on $H_q(\tilde{V}_n(\mathbf{A}))$ is pro trivial for $q \leqslant L-1$.

Consider the spectral sequences (5.1) and the canonical map between them;

$$\begin{array}{c} E^2_{p,q}(\mathbf{A}) = H_p(E_n(\mathbf{A}), H_q(V_n(\mathbf{A}))) \Longrightarrow H_{p+q}(\bigcup_{\sigma\in\Pi_n} BT^\sigma(\mathbf{A})) \\ \downarrow \\ \tilde{E}^2_{p,q}(\mathbf{A}) = H_p(\tilde{E}_n(\mathbf{A}), H_q(\tilde{V}_n(\mathbf{A}))) \Longrightarrow H_{p+q}(\bigcup_{\sigma\in\Pi_n} B\tilde{T}^\sigma(\mathbf{A})). \end{array}$$

For $q \leqslant L-1$, the E^2-terms fit into the extensions

$$\begin{tikzcd} 0 \arrow[r] & H_p(E_n(\mathbf{A})) \otimes H_q(V_n(\mathbf{A})) \arrow[r] \arrow[d] & E^2_{p,q}(\mathbf{A}) \arrow[r] \arrow[d] & \mathrm{Tor}(H_{p-1}(E_n(\mathbf{A})), H_q(V_n(\mathbf{A}))) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & H_p(\tilde{E}_n(\mathbf{A})) \otimes H_q(\tilde{V}_n(\mathbf{A})) \arrow[r] & \tilde{E}^2_{p,q}(\mathbf{A}) \arrow[r] & \mathrm{Tor}(H_{p-1}(\tilde{E}_n(\mathbf{A})), H_q(\tilde{V}_n(\mathbf{A}))) \arrow[r] & 0. \end{tikzcd}$$

By (iv)$_{<L+1}$, the canonical map $H_p(E_n(\mathbf{A})) \to \tilde{H}_p(E_n(\mathbf{A}))$ is a pro isomorphism for $p \leqslant L$. Hence, the canonical map

$$E^2_{p,q}(\mathbf{A}) \to \tilde{E}^2_{p,q}(\mathbf{A})$$

is a pro isomorphism for $p \leqslant L$ and $q \leqslant L-1$. Also, $E^2_{0,q}(\mathbf{A}) \simeq \tilde{E}^2_{0,q}(\mathbf{A})$ for all $q \geqslant 0$, since $H_*(V_n(\mathbf{A})) \simeq H_*(\tilde{V}_n(\mathbf{A}))$. Finally, by Theorem 4.9, the canonical map $E^\infty_i(\mathbf{A}) \to \tilde{E}^\infty_i(\mathbf{A})$ is a pro isomorphism for $n \geqslant 2i$.

Bringing these together, we have:

(1) $E^2_{p,q}(\mathbf{A}) \simeq \tilde{E}^2_{p,q}(\mathbf{A})$ for $p+q = L-1$.

(2) $E^2_{p,q}(\mathbf{A}) \simeq \tilde{E}^2_{p,q}(\mathbf{A})$ for $p+q = L$.

(3) $E^2_{p,q}(\mathbf{A}) \simeq \tilde{E}^2_{p,q}(\mathbf{A})$ for $p+q = L+1$ and $p \geqslant 2$ and $q \geqslant 1$.

(4) $E^\infty_L(\mathbf{A}) \simeq \tilde{E}^\infty_L(\mathbf{A})$ and $E^\infty_{L+1}(\mathbf{A}) \simeq \tilde{E}^\infty_{L+1}(\mathbf{A})$.

Then, by Lemma 5.9 below, we conclude that

$$E^2_{L+1,0}(\mathbf{A}) \to \tilde{E}^2_{L+1,0}(\mathbf{A})$$

is a pro epimorphism, and thus a pro isomorphism. This proves (iv)$_{L+1}$.

Lemma 5.9 ([**Su96**, Remark A.5])**.** *Let $\mathcal{A}$ be an abelian category. Let $f\colon E \to \tilde{E}$ be a morphism of first quadrant homological spectral sequence in $\mathcal{A}$, and let $L \geqslant 0$. Assume that f induces:*

(1) *A monomorphism $E^2_{p,q} \hookrightarrow \tilde{E}^2_{p,q}$ for $p+q = L-1$.*

(2) *An isomorphism $E^2_{p,q} \xrightarrow{\sim} \tilde{E}^2_{p,q}$ for $p+q = L$.*

(3) *An epimorphism $E^2_{p,q} \twoheadrightarrow \tilde{E}^2_{p,q}$ for $p+q = L+1$, $q \geqslant 1$ and $p \geqslant 2$.*

(4) *An isomorphism $E^\infty_L \xrightarrow{\sim} \tilde{E}^\infty_L$ and an epimorphism $E^\infty_{L+1} \twoheadrightarrow \tilde{E}^\infty_{L+1}$.*

Then f induces an epimorphism

$$E^2_{L+1,0} \twoheadrightarrow \tilde{E}^2_{L+1,0}.$$

5.6. Step 3: Covering argument II

Suppose that (iv)$_{\leqslant L+1}$ and (v,vi)$_{<L+1}$ hold. We show (v,vi)$_{L+1}$.

Sublemma 5.10. *For $l \leqslant L+1$ and $n \geqslant 2l+r-2$, let H be a finite subgroup of $\mathrm{GL}_{n+1}(\mathbb{Z})$. Then the conjugate action of H on the image of*

$$H_l(E_n(\mathbf{A})) \to H_l(E_{n+1}(\mathbf{A}))$$

is pro trivial.

Proof. The case $l = 0, 1$ is clear. Suppose that $2 \leqslant l \leqslant L+1$ and $n \geqslant 2l+r-2$.

Note that $\mathrm{GL}_{n+1}(\mathbb{Z})$ is generated by $e_{i,n+1}(1)$, $e_{n+1,i}(1)$, $1 \leqslant i \leqslant n$, and the diagonal matrix $\mathrm{diag}(1,\ldots,1,-1)$. Since H is finite, it suffices to show that each generator acts pro trivially on the image of $H_l(E_n(\mathbf{A})) \to H_l(E_{n+1}(\mathbf{A}))$. It is obvious that $\mathrm{diag}(1,\ldots,1,-1)$ acts trivially on it.

We show the triviality of the conjugate action of $e_{i,n+1}(1)$; that of $e_{n+1,i}(1)$ is similar. By Corollary 2.3, it suffices to show that the action on the image of

$$H_l(E_n(\mathbf{R},\mathbf{A})) \to H_l(E_{n+1}(\mathbf{R},\mathbf{A}))$$

is pro trivial for some unital pro ring $\mathbf{R}$ which contains $\mathbf{A}$ as a two-sided ideal. The inclusion $E_n(\mathbf{R},\mathbf{A}) \hookrightarrow E_{n+1}(\mathbf{R},\mathbf{A})$ factors through

$$\tilde{E}_n(\mathbf{R},\mathbf{A}) := \begin{pmatrix} E_n(\mathbf{R},\mathbf{A}) & * \\ 0 & 1 \end{pmatrix} \subset E_{n+1}(\mathbf{R},\mathbf{A})$$

and it is normalized by $e_{i,n+1}(1)$. Hence, we are reduced to showing that $e_{i,n+1}(1)$ acts pro trivially on the image of $H_l(E_n(\mathbf{R},\mathbf{A})) \to H_l(\tilde{E}_n(\mathbf{R},\mathbf{A}))$. Now, we have a commutative diagram

$$\begin{tikzcd} H_l(\tilde{E}_n(\mathbf{R},\mathbf{A})) \arrow[r, "e_{i,n+1}(1)"] & H_l(\tilde{E}_n(\mathbf{R},\mathbf{A})) \arrow[d, two heads] \\ H_l(E_n(\mathbf{R},\mathbf{A})) \arrow[u, hook] \arrow[r, "\mathrm{id}"] & H_l(E_n(\mathbf{R},\mathbf{A})), \end{tikzcd}$$

and the vertical maps, the canonical inclusion and projection, are pro isomorphisms by $(\mathrm{iv})_{\leqslant L+1}$. This implies the desired pro triviality of the action of $e_{i,n+1}(1)$. □

We consider the hyperhomology spectral sequence

$$E^1_{p,q}(\mathbf{A}) = H_q(E_{n+1}(\mathbf{A}), C_p(\mathsf{SUni}_{\mathbf{A},n+1})) \Rightarrow H_{p+q}(E_{n+1}(\mathbf{A}), C_\bullet(\mathsf{SUni}_{\mathbf{A},n+1})).$$

Note that $C_p(\mathsf{SUni}_{\mathbf{A},n+1})$ decomposes into a direct sum of $E_{n+1}(\mathbf{A})$-submodules $C_p(\mathsf{SUni}^I_{\mathbf{A},n+1})$ with $|I| = p+1$, and that we have a levelwise isomorphism

$$\mathbb{Z}E_{n+1}(\mathbf{A}) \otimes_{\mathbb{Z}E_{n+1}(\mathbf{A})^I} \mathbb{Z} \xrightarrow{\sim} C_p(\mathsf{SUni}^I_{\mathbf{A},n+1}),$$

which sends $\alpha \in E_{n+1}(\mathbf{A})$ to the unimodular function $i \mapsto e_i\alpha$, $i \in I$. Hence,

$$\coprod_{|I|=p+1} H_q(E_{n+1}(\mathbf{A})^I) \simeq E^1_{p,q}(\mathbf{A}).$$

Let Δ^n be the nerve of the partially ordered set $\{1 < 2 < \cdots < n+1\}$. We define level maps $E_{n-p}(\mathbf{A}) \to E_{n+1}(\mathbf{A})^I$ by sending α to $\sigma_I \left(\begin{smallmatrix} \alpha & 0 \\ 0 & 1_{p+1} \end{smallmatrix}\right) \sigma_I^{-1}$, where σ_I is the shuffle permutation $\sigma_I\{n-p+1,\ldots,n+1\} = I$. These maps yield

$$\Psi\colon \Delta^n_p \otimes H_q(E_{n-p}(\mathbf{A})) \simeq \coprod_{|I|=p+1} H_q(E_{n-p}(\mathbf{A})) \to \coprod_{|I|=p+1} H_q(E_{n+1}(\mathbf{A})^I) \simeq E^1_{p,q}(\mathbf{A}).$$

It follows from Theorem 2.6 and $(\mathrm{iv})_{\leqslant L+1}$ that Ψ is a pro isomorphism for $q \leqslant L+1$ and $n-p \geqslant \max(2q+r-2, r+1)$. Furthermore, by Sublemma 5.10 (with $H = \Sigma_{n+1}$), we see that the diagram

$$\begin{tikzcd} \Delta^n_{p+1} \otimes H_q(E_{n-p-1}(\mathbf{A})) \arrow[r, "\Psi"] \arrow[d, "\sum_{k=0}^{p+1}(-1)^k d_k \otimes \delta"'] & E^1_{p+1,q}(\mathbf{A}) \arrow[d, "d^1"] \\ \Delta^n_p \otimes H_q(E_{n-p}(\mathbf{A})) \arrow[r, "\Psi"] & E^1_{p,q}(\mathbf{A}) \end{tikzcd}$$

commutes for $q \leqslant L+1$ and $n-p \geqslant 2q+r-1$, where d_k are the face maps of Δ^n and δ is the canonical map $H_q(E_{n-p-1}(\mathbf{A})) \to H_q(E_{n-p}(\mathbf{A}))$.

Claim 5.11. *For $q \leqslant L$ and $0 < p \leqslant n-2q-r+1$,*

$$E^2_{p,q}(\mathbf{A}) = 0.$$

Proof. Let $q \leqslant L$ and $0 < p \leqslant n-2q-r+1$. We set $F_{p,q}(\mathbf{A}) := \Delta^n_p \otimes H_q(E_{n-p}(\mathbf{A}))$, which we regard as a complex in p with differential $\sum_{k=0}^{p+1}(-1)^k d_k \otimes \delta$. Then it follows from $(\mathrm{vi})_{<L+1}$ that

$$\ker(F_{p,q}(\mathbf{A}) \to F_{p-1,q}(\mathbf{A})) \simeq \ker(\mathbb{Z}\Delta^n_p \to \mathbb{Z}\Delta^n_{p-1}) \otimes H_q(E_{n-p}(\mathbf{A})).$$

Again by $(\mathrm{vi})_{<L+1}$, the canonical map

$$H_q(E_{n-p-1}(\mathbf{A})) \to H_q(E_{n-p}(\mathbf{A}))$$

is a pro epimorphism. Since Δ^n is contractible, we conclude that $H_p(F_{\bullet,q}(\mathbf{A})) = 0$.

Now, we have a pro isomorphism

$$E^2_{p,q}(\mathbf{A}) \simeq H_p(F_{\bullet,q}(\mathbf{A}))$$

for $n-p-1 \geqslant r+1$. Our assumption says $n-p-1 \geqslant 2q+r-2$; hence, in the case $2q+r-2 \geqslant r+1$, the vanishing of $E^2_{p,q}(\mathbf{A})$ follows from that of $H_p(F_{\bullet,q}(\mathbf{A}))$.

It remains to show the case $q=1$. However, in this case,

$$E^1_{p,1}(\mathbf{A}) \underset{\Psi}{\xrightarrow{\sim}} \Delta^n \otimes H_1(E_{n-p}(\mathbf{A})) = 0.$$

This finishes the proof of the claim. □

Suppose that $n \geqslant 2L+r$. Then the E^2-terms with $p+q = L+1$ are zero unless $E^2_{0,L+1}(\mathbf{A})$. Hence, the edge map

$$E^1_{0,L+1}(\mathbf{A}) \to E^\infty_{L+1}(\mathbf{A})$$

is a pro epimorphism. The left-hand side is pro isomorphic to $\Delta^n_0 \otimes H_{L+1}(E_n(\mathbf{A}))$ by the map Ψ. According to Corollary 5.5, $\tilde{H}_i(C_*(\mathsf{SUni}_{\mathbf{A},n+1})) = 0$ for $n \geqslant i+r$. Hence, we have a pro isomorphism

$$E^\infty_{L+1}(\mathbf{A}) = H_{L+1}(E_{n+1}(\mathbf{A}), C_\bullet(\mathsf{SUni}_{\mathbf{A},n+1})) \simeq H_{L+1}(E_{n+1}(\mathbf{A})).$$

By Sublemma 5.10, we see that the edge map

$$\Delta^n_0 \otimes H_{L+1}(E_n(\mathbf{A})) \to H_{L+1}(E_{n+1}(\mathbf{A}))$$

agrees with a sum of copies of the canonical map $\delta\colon H_{L+1}(E_n(\mathbf{A})) \to H_{L+1}(E_{n+1}(\mathbf{A}))$ as a pro morphism. Hence, δ is a pro epimorphism. This proves the first half of $(\mathrm{vi})_{L+1}$.

Next, suppose that $n \geqslant 2L+r+1$. Then we have $E^s_{s,L-s+2}(\mathbf{A}) = 0$ for $s \geqslant 2$ by Claim 5.11. Hence, we have an exact sequence

$$\Delta^n_1 \otimes H_{L+1}(E_{n-1}(\mathbf{A})) \longrightarrow \Delta^n_0 \otimes H_{L+1}(E_n(\mathbf{A})) \longrightarrow H_{L+1}(E_{n+1}(\mathbf{A})) \longrightarrow 0.$$

Since $H_{L+1}(E_{n-1}(\mathbf{A})) \to H_{L+1}(E_n(\mathbf{A}))$ is a pro epimorphism, we conclude that the canonical map

$$H_{L+1}(E_n(\mathbf{A})) \xrightarrow{\sim} H_{L+1}(E_{n+1}(\mathbf{A}))$$

is a pro isomorphism. This proves the second half of $(\mathrm{vi})_{L+1}$.

Finally, since $H_{L+1}(E_{n-1}(\mathbf{A})) \to H_{L+1}(E_n(\mathbf{A}))$ is a pro epimorphism, the action of Σ_n on $H_{L+1}(E_n(\mathbf{A}))$ is pro trivial by Sublemma 5.10. This proves $(\mathrm{v})_{L+1}$

5.7. Step 4: E to V

Suppose that $(\mathrm{i, ii})_{<L-1}$, $(\mathrm{iii})_{\leqslant L-1}$ and $(\mathrm{vi})_{\leqslant L+1}$ hold. We show $(\mathrm{i, ii})_{L-1}$.

Let $n \geqslant 2L+r$. Consider the spectral sequences (5.1) and the canonical map between them;

$$\begin{array}{ccc} {}^{n}E^2_{p,q}(\mathbf{A}) = H_p(E_n(\mathbf{A}), H_q(V_n(\mathbf{A}))) & \Longrightarrow & H_{p+q}\big(\bigcup_{\sigma\in\Pi_n} BT^\sigma(\mathbf{A})\big) \\ \downarrow & & \\ {}^{n+1}E^2_{p,q}(\mathbf{A}) = H_p(E_{n+1}(\mathbf{A}), H_q(V_{n+1}(\mathbf{A}))) & \Rightarrow & H_{p+q}\big(\bigcup_{\sigma\in\Pi_{n+1}} BT^\sigma(\mathbf{A})\big). \end{array}$$

By $(\mathrm{iii})_{\leqslant L-1}$, for $q \leqslant L-1$, the E^2-terms fit into the extensions

$$\begin{tikzcd} 0 \arrow[r] & H_p(E_n(\mathbf{A})) \otimes H_q(V_n(\mathbf{A})) \arrow[r] \arrow[d] & {}^{n}E^2_{p,q}(\mathbf{A}) \arrow[r] \arrow[d] & \mathrm{Tor}(H_{p-1}(E_n(\mathbf{A})), H_q(V_n(\mathbf{A}))) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & H_p(E_{n+1}(\mathbf{A})) \otimes H_q(V_{n+1}(\mathbf{A})) \arrow[r] & {}^{n+1}E^2_{p,q}(\mathbf{A}) \arrow[r] & \mathrm{Tor}(H_{p-1}(E_{n+1}(\mathbf{A})), H_q(V_{n+1}(\mathbf{A}))) \arrow[r] & 0. \end{tikzcd}$$

Hence, it follows from $(\mathrm{i})_{<L-1}$ and $(\mathrm{vi})_{\leqslant L+1}$ that the map

$${}^{n}E^2_{p,q}(\mathbf{A}) \to {}^{n+1}E^2_{p,q}(\mathbf{A})$$

is a pro epimorphism for $q < L-1$ and $p \leqslant L+1$, and it is a pro isomorphism if further $n \geqslant 2p+r-1$. Finally, by Theorem 4.9, the canonical map ${}^{n}E^\infty_i(\mathbf{A}) \to {}^{n+1}E^\infty_i(\mathbf{A})$ is a pro isomorphism for $n \geqslant 2i+1$.

Bringing these together, we have:

(1) ${}^{n}E^2_{p,q}(\mathbf{A}) \simeq {}^{n+1}E^2_{p,q}(\mathbf{A})$ for $p+q = L-1$ and $p \geqslant 1$.

(2) ${}^{n}E^2_{p,q}(\mathbf{A}) \simeq {}^{n+1}E^2_{p,q}(\mathbf{A})$ for $p+q = L$ and $p \geqslant 2$.

(3) ${}^{n}E^2_{p,q}(\mathbf{A}) \twoheadrightarrow {}^{n+1}E^2_{p,q}(\mathbf{A})$ for $p+q = L+1$ and $p \geqslant 3$.

(4) ${}^{n}E^\infty_{L-1}(\mathbf{A}) \simeq {}^{n+1}E^\infty_{L-1}(\mathbf{A})$ and ${}^{n}E^\infty_{L}(\mathbf{A}) \simeq {}^{n+1}E^\infty_{L}(\mathbf{A})$.

Then, by Lemma 5.12 below, we conclude that the canonical map

$${}^{n}E^2_{0,L-1}(\mathbf{A}) \xrightarrow{\sim} {}^{n+1}E^2_{0,L-1}(\mathbf{A})$$

is a pro isomorphism. By $(\mathrm{iii})_{\leqslant L-1}$, the left-hand side (resp. right hand side) is pro isomorphic to $H_{L-1}(V_n(\mathbf{A}))$ (resp. $H_{L-1}(V_{n+1}(\mathbf{A}))$). Hence, we get the second part of $(\mathrm{i})_{L-1}$.

Next, we show (ii)$_{L-1}$. Now, the canonical map

$$H_{L-1}(V_n(\mathbf{A})) \xrightarrow{\sim} H_{L-1}(V_{n+2}(\mathbf{A}))$$

is a Σ_n-equivariant pro isomorphism. Hence, it suffices to show that Σ_{n+2} (and thus Σ_n) acts pro trivially on $H_{L-1}(V_{n+2}(\mathbf{A}))$. Now, the permutation $\tau_{n+1,n+2}$ acts pro trivially on $H_{L-1}(V_{n+2}(\mathbf{A}))$, since it acts trivially on the image of the above map. Since Σ_{n+2} is the normal closure of $\tau_{n+1,n+2}$, Σ_{n+2} also acts pro trivially on $H_{L-1}(V_{n+2}(\mathbf{A}))$.

In Step 1, we have seen that the map (5.4)

$$C_0(\mathsf{SUni}_{\mathbf{A},n}) \otimes H_{L-1}(V_{n-1}(\mathbf{A})) \to H_{L-1}(V_n(\mathbf{A}))$$

sending $f \otimes u \mapsto \sigma_{\{i\}}(\delta u)\sigma_{\{i\}}^{-1}$ ($\operatorname{dom} f = \{i\}$) is a pro epimorphism for $n \geqslant 2L + r$. Now, we know that $\sigma_{\{i\}}(\delta u)\sigma_{\{i\}}^{-1} = \delta u$. Hence, $\delta\colon H_{L-1}(V_{n-1}(\mathbf{A})) \to H_{L-1}(V_n(\mathbf{A}))$ is a pro epimorphism. This completes the proof of (i)$_{L-1}$.

Lemma 5.12 ([Su96, Theorem A.6])**.** *Let $\mathcal{A}$ be an abelian category. Let $f\colon E \to \tilde{E}$ be a morphism of first quadrant homological spectral sequences in $\mathcal{A}$, and let $L > 0$. Assume that f induces:*

(1) *A monomorphism $E^2_{p,q} \hookrightarrow \tilde{E}^2_{p,q}$ for $p + q = L - 1$, $p \geqslant 1$.*
(2) *An isomorphism $E^2_{p,q} \xrightarrow{\sim} \tilde{E}^2_{p,q}$ for $p + q = L$, $p \geqslant 2$.*
(3) *An epimorphism $E^2_{p,q} \twoheadrightarrow \tilde{E}^2_{p,q}$ for $p + q = L + 1$, $p \geqslant 3$.*
(4) *Isomorphisms $E^\infty_{L-1} \xrightarrow{\sim} \tilde{E}^\infty_{L-1}$ and $E^\infty_L \xrightarrow{\sim} \tilde{E}^\infty_L$.*

Then f induces an isomorphism

$$E^2_{0,L-1} \xrightarrow{\sim} \tilde{E}^2_{0,L-1}.$$

5.8. Homology pro stability for GL_n

Now, we prove our main theorem.

Theorem 5.13. *Let $\mathbf{A}$ be a commutative Tor-unital pro ring. Let $r = \max(\mathrm{sr}(\mathbf{A}), 2)$ and $l \geqslant 0$. Then the canonical map*

$$H_l(\mathrm{GL}_n(\mathbf{A})) \to H_l(\mathrm{GL}_{n+1}(\mathbf{A}))$$

is a pro epimorphism for $n \geqslant 2l + r - 2$ and a pro isomorphism for $n \geqslant 2l + r - 1$.

Proof. The case $l = 0$ is clear. The case $l = 1$ is proved in Theorem 2.5. Let $l \geqslant 2$ and $n \geqslant 2l + r - 2$. Then, by Theorem 2.5 and Corollary 2.3, the sequence

$$0 \longrightarrow E_n(\mathbf{A}) \longrightarrow \mathrm{GL}_n(\mathbf{A}) \longrightarrow H_1(\mathrm{GL}(\mathbf{A})) \longrightarrow 0$$

is exact up to pro isomorphisms. Now, we have a morphism of spectral sequences:

$$\begin{array}{ccc} {}^nE^2_{p,q} = H_p(H_1(\mathrm{GL}(\mathbf{A})), H_q(E_n(\mathbf{A}))) & \Longrightarrow & H_{p+q}(\mathrm{GL}_n(\mathbf{A})) \\ \downarrow & & \downarrow \\ {}^{n+1}E^2_{p,q} = H_p(H_1(\mathrm{GL}(\mathbf{A})), H_q(E_{n+1}(\mathbf{A}))) & \Rightarrow & H_{p+q}(\mathrm{GL}_{n+1}(\mathbf{A})). \end{array}$$

Using these spectral sequences, we deduce the theorem from Theorem 5.6 (vi). $\square$

Corollary 5.14. *Let* $\mathbf{B}$ *be a pro ring with a two-sided ideal* $\mathbf{A}$ *and* $r = \max(\mathrm{sr}(\mathbf{A}), 2)$*. Assume that* $\mathbf{A}$ *is commutative and Tor-unital. Then the conjugate action of* $\mathrm{GL}_n(\mathbf{B})$ *on* $H_l(\mathrm{GL}_n(\mathbf{A}))$ *is pro trivial for* $n \geqslant 2l + r - 1$*.*

Proof. Let α (resp. β) be the map $\mathrm{GL}_n \to \mathrm{GL}_{2n}$ given by

$$g \mapsto \begin{pmatrix} g & 0 \\ 0 & 1_n \end{pmatrix} \quad \text{resp. } g \mapsto \begin{pmatrix} 1_n & 0 \\ 0 & g \end{pmatrix}.$$

According to Theorem 5.13, the induced maps

$$\alpha, \beta \colon H_l(\mathrm{GL}_n(\mathbf{A})) \xrightarrow{\sim} H_l(\mathrm{GL}_{2n}(\mathbf{A}))$$

are pro isomorphisms for $n \geqslant 2l + r - 1$.

Write $\mathbf{B} = \{B_m\}_{m \in J}$ and $\mathbf{A} = \{A_m\}_{m \in J}$. For each $m \in J$, choose $s(m) \geqslant m$ such that if $\alpha(a) = 0$ with $a \in H_l(\mathrm{GL}_n(A_{s(m)}))$ then $\iota_{s(m),m}(a) = 0$. Next, choose $t(m) \geqslant s(m)$ such that for every $x \in H_l(\mathrm{GL}_n(A_{t(m)}))$ there exists $y \in H_l(\mathrm{GL}_n(A_{s(m)}))$ with $\iota_{t(m),s(m)}(\alpha(x)) = \beta(y)$. Then, for $g \in \mathrm{GL}_n(B_{t(m)})$ and $x \in H_l(\mathrm{GL}_n(A_{t(m)}))$,

$$\alpha(\iota_{t(m),s(m)}(gx)) = \alpha(\iota_{t(m),s(m)}(g))\beta(y)\beta(y) = \alpha(\iota_{t(m),s(m)}(x)).$$

Hence, $\iota_{t(m),m}(gx) = \iota_{t(m),m}(x)$. This completes the proof. $\square$

Suslin has shown that if a ring A is Tor-unital then, for every ring B which contains A as a two-sided ideal, the conjugate action of $\mathrm{GL}(B)$ on $H_l(\mathrm{GL}(A))$ is trivial, cf. [**Su95**, Corollary 4.5], see also [**SW92**, Corollary 1.6]. Geisser and Hesselholt generalized Suslin's result to a pro setting, cf. [**GH06**, Proposition 1.3]. They stated the result only for pro rings of the form $\{A^m\}$ for some ring A, but their proof works more generally to give the following.

Theorem 5.15 (Suslin, Geisser–Hesselholt)**.** *Let* $\mathbf{B}$ *be a pro ring and* $\mathbf{A}$ *a two-sided ideal of* $\mathbf{B}$*. Assume that* $\mathbf{A}$ *is Tor-unital. Then the conjugate action of* $\mathrm{GL}(\mathbf{B})$ *on* $H_l(\mathrm{GL}(\mathbf{A}))$ *is pro trivial for all* $l \geqslant 0$*.*

By using Theorem 5.15, we can strengthen Theorem 5.13.

Theorem 5.16. *Let* $\mathbf{A}$ *be a commutative Tor-unital pro ring,* $r = \max(\mathrm{sr}(\mathbf{A}), 2)$ *and* $l \geqslant 0$*. Suppose that there exists a unital pro ring* $\mathbf{R}$ *with* $\mathrm{sr}(\mathbf{R}) < \infty$ *which contains* $\mathbf{A}$ *as a two-sided ideal. Then the canonical map*

$$H_l(\mathrm{GL}_n(\mathbf{A})) \to H_l(\mathrm{GL}(\mathbf{A}))$$

is a pro epimorphism for $n \geqslant 2l + r - 2$ *and a pro isomorphism for* $n \geqslant 2l + r - 1$*.*

Proof. Let $\mathbf{R}$ be a unital pro ring as in the statement. Consider the commutative diagram

$$\begin{tikzcd} \mathrm{GL}_n(\mathbf{A}) \arrow[r] \arrow[d] & \mathrm{GL}_n(\mathbf{R}) \arrow[r] \arrow[d] & \overline{\mathrm{GL}}_n(\mathbf{R}/\mathbf{A}) \arrow[d] \\ \mathrm{GL}(\mathbf{A}) \arrow[r] & \mathrm{GL}(\mathbf{R}) \arrow[r] & \overline{\mathrm{GL}}(\mathbf{R}/\mathbf{A}) \end{tikzcd}$$

with exact rows. Now, the second and third maps induce isomorphisms on homology for n large enough. Also, the action of $\mathrm{GL}_n(\mathbf{R})$ on $H_l(\mathrm{GL}_n(\mathbf{A}))$ is pro trivial for

n large enough (Theorem 5.14) and for $n = \infty$ (Theorem 5.15). Consequently, the canonical map

$$H_l(\mathrm{GL}_n(\mathbf{A})) \xrightarrow{\sim} H_l(\mathrm{GL}(\mathbf{A}))$$

is a pro isomorphism for n large enough. Combining it with Theorem 5.13, we get the result. □

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